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# Vertex operator algebras with central charges $164/5$ and $236/7$

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This paper completes the classification problem which was proposed in the previous paper [1] in which we attempted to characterize the minimal models and families obtained by the tensor products and the simple current extensions of minimal models under the condition that the characters of simple modules satisfy modular differential equations of the third order, and a mild condition on vertex operator algebras. In the previous work, several vertex operator algebras which are not the minimal models appeared. Five elevenths of them are identified to well-known vertex operator algebras which are all vertex operator algebras related with orbifold models of lattice vertex operator algebras. However, we were not able to deny the existence of simple, rational vertex operator algebras of CFT and finite type with central charges either $164/5$ or $236/7$ under the condition on which we worked in [1]. The characterization of minimal models with at most two simple modules was achieved in the same paper.

The numbers $164/5$ and $236/7$ were already appeared in the paper of Tuite and Van ([17]) in the different context. However, they were out of reach of our conclusion. Moreover, we solve the conjecture, which was proposed by Hampapura and Mukhi [8], that the $j$-function is expressed by characters of the minimal models.

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## 1. Introduction

In this paper we study a simple, rational vertex operator algebra $V$ (simply VOA) of CFT and finite ($C_2$-cofinite) type, which has further properties are either $164/5$ or $236/7$, (b) the weight one space is trivial, (c) characters of simple modules over $V$ are solutions of a *monic* modular linear differential equations (simply MLDE) of the *third order* of weight 0 (see §3 for the definition of monic MLDEs). In [1], we have shown that there are eleven rational numbers which can be central charges of VOAs satisfying the conditions (b) and (c). Moreover, we have obtained the exact expression of the monic MLDE for each central charge. Three of these numbers *uniquely* correspond to central charges of the minimal models and their tensor product, respectively, and six of them coincide with central charges of $\mathbb{Z}_2$-orbifold models of lattice VOAs and their extensions (which include the moonshine VOA) ([1], [8], [17]). However, it was not known if the remaining two central charges $164/5$ and $236/7$ have corresponding VOAs, respectively. Our principal aim of this paper is to show that a simple, rational VOA of CFT and finite type satisfying the conditions (a)–(c) does not exist. Combining this with the partial classification obtained in [1], we complete a proof that any simple, rational VOA of CFT and of finite type, which satisfies (b) and (c) is isomorphic to one of the minimal models with central charges $1/2$ and $-68/7$ and the two-fold tensor product of the minimal model with the central charge $-22/5$ if it is not a $\mathbb{Z}_2$-orbifold model of a lattice VOA and is not its extensions.

Let $V$ be a simple VOA with a central charge either $164/5$ or $236/7$, which satisfies the condition (b) and (c). Then we can uniquely determine the monic MLDE in (c) as it was written in [1]. Therefore, we can find indicial roots and then solutions of the monic MLDE which would be the characters of simple $V$-modules. It is then well-known that the space of solutions of a monic MLDE is invariant under the usual slash 0 action of the full modular group $\Gamma_1 = SL_2(\mathbb{Z})$. This is closely related to the modular invariance of the space of characters. Then we can determine the square matrix of degree three, which represents the transformation $S : \mathbb{H} \to \mathbb{H}$ $(\tau \mapsto 1/\tau)$ where $\tau \in \mathbb{H}$.

Once the $S$-matrix has been computed, one can obtain the *quantum dimension* of each simple module by Lemma 4.2 and Theorem 5.1 of [4], and then the so-called *global dimension* (which is the sum of square of quantum dimensions). In particular, one knows that the quantum dimension $\operatorname{qdim} M$ for any simple $V$-module $M$ is not less than 1. Proposition 4.5 of [4] now shows that the global dimension of $V$ (denoted by $\operatorname{global}(V)$) is simply written as $\operatorname{global}(V) = 1/(S_{00})^2$. In this paper we find that the value of $S_{00}$ is smaller than 3. However, this contradicts to $\operatorname{global}(V) \geq 3$ as the number of simple modules is at least three, which is also proved in this paper. Thus the theory of quantum and global dimensions developed in [4] allows one to prove non-existence of VOAs which we study.

**Warning** The reader may think that the classification of "unitary" modular tensor categories with rank 3 proved in Section 2 of [18] implies the main results of this paper. However, since our VOAs are not unitary, one cannot apply their result to our problem.

This paper is organized as follows. In Section 2 we give a brief review of basics of VOAs. The notion of vacuum-like vectors introduced in [12], which is used in Section 5, is also explained here. The definitions and the properties of monic MLDEs, and the concept of vector-valued modular forms are presented in Section 3. We recall briefly an important result on the quantum dimensions and the global dimensions of VOAs in Section 4. The explicit expressions of monic MLDEs associated with central charges 164/5 and 236/7 are given in Section 5. In Section 6 and 7 we compute the matrix elements of the $S$-transformation on bases of the spaces of the monic MLDEs which are associated with central charges 164/5 and 236/7, respectively, and obtain the global dimensions. The main theorems (Theorem 8 and Theorem 10) are proved in these sections.

Since the explicit expressions of the monic MLDEs for the central charge 236/7 are quite complicated, they are described in the first part of Appendix. The second part of Appendix is devoted to proofs of two expressions of the $j$-function observed in [8] in terms of solutions of the monic MLDEs (for $c = 164/5$ and $236/5$).

## 2. Vertex operator algebras

In this section we give a brief introduction to the theory of vertex operator algebras (for the complete definition, see [11] and [15]). A vertex operator algebra (simply VOA) is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ equipped with a linear map

$$V \to \mathrm{End}_{\mathbb{C}}(V)[[z, z^{-1}]] \quad \Big(a \mapsto Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}\Big).$$

The vector space $V$ is required to have a so-called vacuum element $\mathbf{1} \in V_0$ and a Virasoro element $\omega \in V_2$ satisfying a number of axioms. One of the axioms demands that $L_n = \omega_{n+1}$ $(n \in \mathbb{Z})$ define a module of the Virasoro algebra over $V$ with a central charge $c \in \mathbb{C}$, i.e.

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} c\, \delta_{m+n,0}. \tag{1}$$

Another axiom asks that $L_0$ is the grading operator. The non-negative integer of an element $v \in V_n$ is said to have an weight $n$ which is denoted by $\mathrm{wt}(v)$. A VOA $V$ is called *of CFT type* when $V_n$ is trivial for any $n < 0$ and $V_0$ is one-dimensional with the basis $\{\mathbf{1}\}$.

A *weak module* of a VOA $V$ is a pair $(M, Y)$ of a vector space and a linear map $Y : V \to \mathrm{End}_{\mathbb{C}}(M)[[z, z^{-1}]]$ satisfying several conditions (for more details, see e.g. [11], [15]). A weak $V$-module $M$ is called a $V$-module if

(a) it is graded by $\mathbb{C}$; $M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$,
(b) for any complex number $\lambda$ there exists a positive number $N$ such that $M_{\lambda+n} = 0$ for any $n + N < 0$,
(c) the endomorphism $a_{(n)}$ has weight $\mathrm{wt}(a) - n - 1$, i.e.,

$$a_{(n)} M_\lambda \subset M_{\lambda+\mathrm{wt}(a)-n-1}$$

for any homogeneous $a \in V$ and $n \in \mathbb{Z}$,

(d) the endomorphism $L_0$ is the grading operator of $M$.

