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# On derived equivalence of general Clifford double mirrors

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We show that the general Clifford double mirrors constructed in [BL18] are derived equivalent.

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## 1. Introduction

Mirror symmetry originated from the observation in physics that different Calabi-Yau threefolds may provide (physical) compactifications of dual string theories. Such Calabi-Yau varieties are called mirror pairs. Mathematically, mirror symmetry is reflected by the relations between mirror pairs on Hodge numbers, derived and Fukaya categories, Gromov-Witten invariants, etc.

In recent years, much of the attention has been drawn to the double mirror phenomenon, that is, two Calabi-Yau varieties both are mirrors to a same Calabi-Yau variety (also called multiple mirror phenomenon in [CK99]). In this scenario, properties of double mirror Calabi-Yaus can be read off from mirror symmetry predictions. For example, their $(p, q)$-stringy Hodge numbers should be the same as they should both equal to the $(n-p, q)$-stringy Hodge number of their common mirror; their derived categories are expected to be equivalent, because according to the homological mirror symmetry conjecture [Kon95], they are both equivalent to the Fukaya category of their common mirror.

These properties have been studied for various known double mirror pairs. For the Batyrev-Borisov double mirrors, the equality of their stringy Hodge numbers is a consequence of [BB96, Theorem 4.15], their derived equivalence are confirmed for the corresponding stacks in [FK17, Theorem 6.3], and their birationality has been proved under some mild assumptions in [Li16, Theorem 4.10]. The analogous results for Berglund-Hübsch-Krawitz mirrors have been established in [CR11, FK19, Sho14, Bor13, Kel13, Cla14].

Mirror symmetry for Calabi-Yau varieties has been generalized to Landau-Ginzburg model (LG-model) and Calabi-Yau (or Fano) correspondence. An LG-model $(X, w)$ consists of a variety $X$ and a regular function $w$ on $X$ which is called a potential. On the level of derived categories, an LG-model corresponds to a derived matrix factorization category which is equivalent to a (relative) singular category of zero locus of $w$. There also exists double mirror phenomenon in this case. In the Givental's LG/Fano setting, for Fano complete intersections in toric varieties, certain Laurent polynomial multiple mirrors are related by a mutation (see [CKP15, Theorem 5.1]). A similar result also appears in [DH16, Theorem 2.24].

Besides above physics considerations, there are a number of sporadic examples [Muk88, Kuz08, CDH$^+$10, Add09, CT15] in the literature involving derived equivalent noncommutative varieties which postulate their connections with double mirror phenomenon. It is this observation that motivates our work [BL18] to uncover the underling toric geometry of such examples and relate them to mirror symmetry.

In [BL18], we work in a slightly more general setting than Batyrev-Borisov construction where we consider a pair of reflexive Gorenstein cones (Definition 2.2). It has been known that decompositions of the degree element of a reflexive cone with coefficients 1 will result in Batyrev-Borisov double mirrors (i.e. two Calabi-Yau complete intersections as double mirrors in the Batyrev-Borisov construction). We generalize this by allowing coefficients 1/2 in the decomposition of degree element, and construct a noncommutative variety (stack) $(\mathcal{S}, \mathcal{B}_0)$ associated to it, where $\mathcal{S}$ is a complete intersection in a Fano toric variety and $\mathcal{B}_0$ is a noncommutative sheaf of algebras. Such construction depends on a choice of a regular simplicial fan satisfying centrality condition (see Section 2 for details) and a parameter $r$ which "counts" how many terms with coefficients 1/2. When $r = 0$, then $\mathcal{B}_0 = 0$, and we are back to the Batyrev-Borisov situation where the noncommutative variety is just a commutative Calabi-Yau variety. However, when $r > 0$, $\mathcal{B}_0$ is non-trivial and $\mathcal{S}$ is no longer Calabi-Yau. In this case, $(\mathcal{S}, \mathcal{B}_0)$ can be viewed as a "noncommutative Calabi-Yau variety" and we call it the (general) Clifford mirror. Here "noncommutative" refers to the fact that the analogous structure sheaf $\mathcal{B}_0$ for $(\mathcal{S}, \mathcal{B}_0)$ is a noncommutative sheaf of algebras (see Definition 2.5), and "Calabi-Yau" means that the corresponding derived category is a Calabi-Yau category.

[BL18, Theorem 6.3] shows that when $r$ achieves its extreme values (i.e. the complete intersection and pure Clifford mirror cases), the corresponding double mirrors are derived equivalent. We conjectured further that for general $r$, the corresponding general Clifford double mirrors should also pass the double mirror tests. Especially, under some appropriate conditions, no matter which $r$ is chosen, they are all derived equivalent ([BL18, Conjecture 7.5]). The goal of the present paper is to give an affirmative answer to that conjecture (Theorem 3.13).

In the recent paper [BW19], Borisov and Wang define a natural modification of stringy Hodge numbers for noncommutative Clifford mirrors of quadric complete intersections, and prove the corresponding equality of Euler characteristics.

We briefly discuss the content of each section. In Section 2, we explain the construction of general Clifford mirror in [BL18]. During that course, we give necessary combinatorial definitions and fix the notation. In Section 3, we prove the derived equivalence of such Clifford double mirrors. It relies on Shipman, Isik, Hirano's result on Knörrer periodicity [Shi12, Isi13, Hir17] and homological variations of GIT quotients [BFK19, HL15]. In Section 4, we study the general Clifford double mirrors of the product type. We give examples of such type and provide heuristic explanations for their derived equivalence.

## 2. The construction of general Clifford double mirrors

In this section, we recall the construction of general Clifford double mirrors given in [BL18, Section 7], and fix the notation used throughout the paper. We begin with some combinatorial definitions.

Let $M$ be a lattice and let $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ be its dual lattice. The natural pairing is given by

$$\langle , \rangle : M \times N \to \mathbb{Z}.$$

Let $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}, N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ be the $\mathbb{R}$-linear extensions of $M, N$. The pairing can be $\mathbb{R}$-linearly extended, and we still use $\langle , \rangle$ to denote this extension.

**Definition 2.1.** *A* rational polyhedral cone $K \subset M_{\mathbb{R}}$ *is a convex cone generated by a finite set of lattice points. We assume that $K \cap (-K) = \{\mathbf{0}\}$. We call the first lattice point of a ray $\rho$ of $K$ a* primitive element *or a* ray generator *of $\rho$.*

**Definition 2.2** ([BB97])**.** *A full-dimensional rational polyhedral cone $K \subset M_{\mathbb{R}}$ is called a* Gorenstein cone *if all the primitive elements of its rays lie on some hyperplane $\langle -, \deg^{\vee} \rangle = 1$ for some* degree element $\deg^{\vee}$ *in $N$. A Gorenstein cone $K \subset M_{\mathbb{R}}$ is called a* reflexive Gorenstein cone *iff its* dual cone $K^{\vee} := \{y \mid \langle x, y \rangle \geq 0$ for all $x \in K\}$ *is also a Gorenstein cone with respect to the dual lattice $N$.*

**Definition 2.3.** *For a pair of dual reflexive Gorenstein cones* $(K, K^\vee)$*, the pairing of their two degree elements* $\langle \deg, \deg^\vee \rangle = k$ *is called the* index *of the pair.*

We consider a pair of reflexive Gorenstein cones $K$ and $K^\vee$ in $M_\mathbb{R}$ and $N_\mathbb{R}$ with degree elements $\deg \in K$ and $\deg^\vee \in K^\vee$ respectively. Suppose that the index of this pair of Gorenstein cones is $k = \langle \deg, \deg^\vee \rangle$. In addition, we consider a generic coefficient function

$$c : K_{(1)} \to \mathbb{C}, \tag{2.1}$$

where $K_{(1)} := \{m \in K \cap M \mid \langle m, \deg^\vee \rangle = 1\}$.

As explained in [BL18, Sections 2 and 7], a decomposition of the degree element $\deg^\vee$ as a summation of lattice elements encompasses the data for toric double mirrors. For example, all the decompositions of $\deg^\vee$ as linear combinations of elements in $K^\vee_{(1)}$ with coefficients 1 correspond to the Batyrev-Borisov double mirrors ([Li16, Theorem 3.4]); and all the decompositions of $\deg^\vee$ as linear combinations of elements in $K^\vee_{(1)}$ with coefficients 1/2 correspond to the pure Clifford double mirrors. Hence it is natural to consider a mix of above coefficients.

We make the following assumptions throughout the paper.

