

The automorphic NS5-brane

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Understanding the implications of $SL(2, \mathbb{Z})$ S-duality for the hypermultiplet moduli space of type II string theories has led to much progress recently in uncovering D-instanton contributions. In this work, we suggest that the extended duality group $SL(3, \mathbb{Z})$, which includes both S-duality and Ehlers symmetry, may determine the contributions of D5 and NS5-branes. We support this claim by automorphizing the perturbative corrections to the “extended universal hypermultiplet,” a five-dimensional universal $SO(3) \backslash SL(3, \mathbb{R})$ subspace which includes the string coupling, overall volume, Ramond zero-form and six-form and NS axion. Using the non-Abelian Fourier expansion of the Eisenstein series attached to the principal series of $SL(3, \mathbb{R})$, worked out many years ago by Vinogradov, Takhtajan and Bump, we extract the contributions of D(-1)-D5 and NS5-brane instantons, corresponding to the Abelian and non-Abelian coefficients, respectively. In particular, the contributions of k NS5-branes can be summarized into a vector of wave functions $\Psi_{k, \ell}$, $\ell = 0, \dots, k - 1$, as expected on general grounds. We also point out that for more general models with a symmetric moduli space $K \backslash G$, the minimal theta series of G generates an infinite series of exponential corrections of the form required for “small” D(-1)-D1-D3-D5-NS5 instanton bound states. As a mathematical spin-off, we make contact with earlier results in the literature about the spherical vectors for the principal series of $SL(3, \mathbb{R})$ and for minimal representations.

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1. Introduction

Understanding quantum corrections to hypermultiplet (HM) moduli spaces in type II Calabi-Yau (CY) compactifications is a long and out-standing problem. One of the main challenges is to include the contributions from NS5-brane instantons which give rise to exponentially suppressed corrections to the moduli space metric of order e^{-1/g_s^2} in the weak-coupling limit. In this work, we propose that these NS5-brane instanton effects can be summed up in terms of a certain $SL(3, \mathbb{Z})$ -invariant Eisenstein series. To motivate this proposal, we begin by discussing some relevant aspects of HM moduli spaces in type II CY compactifications, with particular emphasis on generalized mirror symmetry and dualities.

1.1. Generalized mirror symmetry and quaternion-Kähler geometry

Type II string theory and M-theory canonically associate two quaternion-Kähler (QK) spaces to any CY three-fold X [1],

$$(1.1) \quad \mathcal{M}_C(X) \times \mathcal{M}_K(X),$$

of real dimension $4h + 4$ where $h = h^{1,2}(X)$ and $h = h^{1,1}(X)$, respectively. Both of these spaces $\mathcal{M} = \mathcal{M}_{C,K}(X)$ have a foliation by hypersurfaces $\phi = \text{const}$, such that near $\phi = +\infty$, \mathcal{M} is topologically a fibration

$$(1.2) \quad \tilde{T} \rightarrow \mathcal{M} \rightarrow \mathbb{R}_\phi \times \mathcal{K},$$

where the $2h + 3$ -dimensional fiber \tilde{T} is a circle bundle over a $2h + 2$ -dimensional torus T , and $\mathcal{K} = \mathcal{K}_{C,K}$ is the moduli space of complex structures (respectively, complexified Kähler structures) on X . Given any special Kähler metric on \mathcal{K} , the “c-map” construction [2], or rather its “quantum corrected version” studied in [3, 4], produces a QK metric on this fiber bundle, which agrees with the metric on \mathcal{M} in the “weak coupling” limit $\phi \rightarrow +\infty$, up to exponentially suppressed corrections of order $\exp(-\mathcal{O}(e^{\phi/2}))$ and smaller. An outstanding question is to compute these corrections, which should encode interesting symplectic (resp. algebraic) invariants of X , and provide valuable information about the spectrum of type II string theories on X .

Indeed, the QK spaces $\mathcal{M}_{C,K}(X, Y)$ and associated special Kähler spaces $\mathcal{K}_{C,K}$ appear as the moduli spaces of type IIA and type IIB string theories

compactified on X and $X \times S^1$ (respectively on Y and $Y \times S^1$) as summarized in the table below [1]:

	IIA/ X			IIB/ Y		
$\mathbb{R}^{3,1}$	$\mathcal{K}_{\mathbb{K}}(X)$	\times	$\mathcal{M}_{\mathbb{C}}(X)$	$\mathcal{K}_{\mathbb{C}}(Y)$	\times	$\mathcal{M}_{\mathbb{K}}(Y)$
	\downarrow		\parallel	\downarrow		\parallel
$\mathbb{R}^{2,1} \times S^1$	$\mathcal{M}_{\mathbb{K}}(X)$	\times	$\mathcal{M}_{\mathbb{C}}(X)$	$\mathcal{M}_{\mathbb{C}}(Y)$	\times	$\mathcal{M}_{\mathbb{K}}(Y)$

This table calls for several important comments:

- (i) The vertical arrows indicate the c -map relating the vector multiplet (VM) moduli space \mathcal{K} in $D = 4$ uncompact dimensions to the VM moduli space \mathcal{M} in $D = 3$. In this case, the coordinate ϕ and the twisted torus \tilde{T} in (1.2) correspond to the radius $e^{\phi/2} = R/l_P$ of the circle in 4D Planck units, the Wilson lines $\zeta^\Lambda, \tilde{\zeta}_\Lambda$ of the electric and magnetic vector fields in $D = 4$ along S^1 , and the NUT scalar σ dual to the off-diagonal metric component [2]. The circle $\tilde{T}/T = S^1$ parameterized by σ has first Chern class $c_1 = d\zeta^\Lambda \wedge d\tilde{\zeta}_\Lambda + \chi\omega_{\mathcal{K}}$, where $\omega_{\mathcal{K}}$ is the Kähler class of \mathcal{K} and $\chi = \chi_X$ is the Euler number of X in type IIA, or $\chi = -\chi_Y$ in type IIB [3].
- (ii) The vertical equal signs indicate that the HM moduli space is identical in three and four dimensions; as a matter of fact, on the type IIA side $\mathcal{M}_{\mathbb{C}}(X)$ is also the HM moduli space in M-theory on $\mathbb{R}^{4,1} \times X$, since the radius of the M-theory circle is a VM. The coordinate ϕ and the twisted torus \tilde{T} in (1.2) now correspond to the four-dimensional string coupling $e^{-\phi/2}$, the Wilson lines of the Ramond–Ramond (RR) gauge fields on $H_3(X, \mathbb{Z})$ (respectively $H_{\text{even}}(Y, \mathbb{Z})$), and the Neveu–Schwarz (NS) axion.
- (iii) T-duality along the circle maps IIA/ $X \times S^1$ to IIB on the same CY three-fold $Y \equiv X$ times S^1 and exchanges the VM and HM moduli spaces in $D = 3$ (in particular the radius R/l_P is mapped to the string coupling $e^{\phi/2}$). This justifies the use of the same notation $\mathcal{M}_{\mathbb{C},\mathbb{K}}(\cdot)$ on the IIA and IIB sides.
- (iv) Mirror symmetry identifies IIA/ $X \times S^1$ to IIB/ $\tilde{X} \times S^1$ where $Y \equiv \tilde{X}$ is the mirror CY three-fold to X (in particular $h^{1,2}(X) = h^{1,1}(Y)$, $\chi_X = -\chi_Y$). At the level of the (2,2) SCFT it amounts to the well-supported identity $\mathcal{K}_{\mathbb{C}}(X) = \mathcal{K}_{\mathbb{K}}(\tilde{X})$, but at the “second quantized”

level it requires the more far-reaching identity [5]

$$(1.3) \quad \mathcal{M}_C(X) = \mathcal{M}_K(\tilde{X}), \quad \mathcal{M}_K(X) = \mathcal{M}_C(\tilde{X}).$$

- (v) The c -map construction mentioned above is accurate only in the limit $\phi \rightarrow +\infty$, which corresponds to large radius on the VM side, or small coupling on the HM side. $e^{-\mathcal{O}(e^{\phi/2})}$ corrections away from this limit on the HM side correspond to D-brane instantons already in $D = 4$ [5], while on the VM side they correspond to Euclidean black holes in $\mathbb{R}^3 \times S^1$, whose worldline winds around the circle. In either case, D-instantons correspond mathematically to elements of the derived category $\mathcal{D}(X)$ of coherent sheaves for $\mathcal{M}_K(X)$, or elements in the derived Fukaya category $\mathcal{F}(Y)$ of SLAG submanifolds for $\mathcal{M}_C(Y)$ (see, e.g., [6] for an introduction to these concepts). Thus, equality (1.3) encompasses the homological mirror symmetry conjecture [7].
- (vi) In addition to the $e^{-\mathcal{O}(e^{\phi/2})}$ D-instanton corrections mentioned in (v), one also expects $e^{-\mathcal{O}(e^\phi)}$ corrections, corresponding to Euclidean NS5-brane wrapped on X on the HM side, or to Kaluza–Klein (KK) monopoles (equivalently, Taub–NUT gravitational instantons) with non-zero NUT charge along the circle. These effects are predicted by the presumed growth of the D-instanton series [8], but the mathematical structure underlying them is far from clear at the moment. They are the main subject of this note.
- (vii) Since $\text{IIA}/X \times S^1 = M/X \times T^2$, the VM moduli space $\mathcal{M}_K(X)$ must¹ possess an isometric action of the modular group $\text{SL}(2, \mathbb{Z})$ of the torus T^2 . Equivalently, S-duality of IIB string theory in 10 dimensions implies that $\mathcal{M}_K(Y)$ must have an (identical) isometric action of the modular group $\text{SL}(2, \mathbb{Z})$.

This last observation has been instrumental in the recent progress in understanding $\mathcal{M}_K(X)$ [10–15]: while the metric on $\mathcal{M}_K(Y)$ in the limit of weak coupling and large volume has a continuous isometric action of $\text{SL}(2, \mathbb{R})$ [14, 16, 17], this isometric action is broken by the usual worldsheet instanton corrections to $\mathcal{K}_K(Y)$ and by a “universal” one-loop correction [3,

¹Here, we restrict to cases where, unlike the situation in [9], $\text{SL}(2, \mathbb{Z})$ electric–magnetic duality is not broken to a finite index subgroup by quantum corrections. We are grateful to N. Halmagyi for emphasizing this issue.

[17, 18] proportional to χ_Y . By restoring invariance under a discrete subgroup $\mathrm{SL}(2, \mathbb{Z})$, it is possible to determine the corrections to the QK metric on $\mathcal{M}_{\mathrm{K}}(Y)$ due to D(-1) and D-1 instantons, i.e., to coherent sheaves with support on rational curves in Y [10]. Not surprisingly, these corrections are controlled by the same BPS invariants which determine the tree-level worldsheet instanton corrections. By mirror symmetry, these D(-1)–D1 instantons map to D2-brane instantons wrapping SLAG submanifolds γ in X whose homology class lies in a certain Lagrangian subspace of $H_3(X, \mathbb{Z})$ determined by the large complex structure limit. Using symplectic invariance, the effects of D2-branes wrapping any homology class in $H_3(X, \mathbb{Z})$ were found in [14, 15], to linear order in perturbation around the weak-coupling metric; by mirror symmetry this gives the instanton corrections to $\mathcal{M}_{\mathrm{K}}(Y)$ from arbitrary D5–D3–D1–D(-1) instantons (or from D6–D4–D2–D0 black holes to $\mathcal{M}_{\mathrm{K}}(X)$), or mathematically from any element of the derived category $\mathcal{D}(Y)$ (respectively, $\mathcal{D}(X)$).

A key device in computing instanton corrections to the QK metric on $\mathcal{M} = \mathcal{M}_{\mathrm{K}}(Y)$ is the Lebrun–Salamon theorem [19–21], which relates the QK metric on \mathcal{M} to the complex contact structure on its twistor space \mathcal{Z} . As a consequence, the deformed geometry can be encoded in terms of complex contact transformations between locally flat Darboux patches (which play a similar role as the holomorphic prepotential for special Kähler spaces). The deformed metric can be obtained from the complex coordinates on \mathcal{Z} , also known as contact twistor lines, and from a complex valued (but non-holomorphic) section $e^{\Phi(x^\mu, z)}$ of $H^0(\mathcal{Z}, \mathcal{O}(2))$, known as the contact potential, which determines the Kähler potential on \mathcal{Z} .

Using these twistorial techniques, the QK metric on $\mathcal{M}_{\mathrm{K}}(Y)$ including quantum corrections from all D-instantons was obtained to linear order in [14, 15]. It involves invariants n_γ , presumed to be equal to the generalized Donaldson–Thomas invariants introduced in [22, 23]. An essentially identical structure has emerged in the study of instanton corrections to the (hyperkähler) moduli space of $\mathcal{N} = 2$ Seiberg–Witten theories on $\mathbb{R}^{1,2} \times S^1$ in [24], and in fact directly inspired the construction in [14].

These developments have left out the outstanding problem of computing the subleading $e^{-\mathcal{O}(e^\phi)}$ corrections to $\mathcal{M}_{\mathrm{C}, \mathrm{K}}$ from NS5-brane instantons wrapped on X (or equivalently from KK-monopoles with non-zero NUT charge on S^1). While the $\mathrm{SL}(2, \mathbb{Z})$ symmetry which proved so powerful in determining the D-instanton corrections could in principle be used to convert the D5-instanton corrections into NS5-branes (or D6-branes into KK-monopoles), it is not immediately clear how to covariantize these contributions under the complicated $\mathrm{SL}(2, \mathbb{Z})$ action found in [14].

1.2. Uncovering $\mathrm{SL}(3, \mathbb{Z})$, and the extended universal HM

In this note, we employ a different strategy, and investigate how invariance under a larger discrete group, $\mathrm{SL}(3, \mathbb{Z})$, may constrain the NS5-brane contributions. This discrete symmetry is most easily seen in M-theory on $\mathbb{R}^{1,2} \times T^2 \times X$: indeed, after reduction on a torus T^d of any dimension d and dualization of the KK connection into scalars, the Einstein–Hilbert Lagrangian in $D = d + 3$ dimensions leads to a $\mathrm{SL}(d + 1, \mathbb{R})$ invariant non-linear sigma model in $D = 3$ dimensions: this $\mathrm{SL}(d + 1, \mathbb{R})$ symmetry includes the manifest $\mathrm{SL}(d + 1, \mathbb{R})$ symmetry from diffeomorphism invariance on T^d , and a multiplet of d (non-commuting) $\mathrm{SL}(2, \mathbb{R})$ Ehlers symmetries [25] apparent in the two-step reduction $D = d + 3 \rightarrow D = 4 \rightarrow D = 3$. The discrete subgroup $\mathrm{SL}(d, \mathbb{Z})$ of global diffeomorphisms of the torus should clearly remain a symmetry of the quantum theory, but it is reasonable to assume that a larger discrete subgroup $\mathrm{SL}(d + 1, \mathbb{Z})$ is in fact unbroken quantum mechanically.² In the case at hand with $d = 2$, we postulate that quantum corrections preserve a $\mathrm{SL}(3, \mathbb{Z})$ subgroup of $\mathrm{SL}(3, \mathbb{R})$, larger than the $\mathrm{SL}(2, \mathbb{Z})$ S-duality warranted by diffeomorphism invariance.

At this stage, we should warn the reader against a possible confusion with another $\mathrm{SL}(3, \mathbb{Z})$ symmetry expected by duality to the heterotic $E_8 \times E_8$ on $\mathbb{R}^{1,2} \times T^3 \times K3$: indeed, when (and only when) X admits a $K3$ -fibration with a section, heterotic-type II duality [28] predicts that

$$(1.4) \quad \mathcal{M}(K3) = \mathcal{M}_C(X), \quad \mathcal{M}(T^3) = \mathcal{M}_K(X),$$

where $\mathcal{M}(K3)$ parametrizes the Ricci-flat metric and $E_8 \times E_8$ bundle on $K3$, while $\mathcal{M}(T^3)$ parametrizes the flat metric and $E_8 \times E_8$ bundle on T^3 and the scalars dual to the KK connections and the $U(1)^{16}$ Abelian gauge fields in three dimensions. Just as on the M-theory side, $\mathcal{M}(T^3)$ has an obvious $\mathrm{SL}(3, \mathbb{Z})$ symmetry, enhanced to $\mathrm{SL}(4, \mathbb{Z})$ by Ehlers-type transformations. The M-theory $\mathrm{SL}(3, \mathbb{Z})$ action on $\mathcal{M}_K(X)$ is part of the heterotic $\mathrm{SL}(4, \mathbb{Z})$ action on $\mathcal{M}(T^3)$, but intersects the geometric $\mathrm{SL}(3, \mathbb{Z})$ action only along the $\mathrm{SL}(2, \mathbb{Z})$ S-duality subgroup.

Before discussing how a discrete $\mathrm{SL}(3, \mathbb{Z})$ symmetry can be preserved by quantum corrections, we must understand how the continuous symmetry

²The fact that the discrete Ehlers symmetry is unbroken quantum mechanically in M-theory on $\mathbb{R}^{1,2} \times T^8$ follows by intertwining the geometric $\mathrm{SL}(8, \mathbb{Z})$ and T-duality $\mathrm{SO}(7, 7, \mathbb{Z})$ symmetries, see, e.g., [26]; in the heterotic string on T^6 , Ehlers symmetry is related to S-duality by a sequence of T-dualities [27]. We are not aware of a similar derivation in the $\mathcal{N} = 2$ setting.

group $\mathrm{SL}(3, \mathbb{R})$ acts on the weak coupling, large volume limit of $\mathcal{M}_{\mathrm{K}}(X)$. We shall argue that in this limit, the moduli space $\mathcal{M}_{\mathrm{K}}(X)$ decomposes as a product

$$(1.5) \quad \mathcal{M}_{\mathrm{K}}(X) \sim \frac{\mathrm{SL}(3, \mathbb{R})}{\mathrm{SO}(3)} \times \mathcal{R}_{\mathrm{K}}(X) \times \mathbb{R}^{3h_{1,1}(X)},$$

where $\mathcal{R}_{\mathrm{K}}(X)$ is a space of real dimension $h_{1,1}(X) - 1$, which appears as the VM moduli space in M-theory on $\mathbb{R}^{4,1} \times X$. $\mathrm{SL}(3, \mathbb{R})$ acts on the first factor in (1.5) by the usual non-linear action, leaves the second factor inert and acts linearly on $(\mathbb{R}^3)^{\otimes h_{1,1}(X)}$.

In particular, we claim that the HM moduli space $\mathcal{M}_{\mathrm{K}}(Y)$ in type IIB string theory compactified on Y admits a universal sector $\mathcal{M}_{\mathrm{u}} = \mathrm{SO}(3) \backslash \mathrm{SL}(3, \mathbb{R})$, of real dimension 5, which consists of the 10-dimensional axio-dilaton τ , the overall volume $V = t^3$ of Y in string units, the Wilson line c_0 of the RR six-form potential on Y and the four-dimensional NS axion ψ . Despite the fact that this universal sector does not carry any QK metric, we refer to it as the “extended universal HM,” to distinguish it from the “universal HM” [18, 29–33], which has real dimension 4 and carries, at tree-level, a $\mathrm{SU}(2, 1)$ invariant QK metric. The latter is universal in the sense that it appears as a subfactor in any “ c -map” construction [2]. However, it is unclear whether a finite covolume discrete subgroup of $\mathrm{SU}(2, 1)$ should stay unbroken in general. However, see [34] for a discussion of this possibility when X is a rigid CY three-fold.