A module of $V$, which does not satisfy the condition (d), is called *admissible*. If a $V$-module $M$ is simple, the conditions (b) and (c) show that there is a unique complex number $\lambda$ such that $M = \bigoplus_{n=0}^{\infty} M_{\lambda+n}$ and $M_\lambda \neq 0$. We call this $\lambda$ the *conformal weight* of $M$. A VOA is called *rational* when the number of simple module is finite and any admissible module is completely reducible (see [5] and [20]).

A VOA $V$ is called *of finite type* (or $C_2$-cofinite) if the subspace of $V$, whose elements are linear combinations of $a_{(-2)}b$ for all $a$, $b \in V$, has a finite codimension in $V$. It is known that if $V$ is of finite type, then the number of simple $V$-modules is finite and the central charge of $V$ as well as conformal weights of simple $V$-modules are rational numbers ([2], [16]).

One of interesting simple, rational VOAs of CFT and of finite type would be a series of the minimal model $V = L(c_{p,q}, 0)$ which was studied intensively in [19] (Theorem 4.2) by using works of Feigin and Fuchs ([6], [7]). This VOA is the simple quotient of the Verma module of the Virasoro algebra with the central charge $c_{p,q} = 1 - 6(p-q)^2/pq$ for coprime positive integers $p$ and $q$. It is noteworthy that any simple $V$-module is isomorphic to an irreducible highest weight module $L(c_{p,q}, h_{r,s})$ with the highest weight

$$h_{r,s} = \frac{(rq - sp)^2 - (p-q)^2}{4pq}$$

for $1 \leq r \leq p-1$ and $1 \leq s \leq q-1$ so that the number of simple $V$-modules is equal to $(p-1)(q-1)/2$ (see also [19]).

Let $V$ be a VOA and $M$ a weak $V$-module. An element $v \in M$ is called *vacuum-like* when $Y(a,z)v \in M[[z]]$, i.e., $Y(a,z)v$ has does not have negative exponents of $z$. It is known in [12] (Proposition 3.3) that $v \in M$ is vacuum-like if and only if $L_{-1}v = 0$. The following proposition is proved in [12] (Proposition 3.4).

**Proposition 1** ([12])**.** *Let $V$ be a vertex operator algebra and $M$ a weak $V$-module. Then* $\mathrm{Hom}_V(V, M)$ *is isomorphic to the space of vacuum-like elements of $M$.*

## 3. Monic modular linear differential equations

In this section we give a short explanation of the concepts of vector-valued modular forms and monic modular linear (ordinary) differential equations.

Let $\mathbb{H}$ be the complex upper-half plane. For a non-negative integer $k$ and a holomorphic function $f$ on $\mathbb{H}$, we define the slash action of $\gamma =$

$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1 = SL_2(\mathbb{Z})$ on $f$ by $(f|_k\gamma)(\tau) = (c\tau + d)^{-k} f(\gamma(\tau))$, where $\gamma(\tau) = (a\tau + b)/(c\tau + d)$. We simply write $f|_k\gamma$ instead of $(f|_k\gamma)(\tau)$ if this causes no confusion.

A *vector-valued modular form* (VVMF) of weight $k$ is a column vector ${}^t(f_1, f_2, \ldots, f_n)$ of holomorphic functions on $\mathbb{H}$ such that

(a) ${}^t(f_1, f_2, \ldots, f_n)\big|_k \gamma = \rho(\gamma)\, {}^t(f_1, f_2, \ldots, f_n)$ for any $\gamma \in \Gamma_1$, where $\rho$ is an $n$-dimensional representation of $\Gamma_1$ on $GL_n(\mathbb{C})$,
(b) the component $f_j$ has a Fourier expansion $f_j = q^{\lambda_j} \sum_{i=0}^{\infty} a_i^j q^i$, where $\lambda_j \in \mathbb{R}$ and $q = e^{2\pi i\tau}$ ($i = \sqrt{-1}$, $\tau \in \mathbb{H}$).

Let

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{j=0}^{\infty} \sigma_{2k-1}(j) q^j \quad (k = 1, 2, \ldots)$$

be the (normalized) Eisenstein series of weight $2k$, where $B_m$ is the $m^{\text{th}}$ Bernoulli number and $\sigma_m(n)$ is the division function. Let

$$M_*(\Gamma_1) = \bigoplus_{k=1}^{\infty} M_{2k}(\Gamma_1)$$

be the graded space of modular forms on $\Gamma_1$ and let

$$\mathfrak{d} : M_*(\Gamma_1) \to M_{*+2}(\Gamma_1)$$

be the Serre operation defined by

$$\mathfrak{d}(f) = f' - \frac{k}{12} E_2(f)\,, \quad f' = q\frac{df}{dq} = \frac{1}{2\pi i}\frac{df}{d\tau}$$

for any $f \in M_k(\Gamma_1)$. A *monic modular linear differential equation* (simply monic MLDE) of weight 0 is a linear ordinary differential equation

$$\mathfrak{d}^n(f) + \sum_{j=0}^{n-1} P_j \mathfrak{d}^j(f) = 0\,,$$

where an unknown $f$ is a holomorphic function on $\mathbb{H}$ and $P_i$ is a holomorphic modular form of weight $2(n-i)$. Then [14] (Theorem 3.7 and Theorem 4.3) says:

**Proposition 2.** *Let* ${}^t(f_1, f_2, \ldots, f_n)$ *be a column vector-valued modular form of weight* 0 *whose entries are linearly independent. If* $\lambda_1 < \lambda_2 < \cdots <$

*$\lambda_n$, where $\lambda_j$ is the smallest exponent of $q$ of the Fourier expansion of $f_j$, then $\{f_1, f_2, \ldots, f_n\}$ is a basis of the space of solutions of a monic modular linear differential equation of $n^{th}$ order if and only if*

$$n(n-1) = 12\sum_{j=1}^{n}\lambda_j\,. \tag{2}$$

**Remarks.** (a) If all smallest exponents of $q$ of the Fourier expansions of $f_j$ $(1 \le j \le n)$ satisfy $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, then vector-value function $\mathbb{F} = {}^t(f_1, f_2, \ldots, f_n)$ is called *strictly normalized.*

(b) If the set of entries of a VVMF is linearly independent, then there exists an invertible matrix $A$ such that $A\mathbb{F}$ is strictly normalized. Moreover, the matrix $A$ can be written as a products of elementary matrices.

Since any monic MLDE has a regular singularity point only at $q = 0$ ([14]), one can use the method of Frobenius to obtain solutions of monic MLDEs. The following lemma in [13] (Corollary 2.4) is easily checked.

**Lemma 3.** *Let $\lambda_1$, $\lambda_2$ and $\lambda_3$ be mutually distinct rational numbers. Then there is a unique monic modular linear differential equation of the third order whose indicial roots are $\lambda_1$, $\lambda_2$ and $\lambda_3$.*

## 4. Quantum and global dimensions

In this short section we recall the definitions of quantum dimensions and global dimensions and present a theorem and a proposition which are used in the following sections.