Suppose that

$$\deg^\vee = \frac{1}{2}(s_1 + \cdots + s_{2r}) + t_1 + \cdots + t_{k-r} \tag{2.2}$$

for some $0 \le r \le k$, with $s_i, t_j \in K^\vee \cap N$. The $k + r$ elements $s_i$ and $t_j$ are assumed to be linearly independent. In addition, we assume that there exists a *regular simplicial* fan $\Sigma$ (see [CLS11, Definition 15.2.8]) with support $K^\vee$ such that the following *centrality condition* holds (see [BL18, (7.1)]):

(†) All maximum dimensional cones of $\Sigma$ contain $\{s_1, \dots, s_{2r}, t_1, \dots, t_{k-2r}\}$ as ray generators.

For the existence of such fan under certain conditions, see [BL18, Proposition 2.13]. Once it exists, we fix $\Sigma$ throughout the construction.

We define toric stacks as in [BCS05] (see [FK17, Definition 5.6] or [BL18, §2.2]). Let $K^\vee_{(1)} = \{n_i \mid 1 \le i \le l\}$ and $\rho_i$ be the ray corresponding to $n_i$. For $n \in K^\vee_{(1)}$, let $\mathbf{z}(n)$ denote the corresponding coordinate function. The Cox open subset $U_\Sigma$ of $\mathbb{C}^{K^\vee_{(1)}} \simeq \mathbb{C}^l$ consists of all points $(a_1, \dots, a_l) \in \mathbb{C}^{K^\vee_{(1)}}$ such that $\mathrm{Cone}\{n_i \mid a_i = 0\} \in \Sigma$. Alternatively, let $J := \langle \prod_{\rho_i \not\subset \sigma} \mathbf{z}(n_i) \mid \sigma \in$

$\Sigma\rangle$ be the irrelevant ideal, and $\mathbb{V}(J)$ be the subvariety defined by $J$, then $U_\Sigma = \mathbb{C}^{K^\vee_{(1)}} - \mathbb{V}(J)$. Let $[\mathbf{s},\mathbf{t}]$ denote the set $\{s_1,\dots,s_{2r},t_1,\dots,t_{k-r}\}$. We can similarly consider the subset

$$U_{\overline{\Sigma}} \subset \mathbb{C}^{K^\vee_{(1)}-[\mathbf{s},\mathbf{t}]}$$

that corresponds to the stacky fan $\overline{\Sigma}$ for the group

$$\overline{N} = N/(\mathbb{Z}s_1 + \cdots + \mathbb{Z}s_{2r} + \mathbb{Z}\deg^\vee + \mathbb{Z}t_1 + \cdots + \mathbb{Z}t_{k-r}). \tag{2.3}$$

We have

$$U_\Sigma = U_{\overline{\Sigma}} \times \mathbb{C}^{2r} \times \mathbb{C}^{k-r},$$

where the last two components correspond to $\mathbf{z}(s_i)$ and $\mathbf{z}(t_i)$.

There is a group $\hat{G}$ defined by

$$\hat{G} := \{\lambda : K^\vee_{(1)} \to \mathbb{G}_m \Big| \prod_{n\in K^\vee_{(1)}} \lambda(n)^{\langle m,n\rangle} = 1,\ \text{for all } m \in \operatorname{Ann}(\deg^\vee)\}, \tag{2.4}$$

where $\operatorname{Ann}(\deg^\vee) := \{m \in M \mid \langle m, \deg^\vee\rangle = 0\}$. This group is isomorphic to the product of $\mathbb{G}_m$ (i.e. the $H$ given below) with the group in the definition of the toric stack (i.e. $G'$ in (2.14)). The extra $\mathbb{G}_m$ coming from a dilation action on the fiber is needed in [Isi13, Shi12] (see [FK17, Theorem 3.2]). The group $\hat{G}$ acts naturally on $U_\Sigma$. It has a subgroup $H \subset \hat{G}$ which is isomorphic to $\mathbb{G}_m$ with action

$$\lambda(s_i) = t, \lambda(t_i) = t^2, \lambda(v) = 1,\ \text{for all } v \in K^\vee_{(1)} - [\mathbf{s},\mathbf{t}]. \tag{2.5}$$

It is shown in [BL18, Section 5] that the toric DM stack corresponding to $(\overline{N},\overline{\Sigma})$ can be realized as the quotient of $U_{\overline{\Sigma}}$ by

$$\overline{G} = \hat{G}/H. \tag{2.6}$$

Moreover, there exists a (non-unique) isomorphism $\hat{G} \cong \overline{G} \times H$ ([BL18, Remark 5.1]).

There is a $\overline{G}$-invariant polynomial which is called the *potential*,

$$C(\mathbf{z}) = \sum_{m\in K_{(1)}} c(m) \prod_{n\in K^\vee_{(1)}} \mathbf{z}(n)^{\langle m,n\rangle}. \tag{2.7}$$

The potential $C(\mathbf{z})$ is of total degree 2 with respect to $H \cong \mathbb{G}_m$. It can be further written as

$$C(\mathbf{z}) = C_1(\mathbf{z}) + C_2(\mathbf{z}), \tag{2.8}$$

where $C_1(\mathbf{z})$ is the linear term in $\mathbf{z}(t_i)$, and $C_2(\mathbf{z})$ is the quadratic term in $\mathbf{z}(s_i)$. More precisely,

$$C_1(\mathbf{z}) = \sum_{m\in K_{(1)}\cap \mathrm{Ann}(s_1,\dots,s_{2r})} c(m) \prod_{n\in K_{(1)}^\vee} \mathbf{z}(n)^{\langle m,n\rangle}, \tag{2.9}$$

$$C_2(\mathbf{z}) = \sum_{m\in K_{(1)}\cap \mathrm{Ann}(t_1,\dots,t_{k-r})} c(m) \prod_{n\in K_{(1)}^\vee} \mathbf{z}(n)^{\langle m,n\rangle}. \tag{2.10}$$

If we let

$$\begin{aligned} f_i &= \sum_{\substack{m\in K_{(1)}\\ \langle m,t_i\rangle=1}} c(m) \prod_{n\in K_{(1)}^\vee-\{t_i\}} \mathbf{z}(n)^{\langle m,n\rangle} \\ &= \sum_{\substack{m\in K_{(1)}\\ \langle m,t_i\rangle=1}} c(m) \prod_{n\in K_{(1)}^\vee-[\mathbf{s},\mathbf{t}]} \mathbf{z}(n)^{\langle m,n\rangle}, \end{aligned} \tag{2.11}$$

then

$$C_1(\mathbf{z}) = \sum_{i=1}^{k-r} \mathbf{z}(t_i) f_i. \tag{2.12}$$

Indeed, by the decomposition (2.2) and the choice of $m \in K_{(1)}$, we see that if $\langle m, t_i\rangle = 1$, then $\langle m, s_l\rangle = \langle m, t_j\rangle = 0$ for all $s_l$ and $t_j \neq t_i$. It is straightforward to verify that $C_2(\mathbf{z})$ and $\mathbf{z}(t_i)f_i$ are both $\hat{G}$-semiinvariant section with character

$$\chi(\lambda) = \prod_{n\in K_{(1)}^\vee} \lambda(n)^{\langle \alpha,n\rangle}, \tag{2.13}$$

where $\alpha \in M$ is any lattice point satisfying $\langle \alpha, \deg^\vee\rangle = 1$.

One can check that $\hat{G}$ is generated by $H$ and another subgroup

$$G' := \{\lambda : K_{(1)}^\vee \to \mathbb{G}_m \Big| \prod_{n\in K_{(1)}^\vee} \lambda(n)^{\langle m,n\rangle} = 1, \text{ for all } m \in M\}. \tag{2.14}$$

By definition of $G'$, $\mathbf{z}(t_i)f_i$ is $G'$-invariant, and for any $\lambda \in H \cong \mathbb{G}_m$ (see (2.5)), we have $\lambda \cdot (\mathbf{z}(t_i)f_i) = \lambda(\mathbf{z}(t_i))(\mathbf{z}(t_i)f_i) = \chi(\lambda)(\mathbf{z}(t_i)f_i)$. Hence

$$\chi(\lambda) = \lambda(\mathbf{z}(t_i)), 1 \leq i \leq k-r. \tag{2.15}$$

In particular, this shows that $f_i$ is $\hat{G}$-invariant.