Finally, let us mention that our identification of $\mathrm{SL}(3, \mathbb{Z})$ as the unbroken discrete subgroup of $\mathrm{SL}(3, \mathbb{R})$ is tentative: it is quite possible that only a finite index subgroup of $\mathrm{SL}(3, \mathbb{Z})$ may be unbroken, as it happens with $\mathrm{SL}(2, \mathbb{Z})$ electric–magnetic duality on the VM side. It is rather easy to adapt our considerations to this case, and it may in fact be the key to resolve a shortcoming of our proposal to be discussed presently.

1.3. Summing up NS5-brane instantons

Having postulated that $\mathrm{SL}(3, \mathbb{Z})$ is preserved at the quantum level, we shall demonstrate that this symmetry potentially determines a subset of the NS5-brane corrections, once the tree-level worldsheet instantons and the one-loop correction are given. The adverb “potentially” is in order since our specific proposal (3.24) leads to unexpected terms which blow up at weak coupling

(see (3.33) below). Our approach is very close in spirit to the one taken in [35], where the $SL(3, \mathbb{Z})$ U -duality symmetry of type II string on $\mathbb{R}^{7,1} \times T^2$ was used to determine the contributions of (p, q) strings to R^4 couplings in the effective action. Technically, however, we require the more sophisticated automorphic forms of $SL(3, \mathbb{Z})$ constructed in [36] in the context of BPS membranes.

As mentioned above, quantum corrections to the QK moduli space $\mathcal{M}_K(Y)$ are conveniently encoded in complex coordinates on its twistor space \mathcal{Z} , together with the contact potential $e^{\Phi(x^\mu, z)}$. Taking the conjectured $SL(3, \mathbb{Z})$ -invariance at face value, we shall propose a non-perturbative completion of the contact potential $e^{\Phi(x^\mu, z(x^\mu))}$ restricted to a certain section $z(x^\mu)$ of \mathcal{Z} , in terms of a certain non-holomorphic Eisenstein series $E(g; s_1, s_2)$ attached to the principal continuous series of $SL(3, \mathbb{R})$. Relying on the 30-year old analysis of this Eisenstein series by Vinogradov and Takhtajan [44] and Bump [45], we show that for the special values $(s_1, s_2) = (3/2, -3/2)$ the Fourier expansion of $E(g; s_1, s_2)$ reproduces the correct universal³ tree-level and one-loop corrections to the HM metric. Moreover, the non-Abelian Fourier expansion of $E(g; s_1, s_2)$ predicts an infinite series of exponentially suppressed contributions at weak coupling, of two distinct types:

- (i) the Abelian contributions, of order $e^{-S_{p,q}}$ where $S_{p,q}$ is independent of the NS-axion, given in (3.57) below, can be interpreted as contributions from bound states of p D5 and q D(-1) instantons. In particular, the instanton action $S_{p,q}$ correctly reproduces the mass formula for D0–D6 branes on the type IIA side [37–39]. Via the c -map, the summation measure (3.58) should be related to the D0–D6 bound state degeneracies predicted by the Mac Mahon function [40], but checking this lies beyond the scope of this work.
- (ii) the non-Abelian contributions, of order $e^{-S_{Q,p,k}}$ given in (3.67) below, have a non-trivial dependence on the NS-axion ψ , and can be interpreted as instanton corrections from bound states of Q D(-1) instantons and (p, k) five-branes. Their action (3.67) follows from the D5–D(-1) action (3.57) by S-duality, after subtracting a moduli independent contribution $e^{-2\pi i q d \alpha/k}$ in (3.62). The latter is responsible for

³i.e., depending only on the generalized universal HM moduli, and on the Euler number of Y .

the apparent divergence of (3.67) at $k = 0$. The summation measure (3.69) is obtained from (3.58) by replacing $p \rightarrow d, q \rightarrow Q/d^2$, where $d = \gcd(p, k)$ and multiplying by the phase $e^{2\pi i Q\alpha/(dk)}$. In representation theoretic terms, as explained in Appendix D, this provides the exact real and adelic spherical vectors for the principal continuous series of $\mathrm{SL}(3, \mathbb{R})$ beyond the semi-classical limit obtained in [36].

A general property of the non-Abelian Fourier coefficients, and therefore of the NS5-brane instantons, is that they satisfy a wave function property: namely, the non-Abelian Fourier expansion can be carried out for different choices of polarization, e.g., (3.36) or (3.39), and the corresponding summands $\Psi_{k,\ell}$ and $\tilde{\Psi}_{k,\ell'}$ are related by Fourier transform, Equation (3.40) below. It is tempting to conjecture that the wave function $\Psi_{1,0}$ describing the contribution of one NS5-brane is related to the topological string amplitude, possibly along the lines of [41, 42].

While our main emphasis is on the universal sector, we also speculate on the $\mathrm{SL}(3, \mathbb{Z})$ -invariant completion of the “non-universal” contributions, which include D3- and D1 instantons, and suggest that in the context of “magic” supergravity models with a symmetric HM moduli space (2.25) the minimal theta series associated to $\mathrm{QConf}(J, \mathbb{Z})$ may resum the contributions of “very small instantons,” i.e., those whose charges satisfy $I_4 = \partial I_4 = \partial^2 I_4 = 0$, where I_4 is the quartic invariant for the duality group $\mathrm{Conf}(J)$. In particular, (3.72) should provide the general action for bound states of (p, k) -5 branes, N^a D3-branes, \tilde{N}_a D1-branes and \tilde{Q} D instantons, at least when the D1- and D(-1)-instanton charges are induced from the D3 brane charge via (3.73). Thus, we for the first time provide a physical interpretation of the spherical vector f_K for the minimal representation of any quasiconformal group $\mathrm{QConf}(J, \mathbb{Z})$, which has been known for simply laced groups in the split real form since [46]. From this point of view, the puzzling cubic phase appearing in f_K simply originates from the D-instanton axionic coupling by an $\mathrm{SL}(2, \mathbb{Z})$ transformation, after subtracting out a moduli-independent contribution as in (ii) above.

We now mention some limitations of our proposal. Firstly, in order to obtain the deformed QK metric the contact potential $e^{\Phi(x^\mu, z)}$ should be supplemented by the twistor lines. It would be very interesting to understand how to incorporate $\mathrm{SL}(3, \mathbb{Z})$ -invariance in this context.⁴ In addition, our proposal predicts puzzling perturbative contributions beyond the expected

⁴The $\mathrm{SL}(2, \mathbb{Z})$ action on the instanton-corrected twistor lines has recently been clarified, and is in fact identical to the tree-level action after suitable field redefinitions [43].

tree-level and one-loop terms, which grow like negative genus contributions or diverge faster than linearly at large volume. It is conceivable that these terms could be avoided by postulating invariance under a finite index subgroup of $SL(3, \mathbb{Z})$, or may be attributed to hitherto unknown physical effects. Moreover, our proposal for the $SL(3, \mathbb{Z})$ -invariant completion of non-universal effects is tentative only, and would require a better understanding of the $SL(3, \mathbb{R})$ action on the non-universal sector of the HM moduli space.

1.4. Outline

The rest of this article is organized as follows. In Section 2, we discuss the geometry of the extended universal HM, and work out the decomposition (1.5) in the one-modulus case. In Section 3, we review how the $SL(2, \mathbb{Z})$ symmetry of the HM moduli space $\mathcal{M}_K(X)$ can be restored after including suitable D(-1) and D1-instanton contributions, and show how $SL(3, \mathbb{Z})$ may similarly be restored by including NS5-brane contributions (together with D5 and D(-1)-instantons). Moreover, we identify the NS5-brane contributions as certain non-Abelian Fourier coefficients of the corresponding automorphic form, and comment on their wave function property. In Appendix A, we collect some results on the constant terms of minimal and principal Eisenstein series with respect to certain parabolic subgroups. In Appendix B we give a detailed derivation of the non-Abelian Fourier expansion of the minimal Eisenstein series for $SL(3, \mathbb{Z})$. In Appendix C a certain key integral is computed in the saddle point approximation. Finally, in Appendix D, we give a representation theoretic viewpoint on non-Abelian Fourier expansions, and extract the exact spherical vector for the principal series of $SL(3, \mathbb{R})$.

2. The extended universal HM

In this section, we show that the symmetric space $\mathcal{M}_u = SO(3) \backslash SL(3, \mathbb{R})$ can be viewed as a universal sector of the HM space $\mathcal{M}_K(X)$ in the large volume, weak coupling limit and work out decomposition (1.5) in the one-modulus case.

2.1. $SO(3) \backslash SL(3, \mathbb{R})$ as a HM moduli space

The five-dimensional symmetric space \mathcal{M}_u may be parametrized in the Iwasawa gauge by the coset

$$\begin{aligned}
 (2.1) \quad g &= \left(\nu^{-1/6} \sqrt{\tau_2} \right)^{H_p} \cdot \left(\nu^{-1/3} \right)^{H_q} \cdot e^{\tau_1 E_p} \cdot e^{c_0 E_q} \cdot e^{\psi E} \\
 &= \begin{pmatrix} \nu^{1/6} / \sqrt{\tau_2} & & \\ & \nu^{1/6} \sqrt{\tau_2} & \\ & & \nu^{-1/3} \end{pmatrix} \cdot \begin{pmatrix} 1 & \tau_1 & \psi + \tau_1 c_0 \\ & 1 & c_0 \\ & & 1 \end{pmatrix},
 \end{aligned}$$

where $E_i = \{E, E_p, E_q, H_p, H_q, F_p, F_q, F\}$ form a basis of the Lie algebra of $\mathfrak{sl}(3, \mathbb{R})$, such that any linear combination $\sum \mathcal{E}_i E_i$ with $\mathcal{E}_i \in \mathbb{R}$ is represented in the triplet representation by

$$(2.2) \quad \begin{pmatrix} -\mathcal{H}_p & \mathcal{E}_p & \mathcal{E} \\ -\mathcal{E}_p & -\mathcal{H}_q + \mathcal{H}_p & \mathcal{E}_q \\ -\mathcal{E} & -\mathcal{E}_q & \mathcal{H}_q \end{pmatrix}.$$

The maximal compact subgroup $K = \text{SO}(3)$ is generated by antisymmetric matrices, i.e., by $E_p + F_p, E_q + F_q, E + F$.

The right-invariant metric on \mathcal{M}_u is obtained from the right-invariant form $\theta = dg \cdot g^{-1}$ projected along K via

$$(2.3) \quad ds^2 = \frac{1}{2} \text{Tr}[(\theta + \theta^t)^2] = \frac{d\nu^2}{3\nu^2} + \frac{d\tau_1^2 + d\tau_2^2}{\tau_2^2} + \nu \frac{(d\psi + \tau_1 dc_0)^2 + \tau_2^2 dc_0^2}{\tau_2}.$$

The Killing vectors generating the right action of $\text{SL}(3, \mathbb{R})$ on \mathcal{M}_u are given by

$$\begin{aligned}
 (2.4) \quad E &= \partial_\psi, & E_p &= \partial_{\tau_1} - c_0 \partial_\psi, & E_q &= \partial_{c_0}, \\
 H_p &= 2\tau_1 \partial_{\tau_1} + 2\tau_2 \partial_{\tau_2} + \psi \partial_\psi - c_0 \partial_{c_0}, \\
 H_q &= 2c_0 \partial_{c_0} - 3\nu \partial_\nu + \psi \partial_\psi - \tau_1 \partial_{\tau_1} - \tau_2 \partial_{\tau_2}, \\
 F_p &= -\psi \partial_{c_0} - 2\tau_1 \tau_2 \partial_{\tau_2} + (\tau_2^2 - \tau_1^2) \partial_{\tau_1}, \\
 F_q &= c_0^2 \partial_{c_0} - c_0(3\nu \partial_\nu - \psi \partial_\psi + \tau_1 \partial_{\tau_1} + \tau_2 \partial_{\tau_2}) - \psi \partial_{\tau_1} - (\nu \tau_2)^{-1}(\partial_{c_0} - \tau_1 \partial_\psi), \\
 F &= \psi(\psi \partial_\psi + c_0 \partial_{c_0} - 3\nu \partial_\nu + \tau_1 \partial_{\tau_1} + \tau_2 \partial_{\tau_2}) + c_0[(\tau_1^2 - \tau_2^2) \partial_{\tau_1} + 2\tau_1 \tau_2 \partial_{\tau_2}] \\
 &\quad - (\nu \tau_2)^{-1}[(\tau_1^2 + \tau_2^2) \partial_\psi - \tau_1 \partial_{c_0}].
 \end{aligned}$$

For later reference, we record the Laplace–Beltrami operator on \mathcal{M}_u , equal to the quadratic Casimir of the $\mathfrak{sl}(3, \mathbb{R})$ action (2.4),

$$(2.5) \quad \mathcal{C}_2 = \tau_2^2 (\partial_{\tau_1}^2 + \partial_{\tau_2}^2) + 3\partial_\nu (\nu^2 \partial_\nu) + \frac{1}{\nu \tau_2} (\partial_{c_0} - \tau \partial_\psi) (\partial_{c_0} - \bar{\tau} \partial_\psi).$$

There is also an invariant differential operator \mathcal{C}_3 of third order in derivatives, corresponding to the cubic Casimir given in (A.13).

Parametrization (2.1) was chosen such the $SL(2, \mathbb{R})$ subgroup corresponding to matrices of the form

$$(2.6) \quad \begin{pmatrix} a & b \\ c & d \\ & & 1 \end{pmatrix}, \quad ad - bc = 1,$$

acts by fractional linear transformations on $\tau \equiv \tau_1 + i\tau_2$ and linearly on (c_0, ψ) ,

$$(2.7) \quad \tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} c_0 \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} c_0 \\ \psi \end{pmatrix}, \quad \nu \mapsto \nu.$$

To recognize (2.3) as a HM moduli space metric, let us change variables to $(\phi, t, \zeta, \tilde{\zeta}, \sigma)$ defined by

$$(2.8) \quad \nu = \frac{e^{-3\phi/2}}{t^{3/2}}, \quad \tau_2 = \frac{e^{\phi/2}}{t^{3/2}}, \quad \tau_1 = \zeta, \quad c_0 = \tilde{\zeta}, \quad \psi = -\frac{1}{2}(\sigma + \zeta\tilde{\zeta}).$$

Metric (2.3) becomes

$$(2.9) \quad ds^2 = d\phi^2 + 3\frac{dt^2}{t^2} + e^{-\phi} \left(t^{-3}d\tilde{\zeta}^2 + t^3d\zeta^2 \right) + \frac{1}{4}e^{-2\phi} \left(d\sigma + \tilde{\zeta}d\zeta - \zeta d\tilde{\zeta} \right)^2.$$

This is the standard c -map metric associated to a special Kähler manifold $\mathcal{K}(X)$ with cubic prepotential $F = -\frac{1}{6}\kappa_{abc}z^a z^b z^c$ [2], restricted to the locus $z^a = it r^a, \zeta^a = \tilde{\zeta}_a = 0$, where r^a is a fixed reference value for the Kähler modulus t^a . Comparing (2.8) to [14], we can identify τ as the 10D-type IIB axio-dilaton, $c_0 = -\int_Y A^{(6)} + \dots$ as the RR six-form background, ψ as the 4D Neveu–Schwarz axion, $t^3 \equiv V$ as the volume of Y in string units. The four-dimensional string coupling is then

$$(2.10) \quad g_4 \equiv \frac{1}{\tau_2 \sqrt{8V}} = \frac{1}{\sqrt{8}}e^{-\phi/2}$$

(the factor of $\sqrt{8}$ is conventional), while the Heisenberg Killing vectors act as

$$(2.11) \quad E_p = \partial_\zeta + \tilde{\zeta}\partial_\sigma, \quad E_q = \partial_{\tilde{\zeta}} - \zeta\partial_\sigma, \quad E_k = -2\partial_\sigma.$$

It is perhaps worth noting that although a further restriction to the locus $t = 1$ produces the $SU(2, 1)$ -invariant metric on the universal HM, $SU(2, 1)$ does not act on the five-dimensional manifold (2.9). The stabilizer of the

locus $t = \text{const}$ is the semi-direct product of \mathbb{R}^+ generated by $H_p + H_q$ and the Heisenberg group N :

$$(2.12) \quad \begin{pmatrix} 1 & m & -p + \frac{1}{2}mn \\ & 1 & n \\ & & 1 \end{pmatrix} : (\zeta, \tilde{\zeta}, \sigma) \mapsto (\zeta + m, \tilde{\zeta} + n, \sigma + 2p - n\zeta + m\tilde{\zeta}).$$

On the other hand, the stabilizer of the locus $t^2 + e^{-\phi}\tilde{\zeta}^2/t = \text{const}$ is the semi-direct product of \mathbb{R}^2 generated by E_p, F_q and the $\text{SL}(2, \mathbb{R})$ subgroup

$$(2.13) \quad \begin{pmatrix} A & B \\ & 1 \\ C & D \end{pmatrix}, \quad AD - BC = 1,$$

which acts as

$$(2.14) \quad S \mapsto \frac{AS + B}{CS + D}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad z \mapsto z,$$

where

$$(2.15) \quad S \equiv \frac{1}{2}(-\sigma + \zeta\tilde{\zeta}) + ie^\phi\sqrt{1 + \frac{e^{-\phi}}{t^3}\tilde{\zeta}^2}, \quad z \equiv t^2 + \frac{e^{-\phi}}{t}\tilde{\zeta}^2, \\ \begin{pmatrix} x \\ y \end{pmatrix} \equiv \frac{1}{e^\phi t^3 + \tilde{\zeta}^2} \begin{pmatrix} \tilde{\zeta} \\ \zeta(e^\phi t^3 + \tilde{\zeta}^2) + \frac{1}{2}\tilde{\zeta}(\sigma - \zeta\tilde{\zeta}) \end{pmatrix}.$$

This $\text{SL}(2, \mathbb{R})$ subgroup is just the Ehlers symmetry alluded to in the introduction, written in a somewhat unusual field basis. Note that for $\tilde{\zeta} = 0$, the complex variable S reduces to the four-dimensional axio-dilaton, $S|_{\tilde{\zeta}=0} = -\frac{1}{2}\sigma + ie^\phi$.

In the sequel, we shall assume that physical amplitudes are invariant under $\text{SL}(3, \mathbb{Z})$, the group of integer valued, determinant one matrices. In particular, this includes the $\text{SL}(2, \mathbb{Z})$ subgroup (2.6) with a, b, c, d integer, the $\text{SL}(2, \mathbb{Z})$ subgroup (2.13) with A, B, C, D integer, and the Heisenberg group (2.12) with $m, n, -p + \frac{1}{2}mn$ integer.