Let $V$ be a VOA and $M$ a simple $V$-module. The *trace function* on $M$ is defined by

$$\mathrm{tr}_M(v,\tau) = \mathrm{tr}_M\, o(v)q^{L_0-c/24} \tag{3}$$

for any homogeneous element $v \in V$, where $o(v) = v_{(\mathrm{wt}(v)-1)}$ which is an endomorphism on $M$ that preserves any homogeneous space of $M$. It is proved in [20] (see also [5]) that the series $\mathrm{tr}_M(v,\tau)$ converges for any fixed $v$ and is holomorphic on $\mathbb{H}$ if $V$ is of finite type. Since $o(\mathbf{1}) = \mathrm{id}_M$, the character $\mathrm{ch}_M(\tau)$ of $M$ coincides with $\mathrm{tr}_M(\mathbf{1},\tau)$.

The slash action of $\Gamma_1 = SL_2(\mathbb{Z})$ on the trace functions is defined by

$$(\mathrm{tr}_M\,|_k\gamma)(v,\tau) = (c\tau+d)^{-k}\,\mathrm{tr}_M(v,\gamma(\tau)) \text{ for any } \gamma = \begin{pmatrix} a & b \\ c & c \end{pmatrix} \in \Gamma_1\,,$$

where $k = \mathrm{wt}(v)$. The modular invariance of the space of trace functions are proved in [20] (Theorem 5.3.2).

**Theorem 4.** *Let $V$ be a simple, rational vertex operator algebra of CFT and the finite type and let $M^0$, $M^1$, ..., $M^n$ be the set of inequivalent simple $V$-modules. Let $\gamma$ be an element of $SL_2(\mathbb{Z})$. Then there exist complex numbers $\gamma_{ij}$ such that*

$$\mathrm{tr}_{M^i}|\gamma(v,\tau) = \sum_{j=0}^{n} \gamma_{ij}\,\mathrm{tr}_{M^j}(v,\tau) \tag{4}$$

*for all $v \in V$. Moreover, the complex numbers $\gamma_{ij}$ do not depend on $v \in V$.*

There is a matrix $S = (S_{ij})$ such that

$$\mathrm{tr}_{M^i}(v,-1/\tau) = \tau^{\mathrm{wt}(v)} \sum_{j=0}^{n} S_{ij}\,\mathrm{tr}_{M^j}(v,\tau) \tag{5}$$

for homogeneous $v \in V$. The matrix $S \in GL_{n+1}(\mathbb{C})$ is called the *$S$-matrix* associated with $V$ in the literature.

Let $V$ be a VOA and $M$ a simple $V$-module. Suppose that the characters $\mathrm{ch}_V(\tau)$ and $\mathrm{ch}_M(\tau)$ are holomorphic functions on $\mathbb{H}$. The *quantum dimension* of $M$ (which is originally introduced in [3]) is defined by

$$\mathrm{qdim}_V M = \lim_{y\to +0} \frac{\mathrm{ch}_M(\sqrt{-1}y)}{\mathrm{ch}_V(\sqrt{-1}y)}\,, \tag{6}$$

where $y > 0$ is a real number. Dong, Jiao and Xu have proved the following theorem in [4] (Lemma 4.2, Theorem 5.1).

**Theorem 5.** *Let $V$ be a simple, rational vertex operator algebra of CFT and of finite type and let $M^0$, $M^1$, ..., $M^n$ be the set of inequivalent simple $V$-modules, where $M^0 = V$. Let $\lambda_i$ be a conformal weight of $M^i$.*

(a) *Suppose that $\lambda_i > 0$ for all $1 \le i \le n$. Then $S_{00} \ne 0$ and $\mathrm{qdim}_V M^i = S_{i0}/S_{00}$, where $S = (S_{ij})$ is the $S$-matrix associated with $V$.*
(b) *For any integer $0 \le i \le n$, the quantum dimension of $M^i$ belongs to the set*

$$\{2\cos(\pi/n)\,|\,n \ge 3\} \cup \{a\,|\,2 \le a < \infty\}\,,$$

*where $a$ is an algebraic number. In particular, we have $\mathrm{qdim}_V M^i \ge 1$.*

Suppose that a VOA $V$ has only finitely many simple $V$-modules which are denoted by $M^0, M^1, \ldots, M^n$, where $M^0 = V$, and that $\mathrm{ch}_{M^i}(\tau)$ are holomorphic functions on $\mathbb{H}$. Then the *global dimension* of $V$ is defined by

$$\mathrm{global}(V) = \sum_{j=0}^{n} (\mathrm{qdim}_V M^j)^2 . \tag{7}$$

It then follows from the very definition of the global dimension and Theorem 5 that global($V$) is not smaller than the number of simple $V$-modules.

**Corollary.** *Let $V$ be a VOA satisfying conditions as in Theorem 5. Then the global dimension of $V$ is not smaller than the number of simple $V$-modules.*

In [4] (Proposition 4.5) they found a fairly simple formula of the global dimension.

**Proposition 6** ([4]). *Let $V$ be a simple, rational vertex operator algebra of CFT and finite type. Let $\{M^0, M^1, \ldots, M^n\}$, where $M^0 = V$, be the set of inequivalent simple $V$-modules. If the conformal weight of $M^i$ for any $i > 0$ is positive, then we have* $\mathrm{global}(V) = 1/(S_{00})^2$*, where $S = (S_{ij})$ is the $S$-matrix associated with $V$.*

## 5. Monic modular linear differential equations of third order with the central charges 164/5 and 236/7

Let $V$ be a simple VOA of CFT type. Suppose that $V_1 = 0$ and characters of simple $V$-modules are solutions of a monic MLDE of third order. It was shown in [1] that the central charge of $V$ is an element of the set

$$\{-68/7,\ 1/2,\ -44/5,\ 8,\ 16, 47/2,\ 24,\ 32,\ 164/5,\ 236/7,\ 40\} . \tag{8}$$

In [17] these numbers were found in the different context (cf. [8]).

It was verified that there exists at least one VOA whose central charge is an element of (8) except 164/5 and 236/7. In this paper we show that there does not exist a simple, rational VOA of CFT and of finite type, whose central charge is either 164/5 or 236/7.

The explicit expressions of the monic MLDEs of the third order with the central charges 164/5 and 236/7 are

$$f''' - \frac{1}{2}E_2 f'' + \Big(\frac{1}{2}E_2' - \frac{169}{100}E_4\Big) f' + \frac{1271}{1080}E_6\, f = 0 \tag{9}$$

and

$$f''' - \frac{1}{2}E_2 f'' + \Big(\frac{1}{2}E_2' - \frac{149}{84}E_4\Big) f' + \frac{93869}{74088}E_6\, f = 0\,, \tag{10}$$

respectively. The explicit expressions of solutions of the monic MLDEs (9) and (10) are given in [1], which are homogeneous polynomials of characters of simple modules of the minimal models with the central charges $c_{2,5} = -22/5$ and $c_{2,7} = -68/7$, respectively.

**Remark.** In this paper by means of (2) and the $S$-transformations of (11) and (14) we will give another proof that they are solutions of (9) and (10), respectively.

Now suppose that there exists a simple, rational VOA $V$ which is of CFT and finite type, whose central charge is either $164/5$ or $236/7$ and characters are solutions of a monic MLDE, respectively. The explicit expressions of the $S$-transformations of the space of the solutions of (9) and (10), which will be shown to equal to the $S$-transformation of the spaces of characters of simple $V$-modules (up to similarity transformations) as shown in the proof of Theorem 8, show $\mathrm{global}(V) < 3$ which implies that $V$ is *not* of finite type and rational since $\mathrm{global}(V) \geq 3$ by Theorem 5 (b) as discussed in §6– §7.