We use $f_i$ to define an intersection $Y \subset U_{\overline{\Sigma}}$ by

$$Y = \bigcap_{i=1}^{k-r} \{f_i = 0\}. \tag{2.16}$$

It can be viewed as the zero locus of the section $f := (f_1, \ldots, f_{k-r}) \in H^0(U_\Sigma, \oplus_{i=1}^{k-r} \mathcal{O}_{U_\Sigma})$. Moreover, $Y$ is $\overline{G}$-invariant, and will be shown to be a complete intersection in Proposition 3.5.

We then define $\mathcal{S}$ as the quotient stack $[Y/\overline{G}]$. The even part of sheaf of Clifford algebras $\mathcal{B}_0$ on $\mathcal{S} = [Y/\overline{G}]$ is defined by using the quadratic part $C_2(\mathbf{z})$ of $C(\mathbf{z})$. We formulate its definition as follows.

**Definition 2.4** ([Kuz08, Section 3])**.** *Let the quadratic $C_2(\mathbf{z})$ be a section of $\mathrm{Sym}^2(\oplus_{i=1}^{2r} \mathcal{O}_Y \cdot z_i^\vee)^\vee$, where $z_i^\vee$ are noncommutative variables. Then the even part of sheaf of Clifford algebras over $Y \subseteq U_{\overline{\Sigma}}$ is defined to be the noncommutative locally constant sheaf of algebras*

$$\Big(\mathcal{O}_Y\{z_1^\vee, \ldots, z_{2r}^\vee\}/\langle v \otimes v' + v' \otimes v - 2C_2(v, v'), \ \text{ for all } v, v' \in \oplus_{i=1}^{2r} \mathbb{C} z_i^\vee \rangle\Big)_{\text{even}},$$

*where $C_2(v, v')$ is the quadratic form associated to $C_2(\mathbf{z})$, and* even *refers to elements of even degrees in $z_i^\vee$. It has a natural $\hat{G}$-equivariant structure which descends to $\hat{G}/H = \overline{G}$ (see [BL18, Remark 7.4]). Then $\mathcal{B}_0$ on $[Y/\overline{G}]$ is defined to be above even part of sheaf of Clifford algebras over $Y \subseteq U_{\overline{\Sigma}}$ with $\overline{G}$-equivariant structure.*

**Definition 2.5** ([BL18, Sections 6 and 7])**.** *For arbitrary $r, 0 \leq r \leq 2k$, we call the noncommutative pair $(\mathcal{S}, \mathcal{B}_0)$ a* general Clifford mirror*. The even part of sheaf of Clifford algebras $\mathcal{B}_0$ is viewed as the structure sheaf of this general Clifford mirror. Moreover, Let $\mathrm{D}^b(\mathcal{S}, \mathcal{B}_0)$ be the bounded derived category of coherent sheaves on $\mathcal{S}$ which are also $\mathcal{B}_0$-modules.*

**Remark 2.6.** *This definition generalizes the* pure Clifford mirror *(i.e. $r = 2k$ case) considered in [BL18, Section 5] to a complete intersection in a toric stack. Thus, we call it a* general Clifford mirror*. The word "mirror" in the "general Clifford mirror" refers to the fact that such construction comes* *from mirror symmetry. We do not specify the mirror pair for* $(\mathcal{S}, \mathcal{B}_0)$*. In fact, any pair* $(\mathcal{S}^\vee, \mathcal{B}_0^\vee)$ *associated with the dual Gorenstein cone and the dual data will be its mirror pair.*

The main conjecture of [BL18] is that for a fixed reflexive Gorenstein cone $K$, all the decompositions (2.2) (with $r$ varied) will give double mirrors. In terms of homological mirror symmetry, this can be formulated as follows.

**Conjecture 2.7** ([BL18, Conjecture 7.5])**.** *Under the centrality and appropriate flatness assumptions (see* (3.5)*), for all general Clifford mirrors* $(\mathcal{S}, \mathcal{B}_0)$ *associated to* $K$*, their derived categories* $\mathrm{D}^b(\mathcal{S}, \mathcal{B}_0)$ *are equivalent.*

**Theorem 2.8** ([FK17, Theorem 6.3], [BL18, Theorem 6.3] )**.** *The conjecture 2.7 is true when* $r$ *is* $0$ *or* $2k$*.*

**Remark 2.9.** *For* $r = 0$*, the noncommutative varieties* $(\mathcal{S}, \mathcal{B}_0)$ *are actually the Batyrev-Borisov Calabi-Yau varieties. Hence, Conjecture 2.7 can be viewed as a generalization of Batyrev-Nill's conjecture ([BN08, Conjecture 5.3]) on Batyrev-Borisov double mirrors. This case was first proved by Favero and Kelly in [FK17, Theorem 6.3].*

The main result of this paper (Theorem 3.13) confirms Conjecture 2.7. This is a strong evidence that the construction $(\mathcal{S}, \mathcal{B}_0)$ are indeed double mirrors, and we expect that they should also pass other tests of mirror symmetry.

## 3. Derived equivalence of general Clifford double mirrors

### 3.1. Derived factorization category and Hirano's result

Two technical tools to prove Theorem 3.13 are the results on the relations of derived categories between variation of GIT ([BFK19, HL15]), and Hirano's result [Hir17] analogous to that of Isik and Shipman ([Shi12, Isi13]) in the matrix factorization categories. First let us recall the definition of derived matrix factorization categories in [Pos11, EP15] (see [Hir17, Definition 2.2]).

**Definition 3.1.** *Let* $X$ *be a scheme, and* $G$ *be an affine algebraic group acting on* $X$*. Let* $\chi : G \to \mathbb{G}_m$ *be a character of* $G$*, and* $W : X \to \mathbb{A}^1$ *be a* $\chi$*-semiinvariant function. A* factorization $F$ *of data* $(X, \chi, W, G)$ *is a sequence*

$$F = \left( F_1 \xrightarrow{\phi_1^F} F_0 \xrightarrow{\phi_0^F} F_1(\chi) \right),$$

*where $F_i$ are $G$-equivariant coherent sheaves on $X$ and $\phi_i^F$ are $G$-equivariant homomorphisms for $i = 0, 1$. They satisfy the relations*

$$\phi_0^F \circ \phi_1^F = W \cdot \mathrm{id}_{F_1}, \quad \phi_1^F(\chi) \circ \phi_0^F = W \cdot \mathrm{id}_{F_0}.$$

*A morphism of factorizations $g : E \to F$ is a pair of morphisms $(g_1, g_0)$ that commute with $\phi_i^E$ and $\phi_i^F$. We use $\mathrm{Coh}_G(X, \chi, W)$ to denote this abelian category of factorizations.*

There also exists a notion of chain homotopy between morphisms in $\mathrm{Coh}_G(X, \chi, W)$ and we let $\mathrm{Kcoh}_G(X, \chi, W)$ be the corresponding homotopy category. One can define a natural translation and cone construction in $\mathrm{Kcoh}_G(X, \chi, W)$. These give a triangulated category structure on the homotopy category. Let $\mathrm{Acoh}_G(X, \chi, W)$ be the smallest thick subcategory of $\mathrm{Kcoh}_G(X, \chi, W)$ containing all totalizations of short exact sequences from $\mathrm{Coh}_G(X, \chi, W)$ (see [Hir17, Section 2]).

**Definition 3.2** ([Hir17, Definition 2.10])**.** *The* derived factorization category *of data $(X, \chi, W, G)$ is defined as Verdier quotient*

$$\mathrm{Dcoh}_G(X, \chi, W) := \mathrm{Kcoh}_G(X, \chi, W)/\, \mathrm{Acoh}_G(X, \chi, W).$$

To state Hirano's result ([Hir17, Theorem 4.2]), let us fix the following notation. For consistency, it appears slightly different from that in [Hir17].

Let $U$ be a smooth quasi-projective variety, and $G$ be a reductive affine algebraic group acting on $U$. Let $\chi : G \to \mathbb{G}_m$ be a character, and $C_2 : U \to \mathbb{A}^1$ be a $\chi$-semiinvariant regular function. Suppose that there is a $G$-equivariant locally free sheaf $\mathcal{E}$ over $U$, and a $G$-invariant section $f \in H^0(U, \mathcal{E}^\vee)$. Let $Z$ be the zero locus of $f$. We call $f$ to be a *regular* section if the codimension of $Z$ in $U$ is $\operatorname{rank} \mathcal{E}$ (see [Hir17, Section 4]). Set $\mathcal{E}(\chi) = \mathcal{E} \otimes \mathcal{O}(\chi)$ for the character $\chi$, and

$$V_U(\mathcal{E}(\chi)) := \mathit{Spec}(\mathrm{Sym}^\bullet(\mathcal{E}(\chi)^\vee)) \tag{3.1}$$

the corresponding vector bundle with induced $G$-action. Let $q : V_U(\mathcal{E}(\chi)) \to U$ and $p : V(\mathcal{E}(\chi))|_Z \to Z$ be two natural projections. The regular section $f$ induces a $\chi$-semiinvariant regular function $C_1 : V_U(\mathcal{E}(\chi)) \to \mathbb{A}^1$. Locally, this means that $f = (f_1, \dots, f_{\operatorname{rank} \mathcal{E}})$ is associated with a function $C_1 = \sum_{i=1}^{\operatorname{rank} \mathcal{E}} z_i f_i$, where $z_i$ is an indeterminate such that $g \in G$ acting on $z_i$ equals to $\chi(g) z_i$.

