Communications in
Number Theory and Physics
Volume 13, Number 1, 53–80, 2019

E_n Jacobi forms and Seiberg–Witten curves

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We discuss Jacobi forms that are invariant under the action of the Weyl group of type E_n ($n = 6, 7, 8$). For $n = 6, 7$ we explicitly construct a full set of generators of the algebra of E_n weak Jacobi forms. We first construct $n + 1$ independent E_n Jacobi forms in terms of Jacobi theta functions and modular forms. By using them, we obtain Seiberg–Witten curves of type $\tilde{E}_6$ and $\tilde{E}_7$ for the E-string theory. The coefficients of each curve are E_n weak Jacobi forms of particular weights and indices specified by the root system, realizing the generators whose existence was shown some time ago by Wirthmüller.

1. Introduction and summary

The theory of Jacobi forms was first systematically studied by Eichler and Zagier [1]. A Jacobi form is a holomorphic function of complex variables τ and $\boldsymbol{\mu}$ which has modular properties in τ and quasi-periodicity in $\boldsymbol{\mu}$. Jacobi forms invariant under the action of the Weyl group $W(R)$ of a root system R was investigated by Wirthmüller [2]. Such Jacobi forms, which we call $W(R)$-invariant Jacobi forms or just R Jacobi forms, appear in various contexts in mathematics and physics.

In [2] an inductive construction of the $W(R)$-invariant Jacobi forms (except for $R = E_8$) was also presented. The construction is, however, rather abstract for $R = E_6, E_7$. On the other hand, $W(E_8)$-invariant Jacobi forms were explicitly constructed in the study of the E-string theory [3, 4, 5]. In [4] nine independent E_8 Jacobi forms were first constructed in the course of deriving the Seiberg–Witten curve for the E-string theory. The construction was further refined in [5] in terms of concisely expressed E_8 holomorphic Jacobi forms.

In this paper, we explicitly construct a full set of generators of the algebra of $W(E_n)$-invariant weak Jacobi forms ($n = 7, 6$). We first construct $n + 1$ independent E_n holomorphic Jacobi forms. Most of them are actually

arXiv: 1706.04619

obtained by mere reduction of E_{n+1} Jacobi forms and thus we have only to construct two new Jacobi forms in each E_n case. All these E_n Jacobi forms are explicitly expressed in terms of Jacobi theta functions and modular forms.

Using these Jacobi forms, we next construct Seiberg–Witten curves of type $\tilde{E}_7$ and $\tilde{E}_6$ for the E-string theory. The original Seiberg–Witten curve for the E-string theory is expressed in terms of E_8 Jacobi forms [4, 5]. If we restrict the value of $\boldsymbol{\mu}$ within the E_n root space, the curve can be expressed in terms of the above $n+1$ E_n Jacobi forms. We transform this curve into the form of the general deformation of a singularity of type $\tilde{E}_n$. The coefficients of this new Seiberg–Witten curve are weak Jacobi forms of particular weights and indices specified by the root system E_n. They are identified as generators of the algebra of E_n weak Jacobi forms over the algebra of modular forms. The existence of such generators was shown by Wirthmüller [2].

The main theorem of [2] does not cover the case of $R = E_8$. Very little has been known about generators of the algebra of E_8 Jacobi forms over the algebra of modular forms. We briefly discuss this case and make a conjecture on the overall picture of the algebra of E_8 weak Jacobi forms.

The paper is organized as follows. In section 2, we present the definition of $W(R)$-invariant Jacobi forms and construct $n+1$ independent E_n holomorphic Jacobi forms. In section 3, we construct Seiberg–Witten curves of type $\tilde{E}_7$ and $\tilde{E}_6$ for the E-string theory and present a full set of generators of the algebra of E_n weak Jacobi forms for $n = 7, 6$. We also discuss the case of E_8. There are three appendices, where Seiberg–Witten curves of type $\tilde{E}_n$ at $\tau = i\infty$, our choice of simple roots and fundamental weights, and definitions of special functions are respectively presented.

2. Construction of holomorphic Jacobi forms

2.1. Definitions and generalities

Let L_R be the root lattice of a root system R, and L_R^* the dual lattice of L_R. Let $\varphi_{k,m}(\tau, \boldsymbol{\mu})$ denote a $W(R)$-invariant Jacobi form of weight k and index m ($k \in \mathbb{Z},\ m \in \mathbb{Z}_{>0}$). It is a holomorphic function of τ and $\boldsymbol{\mu}$ ($\operatorname{Im} \tau > 0,\ \boldsymbol{\mu} \in \mathbb{C}^n$) satisfying the following properties [1, 2]:

i) Weyl invariance:

$$\varphi_{k,m}(\tau, w(\boldsymbol{\mu})) = \varphi_{k,m}(\tau, \boldsymbol{\mu}), \qquad w \in W(R). \tag{2.1}$$

ii) Quasi-periodicity:

(2.2)
$$\varphi_{k,m}(\tau, \boldsymbol{\mu} + \tau\boldsymbol{\alpha} + \boldsymbol{\beta}) = e^{-m\pi i(\tau\boldsymbol{\alpha}^2 + 2\boldsymbol{\mu}\cdot\boldsymbol{\alpha})}\varphi_{k,m}(\tau, \boldsymbol{\mu}), \qquad \boldsymbol{\alpha}, \boldsymbol{\beta} \in L_R.$$

iii) Modular properties:

(2.3)
$$\varphi_{k,m}\left(\frac{a\tau + b}{c\tau + d}, \frac{\boldsymbol{\mu}}{c\tau + d}\right) = (c\tau + d)^k \exp\left(m\pi i \frac{c}{c\tau + d}\boldsymbol{\mu}^2\right)\varphi_{k,m}(\tau, \boldsymbol{\mu}),$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

iv) $\varphi_{k,m}(\tau, \boldsymbol{\mu})$ admits a Fourier expansion as

$$\varphi_{k,m}(\tau, \boldsymbol{\mu}) = \sum_{n=0}^{\infty} \sum_{\boldsymbol{w} \in L_R^*} c(n, \boldsymbol{w}) e^{2\pi i(n\tau + \boldsymbol{w}\cdot\boldsymbol{\mu})}. \tag{2.4}$$

To be precise, $\varphi_{k,m}(\tau, \boldsymbol{\mu})$ defined as above is called a weak Jacobi form. If $\varphi_{k,m}(\tau, \boldsymbol{\mu})$ further satisfies the condition that the coefficients $c(n, \boldsymbol{w})$ of the Fourier expansion (2.4) vanish unless $\boldsymbol{w}^2 \leq 2mn$, it is called a holomorphic Jacobi form. If $\varphi_{k,m}(\tau, \boldsymbol{\mu})$ further satisfies the stronger condition that the coefficients $c(n, \boldsymbol{w})$ vanish unless $\boldsymbol{w}^2 < 2mn$, it is called a Jacobi cusp form. In this paper, a Jacobi form means a weak Jacobi form unless otherwise specified.

The condition (2.1) and the form of the Fourier expansion (2.4) imply that $W(R)$-invariant Jacobi forms are closely related to characters of Weyl orbits of the affine R Lie algebra. In our convention, the index coincides with the level of the affine Lie algebra. In fact, we observe that any $W(R)$-invariant Jacobi form of index m can be written as a linear combination of characters of affine Weyl orbits of level-m weights, and vice versa. From this, one can expect that the number of generators of Jacobi forms of index m coincides with the number of fundamental representations at level m.[1] Figure 1 shows the levels of fundamental representations of the affine E_n algebra. From this, we see that generators of R Jacobi forms are of the

[1]Here, "generators" do not mean those for the algebra of $W(R)$-invariant Jacobi forms over the ring of modular forms $\mathbb{C}[E_4, E_6]$. Instead, we consider here a bigger space where we allow meromorphic modular forms as coefficients.

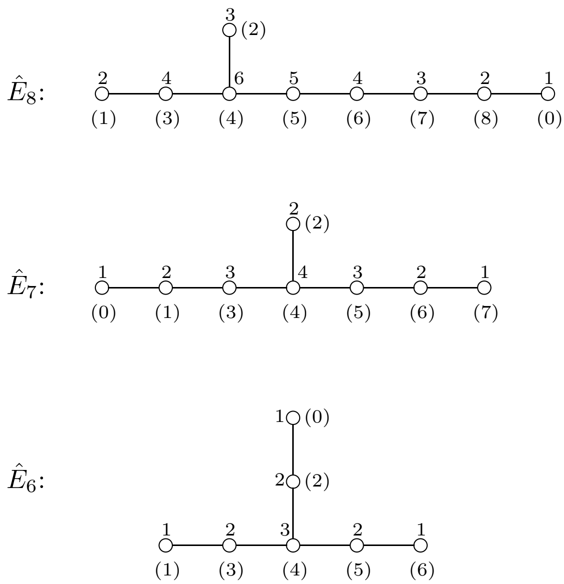

Figure 1: Dynkin diagram for affine E_n: numbers attached to nodes denote the levels of fundamental weights and the numbers in parentheses show their labels.

indices

$$\begin{aligned} 1,2,2,3,3,4,4,5,6 \quad &\text{for} \quad E_8, \\ 1,1,2,2,2,3,3,4 \quad &\text{for} \quad E_7, \\ 1,1,1,2,2,2,3 \quad &\text{for} \quad E_6. \end{aligned} \tag{2.5}$$

Multiple occurrence of the same index means that there are several independent generators of the index. In what follows we will explicitly construct E_n Jacobi forms of these indices.

2.2. E_8 case

Nine independent $W(E_8)$-invariant holomorphic Jacobi forms were constructed in [5]. The summary of the results is shown below.