**Remark.** If we drop the assumption that the spaces of characters are included in the spaces of solutions of MLDEs (9) and (10), respectively, then there are examples of simple, rational VOA $V$ which is of CFT and of finite type, whose central charge is either $164/5$ or $236/7$. Let $L(4/5, 0)$ be the minimal model with central charge $4/5$. Then $L(4/5, 0)^{\otimes 41}$ is a simple, rational VOA of CFT and finite type with the central charge $164/5$. Let $L(6/7, 0)$ and $L(-68/7, 0)$ be the minimal model with central charges $6/7$ and $68/7$, respectively. Then $L(6/7, 0)^{\otimes 62} \otimes L(-68/7, 0)^{\otimes 2}$ is a simple, rational VOA of CFT and of finite type with the central charge $236/7$.

## 6. Central charge 164/5

In this section we will show that there does not exist a simple, rational VOA $V$ which is of CFT and of finite type, whose central charge is $164/5$ and characters are solutions of the MLDE (9).

Let $V$ be a simple VOA of CFT and of finite type with the central charge $164/5$. Suppose that characters of simple $V$-modules are solution of the monic MLDE (9). It is easily seen that the set of the indicial roots of (9) is $\{-41/30,\ 5/6,\ 31/30\}$.

We first present a set of solutions $f_1$, $f_2$ and $f_3$ (unique up a scalar factor) whose leading exponents of Fourier expansions are indicial roots of the monic MLDE (9), which are written in terms of homogeneous polynomials of the functions

$$\begin{aligned} g(q) &= q^{-1/60} \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}\,, \\ h(q) &= q^{11/60} \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}\,. \end{aligned}$$

We now define the functions $f_1$, $f_2$ and $f_3$, respectively, by

$$\begin{aligned} f_1 &= k_1(g,h) = q^{-41/30}\left(1 + 90118q^2 + 53459408q^3 + \cdots\right)\,, \\ f_2 &= k_2(g,h) = 11271q^{-5/6}\left(8 + 2915q + 266160q^2 + \cdots\right)\,, \\ f_3 &= k_1(h,-g) = 5084q^{31/30}\left(121 + 30008q + 2304726q^2 + \cdots\right)\,, \end{aligned} \tag{11}$$

where $k_1$ and $k_2$ are homogeneous polynomials of degree 82 defined by

$$\begin{aligned} k_1(g,h) = g^{12}\big(&g^{70} - 82g^{65}h^5 + 93029g^{60}h^{10} + 46912692g^{55}h^{15} \\ &+ 2556589686g^{50}h^{20} + 28524397164g^{45}h^{25} + 74276556202g^{40}h^{30} \\ &+ 52919401756g^{35}h^{35} + 23300865513g^{30}h^{40} - 10586446246g^{25}h^{45} \\ &+ 28710897349g^{20}h^{50} - 18944773568g^{15}h^{55} + 3063714996g^{10}h^{60} \\ &- 109499192g^5h^{65} + 615164h^{70}\big), \\ k_2(g,h) = g^{11}h^{11}\big(&10168g^{60} + 2983037g^{55}h^5 + 115307662g^{50}h^{10} \\ &+ 958403905g^{45}h^{15} + 1880475660g^{40}h^{20} + 1074772442g^{35}h^{25} \\ &+ 699519268g^{30}h^{30} - 1074772442g^{25}h^{35} + 1880475660g^{20}h^{40} \\ &- 958403905g^{15}h^{45} + 115307662g^{10}h^{50} - 2983037g^5h^{55} \\ &+ 10168h^{60}\big)\,. \end{aligned}$$

These solutions (that will be proved later) are all polynomials (homogeneous of degree 82) in the Rogers–Ramanujan modular functions $g$ and $h$. More precisely,

$$f_1 = g^{12}h^{70}P_{14}(g^5/h^5),\ f_2 = g^{11}h^{71}P_{12}(g^5/h^5),\ f_3 = g^{17}h^{20}P_{14}(-g^5/h^5)\,,$$

where

$$\begin{aligned} P_{14}(t) &= t^{14} - 82t^{13} + 93029t^{12} + 46912692t^{11} + 2556589686t^{10} \\ &\quad + 28524397164t^{9} + 74276556202t^{8} + 52919401756t^{7} \\ &\quad + 23300865513t^{6} - 10586446246t^{5} + 28710897349t^{4} \\ &\quad - 18944773568t^{3} + 3063714996t^{2} - 109499192t + 615164\,, \\ P_{12}(t) &= 10168t^{12} + 2983037t^{11} + 115307662t^{10} + 958403905t^{9} \\ &\quad + 1880475660t^{8} + 1074772442t^{7} + 699519268t^{6} \\ &\quad - 1074772442t^{5} + 1880475660t^{4} - 958403905t^{3} \\ &\quad + 115307662t^{2} - 2983037t + 10168\,. \end{aligned}$$

The functions $h$ and $g$ are characters of $L(-22/5, 0)$ and its simple module $L(-22/5, -1/5)$, respectively. We will now see that $f_i$ is a solution of the monic MLDE (9) in the following.

We first show that the vector-valued function $\mathbb{F} = {}^t(f_1, f_2, f_3)$ is a VVMF. Since $f_i$ has the Fourier expansion, obviously $f_i(\tau + 1)$ is a scalar multiple of $f_i(\tau)$ for each $i$. Therefore it suffices to prove that the vector space spanned by $f_1$, $f_2$ and $f_3$ is invariant under the transformation $S : \mathbb{H} \to \mathbb{H}$ $(\tau \mapsto -1/\tau)$.

It is well-known (cf. [10] (Proposition 6.3)) that the $S$-transformation of $g$ and $h$ are given by

$$\begin{pmatrix} h \\ g \end{pmatrix}\Big|_0 S = \begin{pmatrix} -\sqrt{\left(5+\sqrt{5}\right)/10} & \sqrt{\left(5-\sqrt{5}\right)/10} \\ \sqrt{(5-\sqrt{5})/10)} & \sqrt{\left(5+\sqrt{5}\right)/10} \end{pmatrix} \begin{pmatrix} h \\ g \end{pmatrix}. \tag{12}$$

Then direct computations give (such extensive numerical computation would be impossible without a computer)

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}\Bigg|_0 S = \begin{pmatrix} \left(\sqrt{5}+5\right)/10 & 10\sqrt{5} & \left(5-\sqrt{5}\right)/10 \\ 1/25\sqrt{5} & -1/\sqrt{5} & -1/25\sqrt{5} \\ \left(5-\sqrt{5}\right)/10 & -10\sqrt{5} & \left(\sqrt{5}+5\right)/10 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \tag{13}$$

which shows that $\mathbb{F} = {}^t(f_1, f_2, f_3)$ is a VVMF. Since the leading exponents of Fourier series of $f_1$, $f_2$ and $f_3$ are $-41/30$, $5/6$ and $31/30$, respectively, and $12(-41/30 + 5/6 + 31/30) = 6$, it follows from Proposition 2 that the triple $\{f_1, f_2, f_3\}$ is a basis of the space of solutions of a monic MLDE of third order. Then Lemma 3 shows the following proposition (for a different proof see [1, pp. 25–26]).

**Proposition 7.** *The set $\{f_1, f_2, f_3\}$ given by (11) is a basis of the space of solutions of the monic modular linear differential equation (9).*

**Theorem 8.** *Let $V$ be a simple vertex operator algebra of CFT type whose central charge is 164/5. Suppose that the character of any simple module of $V$ is a solution of the modular linear differential equation (9). Then $V$ is not of finite type and rational.*

*Proof.* Suppose that $V$ is of finite type and rational. The key idea is showing that the consequences of Theorem 5 and eq. (7) give a contradiction.