**Theorem 3.3** ([Hir17, Theorem 4.2]). *Under the above notation, assume that $C_2|_Z$ is flat and $f \in H^0(U, \mathcal{E}^\vee)$ is a regular section, then there is an equivalence*

$$\mathrm{Dcoh}_G(Z, \chi, C_2|_Z) \cong \mathrm{Dcoh}_G(V_U(\mathcal{E}(\chi)), \chi, q^*C_2 + C_1).$$

## 3.2. Proof of the main theorem

Suppose that there are two decompositions as (2.2)

$$\begin{aligned} \deg^\vee &= \frac{1}{2}(s_1 + \ldots + s_{2r}) + t_1 + \ldots + t_{k-r} \\ &= \frac{1}{2}(\tilde{s}_1 + \ldots + \tilde{s}_{2l}) + \tilde{t}_1 + \ldots + \tilde{t}_{k-l}, \end{aligned} \tag{3.2}$$

and respective regular simplicial triangulations $\Sigma, \tilde{\Sigma}$ which satisfy centrality condition (†). Then there exist general Clifford mirrors $(\mathcal{S}, \mathcal{B}_0)$ and $(\tilde{\mathcal{S}}, \tilde{\mathcal{B}}_0)$ as constructed in Section 2. In order to show that they are derived equivalent, we use a derived version of Cayley trick. That is, we associate each derived category to a derived matrix factorization category with the common potential $C(\mathbf{z})$. After this, a homological variation of GIT argument for derived categories will give the desired equivalence.

Let us first work with $\deg^\vee = t_1 + \ldots + t_{k-r} + \frac{1}{2}(s_1 + \ldots + s_{2r})$. The other decomposition can be treated analogously. We will show that the regularity assumption in Theorem 3.3 is satisfied for the generic $Y$ in the general Clifford mirror construction. The proof of the following proposition was explained to us by Lev Borisov.

**Proposition 3.4.** *For each $1 \le i \le k - r$, the linear system*

$$L_i := \{ \sum_{m \in K_{(1)}, \langle m, t_i \rangle = 1} c(n) \prod_{n \in K^\vee_{(1)} - [\mathbf{s}, \mathbf{t}]} \mathbf{z}(n)^{\langle m, n \rangle} \mid c(n) \in \mathbb{C} \}$$

*is base point free on $U_{\overline{\Sigma}}$.*

*Proof.* For $m \in \{m \in K_{(1)} \mid \langle m, t_i \rangle = 1\}$, let $F_m = \prod_{n \in K^\vee_{(1)} - [\mathbf{s}, \mathbf{t}]} \mathbf{z}(n)^{\langle m, n \rangle}$. It suffices to show that there is no common zeros for all monomial functions $F_m$. $F_m$ is non-zero on $(\mathbb{C}^*)^{\# \mathrm{Vert}(\overline{\Sigma})} \subset U_{\overline{\Sigma}}$, hence we only need to consider zero locus on the boundary divisors $\tilde{D}_n := \{\mathbf{z}(n) = 0\} \subset U_{\overline{\Sigma}}$ for $n \in K^\vee_{(1)} - [\mathbf{s}, \mathbf{t}]$. The zero divisor of $F_m$ is

$$\mathrm{div}_0(F_m) = \sum_{n \in K^\vee_{(1)} - [\mathbf{s}, \mathbf{t}]} \langle m, n \rangle \tilde{D}_n.$$

Suppose that there were a closed point $z \in U_{\overline{\Sigma}}$ on which all $F_m$ are zero. By the definition of $U_{\overline{\Sigma}}$, we may assume that $\mathrm{Cone}\{\bar{\rho}_i \in \overline{\Sigma}(1) \mid z_i = 0\} \subset \bar{\sigma}$, where $\overline{\Sigma}(1)$ is the set of rays and $\bar{\sigma} \in \overline{\Sigma}$ is a maximum cone.

Let $n_1, \dots, n_d$ be the preimage in $K^{\vee}_{(1)}$ of the ray generators of $\bar{\sigma}$. Then the preimage $\sigma$ of $\bar{\sigma}$ is the maximum cone with ray generators

$$n_1, \dots, n_d, s_1, \dots, s_{2r}, t_1, \dots, t_{k-r}.$$

Consider the facet $\theta$ of this cone generated by all of the above elements, except $t_i$. This facet $\theta$ lies on a facet of $K^{\vee}$. Indeed, otherwise, it would be in the interior of $K^{\vee}$ but then points on the opposite side of $t_i$ can not be in any cone that contains $\deg^{\vee}$. This contradicts the centrality assumption (†) in Section 2.

The dual face of $\theta$ is generated by some point $m \in K_{(1)}$. We must have $\langle m, t_i \rangle \neq 0$, since otherwise $m$ is orthogonal to all of $\sigma$ and must be $\mathbf{0}$. By the definition of the dual face, $\langle m, n_p \rangle = \langle m, s_q \rangle = 0$ for all $1 \leq p \leq d, 1 \leq q \leq 2k$, and $\langle m, t_j \rangle = 0$ for all $j \neq i$. Therefore, $\langle m, t_i \rangle = \langle m, \deg^{\vee} \rangle = 1$. We claim that if $n_l \in K^{\vee}_{(1)} - [\mathbf{s}, \mathbf{t}] - \{n_1, \dots, n_d\}$, then $z \notin \tilde{D}_{n_l}$. In fact, otherwise, let $\bar{\rho}_l \in \overline{\Sigma}(1)$ be the ray generated by the image of $n_l$, then $z \in \tilde{D}_{n_l}$ implies that $z_l = 0$ and thus $n_l \in \sigma$ is a ray generator by the choice of $\sigma$. This is a contradiction because $n_l \notin \{n_1, \dots, n_d\}$. By $\langle m, n_p \rangle = 0$ for all $1 \leq p \leq d$, we have

$$\mathrm{Supp}(\mathrm{div}_0(F_m)) \subset \mathrm{Supp}(\cup_{n \in K^{\vee}_{(1)} - [\mathbf{s}, \mathbf{t}] - \{n_1, \dots, n_d\}} \tilde{D}_n),$$

and thus $z \notin \mathrm{Supp}(\mathrm{div}_0(F_m))$. This is a contradiction. □

The upshot of the above discussion is the desired regularity property of $f = (f_1, \dots, f_{k-r}) \in H^0(U_{\overline{\Sigma}}, \oplus_{i=1}^{k-r} \mathcal{O}_{\overline{\Sigma}})$ on $U_{\overline{\Sigma}}$ (see Section 2 for notation and (2.15) for the fact that $f_i$ is $\hat{G}$-equivariant).

**Proposition 3.5.** *For a general coefficient function $c$, $f$ is a regular section on $U_{\overline{\Sigma}}$ in the sense of Theorem 3.3, that is, $Y$ in* (2.16) *is a complete intersection of codimension $k - r$.*

*Proof.* By Proposition 3.4, the linear system $L_i$ is base point free. Hence for general coefficients,

$$Y = \bigcap_{1 \leq i \leq k-r} \{f_i = 0\} \subset U_{\overline{\Sigma}}$$

is a complete intersection by Bertini's theorem (for example, see [Mul09, Proposition 6.7]). □

Recall that by definition (2.3),

$$\overline{N} = N/(\mathbb{Z}s_1 + \cdots + \mathbb{Z}s_{2r} + \mathbb{Z}\deg^\vee + \mathbb{Z}t_1 + \cdots + \mathbb{Z}t_{k-r}),$$

and according to the discussion in Section 2, the Cox open set $U_{\overline{\Sigma}}$ can be viewed as a $\hat{G} = \overline{G} \times H$ invariant variety with $H$ acts trivially. We define two $\hat{G}$-equivariant locally free sheaves $\mathcal{F}_L$ and $\mathcal{F}_Q$ on $U_{\overline{\Sigma}}$. Let

$$\mathcal{F}_L := \oplus_{i=1}^{k-r} \mathcal{O}_{U_{\overline{\Sigma}}}$$

be the rank $k-r$ locally free sheaf associated to those $t_i$, or the linear part of the potential. Let

$$\mathcal{F}_Q := \oplus_{i=1}^{2r} \mathcal{O}_{U_{\overline{\Sigma}}}(\chi_i)$$

be the rank $2r$ locally free sheaf associated to $s_i$, or the quadratic part of the potential, where $\chi_i$ is the character $\chi_i(\lambda) = \lambda(\mathbf{z}(s_i))$.