Let us first introduce the following functions

$$\begin{aligned} e_1(\tau) &:= \tfrac{1}{12}\left(\vartheta_3(\tau)^4 + \vartheta_4(\tau)^4\right), \\ e_2(\tau) &:= \tfrac{1}{12}\left(\vartheta_2(\tau)^4 - \vartheta_4(\tau)^4\right), \end{aligned}$$

$$e_3(\tau) := \tfrac{1}{12}\left(-\vartheta_2(\tau)^4 - \vartheta_3(\tau)^4\right), \tag{2.6}$$

and

$$h_0(\tau) := \vartheta_3(2\tau)\vartheta_3(6\tau) + \vartheta_2(2\tau)\vartheta_2(6\tau). \tag{2.7}$$

The simplest E_8 Jacobi form is the theta function of the root lattice L_{E_8}:

$$\Theta_{E_8}(\tau,\boldsymbol{\mu}) := \sum_{\boldsymbol{w}\in L_{E_8}} \exp\left(\pi i\tau \boldsymbol{w}^2 + 2\pi i\boldsymbol{\mu}\cdot\boldsymbol{w}\right) \tag{2.8}$$

$$= \frac{1}{2}\sum_{k=1}^{4}\prod_{j=1}^{8}\vartheta_k(\mu_j,\tau). \tag{2.9}$$

Nine $W(E_8)$-invariant holomorphic Jacobi forms can be constructed as follows:

$$\begin{aligned}
A_1(\tau,\boldsymbol{\mu}) &= \Theta_{E_8}(\tau,\boldsymbol{\mu}), \qquad A_4(\tau,\boldsymbol{\mu}) = A_1(\tau,2\boldsymbol{\mu}),\\
A_m(\tau,\boldsymbol{\mu}) &= \tfrac{m^3}{m^3+1}\left(A_1(m\tau,m\boldsymbol{\mu}) + \tfrac{1}{m^4}\textstyle\sum_{k=0}^{m-1}A_1(\tfrac{\tau+k}{m},\boldsymbol{\mu})\right), \qquad m=2,3,5,\\
B_2(\tau,\boldsymbol{\mu}) &= \tfrac{32}{5}\left(e_1(\tau)A_1(2\tau,2\boldsymbol{\mu}) + \tfrac{1}{2^4}e_3(\tau)A_1(\tfrac{\tau}{2},\boldsymbol{\mu}) + \tfrac{1}{2^4}e_2(\tau)A_1(\tfrac{\tau+1}{2},\boldsymbol{\mu})\right),\\
B_3(\tau,\boldsymbol{\mu}) &= \tfrac{81}{80}\left(h_0(\tau)^2A_1(3\tau,3\boldsymbol{\mu}) - \tfrac{1}{3^5}\textstyle\sum_{k=0}^{2}h_0(\tfrac{\tau+k}{3})^2A_1(\tfrac{\tau+k}{3},\boldsymbol{\mu})\right),\\
B_4(\tau,\boldsymbol{\mu}) &= \tfrac{16}{15}\left(\vartheta_4(2\tau)^4A_1(4\tau,4\boldsymbol{\mu}) - \tfrac{1}{2^4}\vartheta_4(2\tau)^4A_1(\tau+\tfrac{1}{2},2\boldsymbol{\mu})\right.\\
&\qquad \left. - \tfrac{1}{2^2\cdot 4^4}\textstyle\sum_{k=0}^{3}\vartheta_2(\tfrac{\tau+k}{2})^4A_1(\tfrac{\tau+k}{4},\boldsymbol{\mu})\right),\\
B_6(\tau,\boldsymbol{\mu}) &= \tfrac{9}{10}\left(h_0(\tau)^2A_1(6\tau,6\boldsymbol{\mu}) + \tfrac{1}{2^4}\textstyle\sum_{k=0}^{1}h_0(\tau+k)^2A_1(\tfrac{3\tau+3k}{2},3\boldsymbol{\mu})\right.\\
&\qquad - \tfrac{1}{3\cdot 3^4}\textstyle\sum_{k=0}^{2}h_0(\tfrac{\tau+k}{3})^2A_1(\tfrac{2\tau+2k}{3},2\boldsymbol{\mu})\\
&\qquad \left. - \tfrac{1}{3\cdot 6^4}\textstyle\sum_{k=0}^{5}h_0(\tfrac{\tau+k}{3})^2A_1(\tfrac{\tau+k}{6},\boldsymbol{\mu})\right).
\end{aligned} \tag{2.10}$$

A_m, B_m are of weight $4,6$ and index m respectively. If we set $\boldsymbol{\mu}=\mathbf{0}$, these Jacobi forms reduce to ordinary modular forms. The normalization of these Jacobi forms is chosen so that they reduce to the Eisenstein series

$$A_m(\tau,\mathbf{0}) = E_4(\tau), \qquad B_m(\tau,\mathbf{0}) = E_6(\tau). \tag{2.11}$$

For the sake of clarity, the above A_m, B_m are sometimes expressed as $A_m^{E_8}$, $B_m^{E_8}$.

2.3. E_7 case

$W(E_7)$-invariant Jacobi forms can be obtained by reduction of $W(E_8)$-invariant ones. This is done by merely restricting $\boldsymbol{\mu}$ within the E_7 root space orthogonal to the fundamental weight $\boldsymbol{\Lambda}_8^{E_8}$. More specifically, such $\boldsymbol{\mu}$ is parametrized as

$$\boldsymbol{\mu} = \boldsymbol{\mu}^{(7)} := (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu, -\mu). \tag{2.12}$$

See Appendix B for our convention. In what follows in this subsection $\boldsymbol{\mu}$ is always constrained as above.

By reducing the E_8 Jacobi forms given in (2.10) one immediately obtains

$$\begin{aligned} A_m^{E_7}(\tau, \boldsymbol{\mu}) &:= A_m^{E_8}\left(\tau, \boldsymbol{\mu}^{(7)}\right) \qquad (m = 1,2,3,4,5),\\ B_m^{E_7}(\tau, \boldsymbol{\mu}) &:= B_m^{E_8}\left(\tau, \boldsymbol{\mu}^{(7)}\right) \qquad (m = 2,3,4,6). \end{aligned} \tag{2.13}$$

These Jacobi forms cover most of the desired E_7 Jacobi forms whose indices are listed in (2.5), but not all of them. We need to construct in addition at least two new Jacobi forms which are of index one and index two respectively.

Let us start our study with E_7 Jacobi forms of index one. There are two independent E_7 Jacobi forms. They can be expressed as some modular-invariant linear combinations of two level-one affine Weyl orbit characters. At level one, affine Weyl orbit characters are simply given by the theta functions

$$\begin{aligned} \Theta_{E_7}(\tau, \boldsymbol{\mu}) &:= \sum_{\boldsymbol{w} \in L_{E_7}} \exp\left(\pi i \tau \boldsymbol{w}^2 + 2\pi i \boldsymbol{\mu} \cdot \boldsymbol{w}\right),\\ \Theta_{E_7}^{[7]}(\tau, \boldsymbol{\mu}) &:= \sum_{\boldsymbol{w} \in L_{E_7} + \boldsymbol{\Lambda}_7} \exp\left(\pi i \tau \boldsymbol{w}^2 + 2\pi i \boldsymbol{\mu} \cdot \boldsymbol{w}\right). \end{aligned} \tag{2.14}$$

Here, $\boldsymbol{\Lambda}_7 = \boldsymbol{\Lambda}_7^{E_7}$ is a fundamental weight of E_7. (See Appendix B.) In terms of Jacobi theta functions they are expressed as

$$\begin{aligned} \Theta_{E_7} &= \frac{1}{2}\vartheta_2(2\mu, 2\tau) \sum_{k=1}^{2} \prod_{j=1}^{6} \vartheta_k(\mu_j, \tau) + \frac{1}{2}\vartheta_3(2\mu, 2\tau) \sum_{k=3}^{4} \prod_{j=1}^{6} \vartheta_k(\mu_j, \tau),\\ \Theta_{E_7}^{[7]} &= \frac{1}{2}\vartheta_3(2\mu, 2\tau) \sum_{k=1}^{2} (-1)^k \prod_{j=1}^{6} \vartheta_k(\mu_j, \tau) \end{aligned}$$

$$-\frac{1}{2}\vartheta_2(2\mu,2\tau)\sum_{k=3}^{4}(-1)^k\prod_{j=1}^{6}\vartheta_k(\mu_j,\tau). \tag{2.15}$$

Note that Fourier expansions of these theta functions are

$$\begin{aligned}\Theta_{E_7} &= 1+w[{}_{100000}^{\;\;0}]q+w[{}_{000010}^{\;\;0}]q^2+\left(w[{}_{010000}^{\;\;0}]+w[{}_{000002}^{\;\;0}]\right)q^3+\mathcal{O}(q^4),\\ \Theta_{E_7}^{[7]} &= w[{}_{000001}^{\;\;0}]q^{3/4}+w[{}_{000000}^{\;\;1}]q^{7/4}+w[{}_{100001}^{\;\;0}]q^{11/4}+w[{}_{000100}^{\;\;0}]q^{15/4}\\ &\quad+\mathcal{O}(q^{19/4}),\end{aligned} \tag{2.16}$$

where $q := e^{2\pi i\tau}$. The coefficients are expressed in terms of characters of Weyl orbits of finite E_7. They are defined by

$$w[{}_{n_1n_3n_4n_5n_6n_7}^{\;\;\;\;n_2}](\boldsymbol{\mu}) := \sum_{\boldsymbol{v}\in\mathcal{O}(\sum_{j=1}^{7}n_j\boldsymbol{\Lambda}_j)} e^{2\pi i\boldsymbol{v}\cdot\boldsymbol{\mu}}. \tag{2.17}$$

Here, $\mathcal{O}(\boldsymbol{\Lambda})$ denotes the Weyl orbit of weight $\boldsymbol{\Lambda}$. $\boldsymbol{\Lambda}_j$ $(j=1,\ldots,7)$ are the fundamental weights of E_7.

The above theta functions transform nontrivially under modular transformations. The modular properties of the theta functions are as follows:

$$\begin{aligned}\Theta_{E_7}(\tau+1,\boldsymbol{\mu}) &= \Theta_{E_7}(\tau,\boldsymbol{\mu}),\\ \Theta_{E_7}^{[7]}(\tau+1,\boldsymbol{\mu}) &= -i\Theta_{E_7}^{[7]}(\tau,\boldsymbol{\mu}),\end{aligned} \tag{2.18}$$

$$\begin{aligned}\begin{pmatrix}\Theta_{E_7}\left(-\frac{1}{\tau},\frac{\boldsymbol{\mu}}{\tau}\right)\\ \Theta_{E_7}^{[7]}\left(-\frac{1}{\tau},\frac{\boldsymbol{\mu}}{\tau}\right)\end{pmatrix} &= e^{-\frac{7\pi i}{4}}\tau^{\frac{7}{2}}e^{\frac{\pi i}{\tau}\boldsymbol{\mu}^2}\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\\ &\quad\times\begin{pmatrix}\Theta_{E_7}(\tau,\boldsymbol{\mu})\\ \Theta_{E_7}^{[7]}(\tau,\boldsymbol{\mu})\end{pmatrix}.\end{aligned} \tag{2.19}$$

To construct modular-invariant linear combinations of Θ_{E_7} and $\Theta_{E_7}^{[7]}$, let us first look into the case of $A_1^{E_7}$. One can easily derive that $A_1^{E_7}$ is expressed as

$$A_1^{E_7}(\tau,\boldsymbol{\mu}) = \vartheta_3(2\tau)\Theta_{E_7}(\tau,\boldsymbol{\mu})+\vartheta_2(2\tau)\Theta_{E_7}^{[7]}(\tau,\boldsymbol{\mu}). \tag{2.20}$$

The coefficient functions can be interpreted as $\vartheta_3(2\tau)=\Theta_{A_1}(\tau,0)$, $\vartheta_2(2\tau)=\Theta_{A_1}^{[1]}(\tau,0)$ and transform as

$$\vartheta_3(2(\tau+1)) = \vartheta_3(2\tau), \qquad \vartheta_2(2(\tau+1)) = i\vartheta_2(2\tau),$$

$$\begin{pmatrix} \vartheta_3\left(-\frac{2}{\tau}\right) \\ \vartheta_2\left(-\frac{2}{\tau}\right) \end{pmatrix} = e^{-\frac{\pi i}{4}}\tau^{\frac{1}{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \vartheta_3(2\tau) \\ \vartheta_2(2\tau) \end{pmatrix}. \tag{2.21}$$

One can easily check that (2.20) is indeed a modular-invariant combination, i.e. it transforms as in (2.3).

