

Motivic Galois coaction and one-loop Feynman graphs

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Following the work of Brown, we can canonically associate a family of motivic periods — called the motivic Feynman amplitude — to any convergent Feynman integral, viewed as a function of the kinematic variables. The motivic Galois theory of motivic Feynman amplitudes provides an organizing principle, as well as strong constraints, on the space of amplitudes in general, via Brown’s “small graphs principle”. This serves as motivation for explicitly computing the motivic Galois action, or, dually, the coaction of the Hopf algebra of functions on the motivic Galois group. In this paper, we study the motivic Galois coaction on the motivic Feynman amplitudes associated to one-loop Feynman graphs. We study the associated variations of mixed Hodge structures, and provide an explicit formula for the coaction on the four-edge cycle graph — the box graph — with non-vanishing generic kinematics, which leads to a formula for all one-loop graphs with non-vanishing generic kinematics in four-dimensional space-time. We also show how one computes the coaction in some degenerate configurations — when defining the motive of the graph requires blowing up the underlying family of varieties — on the example of the three-edge cycle graph.

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1. Introduction

1.1. Context

Integrals we will be interested in are those of the form

$$(1) \quad \int_{\sigma} \omega_G(m, q),$$

where

$$(2) \quad \omega_G(m, q) = \frac{1}{\Psi_G^{d/2}} \left(\frac{\Psi_G}{\Xi_G} \right)^{N_G - h_G d/2} \Omega_G,$$

G is a Feynman graph, Ψ_G , and $\Xi_G(m, q)$ are certain polynomials in the variables α_e indexed by the edges of G , d , N_G , and h_G are fixed integers,

$$\Omega_G = \sum_{i=1}^{N_G} (-1)^i \alpha_i d\alpha_1 \wedge \dots \wedge \widehat{d\alpha_i} \wedge \dots \wedge d\alpha_{N_G},$$

and the domain of integration σ is given by the real points of the coordinate simplex $\alpha_e \geq 0$. In physics, polynomials Ψ_G and $\Xi_G(m, q)$ are called *Symanzik polynomials*, and are sometimes also denoted by \mathcal{U} and \mathcal{F} , respectively (see Definition 3). For the relationship of (1) with the Feynman integral associated to the graph G see (15) and the discussion thereafter. The polynomial $\Xi_G(m, q)$ depends on parameters. For algebraic values of those parameters these integrals are, when they converge, *periods* in the sense of Kontsevich-Zagier [24]. Up to a factor, which is a special value of the gamma function, the integral (1) is the Feynman integral associated to the graph G , in parametric form.

The fact that Feynman integrals can be viewed as periods enables a motivic approach to studying interesting patterns in their evaluations, such as patterns of multiple zeta values studied by Bloch, Esnault and Kreimer in [9]. When viewed in this light, a new structure satisfied by Feynman integrals arises, as all periods conjecturally carry an action of the motivic Galois group [2]. Further study of this structure led to a very striking *coaction conjecture* [28, Conjecture 1.3], which states that the action of the motivic Galois group should be closed on motivic Feynman amplitudes of primitive log-divergent graphs in ϕ^4 theory, and moreover that the Galois conjugates of such Feynman integrals should be periods associated to subquotient graphs if one allows for non- ϕ^4 primitive log-divergent graphs. The

coaction conjecture is checked numerically therein for hundreds of examples, some of which have 11 loops. This, along with other evidence in different theories, e.g. [30], leads one to speculate that such a structure might exist more generally, possibly after enlarging the space of periods under consideration appropriately. An important reason for studying this structure is that any results of this type combined with easy results for small graphs provide very strong constraints on Feynman integrals to all loop orders. This is referred to as *the small graphs principle* [4, §8.4, §9.3].

We will be working in the category of realizations over a smooth base scheme over \mathbb{Q} , following [15, §1.21] and [4, §7.2]. In order to study the motivic Galois coaction we must first lift the Feynman integrals to *motivic periods* defined as functions on the scheme of isomorphisms between the de Rham and Betti fiber functors of the category of realizations. Moreover, we would like to work in a more general context than the one in [9, 28] by considering Feynman integrals as functions of masses and momenta of particles, which leads us to consider *families of motivic periods*. In [4], Brown provides a lifting of Feynman integrals to families of motivic periods, with mild constraints on the possible kinematics, thereby setting up the prerequisites for studying the Galois theory of a very general class of Feynman integrals. He also explains why we expect the Galois conjugates of motivic lifts of Feynman integrals to be motivic periods associated to subquotient graphs [4, Conjecture 1], and proves this in the “affine case” [4, Theorem 8.11].

In order to go further in this direction, we would like to understand in detail the Galois theory of a family of Feynman integrals where we allow non-trivial masses and momenta. It is a theorem due to Nickel [25] that one-loop integrals in four-dimensional space-time always evaluate to linear combinations of integrals associated to one-loop graphs with four edges, which in turn can be evaluated in terms of dilogarithms. From the perspective of algebraic geometry this was studied in [8]. It is shown there that the fact that these integrals evaluate to dilogarithms is a consequence of the fact that the geometry underlying these integrals carries a mixed Tate Hodge structure with weights 0, 2, and 4, which varies in a family over the space of kinematics. These structures are very well understood in algebraic geometry, and we use this here to study the motivic Galois theory of one-loop integrals with kinematic dependence.

1.2. Contents

In the first section we provide a brief overview of the technical background and terminology. We recall the definition of families of motivic periods which we will be working with, as well as the definition of families of de Rham periods, and the de Rham Galois group. The results will be stated in terms of the coaction of the Hopf algebra of functions on the de Rham Galois group. We also briefly recall the definition of the *motivic Feynman amplitude* $I_G^m(m, q)$, which is the family of motivic periods associated to the Feynman integral $I_G(m, q)$ of a Feynman graph G , depending on internal masses and external momenta, following Brown [4]. We will often drop (m, q) altogether in order to ease notation, but dependence on masses and momenta is assumed throughout the paper. The second section contains general results on the (realizations of) motives of one-loop graphs, and can be regarded as restating the results of [8] in terms of motivic periods. In particular it contains the reduction of the motivic Feynman amplitude of any graph with more than four edges to a linear combination of motivic Feynman amplitudes of four-edge graphs in four space-time dimensions, which amounts to lifting the result of Nickel to motivic periods. We also recall how one computes the semi-simplifications of the associated mixed Hodge structures. In the third section we compute the coaction on the motivic Feynman amplitude on the four-edge one-loop graph with non-vanishing generic kinematics, which gives us the coaction for any one-loop graph with non-vanishing generic kinematics by the results of the previous section. Bearing in mind the definition of the associated motivic Feynman amplitude I_G^m and its de Rham version I_G^{dr} (Definition 9), the definition of the Galois coaction (6), as well as the definitions of the de Rham logarithm $\log^{\text{dr}}(x)$ and the Lefschetz de Rham period \mathbb{L}^{dr} (see 2.4.2), the main result of the third section is the following:

Theorem 1. *Let G be a one-loop Feynman graph with 4 edges (Figure 1), with generic non-vanishing masses and momenta. Let I_G^m be its associated motivic Feynman amplitude in $d = 4$ dimensions. Then the motivic Galois coaction on I_G^m is:*

$$(3) \quad \begin{aligned} \Delta I_G^m &= \sum_{1 \leq j < k \leq 4} I_{G/\{e_j, e_k\}}^m(\theta_{G/\{e_j, e_k\}}^1) \otimes P_{j,k} \log^{\text{dr}}(f_{j,k}) \mathbb{L}^{\text{dr}} + \\ &+ I_G^m \otimes (\mathbb{L}^{\text{dr}})^2 + 1 \otimes I_G^{\text{dr}} \end{aligned}$$

where $I_{G/\{e_j, e_k\}}^m \left(\theta_{G/\{e_j, e_k\}}^1 \right)$ is the motivic Feynman amplitude of the bubble

graph obtained by contracting the edges e_j and e_k of G in $d = 2$ dimensions,

$$f_{j,k} = \frac{\sqrt{(U)_{j,k}^2 - (U)_{j,j}(U)_{k,k} - (U)_{j,k}}}{\sqrt{(U)_{j,k}^2 - (U)_{j,j}(U)_{k,k} + (U)_{j,k}}},$$

where $C, D_{j,k}$ are the matrices of the quadratic forms $\Xi_G, \Xi_{G/\{e_j, e_k\}}$ respectively (see (39) and (34)), $U = C^{-1}$, and $P_{j,k} = \frac{\sqrt{|\det D_{j,k}|}}{8\sqrt{|\det C|}}$.

The motivic periods in the coaction in Theorem 1 are identified with motivic periods of quotient graphs via certain natural homomorphisms in the category of realizations called *the face maps*, while the de Rham periods are determined by using the Gauss-Manin connection. In section 5, in which we study the triangle graph, another approach is used to determine the de Rham periods — the residue homomorphism.

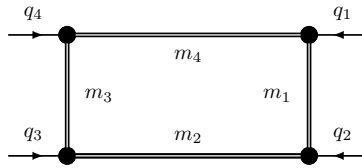


Figure 1: The box graph. Internal edges which have non-vanishing mass are denoted by double lines.

The coaction commutes with specialization to a point in the open subset of the space of kinematics over which the graph motive is defined. For closed subsets outside of the space of generic kinematics, such as those given by the vanishing of some masses, to which Theorem 1 therefore does not apply, one can still apply the methods presented in this paper to compute the Galois coaction, so long as the values of masses and momenta are such that $I_G(m, q)$ converges. However, one should bear in mind that in such cases it might not be possible to interpret the conjugates in the coaction in terms of motivic periods of motives of subquotient graphs. Understanding this subtlety motivates a more detailed study of a couple of special cases in the next section, in particular Theorem 2.

In the fourth section we will concern ourselves with the three-edge graph. The result for the case with generic non-vanishing kinematics is quite similar to the box graph case above, but a few special cases when some of the masses vanish reveal subtleties. The result in one of these special cases is the following:

Theorem 2. *Let G be the Feynman graph with 3 edges and 1 loop, with all internal masses vanishing and non-trivial external momenta. Denote the three external momenta by q_1, q_2, q_3 , and the associated motivic Feynman amplitude by I_G^m . Then the coaction on this motivic Feynman amplitude is:*

$$\begin{aligned} \Delta I_G^m &= a_1 \log^m \left(\frac{q_2^2}{q_3^2} \right) \otimes b_1 \log^{\text{dr}}(f_1) + a_2 \log^m \left(\frac{q_1^2}{q_3^2} \right) \otimes b_2 \log^{\text{dr}}(f_1) + \\ &+ a_3 \log^m \left(\frac{q_2^2}{q_3^2} \right) \otimes b_3 \log^{\text{dr}}(f_2) + a_4 \log^m \left(\frac{q_1^2}{q_3^2} \right) \otimes b_4 \log^{\text{dr}}(f_2) + \\ &+ I_G^m \otimes (\mathbb{L}^{\text{dr}})^2 + 1 \otimes I_G^{\text{dr}} \end{aligned}$$

where

$$f_1 = \frac{q_3^2 - q_2^2 - q_1^2 - \sqrt{q_1^4 + q_2^4 + q_3^4 - 2q_1^2q_2^2 - 2q_2^2q_3^2}}{q_3^2 - q_2^2 - q_1^2 + \sqrt{q_1^4 + q_2^4 + q_3^4 - 2q_1^2q_2^2 - 2q_2^2q_3^2}}$$

and

$$f_2 = f_1 - \left(\frac{q_3^2 + q_2^2 - q_1^2 - \sqrt{q_1^4 + q_2^4 + q_3^4 - 2q_1^2q_2^2 - 2q_2^2q_3^2}}{q_3^2 + q_2^2 - q_1^2 + \sqrt{q_1^4 + q_2^4 + q_3^4 - 2q_1^2q_2^2 - 2q_2^2q_3^2}} \right)$$

for some undetermined constants $a_i, b_i \in k_S$.

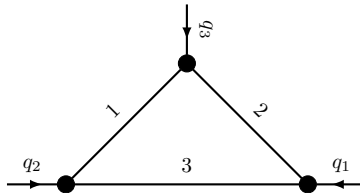


Figure 2: The triangle graph. Internal edges denoted with a single line have vanishing mass.

In the case of the previous theorem one cannot straightforwardly think of the motivic logarithms appearing in the coaction as motivic periods of the motives of quotient graphs of the triangle graph with all internal masses vanishing. This will be clear to physicists, who know that the Feynman integrals associated to the one-loop graphs with two edges and vanishing internal masses are divergent. In order to relate the motivic logarithms in the coaction in Theorem 2 to subquotient graphs, one must consider regularized versions of motivic periods of motives of subquotient graphs [4, Conjecture 1.].