2.2. The universal sector in the one-modulus case

To clarify in what sense $\text{SO}(3)\backslash\text{SL}(3, \mathbb{R})$ is a universal sector of \mathcal{M}_K , we now study decomposition (1.5) in the one-modulus case, when \mathcal{M}_K is the QK manifold $\mathcal{M} = \text{SO}(4)\backslash G_{2(2)}$, obtained by the c -map procedure from a special

Kähler manifold \mathcal{K}_K with prepotential $F = -(X^1)^3/X^0$. The geometry of this symmetric space was studied in detail in [47], whose notations we adhere to.⁵ The QK metric

$$(2.16) \quad ds^2 = 2(u\bar{u} + v\bar{v} + e^1\bar{e}^1 + E_1\bar{E}_1),$$

with

$$(2.17) \quad \begin{aligned} u &= \frac{e^{-\phi/2}}{4t^{3/2}} \left(-d\tilde{\zeta}_0 - zd\tilde{\zeta}_1 + 3z^2d\zeta^1 - z^3d\zeta^0 \right), \\ v &= \frac{1}{2}d\phi - \frac{i}{4}e^{-\phi}(d\sigma - \zeta^0d\tilde{\zeta}_0 - \zeta^1d\tilde{\zeta}_1 + \tilde{\zeta}_0d\zeta^0 + \tilde{\zeta}_1d\zeta^1), \\ e^1 &= \frac{i\sqrt{3}}{2t}dz, \\ E_1 &= -\frac{e^{-\phi/2}}{4\sqrt{3}t^{3/2}} \left(-3d\tilde{\zeta}_0 - d\tilde{\zeta}_1(\bar{z} + 2z) + 3z(2\bar{z} + z)d\zeta^1 - 3\bar{z}z^2d\zeta^0 \right) \end{aligned}$$

has a $G_{2(2)}$ isometric action, and therefore a $SL(3, \mathbb{R}) \subset G_{2(2)}$ isometric action. This action corresponds to right multiplication on the coset representative in the Iwasawa gauge (here $z \equiv b + it$),

$$(2.18) \quad e = t^{-Y_0} \cdot e^{\sqrt{2}bY_+} \cdot e^{-\frac{1}{2}\phi H} \cdot e^{-\frac{1}{\sqrt{2}}\zeta^0 E_{q_0} - \frac{1}{\sqrt{2}}\tilde{\zeta}_0 E_{p^0}} \cdot e^{-\sqrt{\frac{3}{2}}\zeta^1 E_{q_1} - \frac{1}{\sqrt{6}}\tilde{\zeta}_1 E_{p^1}} \cdot e^{-\frac{1}{2}\sigma E},$$

followed by a compensating $SO(4)$ left-action.

The $SL(3, \mathbb{R})$ subgroup of $G_{2(2)}$ is generated by the longest roots with respect to a split Cartan torus. A system of coordinates adapted to the $SL(3)$ action is obtained by choosing instead a coset representative in the (non-Iwasawa) gauge

$$(2.19) \quad e = \nu^{\frac{Y_0}{6} + \frac{H}{4}} \cdot \tau_2^{\frac{Y_0}{2} - \frac{H}{4}} \cdot e^{-\frac{1}{\sqrt{2}}\tau_1 E_{q_0}} \cdot e^{-\frac{1}{\sqrt{2}}c_0 E_{p^0}} \cdot e^{\psi E_k} \cdot e^{\sqrt{\frac{3}{2}}u_1 E_{q_1} - \sqrt{2}u_2 Y_+ + \sqrt{\frac{3}{2}}u_3 F_{p^1}}.$$

In this way, the coordinates $(\nu, \tau_2, \tau_1, c_0, \psi)$ parametrize \mathcal{M}_u as in (2.1), with the same transformations (2.4) as before, while the real coordinates (u_1, u_2, u_3) transform linearly in the triplet representation of $SL(3)$. In these

⁵Except for the following changes of notation: $\tau \rightarrow z, \zeta^\Lambda \rightarrow \zeta^\Lambda/\sqrt{2}, \tilde{\zeta}_\Lambda \rightarrow -\tilde{\zeta}_\Lambda/\sqrt{2}, \sigma \rightarrow -\frac{1}{2}\sigma$.

variables, the metric can be written as

$$(2.20) \quad ds_{\mathcal{M}}^2 = ds_{\mathcal{M}_u}^2 + d\vec{u}^2 + \left(1 + \frac{1}{3}\vec{u}^2\right) (\vec{u} \wedge d\vec{u})^2 + \mathcal{A}^{ijk} u_i u_j du_k,$$

where the contractions of the three-vectors \vec{u} , $d\vec{u}$ and $\vec{u} \wedge d\vec{u}$ are performed with the 3×3 symmetric matrices M , M , M^{-1} , respectively, where $M = g^t g$. Here $\mathcal{A}_{(ij)k}$ are $\text{SL}(3)$ invariant forms on \mathcal{M}_u . The origin of the various terms can be understood by writing it schematically as follows:

$$(2.21) \quad ds_{\mathcal{M}}^2 = (ds_{\mathcal{M}_u} + \mathcal{A} u u du)^2 + d\vec{u}^2 + (\vec{u} \wedge d\vec{u})^2$$

reflecting the decomposition $14 = 8 + 3 + \bar{3} = 5 + 3 + 3 + 3$ of \mathfrak{g}_2 under $\mathfrak{so}(3) \subset \mathfrak{sl}(3)$. Note that there is no translational symmetry along the \vec{u} variables: indeed the triplet of generators $(E_{q_1}, -\sqrt{6}Y_+, F_{p^1})$ differs from $\partial/\partial u_i$ at linear order in u_j .

The relation between the two sets of coordinates can be found by determining the $\text{SO}(4)$ left action needed to cast (2.19) in Iwasawa form (2.18). We suppress the details and quote only the result:

$$(2.22) \quad \begin{aligned} \phi &= -\frac{1}{2} \log \left(\frac{\nu \Delta^2}{\tau_2} \right), & t &= \frac{\Delta^{1/2}}{\nu^{1/6} \tau_2^{1/2} (\hat{u}_3^4 + (\hat{u}_1^2 + \hat{u}_2^2 + 2) \hat{u}_3^2 + 1)}, \\ b &= -\frac{\hat{u}_2 + \hat{u}_1 \hat{u}_3}{\nu^{1/6} \sqrt{\tau_2} (\hat{u}_3^4 + (\hat{u}_1^2 + \hat{u}_2^2 + 2) \hat{u}_3^2 + 1)}, \\ \zeta^0 &= \tau_1 - \frac{\tau_2 \hat{u}_3}{\Delta} (\hat{u}_2 \hat{u}_3 \hat{u}_1^3 - \hat{u}_1^2 + \hat{u}_2 \hat{u}_3 (\hat{u}_2^2 + \hat{u}_3^2 + 3) \hat{u}_1 + \hat{u}_2^2), \\ \zeta^1 &= \frac{\sqrt{\tau_2}}{\nu^{1/6} \Delta} (\hat{u}_2 \hat{u}_3 \hat{u}_1^2 - (\hat{u}_3^2 + 1) \hat{u}_1 + \hat{u}_2 \hat{u}_3 (\hat{u}_2^2 + \hat{u}_3^2 + 1)), \\ \tilde{\zeta}_0 &= c_0 - \frac{\Xi}{(\nu \tau_2)^{1/2} \Delta}, \\ \tilde{\zeta}_1 &= -\frac{3}{\nu^{1/3} \Delta} (\hat{u}_3 (\hat{u}_2^2 + \hat{u}_3^2) \hat{u}_1^2 - \hat{u}_2 \hat{u}_1 + \hat{u}_3 (\hat{u}_2^2 + \hat{u}_3^2 + 1)^2), \\ \sigma &= -2\psi - \tau_1 c_0 + \frac{\tau_2 \hat{u}_3 c_0}{\Delta} (\hat{u}_2 \hat{u}_3 \hat{u}_1^3 - \hat{u}_1^2 + \hat{u}_2 \hat{u}_3 (\hat{u}_2^2 + \hat{u}_3^2 + 3) \hat{u}_1 + \hat{u}_2^2) \\ &\quad - \frac{\tau_1 \Xi - \tau_2 \Xi'}{\Delta \sqrt{\nu \tau_2}}, \end{aligned}$$

where

$$\begin{aligned}
 (2.23) \quad & \hat{u}_1 = \frac{\nu^{1/6}}{\sqrt{\tau_2}} (u_1 + \psi u_3 + \tau_1(u_2 + c_0 u_3)), \quad \hat{u}_2 = \nu^{1/6} \sqrt{\tau_2} (u_2 + c_0 u_3), \\
 & \hat{u}_3 \nu^{-1/3} u_3, \\
 \Delta &= \hat{u}_3^6 + (2\hat{u}_2^2 + 3) \hat{u}_3^4 + (\hat{u}_2^4 + 3\hat{u}_2^2 + 3) \hat{u}_3^2 + \hat{u}_1^2 (\hat{u}_2^2 + \hat{u}_3^2) \hat{u}_3^2 - 2\hat{u}_1 \hat{u}_2 \hat{u}_3 + 1, \\
 \Xi &= \hat{u}_2 \hat{u}_3 (\hat{u}_2^2 + \hat{u}_3^2) \hat{u}_1^2 - (\hat{u}_2^2 - \hat{u}_3^2) \hat{u}_1 \\
 & \quad + \hat{u}_2 \hat{u}_3 (\hat{u}_2^4 + (2\hat{u}_3^2 + 3) \hat{u}_2^2 + \hat{u}_3^4 + 3\hat{u}_3^2 + 3), \\
 \Xi' &= (\hat{u}_3 (\hat{u}_2^2 + 2\hat{u}_3^2) \hat{u}_1^3 - \hat{u}_2 \hat{u}_1^2 + \hat{u}_3 (\hat{u}_2^4 + 3(\hat{u}_3^2 + 1) \hat{u}_2^2 \\
 & \quad + 2\hat{u}_3^4 + 6\hat{u}_3^2 + 3) \hat{u}_1 + \hat{u}_2 \hat{u}_3^2).
 \end{aligned}$$

A similar decomposition holds for any QK space \mathcal{M} given by the c -map of a special Kähler manifold \mathcal{K} with cubic prepotential

$$(2.24) \quad F = -\frac{1}{6} \kappa_{abc} X^a X^b X^c / X^0,$$

where κ_{abc} is the norm form of a Jordan algebra J of degree three. In this case, \mathcal{M} is a symmetric space [48–51]

$$(2.25) \quad \mathcal{M} = [\text{SU}(2) \times \widetilde{\text{Conf}}(J)] \backslash \text{QConf}(J),$$

where $\text{QConf}(J)$ and $\widetilde{\text{Conf}}(J)$ are the quasi-conformal and compact conformal groups associated to J . The root lattice of $\text{QConf}(J)$ admits a two-dimensional projection to the root lattice of $G_{2(2)}$, with a non-trivial multiplicity h for the short roots, and with the zero weights corresponding to the five-dimensional duality group $\text{Str}_0(J)$ together with the non-compact Cartan generators of $\text{SL}(3)$. Using a suitable (non-Iwasawa) gauge, the right-invariant metric can be written as

$$\begin{aligned}
 (2.26) \quad ds_{\mathcal{M}} &= ds_{\mathcal{M}_u}^2 + ds_{\mathcal{R}}^2 + (d\vec{u}^a)^2 + \left(1 + \frac{1}{3}(\vec{u}^a)^2\right) \left(\frac{1}{6} \kappa_{abc} \vec{u}^b \wedge d\vec{u}^c\right)^2 \\
 & \quad + \frac{1}{6} \kappa_{abc} \mathcal{A}^{ijk} u_i^a u_j^b du_k^c,
 \end{aligned}$$

where $a = 1, \dots, h$, where $\mathcal{R} = \text{Aut}(J) \backslash \text{Str}_0(J)$ is the VM space in five dimensions, given by the cubic hypersurface [48],

$$(2.27) \quad \frac{1}{6} \kappa_{abc} r^a r^b r^c = 1,$$

of real dimension $h - 1$. The coordinate t on \mathcal{M}_u is then the overall scale of the Kähler classes, $t^a = t r^a$, while the coordinates u_i^a are related to the RR

Wilson lines by a generalization of (2.22), e.g., to leading order in u_3^a ,

$$(2.28) \quad \begin{aligned} b^a &= -u_2^a + \dots, & \tilde{\zeta}_0 &= c_0 + \frac{1}{6}\kappa_{abc}u_2^a u_2^b (u_1^c + \tau_1 u_2^c) + \dots, \\ \zeta^a &= -(u_1^a + \tau_1 u_2^a) + \dots, & \tilde{\zeta}_a &= \frac{1}{2}\kappa_{abc}[u_2^b (u_1^c + \tau_1 u_2^c) + e^{\phi} t^b u_3^c] + \dots. \end{aligned}$$

These formulae should remain correct in the large volume, weak coupling limit, even when the intersection form κ_{abc} is not the norm form of a Jordan algebra and \mathcal{M}_K not a symmetric space.

3. $SL(3, \mathbb{Z})$ Eisenstein series and NS5-instantons

While $\mathcal{M}_K(X)$ admits an isometric action of $SL(3, \mathbb{R})$ in the strict weak coupling, large volume limit, quantum corrections to the metric generically break all continuous isometries. In this section, we show that a discrete subgroup $SL(3, \mathbb{Z})$ may be restored, provided that quantum corrections take a suitable form.

3.1. QK geometry and contact potential

The QK metric on \mathcal{M} is conveniently encoded in the hyperkähler potential, a $SU(2)$ -invariant, degree one homogeneous function χ on the Swann bundle \mathcal{S} , which provides a Kähler potential for the hyperkähler metric on \mathcal{S} in all complex structures [52, 53]. Here \mathcal{S} is a $\mathbb{R}^4/\mathbb{Z}_2$ bundle over \mathcal{M} (equivalently a \mathbb{C}^\times bundle over the twistor space \mathcal{Z} of \mathcal{M}), which carries a canonical hyperkähler metric with an isometric $SU(2)$ action and homothetic Killing vector κ [52]. Thus, one may choose coordinates x^μ on \mathcal{M} and $(v^b, \bar{v}^b, z, \bar{z})$ on $\mathbb{R}^4/\mathbb{Z}_2$ such that [14, 21, 54],

$$(3.1) \quad \chi = 4|v^b| \frac{(1 + z\bar{z})}{|z|} e^{\text{Re}[\Phi(x^\mu, z)]},$$

where $\Phi(x^\mu, z)$, a complex function holomorphic in z , is known as the “contact potential”. In order to extract the metric on \mathcal{S} or on \mathcal{M} , the hyperkähler potential χ should also be supplemented by the “twistor lines,” i.e., by a set of holomorphic functions $u^i(x^\mu, z)$ on \mathcal{Z} such that (v^b, u^i) provides a set of local complex coordinates on \mathcal{S} . Importantly, any isometry of \mathcal{M} can be combined with a suitable action on $(v^b, \bar{v}^b, z, \bar{z})$ to produce a tri-holomorphic isometry of \mathcal{S} , leaving χ invariant. In the presence of one continuous isometry, $\Phi(x^\mu, z)$ can be taken to be independent of z , but this is not possible

in general. However, using the $SU(2)$ action, it is in principle possible to recover $\Phi(x^\mu, z)$ for any z from the knowledge of its restriction to any section $z(x^\mu)$. In this note, we shall restrict our attention to this “restricted” contact potential Φ for a suitable section $z(x^\mu)$, leaving for future work the determination of the twistor lines and of the contact potential away from the section $z(x^\mu)$.

In type IIB string theory compactified on Y , the contact potential on the HM moduli space $\mathcal{M}_K(Y)$, including the effects of the tree-level $(\alpha')^3$ correction, tree-level world-sheet instantons and one-loop correction was determined in [10, 14, 21]:

$$(3.2) \quad e^{\Phi_{\text{pert}}} = \frac{\tau_2^2}{2} V - \frac{\sqrt{\tau_2}}{16(2\pi)^3} \chi_Y \left[2\zeta(3) \tau_2^{3/2} + \frac{2\pi^2}{3} \tau_2^{-1/2} \right] + \frac{\tau_2^2}{4(2\pi)^3} \sum_{k_a > 0} n_{k_a}^{(0)} \operatorname{Re} \left[\operatorname{Li}_3 \left(e^{2\pi i k_a z^a} \right) + 2\pi k_a t^a \operatorname{Li}_2 \left(e^{2\pi i k_a z^a} \right) \right],$$

where $V \equiv \frac{1}{6} \kappa_{abc} t^a t^b t^c = t^3$ is the volume of Y , $n_{k_a}^{(0)}$ is the BPS invariant in the homology class $k_a \gamma^a \in H_2(Y, \mathbb{Z})$, $\operatorname{Li}_s(x) = \sum_{n=1}^\infty x^n / n^s$ is the polylogarithm and $\zeta(s)$ is Riemann’s zeta function.

In the weak coupling, large volume limit, only the first term in (3.2) remains. The hyperkähler potential χ is then invariant under the $SL(2, \mathbb{R})$ groups (2.6) and (2.13), respectively, provided the prefactor $r^b \equiv v^b |(1 + z\bar{z})/|z|$ in (3.1) transforms as

$$(3.3) \quad r^b \mapsto r^b |c\tau + d|, \quad r^b \mapsto r^b |CS + D|^2 \sqrt{\frac{1 + z^{3/2} x^2 S_2}{1 + \frac{z^{3/2} (Dx - By)^2 S_2}{|CS + D|^2}}},$$

respectively. This invariance is spoiled, however, when the other terms in (3.2) are included. Of course, χ could always be made invariant by adjusting the transformation rule of r^b , but this will in general not lead to a triholomorphic action. For this reason, we do not allow any deformation of the $SL(3, \mathbb{R})$ action on the coordinates x^μ and r^b .⁶ Instead, we allow for deformations of the contact potential $\Phi(x^\mu) \equiv \Phi(x^\mu, z(x^\mu))$, but assume that there exists a choice of section $z(x^\mu)$ such that $\Phi(x^\mu, z(x^\mu))$ retains its tree-level transformation property.

⁶The consistency of this assumption in the case of $SL(2, \mathbb{Z})$ has been checked recently in [43].

In [10], it was shown that an $SL(2, \mathbb{Z})$ subgroup of (2.6) could be restored by adding to the perturbative potential (3.2) a suitable combination of D-instantons and (m, n) -string instantons,

$$\begin{aligned}
 e^{\Phi_{\text{inv}}} &= \frac{\tau_2^2}{2} V + \frac{\sqrt{\tau_2}}{8(2\pi)^3} \sum_{k_a \gamma^a \in H_2^+(Y) \cup \{0\}} n_{k_a}^{(0)} \sum'_{m,n} \frac{\tau_2^{3/2}}{|m\tau + n|^3} \\
 (3.4) \quad &\times (1 + 2\pi|m\tau + n|k_a t^a) e^{-2\pi S_{m,n,k_a}},
 \end{aligned}$$

where $n_0^{(0)} = -\chi_Y/2$,

$$(3.5) \quad S_{m,n,k_a} = k_a|m\tau + n|t^a - ik_a(mc^a + nb^a)$$

and the primed sum runs over pairs of integers $(m, n) \neq (0, 0)$. Thus, $SL(2, \mathbb{Z})$ invariance is powerful enough to determine these types of instanton corrections, which with our current understanding of string theory could not be computed from first principles. As a strong consistency check, it was shown that (3.4) reproduces the expected behavior near the conifold [11].