It is natural to expect that the other modular-invariant linear combination can also be constructed by using polynomials of $\vartheta_3(2\tau), \vartheta_2(2\tau)$ as coefficient functions. One of the simplest candidates for this Jacobi form would be the one which reduces to $E_6(\tau)$ when we set $\boldsymbol{\mu} = \mathbf{0}$. In order for the Jacobi form to be of weight 6, the coefficient functions have to be homogeneous quintics in $\vartheta_3(2\tau), \vartheta_2(2\tau)$. And furthermore, in order to be invariant under the transformation $\tau \to \tau + 1$, the Jacobi form has to take the form

$$\left(c_1\vartheta_3(2\tau)^4 + c_2\vartheta_2(2\tau)^4\right)\vartheta_3(2\tau)\Theta_{E_7} + \left(c_3\vartheta_3(2\tau)^4 + c_4\vartheta_2(2\tau)^4\right)\vartheta_2(2\tau)\Theta_{E_7}^{[7]}. \tag{2.22}$$

The requirement that it reduces to E_6 when $\boldsymbol{\mu} = \mathbf{0}$ immediately determines the unknown coefficients c_j. In this way, we find the combination

$$\begin{aligned} C_1^{E_7}(\tau, \boldsymbol{\mu}) := &\left(\vartheta_3(2\tau)^4 - 5\vartheta_2(2\tau)^4\right)\vartheta_3(2\tau)\Theta_{E_7}(\tau, \boldsymbol{\mu}) \\ &+ \left(\vartheta_2(2\tau)^4 - 5\vartheta_3(2\tau)^4\right)\vartheta_2(2\tau)\Theta_{E_7}^{[7]}(\tau, \boldsymbol{\mu}). \end{aligned} \tag{2.23}$$

One can check that $C_1^{E_7}$ is indeed an E_7 holomorphic Jacobi form of index one. It is clear that $A_1^{E_7}$ and $C_1^{E_7}$ are independent. By construction,

$$C_1^{E_7}(\tau, \mathbf{0}) = E_6(\tau). \tag{2.24}$$

Let us now move on to the construction of a new Jacobi form of index two. This is actually easy. Applying the Hecke transformation of order two to $C_1^{E_7}$, one obtains

$$C_2^{E_7}(\tau, \boldsymbol{\mu}) := \frac{32}{33}\left(C_1^{E_7}(2\tau, 2\boldsymbol{\mu}) + \frac{1}{64}\sum_{k=0}^{1} C_1^{E_7}\left(\frac{\tau + k}{2}, \boldsymbol{\mu}\right)\right). \tag{2.25}$$

The normalization is chosen so that

$$C_2^{E_7}(\tau, \mathbf{0}) = E_6(\tau). \tag{2.26}$$

One can check that $C_2^{E_7}$ is an independent Jacobi form, i.e. it is not expressed as polynomials in $A_1^{E_7}, C_1^{E_7}, A_2^{E_7}, B_2^{E_7}$.

One can also check that $A_3^{E_7}, B_3^{E_7}$ are independent in the same sense. On the other hand, it turns out that $A_4^{E_7}$ is not independent. It is expressed in terms of $A_m^{E_7}$, $B_m^{E_7}$, $C_m^{E_7}$ $(m \leq 3)$ as

$$\begin{aligned} A_4 = \frac{1}{13824E_4^2\Delta}\big(& -448E_4^4A_1A_3 + 448E_4^2E_6C_1A_3 - 1280E_4^2E_6A_1B_3 \\ & + 1280E_4^3C_1B_3 + 216E_4^4A_2^2 - 1440E_4^3A_1^2A_2 + 720E_4^2E_6A_2B_2 \\ & + 288E_4^2C_1^2A_2 + (1275E_4^3 - 675E_6^2)B_2^2 + (-990E_4^3 + 990E_6^2)B_2C_2 \\ & + 360E_4E_6A_1^2B_2 - 2640E_4^2A_1C_1B_2 + 360E_6C_1^2B_2 \\ & + (363E_4^3 - 363E_6^2)C_2^2 - 264E_4E_6A_1^2C_2 + 528E_4^2A_1C_1C_2 \\ & - 264E_6C_1^2C_2 + 1680E_4^2A_1^4 - 96E_4A_1^2C_1^2 - 48C_1^4\big). \end{aligned} \tag{2.27}$$

Here, we have omitted superscript E_7 from the Jacobi forms and introduced

$$\Delta := \eta^{24} = \frac{1}{1728}\left(E_4^3 - E_6^2\right). \tag{2.28}$$

To summarize, we now have eight Jacobi forms

$$A_m^{E_7} \quad (m = 1, 2, 3), \qquad B_m^{E_7} \quad (m = 2, 3, 4), \qquad C_m^{E_7} \quad (m = 1, 2), \tag{2.29}$$

which are of weight $4, 6, 6$ and index m respectively. We checked that they are independent, holomorphic Jacobi forms. Note that

$$A_m^{E_7}(\tau, \mathbf{0}) = E_4(\tau), \qquad B_m^{E_7}(\tau, \mathbf{0}) = C_m^{E_7}(\tau, \mathbf{0}) = E_6(\tau). \tag{2.30}$$

As expected, $A_5^{E_7}, B_6^{E_7}$ are no longer independent and are expressed as polynomials in the eight Jacobi forms (2.29). While these relations are essential to obtain the results in the next section, their concrete expressions are rather lengthy and thus we do not present them here. (In any case, these relations are immediately restored from the results in the next section.)

2.4. E_6 case

As in the E_7 case, $W(E_6)$-invariant Jacobi forms can be obtained by reduction of those for E_7 or E_8. This is done by restricting $\boldsymbol{\mu}$ within the E_6 root space orthogonal to both $\boldsymbol{\Lambda}_7^{E_8}$ and $\boldsymbol{\Lambda}_8^{E_8}$. More specifically, such a vector $\boldsymbol{\mu}$ is parametrized as

$$\boldsymbol{\mu} = \boldsymbol{\mu}^{(6)} := (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu, \mu, -\mu). \tag{2.31}$$

In what follows in this subsection $\boldsymbol{\mu}$ is always constrained as above.

By reducing the E_7 (or E_8) Jacobi forms one immediately obtains

$$
\begin{aligned}
&A_m^{E_6}(\tau,\boldsymbol{\mu}) := A_m^{E_7}\left(\tau,\boldsymbol{\mu}^{(6)}\right) = A_m^{E_8}\left(\tau,\boldsymbol{\mu}^{(6)}\right) \qquad (m=1,2,3),\\
&B_m^{E_6}(\tau,\boldsymbol{\mu}) := B_m^{E_7}\left(\tau,\boldsymbol{\mu}^{(6)}\right) = B_m^{E_8}\left(\tau,\boldsymbol{\mu}^{(6)}\right) \qquad (m=2,3,4),\\
&C_m^{E_6}(\tau,\boldsymbol{\mu}) := C_m^{E_7}\left(\tau,\boldsymbol{\mu}^{(6)}\right) \qquad (m=1,2).
\end{aligned}
\tag{2.32}
$$

These Jacobi forms cover most of the desired E_6 Jacobi forms whose indices are listed in (2.5), but as in the E_7 case, we need to construct at least two new Jacobi forms which are of index one and index two respectively.

There are three affine E_6 Weyl orbit characters at level one: $\Theta_{E_6}, \Theta_{E_6}^{[1]}$ and $\Theta_{E_6}^{[6]}$. They are defined by means of the root lattice E_6 and fundamental weights $\boldsymbol{\Lambda}_1^{E_6}, \boldsymbol{\Lambda}_6^{E_6}$ in the same way as in the E_7 case. (See Appendix B for our convention.) They are expressed in terms of Jacobi theta functions as

$$
\begin{aligned}
\Theta_{E_6}(\tau,\boldsymbol{\mu}) &= \frac{1}{2}\sum_{k=1}^{4}\vartheta_k(3\mu,3\tau)\prod_{j=1}^{5}\vartheta_k(\mu_j,\tau),\\
\Theta_{E_6}^{[1]}(\tau,\boldsymbol{\mu}) &= \frac{1}{2}\sum_{k=1}^{4}\sigma(k)q^{1/6}e^{2\pi i\mu}\vartheta_k(3\mu+\tau,3\tau)\prod_{j=1}^{5}\vartheta_k(\mu_j,\tau),\\
\Theta_{E_6}^{[6]}(\tau,\boldsymbol{\mu}) &= \frac{1}{2}\sum_{k=1}^{4}\sigma(k)q^{1/6}e^{-2\pi i\mu}\vartheta_k(3\mu-\tau,3\tau)\prod_{j=1}^{5}\vartheta_k(\mu_j,\tau),
\end{aligned}
\tag{2.33}
$$

where $\sigma(1)=\sigma(4)=-1$, $\sigma(2)=\sigma(3)=1$. These theta functions are expanded as

(2.34)

$$
\begin{aligned}
\Theta_{E_6} &= 1 + w[{}^{\;\;1}_{00000}]q + w[{}^{\;\;0}_{10001}]q^2 + w[{}^{\;\;0}_{00100}]q^3 + \mathcal{O}(q^4),\\
\Theta_{E_6}^{[1]} &= w[{}^{\;\;0}_{10000}]q^{2/3} + w[{}^{\;\;0}_{00010}]q^{5/3} + \left(w[{}^{\;\;1}_{10000}] + w[{}^{\;\;0}_{00002}]\right)q^{8/3} + \mathcal{O}(q^{11/3}),\\
\Theta_{E_6}^{[6]} &= w[{}^{\;\;0}_{00001}]q^{2/3} + w[{}^{\;\;0}_{01000}]q^{5/3} + \left(w[{}^{\;\;1}_{00001}] + w[{}^{\;\;0}_{20000}]\right)q^{8/3} + \mathcal{O}(q^{11/3}).
\end{aligned}
$$

The modular properties of these theta functions are as follows:

$$
\begin{aligned}
\Theta_{E_6}(\tau+1,\boldsymbol{\mu}) &= \Theta_{E_6}(\tau,\boldsymbol{\mu}),\\
\Theta_{E_6}^{[1]}(\tau+1,\boldsymbol{\mu}) &= e^{4\pi i/3}\Theta_{E_6}^{[1]}(\tau,\boldsymbol{\mu}),
\end{aligned}
$$

$$\Theta_{E_6}^{[6]}(\tau+1,\boldsymbol{\mu}) = e^{4\pi i/3}\Theta_{E_6}^{[6]}(\tau,\boldsymbol{\mu}), \tag{2.35}$$

$$\begin{pmatrix} \Theta_{E_6}\left(-\frac{1}{\tau},\frac{\boldsymbol{\mu}}{\tau}\right) \\ \Theta_{E_6}^{[1]}\left(-\frac{1}{\tau},\frac{\boldsymbol{\mu}}{\tau}\right) \\ \Theta_{E_6}^{[6]}\left(-\frac{1}{\tau},\frac{\boldsymbol{\mu}}{\tau}\right) \end{pmatrix} = i\tau^3 e^{\frac{\pi i}{\tau}\boldsymbol{\mu}^2}\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \end{pmatrix} \times \begin{pmatrix} \Theta_{E_6}(\tau,\boldsymbol{\mu}) \\ \Theta_{E_6}^{[1]}(\tau,\boldsymbol{\mu}) \\ \Theta_{E_6}^{[6]}(\tau,\boldsymbol{\mu}) \end{pmatrix}. \tag{2.36}$$