The vector bundle $V_{U_{\overline{\Sigma}}}(\mathcal{F}_Q)$ has a $\hat{G} = \overline{G} \times H$ action given by

$$(\bar{g}, t) \times (\bar{x}, \mathbf{z}(s_i)) \mapsto (\bar{g} \cdot \bar{x}, t \cdot \mathbf{z}(s_i)), \tag{3.3}$$

where $\bar{g} \in \overline{G}$ acts on $\bar{x} \in U_{\overline{\Sigma}}$ by the action of $\overline{G}$ on $\overline{U}_{\overline{\Sigma}}$, and $t \in H \cong \mathbb{G}_m$ acts by $t \cdot \mathbf{z}(s_i) = t\mathbf{z}(s_i), 1 \leq i \leq 2r$. Then $C_2(\mathbf{z})$ (see (2.10)) is a section of $H^0(U_{\overline{\Sigma}}, \mathrm{Sym}^2 \mathcal{F}_Q)$, and according to the discussion in Section 2, it is $\hat{G}$-semiinvariant with character $\chi$. Similarly, the vector bundle $V_{U_{\overline{\Sigma}}}(\mathcal{F}_L(\chi) \oplus \mathcal{F}_Q)$ has a $\hat{G}$-action given by

$$(\bar{g}, t) \times (\bar{x}, \mathbf{z}(t_j), \mathbf{z}(s_i)) \mapsto (\bar{g} \cdot \bar{x}, t \cdot \mathbf{z}(t_j), t \cdot \mathbf{z}(s_i)),$$

where $\bar{g} \cdot \bar{x}, t \cdot \mathbf{z}(s_i)$ are the same as (3.3), and $t \cdot \mathbf{z}(t_j) = t^2 \mathbf{z}(t_j)$. Recall that by (2.15), we have $\chi(\lambda) = \lambda(\mathbf{z}(t_i))$ which amounts to multiplying by $t^2$ under the identification $\hat{G} = \overline{G} \times H$. Then $f = (f_i)_i$ is a $\hat{G}$-equivariant section of $\mathcal{F}_L$ according to the discussion in Section 2, and $C_1(\mathbf{z}) = \sum_{i=1}^{k-r} \mathbf{z}(t_i) f_i$ in (2.12) is $\hat{G}$-semiinvariant with character $\chi$.

By the previous discussion, we have

$$[U_\Sigma / \hat{G}] \cong [V_{U_{\overline{\Sigma}}}(\mathcal{F}_L(\chi) \oplus \mathcal{F}_Q) / \hat{G}]. \tag{3.4}$$

We also write $C_2(\mathbf{z})$ for the pullback of $C_2(\mathbf{z})$ to this vector bundle. The zero locus of $f$ is exactly $Y$, and $C_2(\mathbf{z})$ restricted to $V(\mathcal{F}_Q|_Y)$ is a quadric section, and hence associated to an even part of sheaf of Clifford algebras whose pullback to $\mathcal{S} = [Y/\overline{G}]$ is $\mathcal{B}_0$.

**Remark 3.6.** *We slightly abuse the above notation: technically, the twist of $\chi$ should be on the locally free sheaf $\mathcal{E}$ which is the pullback of $\mathcal{F}_L$ over the morphism $V_{U_{\overline{\Sigma}}}(\mathcal{F}_Q) \to V_{U_{\overline{\Sigma}}}$. Thus, $V_{U_{\overline{\Sigma}}}(\mathcal{F}_L(\chi) \oplus \mathcal{F}_Q)$ should be written as $V_{V(\mathcal{F}_Q)}(\mathcal{E}(\chi))$ according to the notation in Theorem 3.3.*

We need *flatness* assumption for the quadric fibration defined by $C_2(\mathbf{z})$:

(3.5) *The quadric fibration $\{C_2(\mathbf{z}) = 0\} \subset V_{U_{\overline{\Sigma}}}(\mathcal{F}_Q)$ to $U_{\overline{\Sigma}}$ is flat.*

**Remark 3.7.** *As pointed out in [BL18, Standing Assumption 5.4]. This flatness assumption is crucial for establishing the expected derived equivalence of pure Clifford double mirrors. There are examples that derived equivalence no longer holds when the fibration is not flat. On the other hand, since the fibration is defined by a single equation, the geometric criterion for flatness is that all of the fibers are hypersurfaces.*

**Proposition 3.8** ([BL18, Theorem 6.1])**.** *Under the flatness assumption on quadric fibrations (3.5), the derived category $\mathrm{D}^b(\mathcal{S}, \mathcal{B}_0)$ is equivalent to the derived matrix factorization category $\mathrm{Dcoh}_{\hat{G}}(V(\mathcal{F}_Q|_Y), \chi, C_2(\mathbf{z}))$.*

This result follows the same argument as [BL18, Theorem 6.1] which relies on the results of [Kuz08, BDF+18]. Alternatively, one can use equivariant version of [Kuz08, Theorem 4.2] and [Orl09, Theorem 16].

**Proposition 3.9.** *Under the flatness assumption on quadric fibrations (3.5), there exists derived equivalence*

$$\mathrm{D}^b(\mathcal{S}, \mathcal{B}_0) \cong \mathrm{Dcoh}_{\hat{G}}(V_{U_{\overline{\Sigma}}}(\mathcal{F}_L(\chi) \oplus \mathcal{F}_Q), \chi, C(\mathbf{z})).$$

*Proof.* By Proposition 3.8, we have

$$\mathrm{D}^b(\mathcal{S}, \mathcal{B}_0) \cong \mathrm{Dcoh}_{\hat{G}}(V(\mathcal{F}_Q|_Y), \chi, C_2(\mathbf{z})).$$

The zero locus of $f = (f_i)_i$ on $V_{U_{\overline{\Sigma}}}(\mathcal{F}_Q)$ is $V(\mathcal{F}_Q|_Y)$, and the associated $\chi$-semiinvariant function $V_{U_{\overline{\Sigma}}}(\mathcal{F}_L(\chi) \oplus \mathcal{F}_Q) \to \mathbb{A}^1$ is exactly $C_1(\mathbf{z})$. By Proposition 3.5, $C_1(\mathbf{z})$ is regular. Then by Theorem 3.3 (see Remark 3.6), there exists equivalence

$$\begin{aligned}
&\mathrm{Dcoh}_{\hat{G}}(V(\mathcal{F}_Q|_Y), \chi, C_2(\mathbf{z})) \\
\cong\, &\mathrm{Dcoh}_{\hat{G}}(V_{U_{\overline{\Sigma}}}(\mathcal{F}_L(\chi) \oplus \mathcal{F}_Q), \chi, C_1(\mathbf{z}) + q^* C_2(\mathbf{z})),
\end{aligned}$$

where $q : V_{U_{\overline{\Sigma}}}(\mathcal{F}_L(\chi) \oplus \mathcal{F}_Q) \to V_{U_{\overline{\Sigma}}}(\mathcal{F}_Q)$ is the natural projection. However, $q^* C_2(\mathbf{z})$ is exactly the same as $C_2(\mathbf{z})$ restricted to $V_{U_{\overline{\Sigma}}}(\mathcal{F}_L(\chi) \oplus \mathcal{F}_Q)$ as there are no linear terms in $C_2(\mathbf{z})$. Finally, by the decomposition (2.8), $C(\mathbf{z}) = C_1(\mathbf{z}) + C_2(\mathbf{z})$, we have the desired equivalence. $\square$

By the same argument, we can establish the derived equivalence

$$\mathrm{D}^b(\tilde{\mathcal{S}}, \tilde{\mathcal{B}}_0) \cong \mathrm{Dcoh}_{\hat{G}}(V_{U_{\tilde{\Sigma}}}(\tilde{\mathcal{F}}_L(\chi) \oplus \tilde{\mathcal{F}}_Q), \chi, C(\mathbf{z})),$$

where $\tilde{(\ )}$ represent the analogous construction from the decomposition

$$\deg^\vee = \frac{1}{2}(\tilde{s}_1 + \ldots + \tilde{s}_{2l}) + \tilde{t}_1 + \ldots + \tilde{t}_{k-l},$$

and a regular simplicial fan $\tilde{\Sigma}$ satisfying the centrality condition (†).