A common regularization method in physics is *dimensional regularization*, which involves allowing the number of space-time dimensions to vary, and then considering the Laurent expansion around a fixed value of the dimension of space-time. The interplay of the coaction and dimensional regularization in the one-loop case has been considered in [1]. It is conjectured therein that after dimensionally regularizing, the coaction can indeed be expressed in terms of subquotient graphs on the left hand side of the tensor product. Moreover, it is conjectured that the right hand side of the tensor product should correspond to Cutkosky cuts of the original graph.

Alternatively, an approach taken in [4] is to associate to each graph a larger motive, called the *affine motive* [4, §8.5], which is then proved to capture the motivic periods in the coaction applied to the motivic lift of the Feynman integral associated to the given graph [4, Theorem 8.11].

We note that we aim to make the computation of the coaction involving masses and momenta as explicit as possible. For that reason some moderately lengthy computations and bulky notation can be found in the second half of this paper.

2. Brief overview of the technical background

We recall the definitions, constructions and main results on motivic periods and motivic Feynman amplitudes that we use throughout the article. This is a summary of the results of Brown [3, 4].

2.1. Category of realizations

Let S be a smooth geometrically connected scheme over \mathbb{Q} . And let $\mathcal{H}(S)$ be the category of triples $(\mathbb{V}_B, (\mathcal{V}_{\text{dR}}, \nabla), c)$, where \mathbb{V}_B is a local system of finite dimensional \mathbb{Q} -vector spaces on $S(\mathbb{C})$, and $(\mathcal{V}_{\text{dR}}, \nabla)$ is an algebraic vector bundle on S with a flat connection with regular singularities at infinity. Note that for the definition of regular singularities at infinity one needs a good compactification of S , but it does not depend on the choice of a good compactification [22]. Furthermore \mathbb{V}_B is equipped with an increasing *weight* filtration $W_\bullet \mathbb{V}_B$ of local sub-systems, and $(\mathcal{V}_{\text{dR}}, \nabla)$ is equipped by an increasing *weight* filtration by algebraic sub-vector bundles with a flat connection with regular singularities at infinity $W_\bullet \mathcal{V}_{\text{dR}}$ and a decreasing *Hodge* filtration by algebraic sub-vector bundles $F^\bullet \mathcal{V}_{\text{dR}}$ which satisfy Griffiths transversality

$$\nabla : F^p \mathcal{V}_{\text{dR}} \rightarrow F^{p-1} \mathcal{V}_{\text{dR}} \otimes_{\mathcal{O}_S} \Omega_{S/k}^1.$$

Finally, $c : (\mathcal{V}_{\mathrm{dR}}, \nabla)^{an} \xrightarrow{\sim} \mathbb{V}_{\mathbb{B}} \otimes_{\mathbb{Q}} \mathcal{O}_{S^{an}}$ is an isomorphism of analytic vector bundles with connection which respects the weight filtration, where the connection on the right hand side is one for which the sections of $\mathbb{V}_{\mathbb{B}}$ are flat. We further require that $\mathbb{V}_{\mathbb{B}}$, equipped with the filtrations W and cF , is a graded-polarizable variation of mixed Hodge structures. The morphisms in $\mathcal{H}(S)$ respect the above data.

The category $\mathcal{H}(S)$ is Tannakian, see [14]. It is equipped with the following fiber functors. For any simply connected $Z \subset S(\mathbb{C})$ let

$$\omega_B^Z : \mathcal{H}(S) \rightarrow \mathrm{Vec}_{\mathbb{Q}}$$

be the fiber functor which sends a triple $(\mathbb{V}_{\mathbb{B}}, (\mathcal{V}_{\mathrm{dR}}, \nabla), c)$ to the \mathbb{Q} -vector space of sections $\Gamma(Z, \mathbb{V}_{\mathbb{B}})$. We also consider the fiber functor

$$\omega_{\mathrm{dR}}^{\mathrm{gen}} : \mathcal{H}(S) \rightarrow \mathrm{Vec}_{k_S}$$

where k_S is the function field of S . It sends the vector bundle $(\mathcal{V}_{\mathrm{dR}}, \nabla)$ to its fiber over the generic point of S .

2.2. Families of $\mathcal{H}(S)$ -periods

The *ring of $\mathcal{H}(S)$ -periods* is defined as:

$$\mathcal{P}_{\mathcal{H}(S)}^{\mathrm{m}, Z, \mathrm{gen}} := \mathcal{O}(\mathrm{Isom}_{\mathcal{H}(S)}^{\otimes}(\omega_{\mathrm{dR}}^{\mathrm{gen}}, \omega_B^Z)).$$

It is a (\mathbb{Q}, k_S) -bimodule generated by equivalence classes of triples

$$((\mathbb{V}_{\mathbb{B}}, (\mathcal{V}_{\mathrm{dR}}, \nabla), c), \sigma, \omega),$$

where $\mathcal{V} := (\mathbb{V}_{\mathbb{B}}, (\mathcal{V}_{\mathrm{dR}}, \nabla), c) \in \mathrm{ob}(\mathcal{H}(S))$, $\sigma \in (\omega_B^Z \mathcal{V})^{\vee}$, and $\omega \in \omega_{\mathrm{dR}}^{\mathrm{gen}} \mathcal{V}$, modulo the relations:

$$(4) \quad \begin{aligned} (\mathcal{V}, \lambda_1 \sigma_1 + \lambda_2 \sigma_2, \omega) &\sim \lambda_1 (\mathcal{V}, \sigma_1, \omega) + \lambda_2 (\mathcal{V}, \sigma_2, \omega) \\ (\mathcal{V}, \sigma, \gamma_1 \omega_1 + \gamma_2 \omega_2) &\sim \gamma_1 (\mathcal{V}, \sigma, \omega_1) + \gamma_2 (\mathcal{V}, \sigma, \omega_2), \end{aligned}$$

where $\lambda_1, \lambda_2 \in \mathbb{Q}$, $\gamma_1, \gamma_2 \in k_S$. Furthermore, for any morphism $\phi : \mathcal{V}_1 \rightarrow \mathcal{V}_2$ in the category $\mathcal{H}(S)$ we have:

$$(\mathcal{V}_1, (\omega_B^X(\phi))^t(\sigma_2), \omega_1) \sim (\mathcal{V}_2, \sigma_2, \omega_{\mathrm{dR}}^{\mathrm{gen}}(\phi)(\omega_1)),$$

where $(\omega_B^X(\phi))^t$ denotes the dual morphism to $\omega_B^X(\phi)$. We denote the equivalence classes of triples by $[(\mathbb{V}_{\mathbb{B}}, (\mathcal{V}_{\mathrm{dR}}, \nabla), c), \sigma, \omega]^{\mathrm{m}}$.