3.2. Automorphizing under $SL(3, \mathbb{Z})$

Our aim is to show that similarly, invariance under a discrete subgroup $SL(3, \mathbb{Z})$ of $SL(3, \mathbb{R})$ can be restored by including NS5-brane and D5-brane contributions. For simplicity, we concentrate on the $(\alpha')^3$ and g_s “universal” corrections in the first line of (3.2), which depend only on the extended universal sector $(\nu, \tau_2, \tau_1, c_0, \psi)$ and on the Euler number χ_Y . Factoring out the tree-level contribution, we require that

$$(3.6) \quad e^{\Phi} = \frac{\tau_2^2 V}{2} (1 + E(g)),$$

where $E(g)$ is an $SL(3, \mathbb{Z})$ -invariant function such that, at weak coupling,

$$(3.7) \quad E(g) = -\frac{\chi_Y}{8(2\pi)^3} \left(2\zeta(3) V^{-1} + 2\frac{\pi^2}{3} V^{-1}\tau_2^{-2} + \dots \right).$$

While our knowledge of automorphic forms of $SL(3, \mathbb{Z})$ is rather limited, some general principles and a few explicit examples are well understood. As explained, e.g., in [58], $G(\mathbb{Z})$ -invariant functions on $K \backslash G(\mathbb{R})$ can be constructed from

- (i) a unitary representation ρ of $G(\mathbb{R})$ in an (infinite dimensional) Hilbert space \mathcal{H} ,

- (iia) a “spherical” K -invariant vector⁷ $f_K \in \mathcal{H}$ and
- (iiib) a $G(\mathbb{Z})$ -invariant distribution $f_{\mathbb{Z}}$ on \mathcal{H} .

Moreover, $f_{\mathbb{Z}}$ can often be obtained adelicly from spherical vectors f_p of the representation ρ over the p -adic number field \mathbb{Q}_p for all primes p . With these ingredients, one may write

$$(3.8) \quad E(g) = \langle f_{\mathbb{Z}} | \rho(g^{-1}) | f_K \rangle,$$

which is well defined on $K \backslash G(\mathbb{R}) / G(\mathbb{Z})$, due to the $G(\mathbb{Z})$ - and K -invariance of $f_{\mathbb{Z}}$ and f_K , respectively. Moreover, if ρ is an irreducible representation, such that any G -invariant operator \mathcal{O} acts on \mathcal{H} as a scalar, then $E(g)$ is an eigenmode of \mathcal{O} with the same eigenvalue, now acting as a G -invariant differential operator on $K \backslash G(\mathbb{R})$. In particular, the eigenvalue of $E(g)$ under the Laplace–Beltrami operator (2.5) on $K \backslash G(\mathbb{R})$ is equal to the value of the quadratic Casimir \mathcal{C}_2 in the representation ρ .

In the context of R^4 couplings in eight dimensions [35], an elementary example of a $SL(3, \mathbb{Z})$ -invariant function was constructed (see also [59] for a physics discussion of this type of Eisenstein series):

$$(3.9) \quad E(g; s) = \sum_{m \in \mathbb{Z}^3 \setminus \{(0,0,0)\}} \left(m^t M m \right)^{-s},$$

where $M = g^t g$. The sum converges absolutely when $\text{Re}(s) > 3/2$, and can be analytically continued in the rest of the s -plane, except for a pole at $s = 3/2$. This is the automorphic form associated to the representation ρ on homogeneous functions of degree $-2s$ in three variables — also known as the minimal representation. The spherical vector is just $f_K = (m^t m)^{-s}$, and $f_{\mathbb{Z}}$ is the Dirac distribution on the lattice \mathbb{Z}^3 minus the origin. The values of the quadratic and cubic Casimirs are given by

$$(3.10) \quad \mathcal{C}_2 = \frac{2}{3}s(2s - 3), \quad \mathcal{C}_3 = s(2s - 3)(4s - 3),$$

satisfying the relation

$$(3.11) \quad 4\mathcal{C}_3^2 - 108\mathcal{C}_2^3 - 81\mathcal{C}_2^2 = 0.$$

⁷The term “spherical” requires both K -invariance and suitable decrease at infinity. If ρ does not admit a spherical vector, f_K can be replaced by a vector in the lowest K -type, but (3.8) then leads to a section of a non-trivial homogeneous vector bundle over $K \backslash G(\mathbb{R})$.

The infinitesimal generators of the minimal representation are spelled out in Appendix D.1.

More generally, one may consider the principal Eisenstein series [44, 45, 55–57] (see also [36, 58] for a physics discussion)

$$(3.12) \quad E(g; s_1, s_2) = \frac{1}{4\zeta(2s_1)\zeta(2s_2)} \sum_{m,n} \left(m^t M m\right)^{-s_1} \left(n^t M^{-1} n\right)^{-s_2},$$

where the sum runs over pairs of non-zero integer vectors \vec{m}, \vec{n} such that $m^t n = 0$. The automorphic form (3.12) is attached to the principal continuous representation obtained by induction from the minimal parabolic (or “Borel”) subgroup

$$(3.13) \quad P_{\min} = \left\{ \begin{pmatrix} a_3 & * & * \\ & a_2 & * \\ & & a_1 \end{pmatrix} \mid a_1 a_2 a_3 = 1 \right\}$$

via the character⁸

$$(3.14) \quad \chi_{s_1, s_2}(p) = |a_1|^{\frac{2}{3}(s_1+2s_2)} |a_2|^{\frac{2}{3}(s_1-s_2)} |a_3|^{-\frac{2}{3}(2s_1+s_2)}.$$

In contrast to the minimal representation, the principal series has independent quadratic and cubic Casimirs,

$$(3.15) \quad \begin{aligned} C_2 &= \frac{4}{3}(s_1^2 + s_2^2 + s_1 s_2) - 2(s_1 + s_2), \\ C_3 &= (s_1 - s_2)(2s_2 + 4s_1 - 3)(2s_1 + 4s_2 - 3). \end{aligned}$$

The infinitesimal generators of the continuous principal representation are spelled out in Appendix D.2. The sum in (3.12) converges absolutely for

$$(3.16) \quad \operatorname{Re}(s_1) > 1, \quad \operatorname{Re}(s_2) > 1,$$

and may be meromorphically continued to other values of (s_1, s_2) [44, 56, 60]. Singularities arise at the six lines in the (s_1, s_2) plane where (3.11) is obeyed,

⁸More generally, one could multiply χ by $\prod_{i=1,\dots,3} [\operatorname{sign}(a_i)]^{\epsilon_i}$, and obtain in this way the supplementary continuous series; however the signs ϵ_i must be chosen to be + in order for the $\operatorname{SL}(3, \mathbb{Z})$ Eisenstein series not to vanish. The supplementary series may become relevant if only a finite index subgroup of $\operatorname{SL}(3, \mathbb{Z})$ was unbroken. We are grateful to S. Miller for consultations about this point.

namely

$$(3.17) \quad s_1 = 0, \quad s_1 = 1, \quad s_2 = 0, \quad s_2 = 1, \quad s_1 + s_2 = \frac{1}{2}, \quad s_1 + s_2 = \frac{3}{2},$$

where $E(g; s_1, s_2)$ becomes proportional⁹ to the minimal Eisenstein series $E(g; s)$ (3.9).

Finally, one may also consider representations induced from the maximal parabolic subgroup

$$(3.18) \quad P_{\max} = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix} \right\},$$

and construct

$$(3.19) \quad E(g; s_1, s_2, \phi) = \sum \left(m^t M m \right)^{-s_1 - 2s_2} \phi(\tau_{m,n}, \bar{\tau}_{m,n}),$$

where

$$(3.20) \quad \tau_{m,n} = \frac{m^t M n + i\sqrt{m^t M m \ n^t M n - (m^t M n)^2}}{m^t M m},$$

$\phi(\tau, \bar{\tau})$ is a non-holomorphic modular form of $SL(2, \mathbb{Z})$ with weight $-2s_2$,

$$(3.21) \quad \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d}\right) = |c\tau + d|^{2s_2} \phi(\tau, \bar{\tau}),$$

and the sum runs over pairs of non-zero integer vectors m, n , modded out by the equivalence relation $n \sim n + m$. For $\phi = \tau_2^{-2s_2}$, this reproduces (3.12) up to a normalization factor. According to [61], these induced representations exhaust all irreducible unitary representations of $SL(3, \mathbb{R})$.

In the case at hand, the values of the quadratic and cubic Casimirs can be easily determined by acting with the invariant differential operators \mathcal{C}_2

⁹This can be seen by comparing the constant terms, see Appendix A. The matching of the Abelian and non-Abelian Fourier coefficients appears less obvious but should hold on general grounds. Note that none of the lines in (3.17) lies in the convergence domain (3.16).

and \mathcal{C}_3 in (2.5) on the two perturbative terms in (3.7):

$$(3.22) \quad \mathcal{C}_2 = 3, \quad \mathcal{C}_3 = 0.$$

This determines (s_1, s_2) in (3.12) to be one of the six values (which lie away from the singular locus (3.11))

$$(3.23) \quad \left\{ \left(\frac{3}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, -\frac{3}{2}\right), \left(-\frac{3}{2}, \frac{3}{2}\right), \left(-\frac{1}{2}, -\frac{1}{2}\right), \left(-\frac{1}{2}, \frac{5}{2}\right), \left(\frac{5}{2}, -\frac{1}{2}\right) \right\}.$$

The Weyl group of $SL(3)$ acts by permuting these values, and the resulting Eisenstein series (3.12) are identical, up to an overall (s_1, s_2) -dependent constant. There is no loss of generality in choosing $(s_1, s_2) = (3/2, -3/2)$. Thus, we tentatively propose that the $SL(3, \mathbb{Z})$ invariant function appearing in (3.6) is given by

$$(3.24) \quad E(g) = -\frac{\chi_Y}{4(2\pi)^3} \zeta(3) E(g; 3/2, -3/2),$$

where the principal Eisenstein series on right-hand side is given by (3.12) with¹⁰ $(s_1, s_2) = (3/2, -3/2)$.

More generally, it is natural to conjecture that the worldsheet instanton sum in (3.4) is subsumed into a sum of $SL(3, \mathbb{Z})$ -invariant functions

$$(3.25) \quad \begin{aligned} \tilde{E}(g) = & \frac{\zeta(3)}{2(2\pi)^3} \sum_{k_a \gamma^a \in H_2^+(Y) \cup \{0\}} n_{k_a}^{(0)} \sum_{m, n} \left(\frac{m^t M m \ n^t M n - (m^t M n)^2}{m^t M m} \right)^{3/2} \\ & \times \left(1 + 2\pi k_a r^a \sqrt{m^t M m} \right) e^{-2\pi S_{\vec{m}, k_a}}, \end{aligned}$$

where $n_0^{(0)} = -\chi_Y/2$,

$$(3.26) \quad S_{\vec{m}, k_a} = k_a r^a \sqrt{m^t M m} + i k_a \vec{u}^a \vec{m},$$

and the second sum runs over the same set of integers as in (3.12). In the sequel, we shall restrict ourselves to the universal contributions (3.24) corresponding to $k_a = 0$, leaving a study of (3.25) to future work.

We should stress that we do not know whether (3.12) is the only automorphic form of $SL(3, \mathbb{Z})$ with the infinitesimal parameters (3.22). It seems reasonable however to take it as a working assumption, and see what kind of quantum corrections it predicts.

¹⁰Different choices of (s_1, s_2) simply amount to permuting the various terms in (3.28) and (3.33).

3.3. Perturbative and D-instanton contributions

In order to justify our proposal (3.24), we should check that the perturbative terms in (3.7), and indeed the whole D-instanton series (3.4) predicted on the basis of $SL(2, \mathbb{Z})$ duality, are reproduced in the large volume limit $\nu \rightarrow 0$. The mathematical prescription is to extract the “constant term” with respect to the maximal parabolic subgroup (3.18), i.e., the zeroth Fourier coefficient with respect to (c_0, ψ) ,

$$(3.27) \quad E_{P_{\max}}(\nu, \tau_2, \tau_1) \equiv \int_0^1 dc_0 \int_0^1 d\psi E\left(\nu, \tau_1, \tau_2, c_0, \psi; \frac{3}{2}, -\frac{3}{2}\right).$$

Since (c_0, ψ) transforms as a doublet under (2.6), the result must be invariant under $SL(2, \mathbb{Z}) \subset P_{\max}$. Using the general results due to Langlands [56, 60], or the explicit computation in [44], we find (see (A.7) in Appendix A)

$$(3.28) \quad \begin{aligned} E_{P_{\max}}(\nu, \tau_2, \tau_1) &= \tau_2^{-3/2} V^{-1} E_{3/2}(\tau) - 1080 \zeta(3) \zeta'(-4) \tau_2^{9/2} V^3 E_{-1/2}(\tau) \\ &\quad + 120 \zeta(3) \tau_2^{3/2} V E_{-3/2}(\tau), \end{aligned}$$

where $E_s(\tau)$ is the standard non-holomorphic $SL(2, \mathbb{Z})$ Eisenstein series

$$(3.29) \quad E_s(\tau) = \sum_{(m,n) \neq (0,0)} \left(\frac{\tau_2}{|m + n\tau|^2} \right)^s,$$

whose Fourier decomposition is given by the Chowla–Selberg formula (see, e.g., [59, 62] for a physicist discussion)

$$(3.30) \quad \begin{aligned} E_s(\tau) &= 2\zeta(2s) \tau_2^s + 2\sqrt{\pi} \tau_2^{1-s} \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1) \\ &\quad + \frac{2\pi^s \tau_2^s}{(2\pi)^{1/2-s} \Gamma(s)} \sum_{\tilde{m}_1 \neq 0} \sum_{m_2 \neq 0} |\tilde{m}_1|^{2s-1} \mathcal{K}_{s-1/2}(2\pi\tau_2 |\tilde{m}_1 m_2|) e^{2\pi i \tilde{m}_1 m_2 \tau_1}. \end{aligned}$$

To simplify formulas, we have defined the rescaled (modified) Bessel function:

$$(3.31) \quad \mathcal{K}_t(x) \equiv x^{-t} K_t(x) = \sqrt{\frac{\pi}{2}} x^{-(t+1/2)} e^{-x} \left(1 + \mathcal{O}(1/x) \right).$$

The first term $E_{3/2}(\tau)$ in (3.28) indeed reproduces (3.4) with $k_a = 0$, and therefore the two perturbative terms in (3.7).

To analyze the remaining terms in (3.28), it is useful to go to the weak coupling limit $\tau_2 \rightarrow \infty$, where only the first line of (3.30) contributes for each of the Eisenstein series E_s appearing in (3.28). In effect, this amounts to extracting the constant term

$$(3.32) \quad E_{P_{\min}}(\nu, \tau_2) = \int_0^1 d\tau_1 E_{P_{\max}}(\nu, \tau_2, \tau_1)$$

with respect to the minimal parabolic (or Borel) subgroup (3.13) (see Equation (A.2)):

$$(3.33) \quad \begin{aligned} E_{P_{\min}}(\nu, \tau_2) = & 2\zeta(3) V^{-1} + \frac{2\pi^2}{3} \tau_2^{-2} V^{-1} + \frac{405}{\pi^6} \zeta(3)^2 \zeta(5) V \\ & + 180 \zeta(3) \zeta'(-4) \tau_2^4 V^3 + 180 \zeta(3) \zeta'(-4) \tau_2^4 V + 2\zeta(3) \tau_2^6 V^3. \end{aligned}$$

Multiplying out by the prefactor $\tau_2^2 V/2$ from (3.6), we see that the terms on the second line behave like perturbative contributions with negative genus -2 and -3 , while the last term on the first line behaves like a tree-level contribution which grows up like the square of the volume. These are the puzzling terms mentioned in the introduction. It could be that such divergent terms arise perturbatively (in analogy with the $\log g_s$ term encountered in R^4 couplings [35]), or that proposal (3.24) is too naive. Nevertheless, it is instructive to analyze the implications of our proposal at finite volume and coupling, in the hope that these issues can be resolved in the future with minor changes to our set-up.

3.4. Non-Abelian Fourier expansion and the minimal Eisenstein series

At finite volume and coupling, terms with non-trivial dependence on c_0 and ψ will start contributing, corresponding in the type IIB context to D5- and NS5-brane instantons (or, in the IIA context, to D6- and KK-monopoles winding along S^1). However, due to the non-Abelian nature of the Heisenberg group N in (2.12), it is not possible to diagonalize translations in τ_1, c_0, ψ simultaneously and extract Fourier coefficients indexed unambiguously by D(-1), D5 and NS5-brane charges. Instead, one must decompose the action of N on functions on \mathcal{M}_u into irreducible representations. By the Stone-von Neumann theorem, any irreducible unitary representation of the Heisenberg algebra $[E_p, E_q] = E$ is either

- (i) one-dimensional, with E_p and E_q acting as scalars and $E = 0$;
- (iia) infinite-dimensional and isomorphic to the action on the space of functions of two variables x_0, y via

$$(3.34) \quad E_p = ix_0, \quad E_q = -y \partial_{x^0}, \quad E = iy,$$

- (iib) equivalently to (iia) after Fourier transform, infinite-dimensional and isomorphic to the action on the space of functions of two variables x_0, y via

$$(3.35) \quad E_p = y \partial_{x^0}, \quad E_q = ix_0, \quad E = iy.$$

In practice, this means that any function $\Psi(t, \phi; \zeta, \tilde{\zeta}, \sigma)$ invariant under the Heisenberg group (2.12) can be decomposed into its “Abelian” and “non-Abelian” parts,¹¹ this means that any function $\Psi(t, \phi; \zeta, \tilde{\zeta}, \sigma)$ invariant under the Heisenberg group (2.12) can be decomposed into its “Abelian” and “non-Abelian” parts:

$$(3.36) \quad \Psi(t, \phi; \zeta, \tilde{\zeta}, \sigma) = \sum_{(p,q) \in \mathbb{Z}^2} \Psi_{p,q}(t, \phi) e^{2\pi i(q\zeta - p\tilde{\zeta})} + \sum_{k \in \mathbb{Z} \setminus \{0\}} \sum_{\ell \in \mathbb{Z}/(|k|\mathbb{Z})} \times \left[\sum_{n \in \mathbb{Z} + \frac{\ell}{|k|}} \Psi_{k,\ell} \left(t, \phi; \tilde{\zeta} - n \right) e^{-2\pi i kn \zeta - \pi i k(\sigma - \zeta \tilde{\zeta})} \right].$$

The first line is the contribution from one-dimensional type (i) representations, and corresponds to the Fourier expansion of the constant term

$$(3.37) \quad \Psi(t, \phi, \zeta, \tilde{\zeta}) = \int_0^2 \Psi(t, \phi, \zeta, \tilde{\zeta}, \sigma) d\sigma$$

with respect to the “Abelianized Heisenberg group” $\tilde{N} = N/Z$. Here Z denotes the center of N which coincides with the commutator subgroup $[N, N]$:

$$(3.38) \quad Z = [N, N] = \left\{ \left(\begin{array}{ccc} 1 & & * \\ & 1 & \\ & & 1 \end{array} \right) \right\}.$$

¹¹Such non-Abelian Fourier expansions have been discussed in the mathematics literature for a variety of groups [44, 63, 64]. They also occur in condensed matter physics in discussing Landau levels on the torus [65]. We are indebted to A. Neitzke for numerous discussions on this subject.