**Remark 3.10.** *We emphasize that $C(\mathbf{z})$ is the restriction of the same potential function* (2.7)

$$C(\mathbf{z}) = \sum_{m \in K_{(1)}} c(m) \prod_{n \in K^\vee_{(1)}} \mathbf{z}(n)^{\langle m,n \rangle},$$

*hence we use the same symbol. Moreover, by the definition of $\hat{G}$ (see* (2.4)*) and $\chi$, they do not depend on decompositions.*

Next, we relate the two matrix factorization categories by the result of [BFK19].

**Proposition 3.11.** *There exists derived equivalence*

$$\mathrm{Dcoh}_{\hat{G}}(V_{U_{\Sigma}}(\mathcal{F}_L(\chi) \oplus \mathcal{F}_Q), \chi, C(\mathbf{z})) \cong \mathrm{Dcoh}_{\hat{G}}(V_{U_{\tilde{\Sigma}}}(\tilde{\mathcal{F}}_L(\chi) \oplus \tilde{\mathcal{F}}_Q), \chi, C(\mathbf{z})).$$

*Proof.* This version is stated in [BL18, Theorem 3.2]. By (3.4), the category $D_B(K, c; \Sigma)$ therein is exactly

$$\mathrm{Dcoh}_{\hat{G}}(U_{\Sigma}, \chi, C(\mathbf{z})) \cong \mathrm{Dcoh}_{\hat{G}}(V_{U_{\Sigma}}(\mathcal{F}_L(\chi) \oplus \mathcal{F}_Q), \chi, C(\mathbf{z}))$$

under the current notation. Moreover,

$$\mathrm{Dcoh}_{\hat{G}}(V_{U_{\tilde{\Sigma}}}(\tilde{\mathcal{F}}_L(\chi) \oplus \tilde{\mathcal{F}}_Q), \chi, C(\mathbf{z})) \cong D_B(K, c; \tilde{\Sigma})$$

for the same reason. Recall that $\Sigma, \tilde{\Sigma}$ are regular simplicial fans with support $K^\vee$, and $K^\vee_{(1)}$ contains all the ray generators $\{s_1, \ldots, s_{2r}, t_1, \ldots, t_{k-r}\}$

and $\{\tilde{s}_1, \ldots, \tilde{s}_{2l}, \tilde{t}_1, \ldots, \tilde{t}_{k-l}\}$. There is a bijection between chambers of the secondary fan for the action of (see (2.14))

$$G' = \{\lambda : K^\vee_{(1)} \to \mathbb{G}_m \mid \prod_{n \in K^\vee_{(1)}} \lambda(n)^{\langle m,n \rangle} = 1, \text{ for all } m \in M\}$$

on $\mathbb{C}^{K^\vee_{(1)}}$ and regular triangulations on $K^\vee_{(1)}$ (see [CLS11, Proposition 15.2.9]). Thus $\Sigma, \tilde{\Sigma}$ correspond to chambers of the secondary fan, and are connected by elementary wall-crossings (see [BFK19, Definition 3.5.1]). By the definition of Gorenstein cones, $K^\vee_{(1)}$ lies on the hyperplane $\langle \deg, - \rangle = 1$, which is exactly the condition needed for the derived equivalence (see [FK17, Definition 5.10]). For details, see the proof in [BL18, Theorem 3.2] or [FK17, Theorem 4.4]. □

**Remark 3.12.** *A far more general version of this result is proved by Ballard, Favero, Katzarkov [BFK19] and Halpern-Leistner [HL15] independently, which clarifies the earlier work of Herbst and Walcher [HW12]. The version stated above first appears in [FK17, Theorem 4.4] in terms of singular derived categories. The derived matrix factorization category $D_B(K, c; \Sigma)$ is equivalent to the $\hat{G}$-equivariant singular derived category of $\{C(\mathbf{z}) = 0\}$ therein due to the smoothness of $U_\Sigma$. Otherwise, this needs to be replaced by the $\hat{G}$-equivariant relative singular category (see [EP15]).*

Putting above results together, we have the desired equivalence.

**Theorem 3.13.** *Under the flatness assumption on quadric fibrations (3.5), the general Clifford double mirrors $(\mathcal{S}, \mathcal{B}_0)$ and $(\tilde{\mathcal{S}}, \tilde{\mathcal{B}}_0)$ are derived equivalent.*

*Proof.* By Proposition 3.9 and Remark 3.10, we have

$$\mathrm{D}^b(\mathcal{S}, \mathcal{B}_0) \cong \mathrm{Dcoh}_{\hat{G}}(V_{U_{\overline{\Sigma}}}(\mathcal{F}_L(\chi) \oplus \mathcal{F}_Q), \chi, C(\mathbf{z})),$$
$$\mathrm{D}^b(\tilde{\mathcal{S}}, \tilde{\mathcal{B}}_0) \cong \mathrm{Dcoh}_{\hat{G}}(V_{U_{\overline{\tilde{\Sigma}}}}(\tilde{\mathcal{F}}_L(\chi) \oplus \tilde{\mathcal{F}}_Q), \chi, C(\mathbf{z})).$$

By Proposition 3.11, we have $\mathrm{D}^b(\mathcal{S}, \mathcal{B}_0) \cong \mathrm{D}^b(\tilde{\mathcal{S}}, \tilde{\mathcal{B}}_0)$. □

**Remark 3.14.** *We assume the existence of quadric fibrations and their flatness in Theorem 3.13. In particular, for the two decompositions (3.2) corresponding to $(\mathcal{S}, \mathcal{B}_0)$ and $(\tilde{\mathcal{S}}, \tilde{\mathcal{B}}_0)$ respectively, we have $r > 0, l > 0$. However, if one of $r, l$ equals to $0$ (say $r = 0$), then the noncommutative variety $(\mathcal{S}, \mathcal{B}_0)$ becomes a stack $\mathcal{Y}$ whose coarse moduli space is the Calabi-Yau variety defined by the Batyrev-Borisov complete intersection in a toric Fano variety. There still exists derived equivalence between $\mathrm{D}^b(\mathcal{Y})$ and $\mathrm{D}^b(\tilde{\mathcal{S}}, \tilde{\mathcal{B}}_0)$.*

*In fact,* $\mathrm{D}^b(\mathcal{Y})$ *is also derived equivalent to* $\mathrm{Dcoh}_{\hat{G}}(U_\Sigma, \chi, C(\mathbf{z}))$ *by Shipman and Isik's result [Shi12, Isi13]. Therefore, by Proposition 3.11, we still have* $\mathrm{D}^b(\mathcal{Y}) \cong \mathrm{D}^b(\tilde{\mathcal{S}}, \tilde{\mathcal{B}}_0)$.

## 4. General Clifford mirrors of the product type

One type of general Clifford mirrors can be obtained as a "product" of two general Clifford mirrors, and we call them general Clifford mirrors of the product type. The construction is as follows.