$A_1^{E_6}$ and $C_1^{E_6}$ are expressed in terms of these theta functions as

$$A_1^{E_6}(\tau,\boldsymbol{\mu}) = h_0\Theta_{E_6}(\tau,\boldsymbol{\mu}) + h_1\left(\Theta_{E_6}^{[1]}(\tau,\boldsymbol{\mu}) + \Theta_{E_6}^{[6]}(\tau,\boldsymbol{\mu})\right), \tag{2.37}$$

$$C_1^{E_6}(\tau,\boldsymbol{\mu}) = \left(h_0^3 - 4h_1^3\right)\Theta_{E_6}(\tau,\boldsymbol{\mu}) - 3h_0^2 h_1\left(\Theta_{E_6}^{[1]}(\tau,\boldsymbol{\mu}) + \Theta_{E_6}^{[6]}(\tau,\boldsymbol{\mu})\right), \tag{2.38}$$

where $h_j = h_j(\tau)$. $h_0(\tau)$ was introduced in (2.7) and

$$h_1(\tau) := 3\frac{\eta(3\tau)^3}{\eta(\tau)} = \frac{1}{2}\left(h_0(\tau/3) - h_0(\tau)\right). \tag{2.39}$$

They can be interpreted as $h_0(\tau) = \Theta_{A_2}(\tau,\mathbf{0})$, $h_1(\tau) = \Theta_{A_2}^{[1]}(\tau,\mathbf{0}) = \Theta_{A_2}^{[2]}(\tau,\mathbf{0})$.

By taking account of the above modular properties, the other Jacobi form of index one is found as

$$D_1^{E_6}(\tau,\boldsymbol{\mu}) := \eta(\tau)^8\left(\Theta_{E_6}^{[1]}(\tau,\boldsymbol{\mu}) - \Theta_{E_6}^{[6]}(\tau,\boldsymbol{\mu})\right). \tag{2.40}$$

This is a Jacobi form of weight 7. If we set $\boldsymbol{\mu} = \mathbf{0}$, it vanishes:

$$D_1^{E_6}(\tau,\mathbf{0}) = 0. \tag{2.41}$$

The remaining Jacobi form of weight two can be constructed from $D_1^{E_6}$ by the Hecke transformation of order two. One obtains

$$D_2^{E_6}(\tau,\boldsymbol{\mu}) := D_1^{E_6}(2\tau,2\boldsymbol{\mu}) + \frac{1}{128}\sum_{k=0}^{1} D_1^{E_6}\left(\frac{\tau+k}{2},\boldsymbol{\mu}\right). \tag{2.42}$$

Table 1: Our choice of independent E_n Jacobi forms. The subscripts of the Jacobi forms represent their index.

E_8 :	weight 4	A_1	A_2	A_3	A_4	A_5	
	weight 6		B_2	B_3	B_4		B_6
E_7 :	weight 4	A_1	A_2	A_3			
	weight 6	C_1	B_2 C_2	B_3	B_4		
E_6 :	weight 4	A_1	A_2	A_3			
	weight 6	C_1	B_2				
	weight 7	D_1	D_2				

To summarize, we now have seven Jacobi forms

$$A_m^{E_6} \quad (m=1,2,3), \qquad B_2^{E_6}, \qquad C_1^{E_6}, \qquad D_m^{E_6} \quad (m=1,2), \tag{2.43}$$

which are of weight $4,6,6,7$ respectively and index given by their subscripts. We checked that they are independent. We also checked that $A_m^{E_6}, B_2^{E_6}, C_1^{E_6}$ are holomorphic Jacobi forms, while $D_m^{E_6}$ are Jacobi cusp forms. Note that

$$A_m(\tau,\mathbf{0}) = E_4(\tau), \qquad B_2(\tau,\mathbf{0}) = C_1(\tau,\mathbf{0}) = E_6(\tau), \qquad D_m(\tau,\mathbf{0}) = 0. \tag{2.44}$$

As expected, $C_2^{E_6}, B_3^{E_6}, B_4^{E_6}$ are expressed as polynomials in these Jacobi forms. We will use these relations to obtain the results in the next section. Again, we do not present concrete expressions here, as these relations can easily be restored from the results we will obtain there.

We summarize our choice of independent E_n Jacobi forms in Table 1.

3. Seiberg–Witten curves and generators of weak Jacobi forms

3.1. Generalities

In [2] Wirthmüller proved that for any irreducible root system R excluding E_8, the algebra of $W(R)$-invariant Jacobi forms over the algebra of modular forms $\mathbb{C}[E_4, E_6]$ is generated as the polynomial algebra in some $W(R)$-invariant Jacobi forms

$$\{\alpha_{k(j),m(j)}(\tau,\boldsymbol{\mu})\} \qquad (j=0,1,\ldots,n). \tag{3.1}$$

Here, $\{k(j)\}$ and $\{m(j)\}$ are given respectively by the list of degrees of independent Casimir invariants of R and the list of levels of the fundamental representations of the affine R Lie algebra. In what follows, we explicitly construct $\{\alpha_{k(j),m(j)}\}$ for $R = E_6, E_7$ exploiting the Seiberg–Witten curve for the E-string theory. We also present a similar set of meromorphic functions (i.e. not exactly Jacobi forms) for $R = E_8$.

3.2. E_8 case

In [5] the Seiberg–Witten curve for the E-string theory [4] was expressed in terms of the nine Jacobi forms A_m, B_m given in (2.10). The result is as follows:

$$
\begin{aligned}
y^2 &= 4x^3 - \frac{1}{12}E_4u^4x - \frac{1}{216}E_6u^6 \\
&\quad - \sum_{m=2}^{4}\alpha_{4-6m,m}u^{4-m}x - \sum_{m=1}^{6}\alpha_{6-6m,m}u^{6-m},
\end{aligned}
\tag{3.2}
$$

where

$$
\begin{aligned}
\alpha_{0,1} &= -\frac{4}{E_4}A_1, \\
\alpha_{-6,2} &= \frac{5}{6E_4^2\Delta}\Big(E_4^2B_2 - E_6A_1^2\Big), \qquad \alpha_{-8,2} = \frac{6}{E_4\Delta}\Big(-E_4A_2 + A_1^2\Big), \\
\alpha_{-12,3} &= \frac{1}{108E_4^3\Delta^2}\Big(-7E_4^5A_3 - 20E_4^3E_6B_3 \\
&\qquad - 9E_4^4A_1A_2 + 30E_4^2E_6A_1B_2 + (16E_4^3 - 10E_6^2)A_1^3\Big), \\
\alpha_{-14,3} &= \frac{1}{9E_4^2\Delta^2}\Big(-7E_4^2E_6A_3 - 20E_4^3B_3 - 9E_4E_6A_1A_2 + 30E_4^2A_1B_2 \\
&\qquad + 6E_6A_1^3\Big), \\
\alpha_{-18,4} &= \frac{1}{1728E_4^4\Delta^3}\Big((-5E_4^7 + 5E_4^4E_6^2)B_4 + (80E_4^6 - 80E_4^3E_6^2)A_1B_3 \\
&\qquad + 9E_4^5E_6A_2^2 + 30E_4^6A_2B_2 + 25E_4^4E_6B_2^2 - 48E_4^4E_6A_1^2A_2 \\
&\qquad + (-140E_4^5 + 60E_4^2E_6^2)A_1^2B_2 + (74E_4^3E_6 - 10E_6^3)A_1^4\Big), \\
\alpha_{-20,4} &= \frac{1}{864E_4^3\Delta^3}\Big((E_4^6 - E_4^3E_6^2)A_4 + (56E_4^5 - 56E_4^2E_6^2)A_1A_3 - 27E_4^5A_2^2 \\
&\qquad - 90E_4^3E_6A_2B_2 - 75E_4^4B_2^2 + (180E_4^4 - 36E_4E_6^2)A_1^2A_2
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 240E_4^2E_6A_1^2B_2 + (-210E_4^3 + 18E_6^2)A_1^4\Big), \\
\alpha_{-24,5} &= \frac{1}{72E_4^5\Delta^3}\Big((-21E_4^7 + 21E_4^4E_6^2)A_5 - 294E_4^6A_2A_3 - 770E_4^4E_6B_2A_3 \\
&\quad - 840E_4^4E_6A_2B_3 - 2200E_4^5B_2B_3 + 168E_4^5A_1^2A_3 + 480E_4^3E_6A_1^2B_3 \\
&\quad - 621E_4^5A_1A_2^2 + 3525E_4^4A_1B_2^2 + 1224E_4^4A_1^3A_2 - 240E_4^2E_6A_1^3B_2 \\
&\quad + (-456E_4^3 + 24E_6^2)A_1^5\Big), \\
\alpha_{-30,6} &= \frac{1}{13436928E_4^6\Delta^5}\Big((-20E_4^{12} + 40E_4^9E_6^2 - 20E_4^6E_6^4)B_6 \\
&\quad + (-189E_4^{10}E_6 + 378E_4^7E_6^3 - 189E_4^4E_6^5)A_1A_5 \\
&\quad + (-9E_4^{10}E_6 + 9E_4^7E_6^3)A_2A_4 + (-15E_4^{11} + 15E_4^8E_6^2)B_2A_4 \\
&\quad + (-180E_4^{11} + 180E_4^8E_6^2)A_2B_4 + (-300E_4^9E_6 + 300E_4^6E_6^3)B_2B_4 \\
&\quad + (22E_4^9E_6 - 22E_4^6E_6^3)A_1^2A_4 \\
&\quad + (150E_4^{10} + 120E_4^7E_6^2 - 270E_4^4E_6^4)A_1^2B_4 \\
&\quad + (196E_4^{10}E_6 - 196E_4^7E_6^3)A_3^2 + (1120E_4^{11} - 1120E_4^8E_6^2)A_3B_3 \\
&\quad + (1600E_4^9E_6 - 1600E_4^6E_6^3)B_3^2 \\
&\quad + (-2982E_4^9E_6 + 2982E_4^6E_6^3)A_1A_2A_3 \\
&\quad + (-2520E_4^{10} - 4410E_4^7E_6^2 + 6930E_4^4E_6^4)A_1B_2A_3 \\
&\quad + (3360E_4^{10} - 10920E_4^7E_6^2 + 7560E_4^4E_6^4)A_1A_2B_3 \\
&\quad + (-19800E_4^8E_6 + 19800E_4^5E_6^3)A_1B_2B_3 \\
&\quad + (2016E_4^8E_6 - 2016E_4^5E_6^3)A_1^3A_3 \\
&\quad + (-5920E_4^9 + 7360E_4^6E_6^2 - 1440E_4^3E_6^4)A_1^3B_3 \\
&\quad + (405E_4^9E_6 + 162E_4^6E_6^3)A_2^3 \\
&\quad + (1215E_4^{10} + 1620E_4^7E_6^2)A_2^2B_2 + 4725E_4^8E_6A_2B_2^2 \\
&\quad + (1125E_4^9 + 1500E_4^6E_6^2)B_2^3 + (-9477E_4^8E_6 + 5103E_4^5E_6^3)A_1^2A_2^2 \\
&\quad + (-9180E_4^9 - 5400E_4^6E_6^2)A_1^2A_2B_2 \\
&\quad + (20925E_4^7E_6 - 33075E_4^4E_6^3)A_1^2B_2^2 \\
&\quad + (20304E_4^7E_6 - 9072E_4^4E_6^3)A_1^4A_2 \\
&\quad + (12780E_4^8 + 5400E_4^5E_6^2 + 540E_4^2E_6^4)A_1^4B_2 \\
&\quad + (-11076E_4^6E_6 + 1512E_4^3E_6^3 - 36E_6^5)A_1^6\Big). \qquad (3.3)
\end{aligned}
$$