The second line in (3.36) then corresponds to infinite-dimensional representations of type (ia) with $y = -2\pi k$ and $x_0 = -2\pi kn$. Note that the invariance of (3.36) under shifts $\zeta \mapsto \zeta + 1, \sigma \mapsto \sigma + \tilde{\zeta}$ is immediate since $\sigma - \zeta\tilde{\zeta}$ is invariant and n is integer; under shifts $\tilde{\zeta} \mapsto \tilde{\zeta} + 1, \sigma \mapsto \sigma - \zeta$, the summation variable n must be shifted, but the variation of $\pi ik(\sigma - \zeta\tilde{\zeta})$ and $2\pi ikn$ in the exponential compensate each other, so (3.36) is again invariant.

Equivalently, the same function may be decomposed into representations of types (i) and (ia), as

$$\begin{aligned}
 \Psi(t, \phi; \zeta, \tilde{\zeta}, \sigma) &= \sum_{(p,q) \in \mathbb{Z}^2} \Psi_{p,q}(t, \phi) e^{2\pi i(q\zeta - p\tilde{\zeta})} + \sum_{k \in \mathbb{Z} \setminus \{0\}} \sum_{\ell' \in \mathbb{Z}/(|k|\mathbb{Z})} \\
 (3.39) \quad &\times \left[\sum_{m \in \mathbb{Z} + \frac{\ell'}{|k|}} \tilde{\Psi}_{k,\ell'}(t, \phi; \zeta - m) e^{2\pi i k m \tilde{\zeta} - \pi i k (\sigma + \zeta \tilde{\zeta})} \right].
 \end{aligned}$$

The relation between the two sets of non-Abelian Fourier coefficients follows by Poisson resummation over n , and is given by Fourier transform,

$$(3.40) \quad \tilde{\Psi}_{k,\ell'}(t, \phi; \zeta) = \sum_{\ell=0}^{|k|-1} e^{-2\pi i \frac{\ell \ell'}{k}} \int_{-\infty}^{\infty} \Psi_{k,\ell}(t, \phi; \tilde{\zeta}) e^{2\pi i k \zeta \tilde{\zeta}} d\tilde{\zeta},$$

while the Abelian Fourier coefficients in (3.36) and (3.39) are of course identical. Thus, the non-Abelian Fourier coefficients $\Psi_{k,l}$ and $\tilde{\Psi}_{k,l'}$ exhibit a wave function property, i.e., should really be thought of as a single state, which can be expressed in different polarizations.¹²

Before we proceed to the more relevant case of the full principal series, as an example we give here the non-Abelian Fourier coefficients of the minimal Eisenstein series (3.9), as computed in detail in Appendix B:

$$\begin{aligned}
 \Psi_{0,q} &= \frac{2\pi^s}{\Gamma(s)} (e^\phi/t)^s \mu_{2s-1}(q) \mathcal{K}_{s-\frac{1}{2}} \left(2\pi e^{\phi/2} t^{-3/2} |q| \right), \\
 \Psi_{p,0} &= \frac{2\pi^s}{\Gamma(s)} (t e^\phi)^{\frac{3}{2}-s} \mu_{2-2s}(p) \mathcal{K}_{1-s} \left(2\pi e^{\phi/2} t^{3/2} |p| \right),
 \end{aligned}$$

¹²This property suggests that $\tilde{\Psi}_{k,\ell}$ may be closely related to the topological string amplitude, or rather to its one-parameter generalization advocated in [66].

$$\begin{aligned}
 (3.41) \quad \Psi_{k,\ell} &= \frac{2\pi^s}{\Gamma(s)} (e^\phi/t)^s \mu_{2s-1}(k, \ell) \mathcal{K}_{s-\frac{1}{2}} \left(2\pi |k| e^{\phi/2} \sqrt{e^\phi + t^{-3} \tilde{\zeta}^2} \right), \\
 \tilde{\Psi}_{k,\ell'} &= \frac{2\pi^s}{\Gamma(s)} (t e^\phi)^{\frac{3}{2}-s} \mu_{2-2s}(k, \ell') \mathcal{K}_{1-s} \left(2\pi |k| e^{\phi/2} \sqrt{e^\phi + \zeta^2 t^3} \right), \\
 \Psi_{0,0} &= 2\zeta(2s) (e^\phi/t)^s + \frac{2\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\pi^{-\frac{1}{2}}\Gamma(s)} e^{\frac{\phi}{2}} t^{2s-\frac{3}{2}} \\
 &\quad + \frac{2\Gamma(s-1)\zeta(2s-2)}{\pi^{-1}\Gamma(s)} (t e^\phi)^{\frac{3}{2}-s},
 \end{aligned}$$

where the “instanton measure” for n charges is generally defined by a sum over the common divisors of all charges

$$(3.42) \quad \mu_s(N_1, \dots, N_n) \equiv \sum_{m|N_1, \dots, N_n} |m|^s.$$

These Fourier coefficients are considerably simpler than those arising from the principal Eisenstein series $E(g; s_1, s_2)$ discussed in the next subsection, but they illustrate their general structure. In fact, they arise as the limit $(s_1, s_2) \rightarrow (s, 0)$ of a subset of the coefficients of $E(g; s_1, s_2)$. Were this limit to describe some physical coupling, $\Psi_{0,p}$ and $\Psi_{q,0}$ would correspond to D5- and D(-1)-instanton effects, with instanton actions displayed in (3.48) and (3.47) below, while $\Psi_{k,\ell'}$ would correspond to (p, k) five-branes with $p = km \in \mathbb{Z}$ and vanishing D(-1) charge $Q = 0$, as in (3.57). However, the principal Eisenstein series $E(g; s_1, s_2)$ with $s_2 \neq 0$ displays additional contributions with $pq \neq 0$ and $Q \neq 0$, and considerably more involved instanton measure (3.69). A representation-theoretic point of view on the non-Abelian Fourier expansions (3.36) and (3.39) is provided in Appendix D.

3.5. Principal Eisenstein series and NS5-branes

Let us now turn to the non-Abelian Fourier expansion of the full Eisenstein series $E(g; s_1, s_2)$ in (3.12). As it turns out, this was computed 30 years ago by Vinogradov and Takhtajan [44] and by Bump [45]. In this section, we summarize their results adapted to our conventions, and we identify the instanton configurations responsible for each contribution. We work in terms of the variables $\{\nu, \tau_2, \tau_1, c_0, \psi\}$, which can be converted into the variables $\{t, \phi, \zeta, \tilde{\zeta}, \sigma\}$ used in the previous section using (2.8). The main reason for

this choice is that the variables $\tau = \tau_1 + i\tau_2$ and (c_0, ψ) have simple transformation properties (2.6) under the S-duality group $SL(2, \mathbb{Z}) \subset SL(3, \mathbb{Z})$, which is manifest in the Fourier expansion [44].

3.5.1. Constant terms As already discussed in Section 3.3, the term $\Psi_{0,0}$ in the Fourier expansion, depending only on the “dilaton” parameters ν, τ_2 , corresponds to the “constant terms” with respect to the Borel subgroup (3.32). These terms are discussed in Appendix A, and agree with the analysis of [44]:

$$\begin{aligned}
 (3.43) \quad \Psi_{0,0}(\nu, \tau_2; s_1, s_2) &= \int_0^1 d\tau_1 \int_0^1 dc_0 \int_0^1 d\psi E(\nu, \tau_2, \tau_2, c_0, \psi; s_1, s_2) \\
 &= \nu^{-\frac{2s_1+s_2}{3}} \tau_2^{s_2} + c(s_1) \nu^{-\frac{1}{2} + \frac{s_1-s_2}{3}} \tau_2^{s_3} + c(s_2) \nu^{-\frac{2s_1+s_2}{3}} \tau_2^{1-s_2} \\
 &\quad + c(s_1) c(s_3) \nu^{-\frac{1}{2} + \frac{s_1-s_2}{3}} \tau_2^{1-s_3} + c(s_2) c(s_3) \nu^{\frac{2s_1+s_2}{3}-1} \tau_2^{s_1} \\
 &\quad + c(s_1) c(s_2) c(s_3) \nu^{\frac{2s_1+s_2}{3}-1} \tau_2^{1-s_1},
 \end{aligned}$$

where

$$(3.44) \quad c(s) = \frac{\xi(2s-1)}{\xi(2s)}, \quad \xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$$

and $s_3 = s_1 + s_2 - \frac{1}{2}$. These terms were already discussed in (3.33) for $(s_1, s_2) = (3/2, -3/2)$. We shall not discuss them any further here, except to note that they are consistent with the functional equations (A.7) obeyed by the completed series (A.6).

3.5.2. Abelian Fourier coefficients Let us now proceed to analyze the Abelian Fourier coefficients $\Psi_{p,q}$ with $(p, q) \neq (0, 0)$, starting with the simplest cases $\Psi_{0,q}$ and $\Psi_{p,0}$. As shown in [44],

$$\begin{aligned}
 (3.45) \quad \Psi_{0,q}(\nu, \tau_2) &= \frac{2(2\pi)^{1/2-s_1} c(s_2) c(s_3)}{\xi(2s_1)} \nu^{\frac{s_1+2s_2}{3}-1} \tau_2^{1-s_1} \mu_{1-2s_1}(q) \mathcal{K}_{1/2-s_1}(2\pi|q|\tau_2) \\
 &\quad + \frac{2(2\pi)^{1/2-s_2}}{\xi(2s_2)} \nu^{-\frac{2s_1+s_2}{3}} \tau_2^{1-s_2} \mu_{1-2s_2}(q) \mathcal{K}_{1/2-s_2}(2\pi|q|\tau_2) \\
 &\quad + \frac{2(2\pi)^{1/2-s_3} c(s_1)}{\xi(2s_3)} \nu^{\frac{s_1-s_2-1}{2}} \tau_2^{1-s_3} \mu_{1-2s_3}(q) \mathcal{K}_{1/2-s_3}(2\pi|q|\tau_2),
 \end{aligned}$$

and

$$\begin{aligned}
 \Psi_{p,0}(\nu, \tau_2) &= \frac{2(2\pi)^{1/2-s_1}}{\xi(2s_1)} \nu^{-\frac{2s_1+s_2}{3}} \tau_2^{s_2} \mu_{1-2s_1}(p) \mathcal{K}_{1/2-s_1}(2\pi|p|/\sqrt{\nu\tau_2}) \\
 &+ \frac{2(2\pi)^{1/2-s_2} c(s_1)c(s_3)}{\xi(2s_2)} \nu^{\frac{s_1-s_2}{3} - \frac{1}{2}} \tau_2^{\frac{3}{2}-s_1-s_2} \mu_{1-2s_2}(p) \\
 &\times \mathcal{K}_{1/2-s_2}(2\pi|p|/\sqrt{\nu\tau_2}) \\
 &+ \frac{2(2\pi)^{1/2-s_3} c(s_2)}{\xi(2s_3)} \nu^{-\frac{s_2+2s_1}{3}} \tau_2^{1-s_2} \mu_{1-2s_3}(p) \\
 &\times \mathcal{K}_{1/2-s_3}(2\pi|p|/\sqrt{\nu\tau_2}).
 \end{aligned}
 \tag{3.46}$$

By comparing with (3.42) it is apparent that these have a very similar structure to the minimal Eisenstein series. Using the asymptotic expansion of the Bessel function (3.31), we deduce that in the weak coupling limit $\tau_2 \rightarrow \infty$, the coefficients $\Psi_{0,q}$ contribute to expansion (3.39) by exponentially suppressed contributions of order $e^{-2\pi S_{0,q}}$, with

$$S_{0,q}(\tau) = |q| \tau_2 + i q \tau_1.
 \tag{3.47}$$

This is precisely the instanton action for D(-1) instantons [62,67]. Similarly, the coefficients $\Psi_{p,0}$ encode D5-brane instantons, with classical action

$$S_{p,0}(\nu, \tau_2, c_0) = |p|(\nu\tau_2)^{-1/2} - i p c_0 = |p| \tau_2 V - i p c_0.
 \tag{3.48}$$

From (3.31) of the Bessel function, we may also extract the ‘‘instanton measure’’ $\mu(p, q)$, defined by

$$\Psi_{p,q}(\nu, \tau_2) \sim \tau_2^\alpha \nu^\beta (\text{Re } S_{p,q})^\gamma \mu(p, q) e^{-2\pi(\text{Re } S_{p,q})}
 \tag{3.49}$$

in the weak coupling limit $\tau_2 \rightarrow \infty$, with suitable choices of α, β, γ to absorb the moduli dependence. The prefactors in front of $e^{-2\pi S_{p,q}}$, including the instanton measure, should arise from the external vertices and the fluctuation determinant in the instanton background [62]. For the particular case of D(-1)-D5 instantons with $pq = 0$, we find

$$\mu(0, q) \equiv \frac{c(s_2)c(s_3)}{\xi(2s_1)} \mu_{1-2s_1}(q) + \frac{1}{\xi(2s_2)} \mu_{1-2s_2}(q) + \frac{c(s_1)}{\xi(2s_3)} \mu_{1-2s_3}(q)
 \tag{3.50}$$

and

$$\mu(p, 0) \equiv \frac{1}{\xi(2s_1)} \mu_{1-2s_1}(p) + \frac{c(s_1)c(s_3)}{\xi(2s_2)} \mu_{1-2s_2}(p) + \frac{c(s_2)}{\xi(2s_3)} \mu_{1-2s_3}(p).
 \tag{3.51}$$

We now proceed to analyze the coefficients $\Psi_{p,q}$ for $pq \neq 0$, which we recall were absent for the minimal Eisenstein series. These coefficients may be written as

$$(3.52) \quad \Psi_{p,q}(\nu, \tau_2) = \frac{4\nu^{\frac{s_2-s_1}{6} - \frac{1}{2}} \tau_2^{\frac{s_2-s_1}{2} + \frac{1}{2}}}{\xi(2s_1)\xi(2s_2)\xi(2s_3)} \sum_{d_1|p} \sum_{d_2|\frac{p}{d_1}} d_1^{1-2s_3} d_2^{1-2s_2} \\ \times \sigma_{1-2s_1, 1-2s_3} \left(\frac{p}{d_1 d_2}, |q| \right) (pq)^{s_3 - \frac{1}{2}} \mathcal{I}_{s_1, s_2} (R_{p,q}, \mathbf{x}_{p,q}),$$

where

$$(3.53) \quad R_{p,q} \equiv \left(\frac{p^2 |q|}{\nu} \right)^{1/3}, \quad \mathbf{x}_{p,q} \equiv \tau_2^{-1} \left(\frac{p^2}{\nu q^2} \right)^{1/3} = t^2 \left(\frac{|p|}{|q|} \right)^{2/3}.$$

Here, we have also defined the ‘‘double divisor sum’’

$$(3.54) \quad \sigma_{\alpha, \beta}(n, m) \equiv \sum_{\substack{m=d_1 d_2 d_3, \\ d_1, d_2, d_3 > 0, \gcd(d_3, n)=1}} d_2^\alpha d_3^\beta,$$

and the integral

$$(3.55) \quad \mathcal{I}_{s_1, s_2}(R, \mathbf{x}) \equiv \int_0^\infty K_{s_3-1/2}(2\pi R \mathbf{x}^{-1} \sqrt{1+x}) K_{s_3-1/2}(2\pi R \mathbf{x}^{1/2} \sqrt{1+1/x}) \\ \times x^{\frac{s_2-s_1}{2}} \frac{dx}{x}.$$

At weak 4D coupling $\nu \rightarrow 0$ keeping t fixed, one may use the saddle point approximation of the integral (for details, see Appendix C)

$$(3.56) \quad \mathcal{I}_{s_1, s_2}(R, \mathbf{x}) \sim \frac{\mathbf{x}^{\frac{s_2-s_1+1}{2}}}{\sqrt{6} R^{3/2} (1+\mathbf{x})^{1/4}} \exp \left[-\frac{2\pi R (1+\mathbf{x})^{3/2}}{\mathbf{x}} \right] \\ \times \left(1 + \frac{I_1}{R} + \mathcal{O}(1/R^2) \right).$$

Plugging in the values of $R_{p,q}$ and $\mathbf{x}_{p,q}$ given in (3.53), we find that such terms give exponentially suppressed contributions of order $e^{-2\pi S_{p,q}}$, where

$$(3.57) \quad S_{p,q}(\nu, \tau, c_0) = \left[(\tau_2 V |p|)^{2/3} + (\tau_2 |q|)^{2/3} \right]^{3/2} + i(q\tau_1 - pc_0).$$

We note that the real part of this action is proportional to the mass formula for bound states of D0–D6-branes found in [37–39]. Moreover, in the limit

$q = 0$ or $p = 0$, (3.57) reduces to (3.47) and (3.48). We conclude that general Abelian terms with $pq \neq 0$ correspond to bound states of D(-1)- and D5-brane instantons. Their summation measure, defined as in (3.49), is given by

$$(3.58) \quad \mu(p, q) \equiv \frac{4}{\sqrt{6}} \frac{|p|^{\frac{2s_1+4s_2}{3}-\frac{5}{6}} |q|^{\frac{4s_1+2s_2}{3}-\frac{5}{6}}}{\xi(2s_1)\xi(2s_2)\xi(2s_3)} \sum_{d_1|p} \sum_{d_2|\frac{p}{d_1}} d_1^{1-2s_3} d_2^{1-2s_2} \sigma_{1-2s_1, 1-2s_3} \times \left(\frac{|p|}{d_1 d_2}, |q| \right).$$

According to the general relation between instanton measure and BPS black hole degeneracies proposed in [14], measure (3.58) should be related to the D0–D6 bound state degeneracies, which are known to be encoded in the Mac Mahon function [40]. This would provide a crucial check of our proposal, which we leave for future work.

On the type IIA side, the Abelian terms (p, q) correspond to D2-branes wrapped on the 3-cycle $p\gamma_0 - q\gamma^0$, where γ_0 and γ^0 are the 3-cycles singled out by the large complex structure limit.