Let $M_1, M_2$ be lattices. Suppose that there are two reflexive Gorenstein cones $K_1, K_2$ in $(M_1)_\mathbb{R}, (M_2)_\mathbb{R}$ respectively. Their dual cones are $K_1^\vee, K_2^\vee$ in $(M_1^\vee)_\mathbb{R}, (M_2^\vee)_\mathbb{R}$ with degree elements $\deg_1^\vee, \deg_2^\vee$ respectively. Then there is a rational polyhedral cone

$$K := \{(a;b) \in (M_1 \oplus M_2)_\mathbb{R} \mid a \in K_1, b \in K_2\}. \tag{4.1}$$

It is a reflexive cone with degree element $\deg^\vee = (\deg_1^\vee; \deg_2^\vee)$. Its dual cone is exactly

$$K^\vee = \{(a^\vee; b^\vee) \in (M_1^\vee \oplus M_2^\vee)_\mathbb{R} \mid a^\vee \in K_1^\vee, b^\vee \in K_2^\vee\}, \tag{4.2}$$

and hence $K, K^\vee$ is a pair of reflexive Gorenstein cones whose index is $\langle \deg_1, \deg_1^\vee \rangle + \langle \deg_2, \deg_2^\vee \rangle = k_1 + k_2$. Now let

$$\deg_1^\vee = \frac{1}{2}(s_1 + \ldots + s_{2r}) + t_1 + \ldots + t_{k_1 - r}$$

be a decomposition and $\Sigma_1$ be a regular simplicial fan with support $K_1^\vee$ satisfying the assumptions in Section 2. Similarly, let

$$\deg_2^\vee = \frac{1}{2}(\xi_1 + \ldots + \xi_{2l}) + \eta_1 + \ldots + \eta_{k_2 - l}$$

be a decomposition and $\Sigma_2$ be a regular simplicial fan with support $K_2^\vee$ satisfying the assumptions in Section 2. Then

$$\begin{aligned}\deg^\vee &= (\deg_1^\vee; \deg_2^\vee)\\ &= \frac{1}{2}((s_1;\mathbf{0}) + \ldots + (s_{2r};\mathbf{0}) + (\mathbf{0};\xi_1) + \ldots + (\mathbf{0};\xi_{2l}))\\ &\quad + (t_1;\mathbf{0}) + \ldots + (t_{k_1-r};\mathbf{0}) + (\mathbf{0};\eta_1) + \ldots + (\mathbf{0};\eta_{k_2-l})\end{aligned}$$

is a decomposition, and

$$\Sigma := \{\sigma_1 \times \sigma_2 \mid \sigma \in \Sigma_1, \sigma_2 \in \Sigma_2\}$$

is a simplicial fan with support $K^\vee$ and satisfying the centrality condition (†). If $\Sigma$ is further assumed to be regular, then the decomposition together with $\Sigma$ also satisfy the assumptions in Section 2. Let $(\mathcal{S}, \mathcal{B}_0)$ be the general Clifford mirror associate with the above data. Then $(\mathcal{S}, \mathcal{B}_0)$ is said to be of the product type. As a special case of Theorem 3.13, we have the following result.

**Corollary 4.1.** *Under the above notation and assumptions. Assume the flatness for quadric fibrations* (3.5)*, then the general Clifford mirrors* $(\mathcal{S}, \mathcal{B}_0)$ *of the product type are all derived equivalent.*

Perhaps, the most simple example of the above kind can be built upon the anticanonical hypersurfaces associated with the toric Fano variety defined by a reflexive polytope.

First, we work with one copy of the data and consider the example in [BL18, Section 9.4]. Let $\Delta$ be the 2 dimensional reflexive polytope

$$\Delta = \mathrm{Conv}\{(1,1), (1,-1), (-1,-1), (-1,1)\} \subset (M_1)_{\mathbb{R}},$$

whose dual polytope is

$$\Delta^\vee = \mathrm{Conv}\{(1,0), (0,-1), (-1,0), (0,1)\} \subset (M_1^\vee)_{\mathbb{R}}.$$

Then a pair of reflexive Gorenstein cones can be associated with these polytopes

$$K_1 = \{(a; a\cdot\Delta) \mid a \geq 0\} \subset (M_1)_{\mathbb{R}}, \quad K_1^\vee = \{(b; b\cdot\Delta^\vee) \mid b \geq 0\} \subset (M_1^\vee)_{\mathbb{R}}.$$

The degree element $\deg_1^\vee$ can be written in two different ways:

$$\deg_1^\vee = (1; \mathbf{0}) = \frac{1}{2}(s_1 + s_2),$$

where $s_1 = (1,-1,0), s_2 = (1,1,0)$. For each decomposition, there is a unique regular simplicial fan satisfying assumptions in Section 2.

Let $E$ be an one dimensional variety defined by the expression $\deg_1^\vee = (1;\mathbf{0})$. It is an elliptic curve in $\mathbf{P}_{\Delta^\vee} = \mathbb{P}^1 \times \mathbb{P}^1$ given by equation

$$
\begin{aligned}
f = {} & a_{11}x^{-1}y + a_{12}y + a_{13}xy \\
& + a_{21}x^{-1} + a_{22} + a_{13}x \\
& + a_{31}x^{-1}y^{-1} + a_{32}y^{-1} + a_{33}xy^{-1},
\end{aligned}
\tag{4.3}
$$

where $a_{ij} \in \mathbb{C}, 1 \leq i,j \leq 3$ are generically chosen coefficients.

There is a Clifford mirror $(\mathcal{S}, \mathcal{B}_0)$ associated to $\underline{\deg_1^\vee} = s_1 + s_2$. To be precise, $\overline{M_1^\vee} = \mathbb{Z}^3/\mathbb{Z}^3 \cap (\mathbb{R}s_1 + \mathbb{R}s_2) \cong \mathbb{Z}$, and $\Theta = \overline{(K_1^\vee)_{(1)}} = \mathrm{Conv}(-1,1)$, the toric stack $\mathbf{P}_\Theta$ is actually the smooth toric variety $\mathbb{P}^1$. By straightforward computations, one can find that among the lattice points of $(K_1)_{(1)}$, elements in $\{(1,-1,1),(1,-1,0),(1.-1,-1)\}$ pairing with $s_1$ equal to 2; elements in $\{(1,1,1),(1,1,0),(1,1,-1)\}$ pairing with $s_2$ equal to 2; and elements in $\{(1,0,1),(1,0,0),(1,0,-1)\}$ pairing with both $s_1, s_2$ equal to 1. If we use $z_1, z_2$ for the coordinate of the vector bundle, and $t$ for the coordinate of the base $S = \mathbb{P}^1$, then the quadric fibration can be written as

$$
\begin{aligned}
0 = C(\mathbf{z}) = {} & a_{11}z_1^2t + a_{21}z_1^2 + a_{31}z_1^2t^{-1} \\
& + a_{12}z_1z_2t + a_{22}z_1z_2 + a_{32}z_1z_2t^{-1} \\
& + a_{13}z_2^2t + a_{23}z_2^2 + a_{33}z_2^2t^{-1},
\end{aligned}
\tag{4.4}
$$

where $a_{ij}$ are the same coefficients as in (4.3). $C(\mathbf{z})$ is a family of quadratic forms parametrized by $t$, and because their coefficients are chosen generically, the corank of this quadratic form is less than 2 for any $t$. Hence the derived category of $(\mathcal{S}, \mathcal{B}_0)$ is equivalent to a commutative variety $\tilde{\mathcal{S}}$, which is a ramified double cover of $\mathcal{S}$ ([Kuz08, Corollary 3.14]). Moreover, the ramification locus is determined by where the quadratic form degenerates, that is,

$$
(a_{11}t + a_{21} + a_{31}t^{-1})(a_{13}t + a_{23} + a_{33}t^{-1}) - \frac{1}{4}(a_{12}t + a_{22} + a_{32}t^{-1}) = 0.
$$

Hence, the ramification locus on $\mathbb{P}^1$ is exactly the same as those of $E$ over $S = \mathbb{P}^1$ (i.e. the morphism is the projection of $t$). This shows that $\tilde{\mathcal{S}}, E$ are in fact isomorphic elliptic curves. In particular, $\mathrm{D}^b(\mathcal{S}, \mathcal{B}_0) \cong \mathrm{D}^b(\tilde{\mathcal{S}}) \cong \mathrm{D}^b(E)$.

Now, we consider the general Clifford mirrors of the product type built on two copies of this example. Let $M_1 = M_2, M_1^\vee = M_2^\vee$ and $K_1 = K_2$. Then $K, K^\vee$ constructed in (4.1), (4.2) are reflexive Gorenstein cones in rank 6 lattices. There are three different ways to express $\deg^\vee$ which give double mirrors,

$$\begin{aligned}\deg^\vee &= (t_1;\mathbf{0}) + (\mathbf{0};t_1)\\ &= (t_1;\mathbf{0}) + \frac{1}{2}\left((\mathbf{0};s_1) + (\mathbf{0};s_2)\right)\\ &= \frac{1}{2}\left((s_1;\mathbf{0}) + (s_2;\mathbf{0}) + (\mathbf{0};s_1) + (\mathbf{0};s_2)\right).\end{aligned}$$

The corresponding fans in these cases are regular can be derived from [FK17, Proposition 5.20]. The first expression corresponds to a Batyrev-Borisov complete intersection, which is a product of two elliptic curves $E \times E$; the third expression corresponds to the pure Clifford mirror $(\mathcal{S}_K, (\mathcal{B}_K)_0)$ explored in [BL18]. Their derived equivalence is a consequence of [BL18, Theorem 6.3]. The second expression exhibits the general Clifford mirror $(\mathcal{S}_{\text{gen}}, (\mathcal{B}_{\text{gen}})_0)$ considered in this paper, where the hypersurface $\mathcal{S}_{\text{gen}} = E$ defined by $(t_1;\mathbf{0})$ "parametrizes" pure Clifford mirrors $(\mathcal{S}_t, (\mathcal{B}_t)_0) \simeq (\mathcal{S}, \mathcal{B}_0)$. We had shown that $E$ and $(\mathcal{S}, \mathcal{B}_0)$ have equivalent derived categories. Therefore, it is reasonable to have the derived equivalence

$$\mathrm{D}^b(E \times E) \cong \mathrm{D}^b(\mathcal{S}_{\text{gen}}, (\mathcal{B}_{\text{gen}})_0) \cong \mathrm{D}^b(\mathcal{S}_K, (\mathcal{B}_K)_0)$$

which is the consequence of Theorem 3.13.