Since $E_4(e^{2\pi i/3}) = 0$, it is very likely that the above $\alpha_{k,m}$ have a pole at $\tau = e^{2\pi i/3}$. Apart from this flaw, $\alpha_{k,m}$ satisfy all the conditions required for

$W(E_8)$-invariant weak Jacobi forms (of weight k and index m): By construction they satisfy conditions (2.1)–(2.3). It is also obvious that no fractional powers of q appear in their Fourier expansions. It is known [4] that they are finite at $q = 0$ (see Appendix A for the concrete expressions). Thus, the condition (2.4) is also satisfied.

If we set $\boldsymbol{\mu} = \mathbf{0}$, all $\alpha_{k,m}$ of negative weight vanish:

$$\begin{aligned} \alpha_{0,1}(\tau, \mathbf{0}) &= -4, \\ \alpha_{k,m}(\tau, \mathbf{0}) &= 0 \quad (k < 0). \end{aligned} \tag{3.4}$$

Although Wirthmüller's theorem [2] does not cover the case of $R = E_8$ and the above $\alpha_{k,m}$ are not exactly Jacobi forms, it would still be interesting to examine to what extent the statements of the theorem hold for $R = E_8$.[2] Interestingly, there is a small mismatch between the above $\alpha_{k,m}$ and the generators that would be expected supposing Wirthmüller's theorem held: The theorem would require a generator of weight -2 and index 2 instead of $\alpha_{-6,2}$. In fact such a Jacobi form can easily be constructed as

$$\tilde{\alpha}_{-2,2} := E_4 \alpha_{-6,2}. \tag{3.5}$$

However, if one replaces $\alpha_{-6,2}$ with $\tilde{\alpha}_{-2,2}$ in the generator set, certain $W(E_8)$-invariant Jacobi forms cannot be generated over the ring of modular forms $\mathbb{C}[E_4, E_6]$.[3]

Though the above $\alpha_{k,m}$ themselves are not exactly Jacobi forms, one can still consider the polynomial algebra generated by $\alpha_{k,m}$ over $\mathbb{C}[E_4, E_6]$. To the best of our knowledge, this algebra seems general enough to contain all E_8 weak Jacobi forms whose concrete expressions are known. Therefore we conjecture that the algebra of $W(E_8)$-invariant weak Jacobi forms would be a proper subset of the polynomial algebra generated by $\alpha_{k,m}$ over $\mathbb{C}[E_4, E_6]$. It would be very interesting to investigate this problem in a more mathematically rigorous manner.

3.3. E_7 case

One can reduce the Seiberg–Witten curve presented in the last subsection to the curve that has only $W(E_7)$ symmetry by setting $\boldsymbol{\mu} = \boldsymbol{\mu}^{(7)}$. The

[2] There is an algebro-geometric explanation why E_8 should be exceptional [6].

[3] The author is grateful to Haowu Wang for explaining this point and also indicating some misunderstandings about $W(E_8)$-invariant Jacobi forms in the previous manuscript.

curve can be expressed in terms of the eight E_7 Jacobi forms constructed in section 2.3. It is expected that the elliptic fibration described by this curve develops a degenerate fiber. (It was systematically studied in [7] how special values of $\boldsymbol{\mu}$ correspond to degenerations of the elliptic fibration described by the Seiberg–Witten curve for the E-string theory.) One immediate outcome of expressing the curve in terms of the eight E_7 Jacobi forms is that one can directly see this fiber degeneration as the factorization of the discriminant. For the elliptic curve in the Weierstrass form

$$y^2 = 4x^3 - fx - g, \tag{3.6}$$

the discriminant is given by

$$D = f^3 - 27g^2. \tag{3.7}$$

For the above Seiberg–Witten curve expressed in terms of the eight E_7 Jacobi forms, the discriminant indeed factorizes as

$$D = (u - u_0)^2 P_{10}(u), \tag{3.8}$$

where

$$u_0 = \frac{E_4 C_1 - E_6 A_1}{12 E_4 \Delta} \tag{3.9}$$

and $P_{10}(u)$ is some tenth degree polynomial in u. (In this subsection we omit superscript E_7 from Jacobi forms.) This is not the only peculiar feature of the above reduced curve. We can in fact transform the curve into the form of the general deformation of a singularity of type $\tilde{E}_7$, as we will see below.

The general deformation of a singularity of type $\tilde{E}_7$ takes the form [6]

$$\begin{aligned} y^2 \;=\; & 4ux^3 - \frac{1}{12}E_4u^3x - \frac{1}{216}E_6u^4 \\ & +\alpha_{0,1}u^3 + \alpha_{-2,1}u^2x + \alpha_{-6,2}u^2 + \alpha_{-8,2}ux + \alpha_{-10,2}x^2 \\ & +\alpha_{-12,3}u + \alpha_{-14,3}x + \alpha_{-18,4}. \end{aligned} \tag{3.10}$$

For the moment $\alpha_{k,m}$ are just deformation parameters. We formally assign weights $-6, -4, -9, k$ and indices $1, 1, 2, m$ to $u, x, y, \alpha_{k,m}$ respectively, so that all terms in the equation are of weight -18 and index 4. The elliptic curve (3.10) can be transformed into the Weierstrass form (3.6) with

$$f = \frac{1}{12}E_4u^4 + \sum_{k=1}^{4} f_k u^{4-k}, \qquad g = \frac{1}{216}E_6u^6 + \sum_{k=1}^{6} g_k u^{6-k} \tag{3.11}$$

in the following manner: First, perform a translation of x to remove the quadratic term in x. Next, rescale the variables as $x \to u^{-1}x$, $y \to u^{-1}y$. We then obtain the Weierstrass from with f, g being of the form (3.11). One finds that the discriminant of this curve factorizes as

$$D = u^2 \tilde{P}_{10}(u), \tag{3.12}$$

where $\tilde{P}_{10}(u)$ is some tenth degree polynomial in u. Another peculiar feature of the curve is that the coefficients f_4, g_6 also factorize

$$f_4 = \frac{(\alpha_{-10,2})^2}{12}, \qquad g_6 = -\frac{(\alpha_{-10,2})^3}{216}. \tag{3.13}$$

The locations of the double roots of the discriminants (3.8) and (3.12) imply that the original Seiberg–Witten curve with $\boldsymbol{\mu} = \boldsymbol{\mu}^{(7)}$ is identified with the above obtained curve in the Weierstrass form by the translation $u \to u + u_0$. Indeed, after the translation is applied to the former curve, one can see the factorizations of coefficients as in (3.13) and determine $\alpha_{-10,2}$. Furthermore, by comparing two curves term by term, one can fully determine the coefficients $\alpha_{k,m}$ in (3.10). The results are as follows:

$$\begin{aligned}
\alpha_{0,1} &= \frac{E_4^2 A_1 - E_6 C_1}{432\Delta}, \qquad \alpha_{-2,1} = \frac{E_6 A_1 - E_4 C_1}{36\Delta}, \\
\alpha_{-6,2} &= \frac{1}{82944\Delta^2}\big((-25E_4^3 + 25E_6^2)B_2 + (-11E_4^3 + 11E_6^2)C_2 \\
&\qquad -36E_4E_6A_1^2 + 72E_4^2A_1C_1 - 36E_6C_1^2\big), \\
\alpha_{-8,2} &= \frac{1}{288\Delta^2}\big((E_4^3 - E_6^2)A_2 - E_4^2A_1^2 + 2E_6A_1C_1 - E_4C_1^2\big), \\
\alpha_{-10,2} &= \frac{1}{576E_4\Delta^2}\big((-15E_4^3 + 15E_6^2)B_2 + (11E_4^3 - 11E_6^2)C_2 \\
&\qquad -4E_4E_6A_1^2 + 8E_4^2A_1C_1 - 4E_6C_1^2\big), \\
\alpha_{-12,3} &= \frac{1}{746496E_4\Delta^3}\big((28E_4^6 - 28E_4^3E_6^2)A_3 + (80E_4^4E_6 - 80E_4E_6^3)B_3 \\
&\qquad + (36E_4^5 - 36E_4^2E_6^2)A_1A_2 + (-45E_4^3E_6 + 45E_6^3)A_1B_2 \\
&\qquad + (-75E_4^4 + 75E_4E_6^2)C_1B_2 + (33E_4^3E_6 - 33E_6^3)A_1C_2 \\
&\qquad + (-33E_4^4 + 33E_4E_6^2)C_1C_2 + (-64E_4^4 + 92E_4E_6^2)A_1^3 \\
&\qquad -84E_4^2E_6A_1^2C_1 + (96E_4^3 - 12E_6^2)A_1C_1^2 - 28E_4E_6C_1^3\big), \\
\alpha_{-14,3} &= \frac{1}{15552\Delta^3}\big((7E_4^3E_6 - 7E_6^3)A_3 + (20E_4^4 - 20E_4E_6^2)B_3
\end{aligned}$$

$$
\begin{aligned}
&+(9E_4^3-9E_6^2)C_1A_2+(-30E_4^3+30E_6^2)A_1B_2\\
&+3E_4E_6A_1^3-9E_4^2A_1^2C_1+9E_6A_1C_1^2-3E_4C_1^3\big),
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{-18,4}=\frac{1}{5971968E_4\Delta^4}&\big((10E_4^7-20E_4^4E_6^2+10E_4E_6^4)B_4\\
&+(-56E_4^5E_6+56E_4^2E_6^3)A_1A_3+(56E_4^6-56E_4^3E_6^2)C_1A_3\\
&+(-160E_4^6+160E_4^3E_6^2)A_1B_3+(160E_4^4E_6-160E_4E_6^3)C_1B_3\\
&+(-18E_4^5E_6+18E_4^2E_6^3)A_2^2\\
&+(-105E_4^6+150E_4^3E_6^2-45E_6^4)A_2B_2\\
&+(33E_4^6-66E_4^3E_6^2+33E_6^4)A_2C_2+(-50E_4^4E_6+50E_4E_6^3)B_2^2\\
&+(12E_4^4E_6-12E_4E_6^3)A_1^2A_2+(96E_4^5-96E_4^2E_6^2)A_1C_1A_2\\
&+(-12E_4^3E_6+12E_6^3)C_1^2A_2+(325E_4^5-325E_4^2E_6^2)A_1^2B_2\\
&+(-90E_4^3E_6+90E_6^3)A_1C_1B_2+(-75E_4^4+75E_4E_6^2)C_1^2B_2\\
&+(-33E_4^5+33E_4^2E_6^2)A_1^2C_2+(66E_4^3E_6-66E_6^3)A_1C_1C_2\\
&+(-33E_4^4+33E_4E_6^2)C_1^2C_2-8E_4^3E_6A_1^4\\
&+(-152E_4^4+184E_4E_6^2)A_1^3C_1\\
&-48E_4^2E_6A_1^2C_1^2+(56E_4^3-24E_6^2)A_1C_1^3-8E_4E_6C_1^4\big).
\end{aligned}
\tag{3.14}
$$

Here, A_m, B_m, C_m are the holomorphic Jacobi forms constructed in section 2.3 and we have omitted superscript E_7.