3.5.3. Non-Abelian Fourier coefficients Following [44], the general non-Abelian Fourier coefficients are given by

$$(3.59) \quad \tilde{\Psi}_{k,\ell'}(\nu, \tau) = \frac{4\nu^{\frac{s_2-s_1}{6}-\frac{1}{2}}}{\xi(2s_1)\xi(2s_2)\xi(2s_3)} \sum_{q \in \mathbb{Z}} \sum_{d_1|d} \sum_{d_2|\frac{d}{d_1}} d_1^{1-2s_3} d_2^{1-2s_2} \sigma_{1-2s_1, 1-2s_3} \times \left(\frac{d}{d_1 d_2}, |q| \right) [\tau_2]_{-k,p}^{\frac{s_2-s_1}{2}+\frac{1}{2}} (dq)^{s_3-\frac{1}{2}} \mathcal{I}_{s_1, s_2}(R_{d,q}, \mathbf{x}_{d,q}) e^{-2\pi i q [\tau_1]_{-k,p}},$$

where the notations are as follows: $\mathcal{I}_{s_1, s_2}(R, \mathbf{x})$ is defined in (3.55), $d \equiv \text{gcd}(p, k) > 0$,

$$(3.60) \quad R_{d,q} \equiv \left(\frac{d^2 |q|}{\nu} \right)^{1/3}, \quad \mathbf{x}_{d,q} \equiv [\tau_2]_{-k,p}^{-1} \left(\frac{d^2}{\nu q^2} \right)^{1/3},$$

and the variables $[\tau_1]_{-k,p}$ and $[\tau_2]_{-k,p}$ denote the real and imaginary parts of the image of $\tau = \tau_1 + i\tau_2$ under an $\text{SL}(2, \mathbb{Z})$ transformation of the form (2.7),

$$(3.61) \quad \delta = \begin{pmatrix} \alpha & \beta \\ -k' & p' \end{pmatrix}, \quad \delta \cdot \tau = \frac{\alpha\tau + \beta}{-k'\tau + p'} \equiv [\tau_1]_{-k,p} + i[\tau_2]_{-k,p},$$

where $k' = k/d$ and $p' = p/d$ and α, β are two integers such that $\alpha p' + \beta k' = 1$. Since $k \neq 0$, this is usefully rewritten as

$$(3.62) \quad [\tau_2]_{-k,p} = \frac{d^2 \tau_2}{|p - k\tau|^2}, \quad [\tau_1]_{-k,p} = -\frac{d\alpha}{k} + \frac{d^2(p - k\tau_1)}{k|p - k\tau|^2},$$

where we used $\beta = (1 - p'\alpha)/k'$ to derive the second relation. Defining $Q \equiv d^2q$, (3.60) may therefore be rewritten as

$$(3.63) \quad R_{d,q} = \tau_2 t^2 |Q|^{1/3}, \quad \mathbf{x}_{d,q} = \frac{t^2 |p - k\tau|^2}{|Q|^{2/3}}.$$

In type IIA variables suitable for comparison with (3.39), this becomes

$$(3.64) \quad R_{d,q} = e^{\phi/2} t^{1/2} kn, \quad \mathbf{x}_{d,q} = \frac{e^\phi + t^3(m - \zeta)^2}{n^2 t},$$

where we have used (2.8) and defined

$$(3.65) \quad m = \frac{p}{k} \in \mathbb{Z} + \frac{\ell'}{|k|}, \quad n = \text{sgn}(Q) |Q|^{1/3} / k.$$

In particular, (3.59) depends only on the difference $\zeta - m$. Setting $m = 0$, we therefore have

$$(3.66) \quad \begin{aligned} \tilde{\Psi}_{k,\ell'}(t, \phi; \zeta) &= \frac{4t^{\frac{s_2 - s_1 + 3}{2}} e^\phi d^{2(1-s_1)}}{\xi(2s_1)\xi(2s_2)\xi(2s_3)} \sum_{n: (kn)^3 \in d^2\mathbb{Z}} \sum_{d_1|d} \sum_{d_2|\frac{d}{d_1}} d_1^{1-2s_3} d_2^{1-2s_2} \\ &\times \sigma_{1-2s_1, 1-2s_3} \left(\frac{d}{d_1 d_2}, |q| \right) \left(e^\phi + t^3 \zeta^2 \right)^{\frac{s_1 - s_2 - 1}{2}} |kn|^{s_3 - \frac{1}{2}} \mathcal{I}_{s_1, s_2} \\ &\times \left(e^{\phi/2} t^{1/2} kn, \frac{e^\phi + t^3 \zeta^2}{n^2 t} \right) e^{-ik \frac{t^3 n^3 \zeta}{e^\phi + t^3 \zeta^2} + 2\pi i \frac{k^2 n^3 \alpha}{d}}, \end{aligned}$$

where now $d = \text{gcd}(\ell', k)$.

As before, using the saddle point approximation of the integral $\mathcal{I}(R, \mathbf{x})$ (see Appendix C), we find that $\tilde{\Psi}_{k,\ell'}$ contributes to (3.39) with terms which are exponentially suppressed by $e^{-2\pi S_{Q,p,k}}$, where

$$(3.67) \quad S_{Q,p,k} = \frac{\left[(V\tau_2)^{2/3} |p - k\tau|^2 + \tau_2^{2/3} Q^{2/3} \right]^{3/2}}{|p - k\tau|^2} + iQ \frac{p - k\tau_1}{k|p - k\tau|^2} - i(p c_0 + k \psi),$$

or, in type IIA variables,

$$\begin{aligned}
 S_{m,n,k} = & |k| \frac{e^{\phi/2} [e^\phi + t^3(\zeta - m)^2 + tn^2]^{3/2}}{e^\phi + t^3(\zeta - m)^2} - ik \frac{t^3 n^3 (\zeta - m)}{e^\phi + t^3(\zeta - m)^2} \\
 (3.68) \quad & + \frac{i}{2} k (\sigma + \zeta \tilde{\zeta}) - ikm \tilde{\zeta}.
 \end{aligned}$$

In these expressions, the last two terms originate from the phase factors appearing in the Fourier expansion (3.39). From the saddle point approximation (3.56), we may also extract the summation measure defined as in (3.49),

$$\begin{aligned}
 \mu(Q, p, k) \equiv & \frac{4e^{\frac{2\pi i Q \alpha}{ak}}}{\sqrt{6}} \frac{|d|^{\frac{5}{6}-2s_1} |Q|^{\frac{4s_1+2s_2}{3}-\frac{5}{6}}}{\xi(2s_1)\xi(2s_2)\xi(2s_3)} \sum_{d_1|d} \sum_{d_2|\frac{d}{d_1}} d_1^{1-2s_3} d_2^{1-2s_2} \sigma_{1-2s_1, 1-2s_3} \\
 (3.69) \quad & \times \left(\frac{|d|}{d_1 d_2}, |Q/d^2| \right).
 \end{aligned}$$

Contrary to appearances, the limit $k \rightarrow 0$ is smooth: the apparent singularity cancels between the action and the summation measure, as the two terms in (3.62) combine into

$$(3.70) \quad [\tau_1]_{-k,p} = \frac{d[(\alpha p - \beta k)\tau_1 + \beta p - \alpha k|\tau|^2]}{|p - k\tau|^2} \xrightarrow{k \rightarrow 0} \frac{d}{|p|} (\alpha\tau_1 - \beta) = \tau_1,$$

where in the last step we used the fact that $\alpha = 1, \beta = 0, d = |p|$ when $k = 0$. In this limit, we therefore recover action (3.57) of D(-1)-D5 bound states. Moreover, the term with $k \neq 0$ can be recovered from $(Q, p, k) = (d^2q, d, 0)$ by an $SL(2, \mathbb{Z})$ action. It would be interesting to recover (3.68) from the type IIA five-brane action [68], however by duality it is clear that (3.68) must describe the action of k NS5-branes bound to km D2-branes wrapped on γ_0 and $(kn)^3$ D2-branes wrapped on γ^0 .

Therefore, we conclude that (3.59) describes the contribution of general bound states of Q D(-1) instantons and (p, k) five-branes (equivalently, on the type IIA side, NS5-D2 bound states). Thus, despite the fact that the non-Abelian nature of the Heisenberg group prevents us from defining D(-1) and D5-brane charges unambiguously when the NS5-brane charge k is non-zero, we still find that the general term involves a contribution of D(-1), D5 and NS5-brane instantons, with independent charges Q, p, k .

3.6. The minimal theta series as a NS5-brane partition function

The general non-Abelian contribution (3.67) will undoubtedly remind the cognoscente of the minimal representation of $G_{2(2)}$ constructed in Section 3.5 of [47]. Indeed, under the standard embedding $\mathrm{SL}(3, \mathbb{R}) \subset G_{2(2)}$, the minimal representation of $G_{2(2)}$ belongs to the non-spherical supplementary series of $\mathrm{SL}(3, \mathbb{R})$ with infinitesimal parameters $(s_1, s_2) = (2/3, 2/3)$ [69]; although this representation admits no spherical vector, its lowest K -type is indistinguishable from the spherical vector of the principal representation in the strict classical limit. Moreover, the minimal representation of $G_{2(2)}$ is a special instance of the minimal representation of the quasi-conformal group $\mathrm{QConf}(J)$ attached to any cubic Jordan algebra J , in the case where $J = \mathbb{R}$, $\kappa_{111} = 6$. These minimal representations were constructed in [46] for simply laced Lie groups in the split real form (albeit not using the language of Jordan algebras), and more recently for all simple Lie groups in any non-compact real form, in particular the quaternionic real form, in [49, 50, 70]. The latter can all be reached by analytic continuation from the minimal representation for the split real form constructed in [46]. Their lowest K -type is not known in general yet, but from the $G_{2(2)}$ example it is clear that it will be identical in the strict classical limit to the spherical vector for the split real form, found in [46] and displayed in (D.17) below.

Having recalled this representation-theoretic background, we now assume that the HM moduli space $\mathcal{M}_K(Y)$ is given at tree level by the QK symmetric space (2.25), and that a larger discrete symmetry $G(\mathbb{Z}) = \mathrm{QConf}(J, \mathbb{Z}) \supset \mathrm{SL}(3, \mathbb{Z})$ remains unbroken by quantum corrections, and argue that the minimal theta series of $G(\mathbb{Z})$, i.e., the automorphic form attached to the minimal representation¹³ via (3.8), predicts exponential corrections of the form (3.67), where the D-instanton charge Q is now a composite of D3-brane charges N^a labelling the various four-cycles in Y .

To make this more precise, we use expression (D.17) for the approximate spherical vector f_K , and incorporate the moduli dependence by acting with $\rho(e^{-1})$, where $e \in \mathrm{SO}(4) \backslash G_{2(2)}$ is the Iwasawa coset representative in (2.18) and ρ is the minimal representation obtained in Section 3.5.1 of [47] (or rather, its Fourier transform over $(x_0, x_1) \mapsto (x^0, x^1)$). By further relabeling $y = 2\pi k, x^0 = 2\pi p, x^1 = 2\pi N^1$, we then find that the minimal theta series of $G_{2(2)}$ (3.8) predicts exponentially suppressed contributions of order $e^{-2\pi S}$

¹³Early suggestions that the minimal representation is relevant for black hole counting were made in [66, 71–73].

where

$$(3.71) \quad S = \tau_2 V |p - k\tau| \left(1 + \frac{(\tilde{N}^1/t)^2}{|p - k\tau|^2} \right)^{3/2} - i(pc_0 + k\psi + N^1 c_1) + i \frac{(p - k\tau_1)}{k|p - k\tau|^2} \times (\tilde{N}^1)^3 + \frac{3i}{k} b^1 (\tilde{N}^1)^2 - \frac{i}{k} b^1 (pb^1 - kc^1) (3\tilde{N}^1 + pb^1 - kc^1),$$

where $\tilde{N}^1 = N^1 + kc^1 - pb^1$. More generally, for an arbitrary quasi-conformal group $G = \text{QConf}(J)$, the same procedure based on (D.17) and the results in [49, 50] leads to

$$(3.72) \quad S = \tau_2 V |p - k\tau| \left(1 + 3 \frac{(\tilde{N}^a/t^a)^2}{|p - k\tau|^2} + \frac{1}{12} \frac{(\kappa_{abc} t^a \tilde{N}^b \tilde{N}^c)^2}{V^2 |p - k\tau|^4} + \frac{1}{2^2 \cdot 3^2} \frac{(\kappa_{abc} \tilde{N}^a \tilde{N}^b \tilde{N}^c)^2}{V^2 |p - k\tau|^6} \right)^{1/2} - i(pc_0 + k\psi + N^a c_a) + \frac{i}{6} \frac{(p - k\tau_1)}{k|p - k\tau|^2} \kappa_{abc} \tilde{N}^a \tilde{N}^b \tilde{N}^c + \frac{i}{2k} \kappa_{abc} b^a \tilde{N}^b \tilde{N}^c - \frac{i}{6k} \kappa_{abc} b^a (pb^b - kc^b) (3\tilde{N}^c + pb^c - kc^c),$$

where $\tilde{N}^a = N^a + kc^a - pb^a$. Here and in (3.71), the type IIB variables (c^a, c_a, c_0, ψ) are related to $(\zeta^a, \tilde{\zeta}_a, \tilde{\zeta}_0, \psi)$ by the tree-level mirror map, Equation (3.20) in [14]. Of course, (3.72) reduces to (3.71) upon setting $\kappa_{111} = 6$.

Equation (3.72) is recognized as the action of a (p, k) 5-brane bound to D3-branes wrapping $N^a \gamma_a \in H_4(Y, \mathbb{Z})$, with induced D1-brane charge \tilde{N}_a and D(-1)-instanton charge \tilde{Q} given by

$$(3.73) \quad \tilde{N}_a = \frac{1}{6|p - k\tau|} \kappa_{abc} \tilde{N}^b \tilde{N}^c, \quad \tilde{Q} = \frac{1}{6|p - k\tau|^2} \kappa_{abc} \tilde{N}^a \tilde{N}^b \tilde{N}^c.$$

In particular, by the same token as in (3.62), the last term in (3.72) is just the axionic coupling of \tilde{Q} D instantons for vanishing NS5-brane charge, after rotating to the (p, k) 5-brane duality frame. It would be interesting to compare (3.72) with other studies of NS5-instanton corrections to the HM moduli space based on supergravity (see, e.g., [74, 75] and references therein).

In the limit $k \rightarrow 0$, (3.72) reduces to the usual action of a D5–D3–D1–D(-1) bound state,

$$(3.74) \quad S = \tau_2 \left(p^2 V^2 + 3(\tilde{N}^a/t^a)^2 V^2 + 3(\tilde{N}_a t^a)^2 + \tilde{Q}^2 \right)^{1/2} - i(pc_0 + N^a c_a),$$

with charges $(p, \tilde{N}^a, \tilde{N}_a, \tilde{Q})$ given by

(3.75)

$$\tilde{N}^a = N^a - p b^a, \quad \tilde{N}_a = \frac{1}{6|p|} \kappa_{abc} \tilde{N}^b \tilde{N}^c, \quad \tilde{Q} = \frac{1}{6|p|^2} \kappa_{abc} \tilde{N}^a \tilde{N}^b \tilde{N}^c.$$

However, this is not the most general D5–D3–D1–D(-1) instanton correction, since the D1 and D(-1) charges are determined in terms of the D5 and D3 charges (p, N^a) . This reflects the fact that the Abelian Fourier expansion of the minimal theta series has support on “very small” charges satisfying $I_4 = \partial I_4 = \partial^2 I_4 = 0$, where I_4 is the quartic invariant of the 4D duality group $\text{Conf}(J)$ (these conditions generalize the condition $pq = 0$ for the minimal representation of $\text{SL}(3, \mathbb{R})$). Therefore, one should probably look for automorphic forms in the quaternionic discrete series [47, 76, 77], where this restriction does not apply. It is also conceivable that only “very small” charges may contribute to the HM metric, but this does not seem to be required by supersymmetry.

Nevertheless, it is tempting to conjecture that, in cases where the discrete symmetry $G(\mathbb{Z}) = \text{QConf}(J, \mathbb{Z})$ is unbroken quantum mechanically, the minimal theta series of G , or an automorphic form in the quaternionic discrete series of G , may encode the effects of bound states of NS5-brane and D instantons on the HM moduli space. If so, it should be possible to express them as a sum of $\text{SL}(3, \mathbb{Z})$ invariant contributions as in (3.25) and compute the corresponding invariants $n_{k_a}^{(0)}$. Clearly, more work remains to establish this claim. For example, the minimal representation is naturally understood as a submodule of $H^1(\mathcal{Z}_{\mathcal{M}}, \mathcal{O}(-(h+3)/3))$ [76], whereas deformations of the QK space \mathcal{M} are usually controlled by $H^1(\mathcal{Z}_{\mathcal{M}}, \mathcal{O}(2))$ [21, 78]. Nevertheless, this proposal meshes well with ideas expressed in [66], where the minimal representation was related to the topological string amplitude, and in [41, 42], where the topological amplitude was related to the NS5-brane and D5-brane partition functions. It would also be interesting to extend these considerations to CY three-folds which do not have a symmetric moduli space at tree level.

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Appendix A. Constant terms for $SL(3)$ Eisenstein series

In this section, we summarize the constant terms of the Eisenstein series (3.12) and (3.9) along the parabolic subgroups (3.13) and (3.18), based on the general results of Langlands [56, 60], or the explicit computation in [44, 45]. Our notation follows [36].

In general, the constant term of a Γ -invariant function $E(g)$ on G with respect to a parabolic subgroup $P \subset G$ is defined by the integral

$$(A.1) \quad E_P(g) = \int_{N/\Gamma \cap N} dn E(g),$$

where N is the unipotent radical of P . These coefficients provide the “perturbative” part of the automorphic form ($E(g)$ is said to be cuspidal if its constant terms $E_P(g)$ vanish for all parabolic subgroups).