We first show that $f_1$, $f_2$ and $f_3$ are characters (up to scalar multiples) of the $V$-modules. The monic MLDE (9) has mutually different indicial roots which do not have integral differences. Therefore there is a unique solution (up to a scalar multiple) such that the leading exponent of Fourier expansion is an indicial root. Any character is, by the assumption, a linear combination of $f_1$, $f_2$ and $f_3$ and the indicial roots of (13) do not have integral differences. Since any character is a solution of (13), it is one of $f_1$, $f_2$ and $f_3$ (up to a scalar multiple). (Any character has the Fourier expansion $q^r \sum_{n=0}^{\infty} q^n$.) In particular, the conformal weight of each simple $V$-module is one of $\{0,\ 11/5,\ 12/5\}$. It follows that $\mathrm{ch}_V = f_1$ and $\dim V_1 = 0$ as $f_1 = q^{-41/30}(1 + 90118q^2 + O(q^3))$ by (11) and the leading exponents of Fourier expansions of $\mathrm{ch}_V$ and $f_1$ are $-41/30$ and leading coefficients are 1. Moreover, there are at least three simple $V$-modules.

Secondly, we show that the conformal weights of simple $V$-modules except $V$ are positive (since this is assumed in Theorem 5). Since the conformal weight of a simple $V$-module is non-negative, it suffices to check that any simple $V$-module $M$ with the conformal weight 0 is isomorphic to $V$. Let $M$ be a $V$-module. The character $\mathrm{ch}_M$ is a scalar multiple of $f_1$ since $\mathrm{ch}_M$ is a solution of the monic MLDE (9) and the conformal weight of $M$ is 0 (and then they have the same leading power of Fourier expansions). It hence follows from the Fourier expansion (11) that $\dim M_0 \neq 0$ and $\dim M_1 = 0$, and therefore, the space of vacuum-like elements of $M$ is nontrivial since $L_{-1}M_0 \subset M_1 = 0$. Then Proposition 1 shows that $\mathrm{Hom}_V(V, M) \neq 0$ so that $V$ is isomorphic to $M$ since $M$ is simple. Since from the argument in the previous paragraph, there are at least three simple $V$-modules. Hence the global dimension of $V$ is not smaller than 3 by Proposition 5 and the very definition of global dimensions, while it follows from (5) and (13) that $S_{00} = \left(\sqrt{5} + 5\right)/10$. Hence we have

$$\mathrm{global}(V) = 100/(5 + \sqrt{5})^2 = 5(3 - \sqrt{5})/2 \;=\; 1.90983\cdots < 2$$

by Proposition 7. Thus we have a contradiction. $\square$

## 7. Central charge $c = 236/7$

In this section we will show that there does not exist a simple, rational VOA $V$ which is of CFT and of finite type, whose central charge is 236/7 and characters are solutions of the monic MLDE (10).

Let $V$ be a simple VOA of CFT type with the central charge 236/7. Suppose that characters of simple $V$-modules are solutions of the monic MLDE (10). Since the set of indicial roots of (10) is $\{-59/42, 37/42, 43/42\}$ and the central charge of $V$ is 236/7, as in the arguments given in the proof of Theorem 8, the sets of conformal weights of simple $V$-modules is $\{0, 16/7, 17/7\}$.

Let $a_1$, $a_2$ and $a_3$ be homogeneous polynomials of degree 59 (for the explicit expressions see Appendix A.1). Let $x$, $y$ and $z$ be the characters simple modules of the minimal model $L(c_{2,7}, 0)$ ($c_{2,7} = -68/7$), whose conformal weights are 0, $-2/7$ and $-3/7$, respectively, i.e.

$$x = q^{17/42} \prod_{\substack{n>0 \\ n\not\equiv 0,\pm 1 \\ (\mathrm{mod}\ 7)}} (1-q^n)^{-1},$$

$$y = q^{5/42} \prod_{\substack{n>0 \\ n\not\equiv 0,\pm 2 \\ (\mathrm{mod}\ 7)}} (1-q^n)^{-1},$$

$$z = q^{-1/42} \prod_{\substack{n>0 \\ n\not\equiv 0,\pm 3 \\ (\mathrm{mod}\ 7)}} (1-q^n)^{-1}.$$

We now give solutions of (10) whose leading exponents of the Fourier expansions are indicial roots. The explicit expressions of them are given by

$$\begin{aligned} g_1 &= a_1(x,y,z) = q^{-59/42}\left(1 + 63366q^2 + 46421200q^3 + \cdots\right), \\ g_2 &= a_2(x,y,z) = 31093q^{37/42}\left(23 + 8288q + 774410q^2 + \cdots\right), \\ g_3 &= a_3(x,y,z) = 3422q^{43/42}\left(248 + 67983q + 5611328q^2 + \cdots\right), \end{aligned} \tag{14}$$

where $a_1$, $a_2$ and $a_3$ are defined in Appendix A.1. (We will prove that these are in fact solutions later.) It is known [9, Proposition 2.3] that the functions $x$, $y$ and $z$ have a homogeneous algebraic relation $y^3z - z^3x - x^3y = 0$ which yields

$$a_2(x,y,z) = a_1(-y,z,-x) \quad \text{and} \quad a_3(x,y,z) = -a_1(-x,-z,y). \tag{15}$$

We first show that the vector-valued function ${}^t(g_1, g_2, g_3)$ is a VVMF. Obviously $g_i(\tau+1)$ is a scalar multiple of $g_i(\tau)$ for each $i$. Therefore it suffices to show that the vector space whose basis is $\{g_1, g_2, g_3\}$ is invariant under the transformation $S$. It is well-known [10, Proposition 6.3] that the transformations $S$ of the functions $x$, $y$ and $z$ are given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}\Bigg|_0 S = \frac{2}{\sqrt{7}} \begin{pmatrix} \cos(3\pi/14) & -\cos(\pi/14) & \sin(\pi/7) \\ -\cos(\pi/14) & -\sin(\pi/7) & \cos(3\pi/14) \\ \sin(\pi/7) & \cos(3\pi/14) & \cos(\pi/14) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{16}$$

The function $g_1|_0 S$ is a polynomial in $x$, $y$ and $z$, which is generated by 93 monomials[1] (see Appendix A.2). Moreover, we find

$$g_1|_0\, S = s_1 g_2 + s_2 a_2 + s_3 g_3\,, \tag{17}$$

where $s_1 = 2\cos(3\pi/14)/\sqrt{7}$, $s_2 = 2\cos(\pi/14)/\sqrt{7}$, $s_3 = 2\sin(\pi/7)/\sqrt{7}$. Since the left-hand side of (17) equals to

$$G(x,y,z) = a_1(s_1x - s_2y + s_3z, -s_2x - s_3y + s_1z, s_3x + s_2y + s_1z)$$

by the very definition, it follows from (15) and (16) that $a_2(x, y, z)|_0\, S = G(-z, -x, y)$ and $a_3(x, y\ z) = -G(-y, z, -x)|_0 S$. Therefore, by (15) and eq. (17) we have

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}\Bigg|_0 S = \frac{2}{\sqrt{7}} \begin{pmatrix} \cos(3\pi/14) & \cos(\pi/14) & \sin(\pi/7) \\ \cos(\pi/14) & -\sin(\pi/7) & -\cos(3\pi/14) \\ \sin(\pi/7) & -\cos(3\pi/14) & \cos(\pi/14) \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}. \tag{18}$$

Hence the column vector-valued function ${}^t(g_1, g_2, g_3)$ is a VVMF.