The ring $\mathcal{P}_{\mathcal{H}(S)}^{m,Z,gen}$ is equipped with a *period homomorphism*:

$$(5) \quad \text{per} : \mathcal{P}_{\mathcal{H}(S)}^{m,Z,gen} \rightarrow M_Z(S(\mathbb{C})),$$

where $M_Z(S(\mathbb{C}))$ denotes the ring of multivalued meromorphic functions on $S(\mathbb{C})$ with a prescribed branch along Z . To define it let $\pi : \tilde{S}(\mathbb{C})_Z \rightarrow S(\mathbb{C})$ be the universal cover of $S(\mathbb{C})$ based at Z (recall that Z is simply connected). Since $\tilde{S}(\mathbb{C})_Z$ is simply connected the local system $\pi^*(\mathbb{V}_B^\vee)$ is trivial, and $\sigma \in \Gamma(Z, \mathbb{V}_B^\vee)$ extends to a unique global section $\Gamma(\tilde{S}(\mathbb{C})_Z, \pi^*(\mathbb{V}_B^\vee))$. Let $x \in \tilde{S}(\mathbb{C})_Z$, and $N(x)$ be a small enough neighbourhood of x so that the restriction of π is an isomorphism. Define a local section

$$\sigma_x = (\pi|_{N(x)}^{-1})^* \sigma \in \Gamma(\pi(N(x)), \mathbb{V}_B^\vee).$$

Since $\omega \in \Gamma(\text{Spec}(k_S), \mathcal{V}_{\text{dR}})$, there exists a non-empty affine open $U \subset S$ such that $\omega|_U \in \Gamma(U, \mathcal{V}_{\text{dR}})$. Let $W \subset S$ be an affine open such that $\pi(x) \in W(\mathbb{C})$, and furthermore we can make $\pi(N(x)) \in W(\mathbb{C})$ by making $N(x)$ smaller if necessary. Since S is irreducible $U \cap W \neq \emptyset$. The section $\omega|_{U \cap W}$ can have poles along $W \setminus U$, but we can ‘clear denominators’, i.e., there is $\alpha \in \mathcal{O}_W$ such that $\alpha\omega \in \Gamma(W, \mathcal{V}_{\text{dR}})$. By restriction, and passing to the associated analytic vector bundle we can view $\alpha\omega$ as an element in $\Gamma(\pi(N(x)), \mathcal{V}_{\text{dR}}^{an})$. Then the comparison isomorphism $c : \mathcal{V}_{\text{dR}}^{an} \rightarrow \mathbb{V}_B \otimes_{\mathbb{Q}} \mathcal{O}_{S^{an}}$ gives an element:

$$\text{per}([\mathbb{V}_B, (\mathcal{V}_{\text{dR}}, \nabla), c], \sigma, \alpha\omega]^m) = \sigma_x(c(\alpha\omega)) \in \Gamma(\pi(N(x)), \mathcal{O}_{S^{an}}),$$

which can be viewed as a locally analytic function on $N(x)$. The period homomorphism is defined on $[\mathbb{V}_B, (\mathcal{V}_{\text{dR}}, \nabla), c], \sigma, \omega]^m$ by dividing by the function α . It locally has poles along the zeroes of α .

This ring comes equipped with a *connection*

$$\nabla : \mathcal{P}_{\mathcal{H}(S)}^{m,Z,gen} \rightarrow \mathcal{P}_{\mathcal{H}(S)}^{m,Z,gen} \otimes_{k_S} \Omega_{k_S/k}^1$$

which acts on families of motivic periods by:

$$\nabla[\mathcal{V}, [\sigma], [\omega]]^m = [\mathcal{V}, [\sigma], \nabla[\omega]]^m.$$

For details see [3, §7.4].

2.3. Families of de Rham periods and coaction

We define the ring of *de Rham periods* as:

$$\mathcal{P}_{\mathcal{H}(S)}^{\text{dr},gen} := \mathcal{O}(\text{Aut}_{\mathcal{H}(S)}^{\otimes}(\omega_{\text{dR}}^{gen})).$$

It is spanned, over k_S , by equivalence classes of triples

$$[(\mathbb{V}_B, (\mathcal{V}_{\text{dR}}, \nabla), c), v, \omega]^{\text{dr}},$$

where $v \in (\omega_{\text{dR}}^{gen} \mathcal{V})^{\vee}$, and $\omega \in \omega_{\text{dR}}^{gen} \mathcal{V}$, defined analogously to $\mathcal{H}(S)$ -periods. Furthermore, $\mathcal{P}_{\mathcal{H}(S)}^{\text{dr},Z,gen}$ is a Hopf algebra, and the ring $\mathcal{P}_{\mathcal{H}(S)}^{\text{m},Z,gen}$ has a right *Galois coaction* by $\mathcal{P}_{\mathcal{H}(S)}^{\text{dr},Z,gen}$:

$$\Delta : \mathcal{P}_{\mathcal{H}(S)}^{\text{m},Z,gen} \rightarrow \mathcal{P}_{\mathcal{H}(S)}^{\text{m},Z,gen} \otimes_{k_S} \mathcal{P}_{\mathcal{H}(S)}^{\text{dr},gen},$$

given by the formula:

$$(6) \quad \Delta[\mathcal{V}, \sigma, \omega]^{\text{m}} = \sum_{e_i} [\mathcal{V}, \sigma, e_i]^{\text{m}} \otimes [\mathcal{V}, e_i^{\vee}, \omega]^{\text{dr}}$$

where the $\{e_i\}$ is a basis of $\omega_{\text{dR}}^{gen} \mathcal{V}$, and e_i^{\vee} is the dual basis. Dual to this is the left *Galois action* of the affine group scheme $G_{\mathcal{H}(S)}^{\text{dr},gen} = \text{Spec}(\mathcal{P}_{\mathcal{H}(S)}^{\text{dr},gen})$:

$$G_{\mathcal{H}(S)}^{\text{dr},gen} \times \mathcal{P}_{\mathcal{H}(S)}^{\text{m},Z,gen} \rightarrow \mathcal{P}_{\mathcal{H}(S)}^{\text{m},Z,gen},$$

given by

$$(7) \quad g[\mathcal{V}, \sigma, \omega]^{\text{m}} = (1 \otimes g)\Delta[\mathcal{V}, \sigma, \omega]^{\text{m}} = \sum_{e_i} [\mathcal{V}, \sigma, e_i]^{\text{m}} \otimes g([\mathcal{V}, e_i^{\vee}, \omega]^{\text{dr}}),$$

where $g \in G_{\mathcal{H}(S)}^{\text{dr},gen}(R)$, for R a k_S -algebra.