**Remark 4.2.** *In general, when the reflexive Gorenstein cone does not have above "direct sum" property, the corresponding noncommutative varieties are predictably more complicated. It is interesting to classify the combinatoric data for general Clifford mirrors in low dimensions just as in the Batyrev-Borisov case. Moreover, it is desirable to use the above method to construct examples as [Căl02], where an elliptic threefold without a section is derived equivalent to the twisted derived category of its (small resolution of) relative Jacobian. The twist there should be replaced by the even part of sheaf of Clifford algebras $\mathcal{B}_0$.*

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## References

[Add09] Nicolas Addington. The derived category of the intersection of four quadrics. *arXiv:0904.1764*, 2009.

[BB96] Victor V. Batyrev and Lev A. Borisov. Mirror duality and string-theoretic Hodge numbers. *Invent. Math.*, 126(1):183–203, 1996. MR1408560

[BB97] Victor V. Batyrev and Lev A. Borisov. Dual cones and mirror symmetry for generalized Calabi-Yau manifolds. In *Mirror symmetry, II*, volume 1 of *AMS/IP Stud. Adv. Math.*, pages 71–86. Amer. Math. Soc., Providence, RI, 1997. MR1416334

[BCS05] Lev A. Borisov, Linda Chen, and Gregory G. Smith. The orbifold Chow ring of toric Deligne-Mumford stacks. *J. Amer. Math. Soc.*, 18(1):193–215 (electronic), 2005. MR2114820

[BDF+18] Matthew Ballard, Dragos Deliu, David Favero, M Umut Isik, and Ludmil Katzarkov. On the derived categories of degree $d$ hypersurface fibrations. *Math. Ann.*, 371(1):337–370, 2018. MR3788850

[BFK19] Matthew Ballard, David Favero, and Ludmil Katzarkov. Variation of geometric invariant theory quotients and derived categories. *J. Reine Angew. Math.*, 2019(746):235–303, 2019. MR3895631

[BL18] Lev Borisov and Zhan Li. On Clifford double mirrors of toric complete intersections. *Adv. Math.*, 328:300–355, 2018. MR3771131

[BN08] Victor Batyrev and Benjamin Nill. Combinatorial aspects of mirror symmetry. In *Integer points in polyhedra—geometry, number theory, representation theory, algebra, optimization, statistics*, volume 452 of *Contemp. Math.*, pages 35–66. Amer. Math. Soc., Providence, RI, 2008. MR2405763

[Bor13] Lev A. Borisov. Berglund-Hübsch mirror symmetry via vertex algebras. *Comm. Math. Phys.*, 320(1):73–99, 2013. MR3046990

[BW19] Lev Borisov and Chengxi Wang. On stringy Euler characteristics of Clifford non-commutative varieties. *Adv. Math.*, 349:1117–1150, 2019. MR3945588

[Căl02] Andrei Căldăraru. Derived categories of twisted sheaves on elliptic threefolds. *J. Reine Angew. Math.*, 544:161–179, 2002. MR1887894

[CDH+10] Andrei Căldăraru, Jacques Distler, Simeon Hellerman, Tony Pantev, and Eric Sharpe. Non-birational twisted derived equivalences in abelian GLSMs. *Comm. Math. Phys.*, 294(3):605–645, 2010. MR2585982

[CK99] David A. Cox and Sheldon Katz. *Mirror symmetry and algebraic geometry*, volume 68 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999. MR1677117

[CKP15] Tom Coates, Alexander Kasprzyk, and Thomas Prince. Four-dimensional Fano toric complete intersections. *Proceedings of the Royal Society A*, 471(2175), 2015. MR3303391

[Cla14] Patrick Clarke. A proof of the birationality of certain BHK-mirrors. *Complex Manifolds*, 1:45–51, 2014. MR3333467

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124. American Mathematical Soc., 2011. MR2810322

[CR11] Alessandro Chiodo and Yongbin Ruan. LG/CY correspondence: the state space isomorphism. *Adv. Math.*, 227(6):2157–2188, 2011. MR2807086

[CT15] John R. Calabrese and Richard P. Thomas. Derived equivalent Calabi–Yau threefolds from cubic fourfolds. *Math. Ann.*, pages 1–18, 2015. MR3498907

[DH16] Charles F. Doran and Andrew Harder. Toric degenerations and the Laurent polynomials related to Givental's Landau-Ginzburg models. *Can. J. Math.*, 68(4):784–815, 2016. MR3518994

[EP15] Alexander I. Efimov and Leonid Positselski. Coherent analogues of matrix factorizations and relative singularity categories. *Algebra Number Theory*, 9(5):1159–1292, 2015. MR3366002

[FK17] David Favero and Tyler L Kelly. Proof of a conjecture of Batyrev and Nill. *Amer. J. Math.*, 139(6):1493–1520, 2017. MR3730928

[FK19] David Favero and Tyler L. Kelly. Derived categories of BHK mirrors. *Adv. Math.*, 352:943–980, 2019. MR3975030

[Hir17] Yuki Hirano. Derived Knörrer periodicity and Orlov's theorem for gauged Landau-Ginzburg models. *Compos. Math.*, 153(05):973–1007, 2017. MR3631231

[HL15] Daniel Halpern-Leistner. The derived category of a GIT quotient. *J. Amer. Math. Soc.*, 28(3):871–912, 2015. MR3327537

[HW12] Manfred Herbst and Johannes Walcher. On the unipotence of autoequivalences of toric complete intersection Calabi-Yau categories. *Math. Ann.*, 353(3):783–802, 2012. MR2923950

[Isi13] Mehmet Umut Isik. Equivalence of the derived category of a variety with a singularity category. *Int. Math. Res. Not.*, (12):2787–2808, 2013. MR3071664

[Kel13] Tyler L. Kelly. Berglund-Hübsch-Krawitz mirrors via Shioda maps. *Adv. Theor. Math. Phys.*, 17(6):1425–1449, 2013. MR3262528

[Kon95] Maxim Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 120–139. Birkhäuser, Basel, 1995. MR1403918

[Kuz08] Alexander Kuznetsov. Derived categories of quadric fibrations and intersections of quadrics. *Adv. Math.*, 218(5):1340–1369, 2008. MR2419925

[Li16] Zhan Li. On the birationality of complete intersections associated to nef-partitions. *Adv. Math.*, 299 (2016):71–107, 2016. MR3519464

[Muk88] Shigeru Mukai. Moduli of vector bundles on K3 surfaces, and symplectic manifolds. *Sugaku Expositions*, 1(2):138–174, 1988. MR0922020

[Mul09] Joshua Mullet. Toric Calabi-Yau hypersurfaces fibered by weighted K3 hypersurfaces. *Commun. Anal. Geom.*, 17:107–138, 2009. MR2495835

[Orl09] Dmitri Orlov. Derived categories of coherent sheaves and triangulated categories of singularities. In *Algebra, Arithmetic, and Geometry: In Honor of Yu. I. Manin. Vol. II*, volume 270 of *Progr. Math.*, pages 503–531. Birkhäuser Boston, Inc., Boston, MA, 2009. MR2641200

[Pos11] Leonid Positselski. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. *Mem. Amer. Math. Soc.*, 212(996):vi+133, 2011. MR2830562

[Shi12] Ian Shipman. A geometric approach to Orlov's theorem. *Compos. Math.*, 148(5):1365–1389, 2012. MR2982435

[Sho14] Mark Shoemaker. Birationality of Berglund-Hübsch-Krawitz mirrors. *Comm. Math. Phys.*, 331(2):417–429, 2014. MR3238520

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