For the principal Eisenstein series defined in (3.12), the constant term along the minimal parabolic (3.13) is a sum of six terms,

$$(A.2) \quad \begin{aligned} E_{P_{\min}}(g; s_1, s_2) &= t_1^{\lambda_1+1} t_2^{\lambda_2} t_3^{\lambda_3-1} &+& t_1^{\lambda_2+1} t_2^{\lambda_1} t_3^{\lambda_3-1} \frac{\xi(\lambda_{12})}{\xi(\lambda_{21})} \\ &+ t_1^{\lambda_1+1} t_2^{\lambda_3} t_3^{\lambda_2-1} \frac{\xi(\lambda_{23})}{\xi(\lambda_{32})} &+& t_1^{\lambda_3+1} t_2^{\lambda_1} t_3^{\lambda_2-1} \frac{\xi(\lambda_{13})}{\xi(\lambda_{31})} \frac{\xi(\lambda_{23})}{\xi(\lambda_{32})} \\ &+ t_1^{\lambda_2+1} t_2^{\lambda_3} t_3^{\lambda_1-1} \frac{\xi(\lambda_{12})}{\xi(\lambda_{21})} \frac{\xi(\lambda_{13})}{\xi(\lambda_{31})} &+& t_1^{\lambda_3+1} t_2^{\lambda_2} t_3^{\lambda_1-1} \frac{\xi(\lambda_{12})}{\xi(\lambda_{21})} \frac{\xi(\lambda_{13})}{\xi(\lambda_{31})} \frac{\xi(\lambda_{23})}{\xi(\lambda_{32})} \end{aligned} .$$

Here, $\lambda_1, \lambda_2, \lambda_3$ are defined by

$$(A.3) \quad 2s_1 = 1 + \lambda_2 - \lambda_3, \quad 2s_2 = 1 + \lambda_1 - \lambda_2, \quad \lambda_1 + \lambda_2 + \lambda_3 = 0,$$

λ_{ij} denotes the difference $\lambda_i - \lambda_j$, the real variables t_1, t_2, t_3 are related to the Abelian part of the Iwasawa decomposition (2.1) via

$$(A.4) \quad (t_1, t_2, t_3) = (\nu^{-1/3}, \nu^{1/6} \sqrt{\tau_2}, \nu^{1/6} / \sqrt{\tau_2}) = (e^{\phi/2} \sqrt{t}, 1/t, e^{-\phi/2} \sqrt{t}),$$

and

$$(A.5) \quad \xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$$

is the “completed” Riemann Zeta function, satisfying $\xi(s) = \xi(1-s)$. The Weyl group acts by permutations on $(\lambda_1, \lambda_2, \lambda_3)$, and leaves invariant $\xi(\lambda_{21})\xi(\lambda_{31})\xi(\lambda_{32})E_{P_{\min}}(g; s_1, s_2)$. This invariance extends to the “completed” Eisenstein series

$$(A.6) \quad \mathcal{E}(g; \lambda_i) = \xi(\lambda_{21}) \xi(\lambda_{31}) \xi(\lambda_{32}) E(g; s_1, s_2),$$

which is invariant under permutations of $(\lambda_1, \lambda_2, \lambda_3)$, corresponding in the $(s_1, s_2, s_3 \equiv s_1 + s_2 - \frac{1}{2})$ variables to

$$(A.7) \quad \begin{aligned} (s_1, s_2) &\sim (1-s_2, 1-s_1) \sim (1-s_1, s_3) \sim (s_3, 1-s_2) \\ &\sim (1-s_3, s_1) \sim (s_2, 1-s_3). \end{aligned}$$

On the other hand, the constant terms around the maximal parabolic (3.18) are given by

$$(A.8) \quad \begin{aligned} E_{P_{\max}}(g; s_1, s_2) &= t_1^{\lambda_{12} - \frac{1}{2}\lambda_{32} + \frac{3}{2}} \mathcal{E}_{\lambda_{32}}(\tau) + \frac{\xi(\lambda_{12})}{\xi(\lambda_{21})} t_1^{\lambda_{21} - \frac{1}{2}\lambda_{31} + \frac{3}{2}} \mathcal{E}_{\lambda_{31}}(\tau) \\ &+ \frac{\xi(\lambda_{23})\xi(\lambda_{13})}{\xi(\lambda_{32})\xi(\lambda_{31})} t_1^{\lambda_{31} - \frac{1}{2}\lambda_{21} + \frac{3}{2}} \mathcal{E}_{\lambda_{21}}(\tau), \end{aligned}$$

where $\tau = a_3 + i(t_2/t_3)$. Here $\mathcal{E}_s(\tau)$ is related to the $\mathrm{SL}(2, \mathbb{Z})$ Eisenstein series (3.29) via $\mathcal{E}_{1-2s}(\tau) = E_s(\tau)/(2\zeta(2s))$, and satisfies the functional relation

$$(A.9) \quad \xi(s) \mathcal{E}_s(\tau) = \xi(-s) \mathcal{E}_{-s}(\tau).$$

The minimal Eisenstein series (3.9) can be obtained from the principal Eisenstein series (3.12) by taking the limit

$$(A.10) \quad E(g; s) = 2\zeta(2s) \lim_{s_2 \rightarrow 0} E(g; s_1 = s, s_2)$$

or more generally by keeping the leading term in the limit $\lambda_{ij} \rightarrow 1$ for any $i \neq j$. Thus, its constant terms are given by

$$\begin{aligned}
 \frac{E_{P_{\min}}(g; s)}{2\zeta(2s)} &= \left(\frac{t_1 t_2}{t_3^2}\right)^{\frac{2s}{3}} + \frac{\xi(2s-1)t_2}{\xi(1-2s)t_3} \left(\frac{t_1 t_3}{t_2^2}\right)^{\frac{2s}{3}} \\
 (A.11) \qquad &+ \frac{\xi(2s-2)\xi(2s-1)t_1 t_2}{\xi(1-2s)\xi(2-2s)t_3^2} \left(\frac{t_1 t_3}{t_2^2}\right)^{\frac{2s}{3}}, \\
 \frac{E_{P_{\max}}(g; s)}{2\zeta(2s)} &= \frac{t_1^s E_s(\tau)}{2\zeta(2s)} + t_1^{3-2s} \frac{\xi(2s-2)\xi(2s-1)}{\xi(1-2s)\xi(2-2s)},
 \end{aligned}$$

where in the second line we used the identity $E_0(\tau) = -1$ [59].

It is straightforward if tedious to check that all terms in these expansions have the same values (3.15) of the quadratic and cubic Casimirs, where \mathcal{C}_2 and \mathcal{C}_3 are given by

$$\begin{aligned}
 \mathcal{C}_2 &= -\frac{1}{2}(E_p F_p + F_p E_p + E_q F_q + F_q E_q + F E + E F) \\
 &\quad - \frac{1}{3}(H_p^2 + H_p H_q + H_q^2), \\
 (A.12)
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{C}_3 &= -\frac{1}{4}(3 E_p F_p H_p + 6 E_p F_p H_q - 9 E_p E_q F + 3 E_p H_p F_p + 6 E_p H_q F_p \\
 &\quad - 9 E_p F E_q + 3 F_p E_p H_p + 6 F_p E_p H_q + 9 F_p F_q E + 3 F_p H_p E_p \\
 &\quad + 6 F_p H_q E_p + 9 F_p E F_q - 9 E_q E_p F - 6 E_q F_q H_p - 3 E_q F_q H_q \\
 &\quad - 6 E_q H_p F_q - 3 E_q H_q F_q - 9 E_q F E_p + 9 F_q F_p E - 6 F_q E_q H_p \\
 &\quad - 3 F_q E_q H_q - 6 F_q H_p E_q - 3 F_q H_q E_q + 9 F_q E F_p + 3 H_p E_p F_p \\
 &\quad + 3 H_p F_p E_p - 6 H_p E_q F_q - 6 H_p F_q E_q - 4 H_p H_p H_p - 2 H_p H_p H_q \\
 &\quad - 2 H_p H_q H_p + 2 H_p H_q H_q + 3 H_p F E + 3 H_p E F + 6 H_q E_p F_p \\
 &\quad + 6 H_q F_p E_p - 3 H_q E_q F_q - 3 H_q F_q E_q - 2 H_q H_p H_p + 2 H_q H_p H_q \\
 &\quad + 2 H_q H_q H_p + 4 H_q H_q H_q - 3 H_q F E - 3 H_q E F - 9 F E_p E_q \\
 &\quad - 9 F E_q E_p + 3 F H_p E - 3 F H_q E + 3 F E H_p - 3 F E H_q + 9 E F_p F_q \\
 &\quad + 9 E F_q F_p + 3 E H_p F - 3 E H_q F + 3 E F H_p - 3 E F H_q). \\
 (A.13)
 \end{aligned}$$

Appendix B. Fourier expansion of the minimal Eisenstein series

As an illustration of the general principle explained in Section 3.4, let us compute the non-Abelian Fourier expansion of the minimal Eisenstein series (3.9):

$$(B.1) \quad E(g, s) = \sum' \left[t e^{-\phi} \left(m_1 + \zeta m_2 - \frac{1}{2}(\sigma - \zeta \tilde{\zeta}) m_3 \right)^2 + \frac{1}{t^2} \left(m_2 + \tilde{\zeta} m_3 \right)^2 + t e^{\phi} m_3^2 \right]^{-s},$$

where the sum runs over $(m_1, m_2, m_3) \in \mathbb{Z}^3 \setminus (0, 0, 0)$. We first split

$$(B.2) \quad E(g, s) = E^{(0)} + E^{(1)},$$

where the first term is the contribution with $m_3 = 0$ (and therefore $(m_1, m_2) \in \mathbb{Z}^2 \setminus (0, 0)$) and the second term is the one with $m_3 \neq 0$. The first term is proportional to the standard $SL(2, \mathbb{Z})$ Eisenstein series (3.29),

$$(B.3) \quad E^{(0)} = \nu^{-s/3} \left[2\zeta(2s)\tau_2^s + 2\sqrt{\pi}\tau_2^{1-s} \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta(2s-1) + \frac{2\pi^s \tau_2^s}{(2\pi)^{1/2-s} \Gamma(s)} \right. \\ \left. \times \sum_{\tilde{m}_1 \neq 0} \sum_{m_2 \neq 0} |\tilde{m}_1|^{2s-1} \mathcal{K}_{s-1/2}(2\pi\tau_2|\tilde{m}_1 m_2|) e^{2\pi i \tilde{m}_1 m_2 \tau_1} \right].$$

The last term in the bracket corresponds to the Abelian Fourier coefficient with $(p, q) = (0, \tilde{m}_1 m_2)$, corresponding to D(-1) instantons, while the first two terms reproduce the first two constant terms on the first line of (A.8).

In the second term of (B.2), the sum over (m_1, m_2) runs over \mathbb{Z}^2 without restriction: thus we may perform a Poisson resummation over (m_1, m_2) by using the standard integral representation of the summand (see, e.g., [59]),

$$(B.4) \quad M^{-s} = \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{du}{u^{s+1}} e^{-\frac{\pi}{u} M}.$$

After Poisson resummation we then obtain

$$(B.5) \quad E^{(1)} = \frac{\pi^s \sqrt{t} e^{\phi/2}}{\Gamma(s)} \sum_{\tilde{m}_1, \tilde{m}_2} \sum'_{m_3} \int_0^\infty \frac{du}{u^s} \exp \left[-\frac{\pi u}{t} e^\phi \tilde{m}_1^2 - \pi u t^2 (\tilde{m}_2 - \zeta \tilde{m}_1)^2 - \frac{\pi t}{u} e^\phi m_3^2 - 2\pi i \left(\tilde{\zeta} \tilde{m}_2 m_3 - \frac{1}{2} \tilde{m}_1 m_3 (\sigma + \zeta \tilde{\zeta}) \right) \right].$$

The term with $(\tilde{m}_1, \tilde{m}_2) = (0, 0)$ leads to a Gamma-type integral,

$$(B.6) \quad E^{(1)} = 2\pi \left(t e^\phi \right)^{\frac{3}{2}-s} \frac{\Gamma(s-1)\zeta(2s-2)}{\Gamma(s)} + E^{(2)}$$

while the one with $(\tilde{m}_1, \tilde{m}_2) \neq (0, 0)$ leads to a Bessel function:

$$(B.7) \quad E^{(2)} = \frac{2\pi^s (t e^\phi)^{\frac{3}{2}-s}}{(2\pi)^{s-1}\Gamma(s)} \sum_{\tilde{m}_1, \tilde{m}_2} \sum'_{m_3} |m_3|^{2(1-s)} e^{-2\pi i \tilde{\zeta} \tilde{m}_2 m_3 + i\pi \tilde{m}_1 m_3 (\sigma + \zeta \tilde{\zeta})} \times \mathcal{K}_{1-s} \left(2\pi e^{\phi/2} |m_3| \sqrt{e^\phi \tilde{m}_1^2 + t^3 (\tilde{m}_2 - \zeta \tilde{m}_1)^2} \right)$$

or, in a manifestly $SL(2, \mathbb{Z})$ -invariant form,

$$(B.8) \quad E^{(2)} = \frac{2\pi^s \nu^{-1+\frac{2}{3}s}}{(2\pi)^{s-1}\Gamma(s)} \sum_{\tilde{m}_1, \tilde{m}_2} \sum'_{m_3} |m_3|^{2(1-s)} e^{-2\pi i m_3 (\tilde{m}_1 \psi + \tilde{m}_2 c_0)} \times \mathcal{K}_{1-s} \left(2\pi \frac{|m_3|}{\sqrt{\nu}} \cdot \frac{|\tilde{m}_2 - \tau \tilde{m}_1|}{\sqrt{\tau_2}} \right).$$

The term with $\tilde{m}_1 = 0$ is an Abelian Fourier coefficient with $(p, q) = (\tilde{m}_2 m_3, 0)$, corresponding to D5-brane instantons:

$$(B.9) \quad E^{(3)} = \frac{2\pi^s (t e^\phi)^{\frac{3}{2}-s}}{(2\pi)^{s-1}\Gamma(s)} \sum_{\tilde{m}_2} \sum'_{m_3} |m_3|^{2(1-s)} e^{-2\pi i \tilde{\zeta} \tilde{m}_2 m_3} \mathcal{K}_{1-s} \left(2\pi e^{\phi/2} t^{3/2} |\tilde{m}_2 m_3| \right).$$

The general term with $\tilde{m}_1 \neq 0$ can be recast as (3.39), by identifying

(B.10)

$$\tilde{m}_1 m_3 = -k, \quad \tilde{m}_2 m_3 = -km, \quad m \in \mathbb{Z} + \frac{\ell'}{|k|},$$

(B.11)

$$\tilde{\Psi}_{k,\ell'}(\zeta) = \left(\sum_{d|k} d^{2(1-s)} \right) \frac{2\pi^s (t e^\phi)^{\frac{3}{2}-s}}{(2\pi)^{s-1} \Gamma(s)} \mathcal{K}_{1-s} \left(2\pi|k| e^{\phi/2} \sqrt{e^\phi + \zeta^2 t^3} \right).$$

B.1. Dual expansion

Alternatively, one may arrive at the non-Abelian Fourier expansion (3.36) by returning to (B.1), extracting the term with $m_2 = m_3 = 0$ and performing a Poisson resummation over the single variable m_1 :

(B.12)

$$E(g, s) = 2\zeta(2s) e^{s\phi} t^{-s} + \sum_{\tilde{m}_1} \sum_{m_2, m_3}^{\prime} \frac{e^{\phi/2}}{\sqrt{t}} \frac{\pi^s}{\Gamma(s)} \int_0^\infty \frac{du}{u^{s+\frac{1}{2}}} \exp \left[-\frac{\pi u}{t} e^\phi \tilde{m}_1^2 - \frac{\pi}{ut^2} \left(m_2 + \tilde{\zeta} m_3 \right)^2 - \frac{\pi t}{u} e^\phi m_3^2 + 2\pi i \tilde{m}_1 \left(\zeta m_2 - \frac{1}{2}(\sigma - \zeta \tilde{\zeta}) m_3 \right) \right].$$

For $\tilde{m}_1 = 0$, one may similarly extract the term with $m_3 = 0$, Poisson resum over m_2 and extract the term $\tilde{m}_2 = 0$, to get

(B.13)

$$\frac{2\pi^s \Gamma(s - \frac{1}{2}) e^{\phi/2} t^{2s-\frac{3}{2}}}{\pi^{s-\frac{1}{2}} \Gamma(s)} \zeta(2s - 1) + \frac{2\pi^s \Gamma(s - 1) (t e^\phi)^{\frac{3}{2}-s}}{\pi^{s-1} \Gamma(s)} \zeta(2s - 2) + \frac{2\pi^s (e^\phi t)^{\frac{3}{2}-s}}{(2\pi)^{s-1} \Gamma(s)} \sum_{\tilde{m}_2} \sum_{m_3}^{\prime} |m_3|^{2(1-s)} \mathcal{K}_{1-s} \left(2\pi e^{\phi/2} t^{3/2} |\tilde{m}_2 m_3| \right) e^{-2\pi i \tilde{\zeta} \tilde{m}_2 m_3}.$$

The last term is the Abelian Fourier coefficient with $(p, q) = (\tilde{m}_2 m_3, 0)$, identical to (B.9), corresponding to D5-instantons. For $\tilde{m}_1 \neq 0$, the integral over u is of Bessel type, leading to

$$\frac{2\pi^s e^{s\phi}}{t^s (2\pi)^{1/2-s} \Gamma(s)} \sum_{\tilde{m}_1} \sum_{m_2, m_3}^{\prime} |\tilde{m}_1|^{2s-1} e^{2\pi i \tilde{m}_1 (\zeta m_2 - \frac{1}{2}(\sigma - \zeta \tilde{\zeta}) m_3)} \times \mathcal{K}_{s-1/2} \left(\frac{2\pi e^{\phi/2}}{t^{3/2}} |\tilde{m}_1| \sqrt{\left(m_2 + \tilde{\zeta} m_3 \right)^2 + t^3 e^\phi m_3^2} \right).$$

(B.14)

For $m_3 = 0$, this reduces to the Abelian Fourier coefficient $(p, q) = (0, \tilde{m}_1 m_2)$, corresponding to D(-1)-instantons, identical to the last term in (B.3),

$$(B.15) \quad \frac{2 \pi^s e^{s\phi}}{t^s (2\pi)^{1/2-s} \Gamma(s)} \sum'_{\tilde{m}_1} \sum'_{m_2} |\tilde{m}_1|^{2s-1} \mathcal{K}_{s-1/2} \left(\frac{2\pi e^{\phi/2}}{t^{3/2}} |\tilde{m}_1 m_2| \right) e^{2\pi i \tilde{m}_1 m_2 \zeta}.$$

For the general term with $\tilde{m}_1 \neq 0$ and $m_3 \neq 0$, identifying

$$(B.16) \quad \tilde{m}_1 m_3 = k, \quad \tilde{m}_1 m_2 = -kn, \quad n \in \mathbb{Z} + \frac{l}{|k|},$$

we recognize the non-Abelian Fourier coefficient (3.36) with

$$(B.17) \quad \Psi_{k,\ell}(\tilde{\zeta}) = \left(\sum_{d|k} d^{2s-1} \right) \frac{2 \pi^s e^{s\phi}}{(2\pi)^{1/2-s} \Gamma(s) t^s} \mathcal{K}_{s-1/2} \left(2\pi |k| e^{\phi/2} \sqrt{e^\phi + t^{-3} \tilde{\zeta}^2} \right).$$

Thus, we have reproduced the non-Abelian Fourier coefficients summarized in (3.41).

Appendix C. Asymptotic expansion of the integral \mathcal{I}

In this appendix, we discuss the properties of an integral which is a key ingredient for the Fourier expansion, as it enters coefficients (3.52) and (3.59):

$$(C.1) \quad \begin{aligned} \mathcal{I}_{s_1, s_2}(R, \mathbf{x}) &= \int_0^\infty K_{s_3-1/2}(2\pi R \mathbf{x}^{-1} \sqrt{1+x}) K_{s_3-1/2}(2\pi R \mathbf{x}^{1/2} \sqrt{1+1/x}) \\ &\times x^{\frac{s_2-s_1}{2}} \frac{dx}{x}. \end{aligned}$$

First, we note the functional equation

$$(C.2) \quad \mathcal{I}_{s_1, s_2}(R, \mathbf{x}) = \mathcal{I}_{s_2, s_1} \left(\frac{R}{\sqrt{\mathbf{x}}}, \frac{1}{\mathbf{x}} \right),$$

which, at the origin of moduli space $\tau_2 = \nu = 1$, amounts to exchanging (p, q) in (3.53).