2.4. Motivic periods

We are particularly interested in objects of $\mathcal{H}(S)$ of a prescribed geometric origin. By this we mean the following. Recall that a simple normal crossing divisor is a normal crossing divisor such that each of its irreducible components is smooth. Assume, as before, that S is a smooth geometrically connected scheme over \mathbb{Q} , and let $D \subset X$ be a family of simple normal

crossing divisors relative to a smooth morphism $\pi : X \rightarrow S$ which, on the underlying analytic varieties, we assume to be a locally trivial fibration of stratified analytic varieties for the stratification on X induced by D – see [21] and [29, Ch. IV]. Let $j : X \setminus D \hookrightarrow X$ be the inclusion, and denote by D_i the irreducible components of D , for $i \in I$. Denote by $D_J = \cap_{j \in J} D_j$, for $\emptyset \neq J \subset I$, and let $D_\emptyset = X$.

Because π is a locally trivial fibration, the sheaf

$$(8) \quad H_{\mathbb{B}}^n(X, D)_{/S} := R^n \pi_* j_! \mathbb{Q},$$

where \mathbb{Q} is the constant sheaf on $(X \setminus D)(\mathbb{C})$, is a local system over $S(\mathbb{C})$ with its analytic topology. It is computed by the hypercohomology of the complex of sheaves:

$$(9) \quad \mathbb{Q}_{D_\bullet/S} : \mathbb{Q}_X \rightarrow \bigoplus_{|J|=1} \mathbb{Q}_{D_J} \rightarrow \bigoplus_{|J|=2} \mathbb{Q}_{D_J} \rightarrow \dots$$

where \mathbb{Q}_{D_J} is the constant sheaf \mathbb{Q} on $D_J(\mathbb{C})$ extended by 0 to $X(\mathbb{C})$.

Let $\Omega_{D_\bullet/S}^\bullet$ denote the sheaf on X which is the direct image of the corresponding sheaves of Kähler differentials on D_J , and which vanishes outside of D_J . Consider the double complex of sheaves on X

$$(10) \quad \Omega_{D_\bullet/S}^\bullet : \Omega_{X/S}^\bullet \rightarrow \bigoplus_{|J|=1} \Omega_{D_J/S}^\bullet \rightarrow \dots \rightarrow \bigoplus_{|J|=d} \Omega_{D_J/S}^\bullet$$

where d is the relative dimension of X over S , and horizontal maps are pullbacks along inclusions $D_J \hookrightarrow D_{J \setminus i_j}$ multiplied by $(-1)^k$ if i_j is the k th element of J . We will denote such restrictions, including the signs, by $r_{i_j}^J$ and write simply r_{i_j} when $J = \{i_j\}$. Then define

$$H_{\text{dR}}^n(X, D)_{/S} = \mathbb{R}^n \pi_*(\text{Tot}^\bullet(\Omega_{D_\bullet/S}^\bullet)),$$

where Tot^\bullet denotes the total complex. It has a flat connection

$$\nabla : H_{\text{dR}}^n(X, D)_{/S} \rightarrow H_{\text{dR}}^n(X, D)_{/S} \otimes \Omega_{S/k}^1$$

by a relative version of [23].

By [18, Proposition 2.28], using the assumption that π is topologically trivial, we have an isomorphism:

$$c^{-1} : H_{\mathbb{B}}^n(X, D)_{/S} \otimes_{\mathbb{Q}} \mathcal{O}_S^{an} \xrightarrow{\sim} (H_{\text{dR}}^n(X, D)_{/S})^{an}.$$

A gap in the literature seems to be that $H_{\mathbb{B}}^n(X, D)_{/S}$, with its weight filtration and a Hodge filtration coming from $cH_{\text{dR}}^n(X, D)_{/S}$, should be a variation of mixed Hodge structure. Putting everything together, and admitting the above stated claims, we get an element

$$(11) \quad (H_{\mathbb{B}}^n(X, D)_{/S}, H_{\text{dR}}^n(X, D)_{/S}, c)$$

of the category $\mathcal{H}(S)$, which we denote $H^n(X, D)_{/S}$. We refer to the $\mathcal{H}(S)$ -periods associated to such objects of $\mathcal{H}(S)$ as *families of motivic periods*.

For the purposes of this paper we will define explicitly an open S as a complement of a certain closed subset of an irreducible affine algebraic variety over which $H_{\mathbb{B}}^n(X, D)_{/S}$ will be a local system, and we will work with a family of divisors $D \subset X$ which is simple normal crossing over the generic point of the said irreducible affine algebraic variety. Therefore we will only consider the double complex (10) over the generic point.

Definition 1. Let $D^I = \cup_{j \notin I} D_j$, and $k = |I|$. *Face maps* are morphisms in the category $\mathcal{H}(S)$

$$(12) \quad H^{n-k}(D_I, D^I \cap D_I)_{/S} \rightarrow H^n(X, D)_{/S},$$

defined by the inclusions of complexes

$$\Omega_{D^I_{/S}}^{\bullet-k} \rightarrow \Omega_{D_{/S}}^{\bullet} \quad \text{and} \quad \mathbb{Q}_{D^I_{/S}}[-k] \rightarrow \mathbb{Q}_{D_{/S}},$$

on the de Rham and Betti realizations respectively.

2.4.1. Mixed Tate Hodge structures In this paper we will focus on a special case of families of motivic periods coming from objects of $\mathcal{H}(S)$ associated to variations of mixed Tate Hodge structures. To make this precise we recall a few definitions. A *Tate Hodge structure* $\mathbb{Q}(-m)$ is the pure Hodge structure of weight $2m$ defined by

$$H_{\mathbb{Q}} = (2\pi i)^{-m} \mathbb{Q}, \quad H_{\mathbb{C}} = H^{m,m}(\mathbb{Q}(-m)_{\mathbb{C}}).$$

We can pull back $\mathbb{Q}(-m)$ to S via the structure map $S \rightarrow \text{Spec}(\mathbb{Q})$ to obtain a constant *variation of Tate structure*, denoted $\mathbb{Q}(-m)_{/S}$.