In contrast to the E_8 case, the above $\alpha_{k,m}$ are genuine $W(E_7)$-invariant weak Jacobi forms (of weight k and index m). This can be shown as follows: By construction they satisfy conditions (2.1)–(2.3) and no fractional powers of q appear in their Fourier expansions. We checked explicitly that they are finite at $q=0$. We present the concrete expressions of $\alpha_{k,m}$ at $q=0$ in Appendix A. On the other hand, it is less trivial to show that $\alpha_{k,m}$ are holomorphic in τ. As the expressions of $\alpha_{-10,2}, \alpha_{-12,3}$ and $\alpha_{-18,4}$ contain E_4 in the denominator, these generators may have a pole at $\tau=e^{2\pi i/3}$. By carefully examining the structure of these expressions, one finds that these generators can be written as

$$
\begin{aligned}
\alpha_{-10,2}&=\frac{E_6X+\cdots}{576\Delta^2},\\
\alpha_{-12,3}&=\frac{3E_6^2A_1X+\cdots}{746496\Delta^3},\\
\alpha_{-18,4}&=\frac{E_6^2(-E_6A_2+2A_1C_1)X+\cdots}{1990656\Delta^4},
\end{aligned}
\tag{3.15}
$$

where "$\cdots$" are some polynomials in A_m, B_m, C_m, E_k and

$$X := \frac{15E_6B_2 - 11E_6C_2 - 4C_1^2}{E_4}. \tag{3.16}$$

Clearly, potential divergence can arise only through X. Therefore the proof boils down to showing that X is regular at $\tau = e^{2\pi i/3}$. This can be done as follows: The relation (2.27) can be rewritten as

$$\begin{aligned} 3X^2 - 24A_1^2X \;=\; & -13824\Delta A_4 - 448E_4^2A_1A_3 + 448E_6C_1A_3 - 1280E_6A_1B_3 \\ & +1280E_4C_1B_3 + 216E_4^2A_2^2 - 1440E_4A_1^2A_2 + 720E_6A_2B_2 \\ & +288C_1^2A_2 + 1275E_4B_2^2 - 990E_4B_2C_2 - 2640A_1C_1B_2 \\ & +363E_4C_2^2 + 528A_1C_1C_2 + 1680A_1^4. \end{aligned} \tag{3.17}$$

Since the right-hand side is holomorphic in τ, X has to be regular at $\tau = e^{2\pi i/3}$. Hence, we have shown that all $\alpha_{k,m}$ are indeed $W(E_7)$-invariant weak Jacobi forms.

The above $\alpha_{k,m}$ satisfy all the conditions required for the generators in the Wirthmüller's theorem explained in section 3.1. Thus we conclude that they give a full set of generators of the algebra of $W(E_7)$-invariant weak Jacobi forms over the algebra of modular forms $\mathbb{C}[E_4, E_6]$.

If we set $\boldsymbol{\mu} = \mathbf{0}$, the generators become

$$\begin{aligned} \alpha_{0,1}(\tau, \mathbf{0}) \;&=\; 4, \\ \alpha_{k,m}(\tau, \mathbf{0}) \;&=\; 0 \quad (k < 0). \end{aligned} \tag{3.18}$$

3.4. E_6 case

In the same way as in the E_7 case, one can reduce the Seiberg–Witten curve for the E-string theory to the curve that has only $W(E_6)$ symmetry and transform it into the form of the deformed singularity of type $\tilde{E}_6$.

The general deformation of a singularity of type $\tilde{E}_6$ takes the form [6]

(3.19)
$$\begin{aligned} uy^2 = 4x^3 - \frac{1}{12}E_4u^2x - \frac{1}{216}E_6u^3 & \\ + \alpha_{0,1}u^2 + \alpha_{-2,1}ux + \alpha_{-5,1}xy + \alpha_{-6,2}u + \alpha_{-8,2}x + \alpha_{-9,2}y + \alpha_{-12,3}. & \end{aligned}$$

One can formally assign weights $-6, -4, -3, k$ and indices $1, 1, 1, m$ to $u, x, y, \alpha_{k,m}$ respectively, so that all terms in the equation are of weight -12

and index 3. The curve (3.19) can be transformed into the Weierstrass form (3.6) with (3.11) as follows: First, perform a translation of y to eliminate the linear terms in y. Next, perform a translation of x to eliminate the quadratic terms in x. Finally, rescale the variables as $x \to u^{-1}x,\ y \to u^{-2}y$.

Next, we reduce the original Seiberg–Witten curve in section 3.2: We first set $\boldsymbol{\mu} = \boldsymbol{\mu}^{(6)}$, then rewrite it in terms of the seven E_6 Jacobi forms constructed in section 2.4, and finally replace u by $u + u_0$. Here, u_0 is given in (3.9). By comparing this curve with the above curve in the Weierstrass form, we are able to determine all the coefficients $\alpha_{k,m}$ in (3.19). The results are as follows:

$$
\begin{aligned}
\alpha_{0,1} &= \frac{E_4^2A_1 - E_6C_1}{432\Delta}, \qquad \alpha_{-2,1} = \frac{E_6A_1 - E_4C_1}{36\Delta}, \qquad \alpha_{-5,1} = \frac{2iD_1}{\Delta},\\
\alpha_{-6,2} &= \frac{1}{10368\Delta^2}\big((-5E_4^3 + 5E_6^2)B_2\\
&\qquad -5E_4E_6A_1^2 + 10E_4^2A_1C_1 - 5E_6C_1^2 + 72E_4D_1^2\big),\\
\alpha_{-8,2} &= \frac{1}{288\Delta^2}\big((E_4^3 - E_6^2)A_2 - E_4^2A_1^2 + 2E_6A_1C_1 - E_4C_1^2\big),\\
\alpha_{-9,2} &= \frac{i}{108E_4\Delta^2}\big((-8E_4^3 + 8E_6^2)D_2 - 3E_4^2A_1D_1 + 3E_6C_1D_1\big),\\
\alpha_{-12,3} &= \frac{1}{186624E_4^2\Delta^3}\big((7E_4^7 - 14E_4^4E_6^2 + 7E_4E_6^4)A_3\\
&\qquad + (9E_4^6 - 9E_4^3E_6^2)A_1A_2\\
&\qquad + (-9E_4^4E_6 + 9E_4E_6^3)C_1A_2 + (30E_4^4E_6 - 30E_4E_6^3)A_1B_2\\
&\qquad + (-30E_4^5 + 30E_4^2E_6^2)C_1B_2 + (1152E_4^3E_6 - 1152E_6^3)D_1D_2\\
&\qquad + (-16E_4^5 + 23E_4^2E_6^2)A_1^3 - 21E_6E_4^3A_1^2C_1\\
&\qquad + (30E_4^4 - 9E_4E_6^2)A_1C_1^2 - 7E_4^2E_6C_1^3\\
&\qquad + (432E_4^3 - 432E_6^2)C_1D_1^2\big).
\end{aligned}
\tag{3.20}
$$

Here, A_m, B_2, C_1, D_m are the Jacobi forms constructed in section 2.4 and we have omitted superscript E_6.

When $\boldsymbol{\mu} = \boldsymbol{\mu}^{(6)}$, $\alpha_{k,m}^{E_7}$ are expressed as polynomials in $\alpha_{k,m}^{E_6}$. The relations are extremely simple:

$$
\begin{aligned}
&\alpha_{0,1}^{E_7} = \alpha_{0,1}^{E_6}, && \alpha_{-2,1}^{E_7} = \alpha_{-2,1}^{E_6}, && \alpha_{-6,2}^{E_7} = \alpha_{-6,2}^{E_6},\\
&\alpha_{-8,2}^{E_7} = \alpha_{-8,2}^{E_6}, && \alpha_{-10,2}^{E_7} = \frac{1}{4}\left(\alpha_{-5,1}^{E_6}\right)^2, && \alpha_{-12,3}^{E_7} = \alpha_{-12,3}^{E_6},\\
&\alpha_{-14,3}^{E_7} = \frac{1}{2}\alpha_{-5,1}^{E_6}\alpha_{-9,2}^{E_6}, && \alpha_{-18,4}^{E_7} = \frac{1}{4}\left(\alpha_{-9,2}^{E_6}\right)^2.
\end{aligned}
\tag{3.21}
$$

This is in agreement with the description of the generators of E_7 Jacobi forms in [2].

In the same way as in the E_7 case, one can show that $\alpha_{k,m}$ given in (3.20) are genuine $W(E_6)$-invariant weak Jacobi forms (of weight k and index m). We present the concrete expressions of them at $q=0$ in Appendix A. The expressions of $\alpha^{E_6}_{-9,2}$ and $\alpha^{E_6}_{-12,3}$ contain E_4 in the denominator and thus they may have a pole at $\tau = e^{2\pi i/3}$. However, since all $\alpha^{E_7}_{k,m}$ are genuine Jacobi forms, it is clear from (3.21) that $\alpha^{E_6}_{-9,2}$ and $\alpha^{E_6}_{-12,3}$ are in fact regular at $\tau = e^{2\pi i/3}$.

The above $\alpha_{k,m}$ satisfy all the conditions required for the generators in the Wirthmüller's theorem explained in section 3.1. Thus we conclude that they give a full set of generators of the algebra of $W(E_6)$-invariant weak Jacobi forms over the algebra of modular forms $\mathbb{C}[E_4, E_6]$.