**Proposition 9.** *The set of functions $\{g_1, g_2, g_3\}$ defined by (14) is a basis of the space of solutions of the monic modular linear differential equation (10).*

*Proof.* Since ${}^t(g_1, g_2, g_3)$ is a VVMF and the leading exponents of the $q$-expansions of functions $g_1$, $g_2$ and $g_3$ are $-59/42$, $37/42$ and $43/42$, respectively, we have $12(-59/42 + 37/42 + 43/42) = 6$. Then Proposition 2 yields that $\{g_1, g_2, g_3\}$ is a basis of the space of solutions of a monic MLDE of third order. Moreover, it follows from Lemma 3 that this MLDE coincides with the monic MLDE (10). □

[1]It follows from (16) that $x|_0S$, $y|_0S$ and $z|_0S$ are expressed as polynomials of $x$, $y$ and $z$. If $x$, $y$ and $z$ have not had any algebraic relation, then $g_1|_0S$ was written as linear combinations of 1824 monomials.

**Remark.** Another poof of Proposition 9 is given in ($\ell$) of [1].

**Theorem 10.** *Let $V$ be a simple vertex operator algebra of CFT type with the central charge $236/7$. Suppose that the characters of simple $V$-modules are solutions of the monic modular linear differential equation (10). Then $V$ is not of finite type and rational.*

*Proof.* Suppose that $V$ is rational and of finite type. Since $V$ is of CFT type and its central charge is $236/7$, the character of $V$ coincides with $g_1$. Because the $S$-transformation of $g_1$ is a linear combination of the characters of the simple modules of $V$ by the modular invariance property and (17), the arguments as in the proof of Theorem 8 show that there are at least 3 simple $V$-modules and that the conformal weights of simple $V$-modules are non-negative and any simple $V$-module with conformal weight 0 is isomorphic to $V$. The very definition of the global dimension and Proposition 5 show that $\mathrm{global}(V) \geq 3$. However, the entry $S_{00}$ of the $S$-matrix is equal to $2\cos\left(3\pi/14\right)/\sqrt{7}$ by (5) and (18). Hence it follows from Proposition 7 that

$$\mathrm{global}(V) = \frac{7}{4\cos^2(3\pi/14)} = 2.86294\ldots < 2.9\,.$$

Thus we have a contradiction. $\square$

## Appendix A. Homogeneous polynomials appeared in the $c = 236/7$ monic modular linear differential equation

In this appendix we give the explicit expressions of polynomials which appear in §7 and give the $S$-matrix.

### A.1. Expressions of polynomials $a_1$, $a_2$ and $a_3$

The polynomials $a_1$, $a_2$ and $a_3$ in $x$, $y$ and $z$ of degree 59 which appeared in §7 are explicitly expressed as

(19)
$$\begin{aligned}
&a_1(x,y,z)\\
&= 2190849987347x^{58}y + 2190849987347x^{56}z^3 + 8816184633328x^{53}y^2z^4\\
&\quad + 465452872955x^{51}y^8 + 17330415570670x^{51}yz^7\\
&\quad + 10705080924689x^{49}z^{10} + 20273356011456x^{46}y^2z^{11}\\
&\quad + 97883562370x^{44}y^{15} + 61661154366700x^{44}yz^{14}
\end{aligned}$$

$$
\begin{aligned}
&+ 47658393772643x^{42}z^{17} + 139841916769201x^{39}y^2z^{18}\\
&- 109424817575x^{37}y^{22} + 320520742923731x^{37}yz^{21}\\
&+ 217896152319363x^{35}z^{24} + 361856157239137x^{32}y^2z^{25}\\
&+ 10067353726x^{30}y^{29}470476510477120x^{30}yz^{28}\\
&+ 252772915072319x^{28}z^{31} + 223747642357998x^{25}y^2z^{32}\\
&- 215505583x^{23}y^{36} + 149102376058101x^{23}yz^{35}\\
&+ 52937745467620x^{21}z^{38} + 20641842052772x^{18}y^2z^{39}\\
&+ 715139x^{16}y^{43} + 5462274021285x^{16}yz^{42}\\
&+ 829805597999x^{14}z^{45} + 80972731266x^{11}y^2z^{46}\\
&+ 3431399762x^9yz^{49} + 42913178x^7z^{52} + 64900x^4y^2z^{53}\\
&- 59x^2yz^{56} + z^{59},
\end{aligned}
$$

(20)

$$
\begin{aligned}
&a_2(x,y,z)\\
&= -x^{59} + 882794444359x^{56}y^2z + 4543893054975x^{54}yz^4\\
&\quad + 138258169436x^{52}y^7 + 3661098610557x^{52}z^7 + 10224524748288x^{49}y^2z^8\\
&\quad + 31490183598954x^{47}yz^{11} + 6924887466x^{45}y^{14} + 24043962951905x^{45}z^{14}\\
&\quad + 80499190812167x^{42}y^2z^{15} + 228094024607248x^{40}yz^{18}\\
&\quad - 59881352148x^{38}y^{21} + 166226386774472x^{38}z^{21}\\
&\quad + 352186560279214x^{35}y^2z^{22} + 587928082399742x^{33}yz^{25}\\
&\quad + 6892739546x^{31}y^{28} + 354743600999784x^{31}z^{28}\\
&\quad + 417326748220400x^{28}y^2z^{29} + 377551394875116x^{26}yz^{32}\\
&\quad - 182567122x^{24}y^{35} + 165247049735260x^{24}z^{35}\\
&\quad + 94762510467036x^{21}y^2z^{36} + 39200808461423x^{19}yz^{39}\\
&\quad + 848656x^{17}y^{42} + 9350127088939x^{17}z^{42}\\
&\quad + 1876091330673x^{14}y^2z^{43} + 216146813939x^{12}yz^{46}\\
&\quad + 11583044197x^{10}z^{49} + 219081278x^7y^2z^{50} + 715139x^5yz^{53}
\end{aligned}
$$

and

$$
\begin{aligned}
(21)\quad & a_3(x,y,z)\\
&= -1282552304527x^{56}y^3 - 5134394452787x^{54}y^2z^3\\
&\quad - 6766778252144x^{52}yz^6 - 2914936103884x^{50}z^9
\end{aligned}
$$

$$
\begin{aligned}
&- 368294187889x^{49}y^{10} + 6031840984522x^{47}y^{2}z^{10}\\
&+ 28126445594091x^{45}yz^{13} + 21748959064557x^{43}z^{16}\\
&+ 61766535503281x^{40}y^{2}z^{17} + 150382341083241x^{38}yz^{20}\\
&+ 104928152458177x^{36}z^{23} + 51886767247x^{35}y^{24}\\
&+ 190254165627419x^{33}y^{2}z^{24} + 269302315887115x^{31}yz^{27}\\
&+ 150722349577506x^{29}z^{30} - 3132177486x^{28}y^{31}\\
&+ 147021943645516x^{26}y^{2}z^{31} + 109111294527183x^{24}yz^{34}\\
&+ 41709068640197x^{22}z^{37} + 42653165x^{21}y^{38}\\
&+ 18683198910349x^{19}y^{2}z^{38} + 5796683914336x^{17}yz^{41}\\
&+ 1045910881484x^{15}z^{44} - 65018x^{14}y^{45} + 133937600144x^{12}y^{2}z^{45}\\
&+ 8366006362x^{10}yz^{48} + 186810402x^{8}z^{51}\\
&- 144799582921x^{42}y^{17} - 59x^{7}y^{52} + 848656x^{5}y^{2}z^{52} - y^{59}\,.
\end{aligned}
$$