A mixed Tate Hodge structures are mixed Hodge structures H such that the weight graded quotients are $gr_{2m}^W H = \bigoplus \mathbb{Q}(-m)$ and $gr_{2m+1}^W H = 0$. They are extensions of Tate Hodge structures. We will be working with objects of $\mathcal{H}(S)$ such that their fibers over $S(\mathbb{C})$ are mixed Tate Hodge

structures. In this particular case, the choice of terminology where we refer to “motivic periods” is justified by the fact that the Hodge realization functor is fully faithful on mixed Tate motives over number fields [19].

2.4.2. Examples Let $S = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $X = S \times \mathbb{G}_m$, and $\pi : X \rightarrow S$ the projection. Let x be the coordinate on S and y a coordinate on \mathbb{G}_m . Define $D = \{y = 1\} \cup \{y = x\}$. We consider the object

$$\mathcal{V} = (H_{\mathbb{B}}^1(X, D)_{/S}, H_{\text{dR}}^1(X, D)_{/S}, c) \in \mathcal{H}(S).$$

We can choose Z to be the real interval $(0, 1)$ of $S(\mathbb{C})$, and define for all $x \in Z$ a cycle $\sigma_x \in \mathbb{G}_m(\mathbb{C})$ — a straight line path from 1 to x , which defines a class in $(\omega_B^Z(H^1(X, D)_{/S}))^\vee$. Note that in this case the differential form $\frac{dy}{y}$ defines an element of $\Gamma(S, (H_{\text{dR}}^1(X, D)_{/S}))$, and we do not have to restrict to working over the generic point only. We define the motivic logarithm as a family of motivic periods

$$\log^m(x) = \left[\mathcal{V}, [\sigma_x], \left[\frac{dy}{y} \right] \right]^m \in \mathcal{P}_{\mathcal{H}(S)}^{m, Z, \text{gen}}$$

Its period is the logarithm

$$\text{per}(\log^m(x)) = \int_{\sigma_x} \frac{dy}{y} = \log(x)$$

for $x \in Z$. Later in this paper we will consider logarithms over a higher dimensional, but still irreducible affine, base than $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, in which case we must extend the definition of the motivic logarithm given above. However note that we will always be given a subset Z of the complex points of our base over which the branch of the logarithm is unambiguous, and the definition is thus extended in an obvious way.

In order to lift $2\pi i$ to its motivic version it is enough to define it over $S = \text{Spec}(\mathbb{Q})$, and it can then be pulled back to a constant family over $S' \rightarrow S$ for some S' . Consider the object $H = (H_{\mathbb{B}}^1(\mathbb{G}_m), H_{\text{dR}}^1(\mathbb{G}_m), c) \in \text{ob}(\mathcal{H})$, where $\mathcal{H} = \mathcal{H}(\text{Spec}(\mathbb{Q}))$, and let γ_0 be a positively oriented circle around 0. We have that $[\gamma_0] \in H_{\mathbb{B}}^1$, and

$$\text{per} \left(\left[H, [\gamma_0], \left[\frac{dt}{t} \right] \right]^m \right) = \int_{\gamma_0} \frac{dt}{t} = 2\pi i$$

We denote

$$\mathbb{L}^m = \left[H, [\gamma_0], \left[\frac{dt}{t} \right] \right]^m \in \mathcal{P}_{\mathcal{H}}^m,$$

where we have dropped Z, gen from the notation since we are working over a point, and we refer to it as the *Lefschetz motivic period*. We denote the constant family of motivic periods obtained by pulling back \mathbb{L}^m to S via its structure morphism by the same symbol.

We will also consider the de Rham versions of these motivic periods. We choose a basis of the de Rham realization $\left\{ \left[\frac{dy}{y} \right], \left[\frac{dy}{1-x} \right] \right\}$, and denote its dual basis $\left\{ \left[\frac{dy}{y} \right]^\vee, \left[\frac{dy}{1-x} \right]^\vee \right\}$. For the de Rham logarithm define

$$\log^{\text{dr}}(x) = \left[\mathcal{V}, \left[\frac{dy}{1-x} \right]^\vee, \left[\frac{dy}{y} \right]^\vee \right]^{\text{dr}}.$$

This definition of the de Rham logarithm is motivated by it being the image of the motivic logarithm, defined earlier, under the *de Rham projection* – see [6, §4, Example 4.5.1]. For the Lefschetz de Rham period define:

$$(13) \quad \mathbb{L}^{\text{dr}} := \left[H, \left[\frac{dt}{t} \right]^\vee, \left[\frac{dt}{t} \right]^\vee \right]^{\text{dr}}$$

where H is as above. This can also be pulled back to a constant family of de Rham periods on S , which we denote by \mathbb{L}^{dr} as well.

2.5. Feynman integrals and their motivic lifts

2.5.1. Feynman integrals

Definition 2. A Feynman graph is a multigraph G , defined by a triple:

$$(V_G, E_G, E_G^{\text{ext}})$$

where V_G are the vertices, E_G is the set of internal edges of the graph which are not oriented, with the endpoints of each element of E_G encoded by a map $\partial : E_G \rightarrow \text{Sym}^2 V_G$, and E_G^{ext} is the set of external half-edges, with the endpoint of each element of E_G^{ext} encoded by a map $\partial : E_G^{\text{ext}} \rightarrow V_G$. We will only consider connected graphs.

To each internal edge $e \in E_G$ we assign its *particle mass* $m_e \in \mathbb{R}$. To each external edge $i \in E_G^{\text{ext}}$ we assign a *momentum*, which is a vector $q_i \in \mathbb{R}^d$ where $d \geq 0$ is the dimension of space-time. A condition on momenta $\sum_{i \in E_G^{\text{ext}}} q_i = 0$, called *momentum conservation*, is assumed. We write G/e_i for the graph with the edge e_i contracted, and G/e_I for the graph with the

edges $\{e_i\}_{i \in I}$ contracted. Denote by M the number of non-zero masses of a Feynman graph and by F the number of external momenta.