If we set $\boldsymbol{\mu} = \mathbf{0}$, the generators become

$$\begin{aligned} \alpha_{0,1}(\tau,\mathbf{0}) &= 4, \\ \alpha_{k,m}(\tau,\mathbf{0}) &= 0 \quad (k<0). \end{aligned} \tag{3.22}$$

In [8] E_6 Jacobi forms were used in the study of the flat structure for the elliptic singularity of type $\tilde{E}_6$. The generators specified in [8] (up to the overall factor $e(-mt)$) are expressed in terms of our generators as

$$\begin{aligned} &\varphi_0 = 18\alpha_{0,1}, && \varphi_1 = \frac{3}{2}\alpha_{-2,1}, \quad \varphi_2 = -\frac{i}{2}\alpha_{-5,1}, \\ &\varphi_3 = -9\alpha_{-6,2} - \frac{5}{64}E_4\left(\alpha_{-5,1}\right)^2, && \varphi_4 = -3\alpha_{-8,2}, \quad \varphi_5 = -3i\alpha_{-9,2}, \\ &\varphi_6 = 27\alpha_{-12,3} - \frac{159}{16}\alpha_{-2,1}\left(\alpha_{-5,1}\right)^2. \end{aligned} \tag{3.23}$$

Appendix A. Seiberg–Witten curves at $q = 0$

In this appendix we present the Seiberg–Witten curves of type $\tilde{E}_n$ at $q=0$ ($\tau = i\infty$). These Seiberg–Witten curves describe the low-energy theory of 5d SU(2) $N_{\mathrm{f}} = 7$ gauge theory on $\mathbb{R}^4 \times S^1$. The $\tilde{E}_8$ curve below is the 5d E_8 curve in [4]. The $\tilde{E}_n$ $(n=7,6)$ curves below are not equivalent to the 5d E_n curves in [4]: The former curves give a degenerate fiber at $u=0$ while the latter curves give a degenerate fiber at $u=\infty$. Physically, the former curves describe special cases of 5d SU(2) $N_{\mathrm{f}} = 7$ theory while the latter ones describe 5d SU(2) $N_{\mathrm{f}} = n-1$ theories.

For each E_n let us define

$$\alpha^{(0)}_{k,m}(\boldsymbol{\mu}) := \alpha_{k,m}(\tau = i\infty, \boldsymbol{\mu}) \tag{A.1}$$

and the Weyl orbit character associated with the fundamental weight $\boldsymbol{\Lambda}_j$

$$w_j(\boldsymbol{\mu}) := \sum_{\boldsymbol{v}\in\mathcal{O}(\boldsymbol{\Lambda}_j)} e^{2\pi i \boldsymbol{v}\cdot\boldsymbol{\mu}}. \tag{A.2}$$

- $\tilde{E}_8$ curve:

$$\begin{aligned} y^2 \;=\;& 4x^3 - \frac{1}{12}u^4 x - \frac{1}{216}u^6 \\ & - \sum_{m=2}^{4} \alpha^{(0)}_{4-6m,m} u^{4-m} x - \sum_{m=1}^{6} \alpha^{(0)}_{6-6m,m} u^{6-m}, \end{aligned} \tag{A.3}$$

where

$$\begin{aligned} \alpha^{(0)}_{0,1} =& -4, \qquad \alpha^{(0)}_{-6,2} = -\frac{1}{18}w_1 - 3w_8 + 840, \\ \alpha^{(0)}_{-8,2} =& -\frac{2}{3}w_1 + 12w_8 - 1440, \\ \alpha^{(0)}_{-12,3} =& -\frac{1}{6}w_2 - 4w_7 - 8w_1 + 528w_8 - 79680, \\ \alpha^{(0)}_{-14,3} =& -2w_2 + 96w_1 - 1152w_8 + 103680, \\ \alpha^{(0)}_{-18,4} =& \frac{2}{9}w_1^2 - \frac{1}{3}w_3 - \frac{16}{3}w_6 - 24w_1w_8 - 120w_8^2 \\ & + \frac{424}{3}w_2 + 1272w_7 + 4608w_1 - 25920w_8 + 3939840, \\ \alpha^{(0)}_{-20,4} =& \frac{4}{3}w_1^2 - 4w_3 - 16w_6 - 48w_1w_8 - 144w_8^2 \\ & + 400w_2 + 1440w_7 + 1728w_1 + 41472w_8 - 2073600, \\ \alpha^{(0)}_{-24,5} =& \frac{2}{3}w_1w_2 - 4w_5 - 16w_1w_7 + 64w_2w_8 + 288w_7w_8 - 96w_1^2 - 60w_3 \\ & - 160w_6 + 3456w_8^2 + 800w_2 - 24480w_7 - 108480w_1 + 933120w_8 \\ & - 97873920, \\ \alpha^{(0)}_{-30,6} =& -\frac{8}{27}w_1^3 + w_2^2 + \frac{4}{3}w_1w_3 - 4w_4 - \frac{32}{3}w_1w_6 - 48w_1^2w_8 + 48w_2w_7 \\ & + 288w_7^2 - 40w_3w_8 - 480w_6w_8 - 2592w_1w_8^2 - 9792w_8^3 \end{aligned}$$

$$
\begin{aligned}
&+\frac{1124}{3}w_1w_2+548w_5+6688w_1w_7+1884w_2w_8+25632w_7w_8\\
&+24576w_1^2+12920w_3+88320w_6+578688w_1w_8+1714176w_8^2\\
&-1694400w_2-8460000w_7-30102720w_1-104198400w_8\\
&+721612800. \qquad \text{(A.4)}
\end{aligned}
$$

- $\tilde{E}_7$ curve:

$$
\begin{aligned}
y^2 \;=\;& 4ux^3-\frac{1}{12}u^3x-\frac{1}{216}u^4\\
&+\alpha^{(0)}_{0,1}u^3+\alpha^{(0)}_{-2,1}u^2x+\alpha^{(0)}_{-6,2}u^2+\alpha^{(0)}_{-8,2}ux+\alpha^{(0)}_{-10,2}x^2\\
&+\alpha^{(0)}_{-12,3}u+\alpha^{(0)}_{-14,3}x+\alpha^{(0)}_{-18,4}, \qquad \text{(A.5)}
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha^{(0)}_{0,1} &= \frac{1}{36}w_7+\frac{22}{9}, \qquad \alpha^{(0)}_{-2,1}=\frac{1}{3}w_7-\frac{56}{3},\\
\alpha^{(0)}_{-6,2} &= -\frac{1}{16}w_7^2+\frac{26}{9}w_1+\frac{5}{36}w_2+\frac{1}{36}w_6-6w_7+67,\\
\alpha^{(0)}_{-8,2} &= -\frac{1}{2}w_7^2-\frac{32}{3}w_1+\frac{4}{3}w_2+\frac{2}{3}w_6+32w_7-152,\\
\alpha^{(0)}_{-10,2} &= -w_7^2+32w_1-4w_2+4w_6-32w_7+176,\\
\alpha^{(0)}_{-12,3} &= \frac{7}{108}w_7^3-\frac{32}{9}w_1w_7-\frac{7}{18}w_2w_7-\frac{1}{9}w_6w_7+\frac{46}{9}w_7^2\\
&\quad -\frac{1244}{9}w_7+\frac{736}{9}w_1-\frac{77}{9}w_2+\frac{10}{3}w_3+\frac{1}{3}w_5+\frac{104}{9}w_6+\frac{13216}{27},\\
\alpha^{(0)}_{-14,3} &= \frac{1}{3}w_7^3+\frac{64}{3}w_1w_7-\frac{2}{3}w_2w_7-\frac{4}{3}w_6w_7-8w_7^2\\
&\quad -\frac{896}{3}w_1-\frac{116}{3}w_2-8w_3+4w_5-\frac{160}{3}w_6-80w_7-\frac{3968}{3},\\
\alpha^{(0)}_{-18,4} &= -\frac{1}{36}w_7^4-\frac{4}{9}w_1w_7^2+\frac{2}{9}w_2w_7^2+\frac{1}{9}w_6w_7^2-\frac{16}{9}w_7^3+64w_1^2+16w_1w_6\\
&\quad -w_2^2-\frac{896}{9}w_1w_7-\frac{326}{9}w_2w_7-\frac{8}{3}w_3w_7-\frac{2}{3}w_5w_7-\frac{64}{9}w_6w_7\\
&\quad -\frac{596}{3}w_7^2+\frac{29888}{9}w_1+\frac{1184}{9}w_2+\frac{208}{3}w_3+4w_4-\frac{32}{3}w_5\\
&\quad +\frac{5632}{9}w_6+\frac{2816}{9}w_7+\frac{111488}{9}. \qquad \text{(A.6)}
\end{aligned}
$$

• $\tilde{E}_6$ curve:

$$\begin{aligned} uy^2 &= 4x^3 - \frac{1}{12}u^2x - \frac{1}{216}u^3 \\ &\quad + \alpha^{(0)}_{0,1}u^2 + \alpha^{(0)}_{-2,1}ux + \alpha^{(0)}_{-5,1}xy + \alpha^{(0)}_{-6,2}u + \alpha^{(0)}_{-8,2}x + \alpha^{(0)}_{-9,2}y + \alpha^{(0)}_{-12,3}, \end{aligned} \tag{A.7}$$

where

$$\begin{aligned} \alpha^{(0)}_{0,1} &= \frac{1}{36}w_1 + \frac{1}{36}w_6 + \frac{5}{2}, \qquad \alpha^{(0)}_{-2,1} = \frac{1}{3}w_1 + \frac{1}{3}w_6 - 18, \\ \alpha^{(0)}_{-5,1} &= i\left(2w_1 - 2w_6\right), \\ \alpha^{(0)}_{-6,2} &= -\frac{1}{16}w_1^2 - \frac{1}{16}w_6^2 - \frac{7}{72}w_1w_6 + 3w_2 + \frac{1}{6}w_3 + \frac{1}{6}w_5 - \frac{10}{3}w_1 - \frac{10}{3}w_6 \\ &\quad + 54, \\ \alpha^{(0)}_{-8,2} &= -\frac{1}{2}w_1^2 - \frac{1}{2}w_6^2 - \frac{1}{3}w_1w_6 - 12w_2 + 2w_3 + 2w_5 + 20w_1 + 20w_6 \\ &\quad - 108, \\ \alpha^{(0)}_{-9,2} &= i\left(-\frac{1}{3}w_1^2 + \frac{1}{3}w_6^2 + 2w_3 - 2w_5 - 14w_1 + 14w_6\right), \\ \alpha^{(0)}_{-12,3} &= \frac{7}{108}w_1^3 + \frac{7}{108}w_6^3 + \frac{1}{12}w_1^2w_6 + \frac{1}{12}w_1w_6^2 - 2w_1w_2 - 2w_6w_2 \\ &\quad - \frac{1}{2}w_1w_3 - \frac{1}{2}w_6w_5 - \frac{1}{6}w_1w_5 - \frac{1}{6}w_3w_6 + \frac{11}{6}w_1^2 + \frac{11}{6}w_6^2 \\ &\quad + 15w_1w_6 + 4w_4 - 12w_2 - 6w_3 - 6w_5 - 60w_1 - 60w_6 - 72. \end{aligned} \tag{A.8}$$