### A.2. $S$-transformation of $a_1(x, y, z)$

Let $c_1 = 2\cos(3\pi/14)/\sqrt{7}$, $c_2 = 2\cos(\pi/14)/\sqrt{7}$ and $c_3 = 2\sin(\pi/7)/\sqrt{7}$. Then the function $a_1(x, y, z)|_0 S$ is written in terms of $x$, $y$ and $z$ by

$$
\begin{aligned}
&a_1(x, y, z)|_0 S \qquad (22)\\
&= c_1(2190849987347x^{58}y + 2190849987347x^{56}z^{3} + 8816184633328x^{53}y^{2}z^{4}\\
&+ 465452872955x^{51}y^{8} + 17330415570670x^{51}yz^{7} + 10705080924689x^{49}z^{10}\\
&+ 20273356011456x^{46}y^{2}z^{11} + 97883562370x^{44}y^{15} + 61661154366700x^{44}yz^{14}\\
&+ 47658393772643x^{42}z^{17} + 139841916769201x^{39}y^{2}z^{18} - 109424817575x^{37}y^{22}\\
&+ 320520742923731x^{37}yz^{21} + 217896152319363x^{35}z^{24} + 361856157239137x^{32}y^{2}z^{25}\\
&+ 10067353726x^{30}y^{29} + 470476510477120x^{30}yz^{28} + 252772915072319x^{28}z^{31}\\
&+ 223747642357998x^{25}y^{2}z^{32} - 215505583x^{23}y^{36} + 149102376058101x^{23}yz^{35}\\
&+ 52937745467620x^{21}z^{38} + 20641842052772x^{18}y^{2}z^{39} + 715139x^{16}y^{43}\\
&+ 5462274021285x^{16}yz^{42} + 829805597999x^{14}z^{45} + 80972731266x^{11}y^{2}z^{46}\\
&+ 3431399762x^{9}yz^{49} + 42913178x^{7}z^{52} + 64900x^{4}y^{2}z^{53} - 59x^{2}yz^{56} + z^{59})\\
&+ c_2(-x^{59} + 882794444359x^{56}y^{2}z + 4543893054975x^{54}yz^{4} + 138258169436x^{52}y^{7}\\
&+ 3661098610557x^{52}z^{7} + 10224524748288x^{49}y^{2}z^{8} + 31490183598954x^{47}yz^{11}\\
&+ 6924887466x^{45}y^{14} + 24043962951905x^{45}z^{14} + 80499190812167x^{42}y^{2}z^{15}\\
&+ 228094024607248x^{40}yz^{18} - 59881352148x^{38}y^{21} + 166226386774472x^{38}z^{21}\\
&+ 352186560279214x^{35}y^{2}z^{22} + 587928082399742x^{33}yz^{25} + 6892739546x^{31}y^{28}\\
&+ 354743600999784x^{31}z^{28} + 417326748220400x^{28}y^{2}z^{29} + 377551394875116x^{26}yz^{32}
\end{aligned}
$$

$$
\begin{aligned}
&- 182567122x^{24}y^{35} + 165247049735260x^{24}z^{35} + 94762510467036x^{21}y^2z^{36}\\
&+ 39200808461423x^{19}yz^{39} + 848656x^{17}y^{42} + 9350127088939x^{17}z^{42}\\
&+ 1876091330673x^{14}y^2z^{43} + 216146813939x^{12}yz^{46} + 11583044197x^{10}z^{49}\\
&+ 219081278x^7y^2z^{50} + 715139x^5yz^{53}) + c_3(-1282552304527x^{56}y^3\\
&- 5134394452787x^{54}y^2z^3 - 6766778252144x^{52}yz^6 - 2914936103884x^{50}z^9\\
&- 368294187889x^{49}y^{10} + 6031840984522x^{47}y^2z^{10} + 28126445594091x^{45}yz^{13}\\
&+ 21748959064557x^{43}z^{16} - 144799582921x^{42}y^{17} + 61766535503281x^{40}y^2z^{17}\\
&+ 150382341083241x^{38}yz^{20} + 104928152458177x^{36}z^{23} + 51886767247x^{35}y^{24}\\
&+ 190254165627419x^{33}y^2z^{24} + 269302315887115x^{31}yz^{27} + 150722349577506x^{29}z^{30}\\
&- 3132177486x^{28}y^{31} + 147021943645516x^{26}y^2z^{31} + 109111294527183x^{24}yz^{34}\\
&+ 41709068640197x^{22}z^{37} + 42653165x^{21}y^{38} + 18683198910349x^{19}y^2z^{38}\\
&+ 5796683914336x^{17}yz^{41} + 1045910881484x^{15}z^{44} - 65018x^{14}y^{45}\\
&+ 133937600144x^{12}y^2z^{45} + 8366006362x^{10}yz^{48} + 186810402x^8z^{51}\\
&- 59x^7y^{52} + 848656x^5y^2z^{52} - y^{59})\,.
\end{aligned}
$$

## Appendix B. Expressions of the $j$-function in terms of solutions of monic modular linear differential equations

We denote the $j$-function by simply $j$. Then it is conjectured in (3.8) and Table 2 of [8] that

$$j = 1728E_4^3/(E_4^3 - E_6^2) = q^{-1} + 744 + 196884q + O(q^2)$$

is expressed in terms of solutions of the MLDE (9), (10), i.e. the characters of simple modules of $L(-22/5, 0) \otimes L(-22/5, 0)$ and $L(-68/7, 0)$.

**Conjecture.** We have

$$j - 744 = h^2 f_1 - 50ghf_2 + g^2 f_3 = xg_1 - yg_2 + zg_3\,, \tag{23}$$

respectively.

In [8] they checked these relations numerically using Fourier expansions. Here we give a rigorous proof of the formula (23). The first and the second equalities are proved as Theorem 11 and Theorem 12, respectively.

**Theorem 11.** *We have $j - 744 = h^2 f_1 - 50ghf_2 + g^2 f_3$. In particular, $j - 744$ is a homogeneous polynomial in $g$ and $h$ of degree* 84.

*Proof.* The first three terms of the Fourier expansion of the right-hand side is $q^{-1}+196884q+O(q^2)$, which is equal to the first three terms of the Fourier expansion $j-744$. Therefore the difference $j-744-(h^2f_1-50ghf_2+g^2f_3)$ is holomorphic and zero at $\tau=+i\infty$. Since $j-744$ is a modular function, it suffices to show that the right-hand side is also a modular function.