Definition 3. Let G be a Feynman graph. Associate to each internal edge $e \in E_G$ a variable α_e . Then the *first Symanzik polynomial* is a homogeneous polynomial defined to be

$$\Psi_G = \sum_{T \subset G} \prod_{e \notin T} \alpha_e$$

where the sum is over all spanning trees T of the graph G . We also define the following homogeneous polynomial

$$\Phi_G(q) = \sum_{T_1 \cup T_2 \subset G} (q^{T_1})^2 \prod_{e \notin T_1 \cup T_2} \alpha_e$$

where the sum ranges over all spanning 2-trees $T = T_1 \cup T_2$ of G . A spanning 2-tree of a graph G is a subgraph with 2 connected components, each of which is a tree, with the minimal number of edges such that its vertices include all of the vertices of the original graph G . We define $q^{T_1} = \sum_{i \in E_G^{ext}} q_i$ as the sum of all incoming momenta entering T_1 . By momentum conservation $q^{T_1} = -q^{T_2}$. We denote the Euclidean scalar product of the vector $q \in \mathbb{R}^d$ with itself by q^2 . The *second Symanzik polynomial*, also homogeneous, is then defined to be:

$$(14) \quad \Xi_G(m, q) = \Phi_G(q) + \left(\sum_{e \in E_G} m_e^2 \alpha_e \right) \Psi_G.$$

To abbreviate the dependencies of Ξ_G on momenta and masses in the above definitions we write $q := \{q_1, \dots, q_F\}$ and $m := \{m_1, \dots, m_M\}$.

Let G be a Feynman graph with N_G edges, h_G loops. In parametric form, the Feynman integral which is of interest in physics is the following projective integral:

$$(15) \quad \Gamma \left(N_G - h_G \frac{d}{2} \right) \int_{\sigma} \omega_G(m, q)$$

where

$$(16) \quad \omega_G(m, q) = \frac{1}{\Psi_G^{d/2}} \left(\frac{\Psi_G}{\Xi_G} \right)^{N_G - h_G d/2} \Omega_G$$

and Ψ_G , and Ξ_G are the first and second Symanzik polynomials of G , respectively.

The domain of integration is:

$$\sigma = \{[\alpha_1 : \dots : \alpha_{N_G}] : \alpha_i \geq 0\} \subset \mathbb{P}^{N_G-1}(\mathbb{R})$$

and

$$(17) \quad \Omega_G = \sum_{i=1}^{N_G} (-1)^i \alpha_i d\alpha_1 \wedge \dots \wedge \widehat{d\alpha_i} \wedge \dots \wedge d\alpha_{N_G}$$

where $\widehat{d\alpha_i}$ means that we omit $d\alpha_i$. The derivation of this form of the Feynman integral from its momentum space representation using the Schwinger trick is nicely explained in [27]. Also see [9].

Remark 1. It is important to note that the integral (15) can diverge, for example when $N_G = h_G \frac{d}{2}$, in which case the Γ prefactor will have a pole. It is crucial to make sense of this situation in physics, which is where the theory of renormalization comes in. For a detailed treatment of this, in the context of the parametric representation of Feynman integrals, which is suitable for an algebro-geometric approach see [7]. In this paper we will only consider the cases when (15) converges. We note, however, that in [4], it was already speculated that the theory of the cosmic Galois group should extend to the setting of [7].

Definition 4. We will consider the following projective integral:

$$(18) \quad I_G(m, q) = \int_{\sigma} \omega_G(m, q).$$

Therefore, since we are considering cases in which the (15) converges, we have simply dropped the prefactor which is a value of the Gamma function.

2.5.2. Motivic Feynman amplitudes In order to study their Galois theory we need to lift the functions $I_G(m, q)$ to families of motivic periods. Concretely, we need an element $mot_G \in \text{Ob}(\mathcal{H}(S))$, and a family of motivic periods

$$[mot_G, [\sigma], [\omega]]^m \in \mathcal{P}_{\mathcal{H}(S)}^{m, Z, gen},$$

for a certain $S \subset K_{F, M}^{gen}$ to be defined immediately below, and $Z \subset S(\mathbb{C})$ as in 2.4, such that

$$\text{per}([mot_G, [\sigma], [\omega]]^m) = I_G(m, q).$$

defines a multi-valued meromorphic function on $S(\mathbb{C})$.

The graph polynomial (14), and therefore the Feynman integral, are invariant under the action of the orthogonal group in d dimensions. Therefore they only depend on $s_{i,j} = s_{j,i} := q_i \cdot q_j$, the Euclidean product of $q_i, q_j, 1 \leq i \leq j \leq F$. Recall that they also satisfy momentum conservation by assumption. Furthermore, if a graph G has a vertex $v \in V$ such that the incoming momentum is non-trivial, i.e., $q^{\{v\}} \neq 0$, then the polynomial $\phi_G(q) \neq 0$ if $s_I := \sum_{i,j \in I} s_{i,j} \neq 0$ for all $I \subsetneq \{1, \dots, F\}$ [4, Lemma 1.12]. If the previous condition holds along with $s_I + m_j^2 \neq 0$ for all $I \subsetneq \{1, \dots, F\}$ and $j \in \{1, \dots, |E_G|\}$ then the polynomial $\Xi_G(m, q) = 0$ if and only if all internal masses and all external momenta are trivial [4, Lemma 1.13]. This leads to the following definition.

Definition 5. Define $K_{F,M} = \mathbb{A}^{\binom{F}{2}} \times \mathbb{G}_m^M$ to be the *space of kinematics*. Let $K_{F,M}^{gen} \subset K_{F,M}$ be the open complement of the union of spaces $s_I + m_j^2 = 0$, where $s_I = \sum_{i,j \in I} s_{i,j}$, for $I \subsetneq \{1, \dots, F\}$, and $j \in \{0, 1, \dots, M\}$, and we set $m_0 = 0$. We also define $U_{F,M}^{gen} \subset K_{F,M}^{gen}(\mathbb{C})$ to be the region in $K_{F,M}^{gen}(\mathbb{C})$ where $\text{Re}(s_I) > 0$, and $\text{Re}(m_j^2) > 0$, called the *extended Euclidean sheet*. Denote by $k_S = \text{Frac}(\mathcal{O}(K_{F,M}))$ the field of fractions of $\mathcal{O}(K_{F,M})$. Note that $k_S \cong \mathbb{Q}((s_{i,j})_{1 \leq i \leq j \leq F}, (m_k)_{1 \leq k \leq M})$.