Appendix B. Simple roots and fundamental weights of E_n

Let $\{\mathbf{e}_j\}$ $(j = 1, 2, \ldots, 8)$ be the orthonormal basis of $\mathbb{C}^8$.
• The simple roots of E_8:

$$\begin{aligned} \boldsymbol{\alpha}^{E_8}_1 &= \tfrac{1}{2}\left(\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 - \mathbf{e}_6 - \mathbf{e}_7 + \mathbf{e}_8\right), \\ \boldsymbol{\alpha}^{E_8}_2 &= \mathbf{e}_1 + \mathbf{e}_2, \\ \boldsymbol{\alpha}^{E_8}_j &= -\mathbf{e}_{j-2} + \mathbf{e}_{j-1} \quad (j = 3, 4, \ldots, 8). \end{aligned} \tag{B.1}$$

• The fundamental weights of E_8:

$$\begin{aligned} \boldsymbol{\Lambda}^{E_8}_1 &= 2\mathbf{e}_8, \\ \boldsymbol{\Lambda}^{E_8}_2 &= \tfrac{1}{2}\mathbf{e}_1 + \tfrac{1}{2}\mathbf{e}_2 + \tfrac{1}{2}\mathbf{e}_3 + \tfrac{1}{2}\mathbf{e}_4 + \tfrac{1}{2}\mathbf{e}_5 + \tfrac{1}{2}\mathbf{e}_6 + \tfrac{1}{2}\mathbf{e}_7 + \tfrac{5}{2}\mathbf{e}_8, \end{aligned}$$

$$
\begin{aligned}
\mathbf{\Lambda}_3^{E_8} &= -\tfrac{1}{2}\mathbf{e}_1+\tfrac{1}{2}\mathbf{e}_2+\tfrac{1}{2}\mathbf{e}_3+\tfrac{1}{2}\mathbf{e}_4+\tfrac{1}{2}\mathbf{e}_5+\tfrac{1}{2}\mathbf{e}_6+\tfrac{1}{2}\mathbf{e}_7+\tfrac{7}{2}\mathbf{e}_8,\\
\mathbf{\Lambda}_4^{E_8} &= \mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_5+\mathbf{e}_6+\mathbf{e}_7+5\mathbf{e}_8,\\
\mathbf{\Lambda}_5^{E_8} &= \mathbf{e}_4+\mathbf{e}_5+\mathbf{e}_6+\mathbf{e}_7+4\mathbf{e}_8,\\
\mathbf{\Lambda}_6^{E_8} &= \mathbf{e}_5+\mathbf{e}_6+\mathbf{e}_7+3\mathbf{e}_8,\\
\mathbf{\Lambda}_7^{E_8} &= \mathbf{e}_6+\mathbf{e}_7+2\mathbf{e}_8,\\
\mathbf{\Lambda}_8^{E_8} &= \mathbf{e}_7+\mathbf{e}_8.
\end{aligned} \tag{B.2}
$$

• The simple roots of E_7:

$$\boldsymbol{\alpha}_j^{E_7} := \boldsymbol{\alpha}_j^{E_8} \quad (j=1,2,\ldots,7). \tag{B.3}$$

• The fundamental weights of E_7:

$$
\begin{aligned}
\mathbf{\Lambda}_1^{E_7} &= -\mathbf{e}_7+\mathbf{e}_8,\\
\mathbf{\Lambda}_2^{E_7} &= \tfrac{1}{2}\mathbf{e}_1+\tfrac{1}{2}\mathbf{e}_2+\tfrac{1}{2}\mathbf{e}_3+\tfrac{1}{2}\mathbf{e}_4+\tfrac{1}{2}\mathbf{e}_5+\tfrac{1}{2}\mathbf{e}_6-\mathbf{e}_7+\mathbf{e}_8,\\
\mathbf{\Lambda}_3^{E_7} &= -\tfrac{1}{2}\mathbf{e}_1+\tfrac{1}{2}\mathbf{e}_2+\tfrac{1}{2}\mathbf{e}_3+\tfrac{1}{2}\mathbf{e}_4+\tfrac{1}{2}\mathbf{e}_5+\tfrac{1}{2}\mathbf{e}_6-\tfrac{3}{2}\mathbf{e}_7+\tfrac{3}{2}\mathbf{e}_8,\\
\mathbf{\Lambda}_4^{E_7} &= \mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_5+\mathbf{e}_6-2\mathbf{e}_7+2\mathbf{e}_8,\\
\mathbf{\Lambda}_5^{E_7} &= \mathbf{e}_4+\mathbf{e}_5+\mathbf{e}_6-\tfrac{3}{2}\mathbf{e}_7+\tfrac{3}{2}\mathbf{e}_8,\\
\mathbf{\Lambda}_6^{E_7} &= \mathbf{e}_5+\mathbf{e}_6-\mathbf{e}_7+\mathbf{e}_8,\\
\mathbf{\Lambda}_7^{E_7} &= \mathbf{e}_6-\tfrac{1}{2}\mathbf{e}_7+\tfrac{1}{2}\mathbf{e}_8.
\end{aligned} \tag{B.4}
$$

• The simple roots of E_6:

$$\boldsymbol{\alpha}_j^{E_6} := \boldsymbol{\alpha}_j^{E_8} \quad (j=1,2,\ldots,6). \tag{B.5}$$

• The fundamental weights of E_6:

$$
\begin{aligned}
\mathbf{\Lambda}_1^{E_6} &= -\tfrac{2}{3}\mathbf{e}_6-\tfrac{2}{3}\mathbf{e}_7+\tfrac{2}{3}\mathbf{e}_8,\\
\mathbf{\Lambda}_2^{E_6} &= \tfrac{1}{2}\mathbf{e}_1+\tfrac{1}{2}\mathbf{e}_2+\tfrac{1}{2}\mathbf{e}_3+\tfrac{1}{2}\mathbf{e}_4+\tfrac{1}{2}\mathbf{e}_5-\tfrac{1}{2}\mathbf{e}_6-\tfrac{1}{2}\mathbf{e}_7+\tfrac{1}{2}\mathbf{e}_8,\\
\mathbf{\Lambda}_3^{E_6} &= -\tfrac{1}{2}\mathbf{e}_1+\tfrac{1}{2}\mathbf{e}_2+\tfrac{1}{2}\mathbf{e}_3+\tfrac{1}{2}\mathbf{e}_4+\tfrac{1}{2}\mathbf{e}_5-\tfrac{5}{6}\mathbf{e}_6-\tfrac{5}{6}\mathbf{e}_7+\tfrac{5}{6}\mathbf{e}_8,\\
\mathbf{\Lambda}_4^{E_6} &= \mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_5-\mathbf{e}_6-\mathbf{e}_7+\mathbf{e}_8,\\
\mathbf{\Lambda}_5^{E_6} &= \mathbf{e}_4+\mathbf{e}_5-\tfrac{2}{3}\mathbf{e}_6-\tfrac{2}{3}\mathbf{e}_7+\tfrac{2}{3}\mathbf{e}_8,\\
\mathbf{\Lambda}_6^{E_6} &= \mathbf{e}_5-\tfrac{1}{3}\mathbf{e}_6-\tfrac{1}{3}\mathbf{e}_7+\tfrac{1}{3}\mathbf{e}_8.
\end{aligned} \tag{B.6}
$$

Appendix C. Special functions

The Jacobi theta functions are defined as

$$\begin{aligned}\vartheta_1(z,\tau) &:= i\sum_{n\in\mathbb{Z}}(-1)^n y^{n-1/2}q^{(n-1/2)^2/2},\\ \vartheta_2(z,\tau) &:= \sum_{n\in\mathbb{Z}} y^{n-1/2}q^{(n-1/2)^2/2},\\ \vartheta_3(z,\tau) &:= \sum_{n\in\mathbb{Z}} y^n q^{n^2/2},\\ \vartheta_4(z,\tau) &:= \sum_{n\in\mathbb{Z}}(-1)^n y^n q^{n^2/2},\end{aligned} \tag{C.1}$$

where

$$y = e^{2\pi i z}, \qquad q = e^{2\pi i\tau}. \tag{C.2}$$

We often use the following abbreviated notation

$$\vartheta_k(\tau) := \vartheta_k(0,\tau). \tag{C.3}$$

The Dedekind eta function is defined as

$$\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{C.4}$$

The Eisenstein series are given by

$$E_{2n}(\tau) = 1 - \frac{4n}{B_{2n}}\sum_{k=1}^{\infty}\frac{k^{2n-1}q^k}{1-q^k} \tag{C.5}$$

for $n \in \mathbb{Z}_{>0}$. The Bernoulli numbers B_k are defined by

$$\frac{x}{e^x-1} = \sum_{k=0}^{\infty}\frac{B_k}{k!}x^k. \tag{C.6}$$

We often abbreviate $\eta(\tau)$, $E_{2n}(\tau)$ as η, E_{2n} respectively.

Modular properties of the above functions are as follows:

$$\vartheta_1(z,\tau+1) = e^{\frac{\pi i}{4}}\vartheta_1(z,\tau), \qquad \vartheta_1(\tfrac{z}{\tau},-\tfrac{1}{\tau}) = e^{-\frac{3\pi i}{4}}\tau^{\frac{1}{2}}e^{\frac{\pi i}{\tau}z^2}\vartheta_1(z,\tau),$$

$$\begin{aligned}
\vartheta_2(z,\tau+1) &= e^{\frac{\pi i}{4}}\vartheta_2(z,\tau), & \vartheta_2(\tfrac{z}{\tau},-\tfrac{1}{\tau}) &= e^{-\frac{\pi i}{4}}\tau^{\frac{1}{2}}e^{\frac{\pi i}{\tau}z^2}\vartheta_4(z,\tau),\\
\vartheta_3(z,\tau+1) &= \vartheta_4(z,\tau), & \vartheta_3(\tfrac{z}{\tau},-\tfrac{1}{\tau}) &= e^{-\frac{\pi i}{4}}\tau^{\frac{1}{2}}e^{\frac{\pi i}{\tau}z^2}\vartheta_3(z,\tau),\\
\vartheta_4(z,\tau+1) &= \vartheta_3(z,\tau), & \vartheta_4(\tfrac{z}{\tau},-\tfrac{1}{\tau}) &= e^{-\frac{\pi i}{4}}\tau^{\frac{1}{2}}e^{\frac{\pi i}{\tau}z^2}\vartheta_2(z,\tau),\\
\eta(\tau+1) &= e^{\frac{\pi i}{12}}\eta(\tau), & \eta(-\tfrac{1}{\tau}) &= e^{-\frac{\pi i}{4}}\tau^{\frac{1}{2}}\eta(\tau),\\
E_{2n}(\tau+1) &= E_{2n}(\tau), & E_{2n}(-\tfrac{1}{\tau}) &= \tau^{2n}E_{2n}(\tau) \quad (n\geq 2).
\end{aligned} \tag{C.7}$$

Acknowledgments

The author would like to thank Jie Gu for stimulating questions, which inspired him to start this work. The author would also like to thank Haowu Wang for valuable comments and discussions. This work was supported in part by JSPS KAKENHI Grant Number 26400257 and JSPS Japan–Russia Research Cooperative Program.

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Received 28 June 2017
Accepted 15 August 2018