It follows from (12) and (13) that

$$\begin{aligned}
&\left(h^2f_1-50hgf_2+g^2f_3\right)|_0S\\
&=(h^2|_0S, hg|_0S, g^2|_0S)\begin{pmatrix}1&0&0\\0&-50&0\\0&0&1\end{pmatrix}\begin{pmatrix}f_1|_0S\\f_2|_0S\\f_3|_0S\end{pmatrix}\\
&=(h^2,hg,g^2)A\begin{pmatrix}f_1\\f_2\\f_3\end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
A=\frac{1}{10^2}&\begin{pmatrix}5+\sqrt{5}&-2\sqrt{5}&5-\sqrt{5}\\-4\sqrt{5}&-2\sqrt{5}&4\sqrt{5}\\5-\sqrt{5}&2\sqrt{5}&5+\sqrt{5}\end{pmatrix}\begin{pmatrix}1&0&0\\0&-50&0\\0&0&1\end{pmatrix}\\
&\times\begin{pmatrix}\sqrt{5}+5&100\sqrt{5}&5-\sqrt{5}\\\frac{2}{5\sqrt{5}}&-2\sqrt{5}&-\frac{2}{5\sqrt{5}}\\5-\sqrt{5}&-100\sqrt{5}&\sqrt{5}+5\end{pmatrix}\\
=&\begin{pmatrix}1&0&0\\0&-50&0\\0&0&1\end{pmatrix}.
\end{aligned}$$

Hence we have $\left(h^2f_1-50hgf_2+g^2f_3\right)|_0S=h^2f_1-50hgf_2+g^2f_3$. □

Secondly, we prove the second equality of (23). As in the proof of the first equality at least the first three terms of the Fourier expansions of both hand sides are equal. Therefore it suffices to show that the right-hand side is a modular function since the difference between both is a holomorphic cusp form. It can be verified by (16) and (18) that $xg_1-yg_2+zg_3$ and its $S$-transformation are equal to

$$(xg_1-yg_2+zg_3)|_0S=(x|_0S,y|_0S,z|_0S)\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}g_1|_0S\\g_2|_0S\\g_3|_0S\end{pmatrix}$$

$$= (x, y, z)B \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix},$$

where the matrix $B$ is

$$\frac{4}{7}\begin{pmatrix} \cos(3\pi/14) & -\cos(\pi/14) & \sin(\pi/7) \\ -\cos(\pi/14) & -\sin(\pi/7) & \cos(3\pi/14) \\ \sin(\pi/7) & \cos(3\pi/14) & \cos(\pi/14) \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} \cos(3\pi/14) & \cos(\pi/14) & \sin(\pi/7) \\ \cos(\pi/14) & -\sin(\pi/7) & -\cos(3\pi/14) \\ \sin(\pi/7) & -\cos(3\pi/14) & \cos(\pi/14) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, we have proved the following theorem.

**Theorem 12.** *We have $j - 744 = xg_1 - yg_2 + zg_3$. In particular, $j - 744$ is a homogeneous polynomial in $x$, $y$ and $z$ of degree* 60*.*

**Remark.** The functions $\eta^{2/5}h$ and $\eta^{2/5}g$ are holomorphic modular forms of weight 1/5 on the principal congruence subgroup $\Gamma(5)$, and the functions $\eta^{4/7}x$, $\eta^{4/7}y$ and $\eta^{4/7}z$ are holomorphic modular forms of weight 2/7 on $\Gamma(7)$, where $\eta$ is the Dedekind eta function. Thus the both-sides of (23.1) and (23.2) multiplied by powers of $\eta$ are modular forms of weight 84/5 on $\Gamma(5)$ and of weight 120/7 on $\Gamma(7)$, respectively. Since the generators and relations of the rings of holomorphic modular forms of weight $(1/5)\mathbb{Z}$ on $\Gamma(5)$ and of weight $(2/7)\mathbb{Z}$ on $\Gamma(7)$ are determined in [9, Lemma 1.7], one can prove Theorems 11 and 12 by comparing finite number of Fourier coefficients.

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## References

[1] Y. Arike, K. Nagatomo, Y. Sakai, Characterization of the simple Virasoro vertex operator algebras with 2 and 3-dimensional space of characters, to appear in Contempo. Math., **695**, 175–204 (2017). MR3709712

[2] G. Anderson, G. Moore, Rationality in conformal field theory, Comm. Math. Phys., **117**, no. 3, 441–450 (1988). MR0953832

[3] R. Dijkgraaf, E. Verlinde, Modular invariance and the fusion algebra, Conformal field theories and related topics (Annecy-le-Vieux, 1988). Nuclear Phys. B Proc. Suppl. **5B**, 87–97 (1988). MR1002959

[4] C. Dong, X. Jiao, F. Xu, Quantum dimensions and quantum Galois theory, Trans. Amer. Math. Soc., **365**, no. 12, 6441–6469 (2013). MR3105758

[5] C. Dong, H. Li, G. Mason, Modular-invariance of trace functions in orbifold theory and generalized moonshine, Commun. Math. Phys., **214**, 1–56 (2000). MR1794264

[6] B. L. Feigin, D. B. Fuchs, Representations of the Virasoro algebra, Representation of Lie groups and related topics, 465–554, Adv. Stud. Contemp. Math., **7**, Gordon and Breach, New York, 1990. MR1104280

[7] B. L. Feigin, D. B. Fuchs, Verma modules over the Virasoro algebra. Topology (Leningrad, 1982), 230–245, Lecture Notes in Math., **1060**, Springer, Berlin, 1984. MR0770243

[8] H. R. Hampapura, S. Mukhi, Two-dimensional RCFT's without Kac-Moody symmetry, JHEP, **07**, 138 (2016). MR3550233

[9] T. Ibukiyama, Modular forms of rational weights and modular varieties, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, **70**, 315–339 (2000). MR1809555

[10] K. Iohara, Y. Koga, Representation theory of the Virasoro algebra, Springer-Verlag London, 2011. MR2744610

[11] J. Lepowsky, H. Li, Introduction to vertex operator algebras and their representations, Prog. Math., Birkhäuser, Boston, 2004. MR2023933

[12] H. Li, Symmetric invariant bilinear forms on vertex operator algebras, J. Pure Appl. Algebra, **96**, 279–297 (1994). MR1303287

[13] C. Marks, Irreducible vector-valued modular forms of dimension less than six, Illinois J. Math. **55**, 1267–1297 (2011). MR3082869

[14] G. Mason, Vector-valued modular forms and linear differential operators, Int. J. Number Theory, **3**, 377–390 (2007). MR2352826

[15] A. Matsuo, K. Nagatomo, Axioms for a vertex algebra and the locality of quantum Fields, MSJ Memoirs, **4**, Mathematical Society of Japan, Japan (1999). MR1715197

[16] M. Miyamoto, Modular invariance of vertex operator algebras satisfying $C_2$-cofiniteness, Duke Math. J., **122**, no. 1, 51–91 (2004). MR2046807

[17] M. P. Tuite, and H. D. Van, On exceptional vertex operator (super) algebras, In: Mason G., Penkov I., Wolf J. A. (eds), *Developments and Retrospectives in Lie Theory: Algebraic Methods*, Dev. Math., **38**, 351–384. Springer, Cham (2014). MR3308791

[18] E. Rowell, R. Stong, Z. Wang, On classification of modular tensor categories, Comm. Math. Phys., **292**, no. 2, 343–389 (2009). MR2544735

[19] W. Wang, Rationality of Virasoro vertex operator algebras, IMRN, No. 7, 197–211 (1993). MR1230296

[20] Y. Zhu, Modular invariance of characters of vertex operator algebras, J. Amer. Math. Soc., **9**, 237–302 (1996). MR1317233

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