In order to find the semi-classical interpretation of the Fourier coefficients, we shall be interested in the limit $R \rightarrow \infty$ keeping \mathbf{x} fixed. In this regime, integral (C.1) can be evaluated in the saddle point approximation.

For large argument, the Bessel function may be approximated by

$$(C.3) \quad K_s(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \left(1 + \frac{4s^2 - 1}{8x} + \mathcal{O}(1/x^2) \right).$$

To leading order, the integral then simplifies to

$$(C.4) \quad \mathcal{I}_{s_1, s_2}(R, \mathbf{x}) \sim \frac{\mathbf{x}^{1/4}}{4R} \int_0^\infty \frac{x^{\frac{s_2 - s_1}{2} - \frac{3}{4}} e^{-2\pi S(x)}}{\sqrt{1+x}} dx,$$

where $S(x)$ is given by

$$(C.5) \quad S(x) = R \left(\mathbf{x}^{-1} \sqrt{1+x} + \mathbf{x}^{1/2} \sqrt{1 + \frac{1}{\mathbf{x}}} \right).$$

The exponent is extremized at $x = \mathbf{x}$, with

$$(C.6) \quad S(\mathbf{x}) = \frac{R(1 + \mathbf{x})^{3/2}}{\mathbf{x}}, \quad \partial_x^2 S(\mathbf{x}) = \frac{3R}{4\mathbf{x}^2 \sqrt{1 + \mathbf{x}}}.$$

The saddle point approximation is then

$$(C.7) \quad \begin{aligned} \mathcal{I}_{s_1, s_2}(R, \mathbf{x}) &\sim \frac{\mathbf{x}^{\frac{s_2 - s_1 - 1}{2}}}{4R\sqrt{1 + \mathbf{x}}} \left[\frac{1}{2} \partial_x^2 S(\mathbf{x}) \right]^{-1/2} \exp(-2\pi S(\mathbf{x})) \left(1 + \frac{I_1}{R} + \mathcal{O}(1/R^2) \right) \\ &\sim \frac{\mathbf{x}^{\frac{s_2 - s_1 + 1}{2}}}{\sqrt{6} R^{3/2} (1 + \mathbf{x})^{1/4}} \exp\left(-\frac{2\pi R(1 + \mathbf{x})^{3/2}}{\mathbf{x}}\right) \left(1 + \frac{I_1}{R} + \mathcal{O}(1/R^2) \right). \end{aligned}$$

The subleading term I_1 can be computed by exponentiating the prefactor,¹⁴

$$(C.8) \quad \tilde{S}(x) = S(x) - \frac{1}{2\pi} \log \left[x^{\frac{s_2 - s_1}{2} - \frac{3}{4}} / \sqrt{1+x} \right]$$

and expanding around the perturbed saddle point,

$$(C.9) \quad x = \mathbf{x} - \frac{\mathbf{x}}{6\pi R \sqrt{1 + \mathbf{x}}} (3 + 5\mathbf{x} + 2(1 + \mathbf{x})(s_1 - s_2)) + \frac{1}{R^2} \delta x.$$

¹⁴Note that the subleading term in the Bessel function does not contribute at this order.

We find

$$\begin{aligned}
 \text{(C.10)} \quad \tilde{S}(x) &= \tilde{S}(\mathbf{x}) + \frac{3\delta x^2}{8\mathbf{x}^2\sqrt{\mathbf{x}+1}} - \frac{\delta x^3(7\mathbf{x}+5)}{16\left(\mathbf{x}^3(\mathbf{x}+1)^{3/2}\sqrt{R}\right)} + \frac{\delta x^2}{128\pi R\mathbf{x}^4(\mathbf{x}+1)^{5/2}} \\
 &\times \left(\pi\delta x^2(59\mathbf{x}^2+85\mathbf{x}+35) + 4\mathbf{x}^2(\mathbf{x}+1)^{3/2}(2s_1(5\mathbf{x}+3) \right. \\
 &\left. - 2s_2(5\mathbf{x}+3) + 25\mathbf{x}+9) \right) + \dots
 \end{aligned}$$

Expanding the non-Gaussian piece and performing the Gaussian integration term by term, we find that the leading quantum correction is given by

$$\begin{aligned}
 \text{(C.11)} \quad I_1 &= \frac{2048\mathbf{x}^8(\mathbf{x}+1)^{7/2}}{2187\pi R} \left(-\frac{81(2s_1(5\mathbf{x}+3) - 2s_2(5\mathbf{x}+3) + 25\mathbf{x}+9)}{128\mathbf{x}^4(\mathbf{x}+1)^3} \right. \\
 &\left. - \frac{27(59\mathbf{x}^2+85\mathbf{x}+35)}{16\mathbf{x}^2(\mathbf{x}+1)^{7/2}} + \frac{15(7\mathbf{x}+5)^2}{(\mathbf{x}+1)^3} \right).
 \end{aligned}$$

Appendix D. Non-Abelian Fourier expansions and representation theory

In this appendix we take a representation theoretic viewpoint on the non-Abelian Fourier expansions discussed in Section 3.4. The starting point is that, on the non-Abelian Fourier expansion (3.36), the Heisenberg algebra acts as

$$\text{(D.1)} \quad E_p = -2\pi i k n, \quad E_q = -\partial_n, \quad E = -2\pi i k,$$

while on the dual Fourier expansion (3.39),

$$\text{(D.2)} \quad E_p = \partial_m, \quad E_q = 2\pi i k m, \quad E = 2\pi i k.$$

More generally, when $\Psi(t, \phi; \zeta, \tilde{\zeta}, \sigma)$ is an automorphic form for $SL(3, \mathbb{Z})$, the non-Abelian expansions (3.36) and (3.39) correspond to two different choices of polarization in writing Ψ as a matrix element (3.8), where either (E_p, E) or (E_q, E) have been diagonalized. Note that with the exception of the minimal representation, (E_p, E) do not form a complete basis of commuting operators, which is responsible for the appearance of additional quantum numbers such as q in (3.59).

D.1. Minimal Eisenstein series

Using this observation, it is easy to see that the non-Abelian Fourier expansion of the minimal Eisenstein series (3.9) in polarization (3.39) can be written as an inner product (3.8) of a $G(\mathbb{Z})$ invariant vector $f_{\mathbb{Z}}$ with the transformed spherical vector $\rho(g^{-1})f_K$, where ρ acts on functions of two variables $y = 2\pi k$ and $x_0 = 2\pi km$ via

$$(D.3) \quad \begin{aligned} E_p &= y\partial_{x_0}, & F_p &= -x_0\partial_y, \\ E_q &= ix_0, & F_q &= -i(x_0\partial_{x_0} + y\partial_y + (3-2s))\partial_{x_0} \\ E &= iy, & F &= -i(x_0\partial_{x_0} + y\partial_y + (3-2s))\partial_y \\ H_p &= x_0\partial_{x_0} - y\partial_y, & H_q &= -2x_0\partial_{x_0} - y\partial_y - (3-2s), \end{aligned}$$

and the spherical vector f_K and the $G_{\mathbb{Z}}$ invariant distribution are given by [58]

$$(D.4) \quad f_K = \mathcal{K}_{1-s}\left(\sqrt{y^2 + x_0^2}\right), \quad f_{\mathbb{Z}} = \mu_{2-2s}(y, x_0).$$

Similarly, the non-Abelian Fourier expansion in the polarization (3.36) can be written as (3.8) where ρ is now the representation on functions of two variables $y = -2\pi k$ and $x_0 = -2\pi kn$,

$$(D.5) \quad \begin{aligned} E_p &= ix_0, & F_p &= -i(x_0\partial_{x_0} + y\partial_y + 2s)\partial_{x_0} \\ E_q &= -y\partial_{x_0}, & F_q &= x_0\partial_y, \\ E &= iy, & F &= -i(x_0\partial_{x_0} + y\partial_y + 2s)\partial_y \\ H_q &= x_0\partial_{x_0} - y\partial_y, & H_p &= -2x_0\partial_{x_0} - y\partial_y - 2s, \end{aligned}$$

obtained from the previous one by Fourier transform over y and x_0 . The spherical vector is now

$$(D.6) \quad f_K = \mathcal{K}_{s-\frac{1}{2}}\left(\sqrt{y^2 + x_0^2}\right), \quad f_{\mathbb{Z}} = \mu_{2s-1}(y, x_0),$$

in agreement with $\Psi_{k,\ell}$ in (3.41). The spherical vector f_K in a representation where the Heisenberg algebra takes the canonical form (3.35) or (3.34) is sometimes called the “generalized Whittaker vector” in the mathematics literature.

D.2. Principal Eisenstein series

The principal Eisenstein series (3.12) is attached to the general continuous representation

$$\begin{aligned} E_q &= -\partial_x + w\partial_v, & F_q &= -x^2\partial_x - v\partial_w + 2s_1 x, \\ E_p &= \partial_w, & F_p &= w^2\partial_w + vw\partial_v - (v + xw)\partial_x + 2s_2 w, \\ E &= \partial_v, & F &= v^2\partial_v + vw\partial_w + x(v + xw)\partial_x + 2(s_1 + s_2)v - (1 - 2s_1)xw, \\ H_q &= 2x\partial_x + v\partial_v - w\partial_w + 2s_1, & H_p &= -x\partial_x + v\partial_v + 2w\partial_w + 2s_2 \end{aligned}$$

with spherical vector

$$(D.7) \quad f_K = [1 + x^2 + (v + xw)^2]^{-s_1} [1 + v^2 + w^2]^{-s_2}.$$

Equivalently, it can be written in such a way that the Heisenberg generators are represented as in (3.35), and the $SL(2, \mathbb{Z})$ action acts linearly on (y, x_0) :

$$(D.8) \quad \begin{aligned} E_p &= y\partial_{x_0}, & E_q &= ix_0, & E &= iy, & F_p &= -x_0\partial_y + i\frac{x_1^3}{y^2}, \\ H_p &= x_0\partial_{x_0} - y\partial_y, & H_q &= -2x_0\partial_{x_0} - y\partial_y - x_1\partial_{x_1} + 2(2s_1 + s_2 - 3), \\ F_q &= -i(x_0\partial_{x_0} + y\partial_y + x_1\partial_{x_1} + 2)\partial_{x_0} + (4 - 4s_1 - 2s_2) \left(\frac{y}{9x_1} \partial_{x_1}^2 - i\partial_0 \right) \\ &\quad + \frac{2}{27}(3s_1 - 2)(6s_1 + 6s_2 - 7) \frac{y}{x_1^2} \partial_{x_1} + \frac{y}{27} \partial_{x_1}^3 \end{aligned}$$

and $F = [F_p, F_q]$. For $(s_1, s_2) = 0$, this reproduces the representation obtained in [36] by restricting the minimal representation of E_6 constructed in [46] to singlets of the first two factors in the maximal subgroup $SL(3) \times SL(3) \times SL(3) \subset E_6$. For $(s_1, s_2) = (2/3, 2/3)$ one recovers instead the minimal representation of G_2 considered in [47, 79].

We can now frame the non-Abelian Fourier expansion in the general framework (3.8), and determine the real spherical vector for the principal series away from the semi-classical limit. For this purpose, we change of variables to

$$(D.9) \quad y = -k, \quad x_0 = p, \quad x_1 = (d^2q)^{1/3},$$

and work at the origin of moduli space where $\tau_1 = 0, \tau_2 = \nu = 1$, such that

$$(D.10) \quad [\tau_2]_{-k,p} = \frac{d^2}{y^2 + x_0^2}, \quad R_{d,q} = x_1, \quad \mathbf{x}_{d,q} = \frac{y^2 + x_0^2}{x_1^2}.$$

Moreover, the phase factor in the non-Abelian term of (3.59) becomes

$$(D.11) \quad e^{-2\pi i q[\tau_1]_{-k,p}} = e^{\frac{2\pi i q d \alpha}{k}} e^{\frac{2\pi i p q}{k(k^2+p^2)}} = e^{\frac{2\pi i x_1^3 \alpha}{dk}} e^{-\frac{2\pi i x_0 x_1^3}{y(y^2+x_0^2)}}.$$

We can therefore write the non-Abelian part of the expansion at the origin of moduli space as the overlap

$$(D.12) \quad E_{NA}(1; s_1, s_2) = \sum_{(y, x_0, x_1^3) \in \mathbb{Z}^* \times \mathbb{Z} \times \mathbb{Z}} f_{\mathbb{Z}}(y, x_0, x_1) f_K(y, x_0, x_1) + \dots,$$

where the real spherical vector is given by

$$(D.13) \quad f_K(y, x_0, x_1) = (y^2 + x_0^2)^{\frac{1}{2}(s_1 - s_2 - 1)} x_1^{3(s_1 + s_2 - 1)} e^{-\frac{2\pi i x_0 x_1^3}{y(y^2 + x_0^2)}} \int_0^\infty K_{s_3 - \frac{1}{2}} \\ \times \left(\frac{2\pi x_1^3}{y^2 + x_0^2} \sqrt{1+x} \right) K_{s_3 - \frac{1}{2}} \left(2\pi \sqrt{(y^2 + x_0^2) \left(1 + \frac{1}{x}\right)} \right) x^{\frac{s_2 - s_1}{2}} \frac{dx}{x},$$

the summation measure (or “adelic spherical vector”) is

$$(D.14) \quad f_{\mathbb{Z}}(y, x_0, x_1) \equiv \frac{4e^{\frac{2\pi i x_1^3 \alpha}{dk}}}{\sqrt{6}} \frac{|d|^{\frac{5}{6} - 2s_1} |x_1|^{4s_1 + 2s_2 - \frac{5}{2}}}{\xi(2s_1)\xi(2s_2)\xi(2s_3)} \sum_{d_1|d} \sum_{d_2|\frac{d}{d_1}} d_1^{1-2s_3} d_2^{1-2s_2} \\ \times \sigma_{1-2s_1, 1-2s_3} \left(\frac{d}{d_1 d_2}, \frac{x_1^3}{d^2} \right),$$

and the ellipses stand for degenerate contributions with support at $y = 0$. In (D.14), we recall that $d \equiv \gcd(y, x_0)$, and that $f_{\mathbb{Z}}(y, x_0, x_1)$ vanishes unless d^2 divides x_1^3 . We note that the real and p -adic spherical vectors for the principal series of $SL(n, \mathbb{R})$ for any n have been obtained in [81–84]. It would be interesting to see how (D.14) emerges as a product of the p -adic spherical vectors over all primes.

The spherical vector simplifies considerably in the limit where y, x_0, x_1 are scaled to infinity with fixed ratio: in this case the saddle point approximation (C.7) becomes

$$(D.15) \quad \mathcal{I}(y, x_0, x_1) \sim \frac{(y^2 + x_0^2)^{\frac{s_2 - s_1 + 1}{2}} x_1^{s_1 - s_2 - 2}}{(y^2 + x_0^2 + x_1^2)^{1/4}} \exp \left[-\frac{2\pi(y^2 + x_0^2 + x_1^2)^{3/2}}{y^2 + x_0^2} \right],$$

and the spherical vector simplifies to

$$(D.16) \quad f_K(y, x_0, x_1) \sim \frac{x_1^{4s_1+2s_2-5}}{(y^2+x_0^2+x_1^2)^{1/4}} \exp \left[-\frac{2\pi(y^2+x_0^2+x_1^2)^{3/2}}{y^2+x_0^2} - \frac{2\pi i x_0 x_1^3}{y(y^2+x_0^2)} \right].$$

As a consistency check, we note that in the special case $(s_1, s_2) = (0, 0)$ (i.e., $(\lambda_{23}, \lambda_{21}) = (1, 1)$), this result agrees with the semi-classical spherical vector of the principal series representation of $SL(3, \mathbb{R})$ obtained by restricting the minimal representation of E_6 singlets of the first two factors in the maximal subgroup $SL(3) \times SL(3) \times SL(3) \subset E_6$. [36]

Moreover, we note that (D.16) is in fact a special case of the general formula for the spherical vector (or lowest K -type) of the minimal representation of any group G viewed as a quasi-conformal group $G = QConf(J)$ [46],

$$(D.17) \quad f_K \sim \exp \left[-\sqrt{y^2+x_0^2} \left(1 + 3 \frac{(x^a)^2}{y^2+x_0^2} + \frac{1}{12} \frac{(\kappa_{abc} x^b x^c)^2}{(y^2+x_0^2)^2} + \frac{1}{2^2 \cdot 3^2} \right. \right. \\ \left. \left. \times \frac{(\kappa_{abc} x^a x^b x^c)^2}{(y^2+x_0^2)^3} \right)^{1/2} + i \frac{x_0}{6y(y^2+x_0^2)} \kappa_{abc} x^a x^b x^c \right].$$

Indeed, (D.17) reduces to (D.16) in the one-modulus case with $\kappa_{111} = 6$, corresponding to $G = G_{2(2)}$. This is in accord with the fact that the minimal representation of $G_{2(2)}$ is an irreducible representation of $SL(3, \mathbb{R}) \subset G_{2(2)}$ in the non-spherical supplementary series (see discussion in Section 3.6). Note that the exact lowest K -type of the minimal representation of G_2 was found in [47], Equation (3.119): it would be interesting to see if the integral in (D.13) can be similarly evaluated in closed form. Moreover, the exact spherical vector of the minimal representation of any simply-laced group G in its split form was found in [46]. It would be interesting to see what representation of $SL(3, \mathbb{Z})$ is obtained in the $G_5 = Str_0(J)$ invariant sector, and see how (D.13) is reproduced.

We conclude with a comment on the ‘‘Abelian limit’’ $y \rightarrow 0$, which is needed to properly extract the Abelian Fourier coefficients $\Psi_{p,q}$ in (3.52). As already discussed in (3.70), the phase factor in f_K is singular, but so is the measure $f_{\mathbb{Z}}$, and the two singularities cancel. Thus, we may define $\tilde{f}(0, x_0, x_1)$ as the $y \rightarrow 0$ limit of $f(y, x_0, x_1)$ after removing the singular phase [80],

$$(D.18) \quad \tilde{f}(0, x_0, x_1) \equiv \lim_{y \rightarrow 0} \left(\exp \left[\frac{2\pi i x_0 x_1^3}{y(y^2+x_0^2)} \right] f(y, x_0, x_1) \right),$$

and perform the opposite operation for the dual vector. In particular, the spherical vector (D.13) reduces in this limit to

$$(D.19) \quad \begin{aligned} \tilde{f}_K(0, x_0, x_1) = & x_0^{s_1-s_2-1} x_1^{3(s_1+s_2-1)} \int_0^\infty K_{s_3-\frac{1}{2}} \left(\frac{2\pi x_1^3}{x_0^2} \sqrt{1+x} \right) \\ & \times K_{s_3-\frac{1}{2}} \left(2\pi x_0 \sqrt{1+1/x} \right) x^{\frac{s_2-s_1}{2}} \frac{dx}{x}. \end{aligned}$$

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