The lifting of Feynman integrals to families of motivic periods is provided by Brown in [4] for Feynman graphs of any loop order. Key results for this lifting are certain factorization properties of Symanzik polynomials, which lead to the concept of *motivic subgraphs*. We recall the definition.

Definition 6. A subgraph γ is called *mass-momentum spanning* if it contains all internal edges $e \in E_\gamma$ which have non-vanishing mass m_e , $\partial E_G^{ext} \subset V_\gamma$, and all the vertices ∂E_G^{ext} all lie in a single connected component of γ .

A subgraph spanned by the edges $\gamma \subset E_G$, which we also denote by γ , is called *motivic* if, for every subgraph spanned by the edges $\gamma' \subsetneq \gamma$, which is mass-momentum spanning in γ , one has $h_{\gamma'} < h_\gamma$, where we denote by h_γ the loop number of γ .

In other words, a subgraph $\gamma \subset G$ is motivic if and only if removing any edge of γ makes it non-mass-momentum spanning or reduces its loop number h_γ .

Recall that α_e are projective coordinates. Let $\Delta := \bigcup_{e \in E_G} \Delta_e \subset \mathbb{P}^{N_G-1}$, where $\Delta_e := V(\alpha_e) \subset \mathbb{P}^{N_G-1}$, and note that $\partial \sigma_G \subset \Delta$. Denote $\Delta_\gamma = \bigcup_{e \in E_\gamma} \Delta_e \subset \mathbb{P}^{N_G-1}$, for a subgraph $\gamma \subset G$, and let $\Delta_\emptyset = \mathbb{P}^{N_G-1}$. These schemes are defined a priori over $\text{Spec}(\mathbb{Q})$, and we write, by abuse of notation, $\mathbb{P}^{N_G-1}, \Delta$, and Δ_γ for their base change to $K_{F,M}^{gen}$.

By [4, Proposition 6.2] a linear subspace Δ_γ corresponding to a motic subgraph γ is contained in $V(\Xi_G(m, q))$, and if γ is not mass-momentum spanning then Δ_γ is also contained in $V(\Psi_G)$, over each fiber over $K_{F,M}^{gen}$. Therefore, over each fiber, the boundary of the domain of integration meets the singularities of the integrand ω_G , causing potential divergences. We blow up \mathbb{P}^{N_G-1} along the linear subspaces corresponding to motic subgraphs, at first blowing up those linear subspaces of dimension 0, then the strict transforms of linear subspaces of dimension one etc. Denote the blow-up by $\pi_G : P^G \rightarrow \mathbb{P}^{N_G-1}$. Let $X_G = V(\Xi_G(m, q)) \cup V(\Psi_G)$, $X'_G = V(\Xi_G(m, q))$, and Y_G, Y'_G be their strict transforms. Let $D = \pi_G^{-1}(\Delta)$.

We recall the definition of a Landau variety following [29, Ch. IV], based on stratified Morse theory [21]. Consider the underlying analytic varieties of the pair $(P^G, D \cup Y_G)$. Then the divisor $Y = D \cup Y_G$ gives rise to a stratification of P^G

$$P^G \supset Y \supset Y^{(1)} \supset Y^{(2)} \supset \dots$$

where $Y^{(k)}$ is the skeleton of Y of codimension k . The open strata $Y^{(k)} \setminus Y^{(k+1)}$ are smooth, and the boundary of each irreducible component of $Y^{(k)} \setminus Y^{(k+1)}$, denoted A_k , has the property that the boundary $\overline{A_k} \setminus A_k$ is a union of strata of lower dimension. Furthermore they satisfy Whitney's conditions A and B. Define the critical set of A_k to be the set where π fails to be submersive:

$$cA_k := \{x \in A_k \mid \text{rk } T_x \pi < \dim K_{F,M}^{gen}\}.$$

Definition 7. Define the Landau variety L_G to be the codimension 1 part of $\pi(\cup_i cA_i)$, where the union is over all the strata of Y .

Note that we have been working with underlying analytic varieties of P^G, Y , and by [29, Lemma 5.2, Corollary 5.3] each $\pi(\cup_i cA_i)$ is a complex analytic set. In fact, the same proof as that of Lemma 5.2 tells us that it is in fact an algebraic variety, since it is expressed therein in terms of minors of a matrix of partial derivatives of local equations for X_G and D , which are algebraic. Then by Thom's Isotopy theorem π is a locally trivial fibration of stratified analytic varieties on the complement of L_G .

Let $S = K_{F,M}^{gen} \setminus L_G$. Note that P^G is smooth and $D \cup Y_G$ is generically a simple normal crossing divisor in P^G . This means that we are in the situation set up in 2.4.

Definition 8. The *graph motive* of a Feynman graph G is defined as the triple

$$\text{mot}_G = ((\text{mot}_G)_B, (\text{mot}_G)_{\text{dR}}, c) \in \text{ob}(\mathcal{H}(S)),$$

where

$$(mot_G)_B = H_B^{N_G-1} (P^G \setminus Y_G, D \setminus Y_G \cap D)_{/S}$$

and

$$(mot_G)_{dR} = H_{dR}^{N_G-1} (P^G \setminus Y_G, D \setminus Y_G \cap D)_{/S}.$$

We also define the *restricted graph motive*

$$mot'_G = ((mot'_G)_B, (mot'_G)_{dR}, c) \in \text{ob}(\mathcal{H}(S)),$$

where

$$(mot'_G)_B = H_B^{N_G-1} (P^G \setminus Y'_G, D \setminus Y'_G \cap D)_{/S}$$

and

$$(mot'_G)_{dR} = H_{dR}^{N_G-1} (P^G \setminus Y'_G, D \setminus Y'_G \cap D)_{/S}.$$

Next we write down a canonical Betti class over $Z = U_{F,M}^{gen}$, and a de Rham class over the generic point. Let $\pi_G : P^G \rightarrow \mathbb{P}^{N_G-1}$ be the blow up along Δ_γ where these are viewed as schemes over $\text{Spec}(\mathbb{Q})$, and let $\tilde{\sigma}$ be the closure in the analytic topology of $\pi_G^{-1}(\mathring{\sigma})$, where $\mathring{\sigma}$ is the real open simplex given by $\alpha_e > 0$. We define a constant family of manifolds with corners over $U_{F,M}^{gen}$:

$$\sigma_G = \tilde{\sigma} \times U_{F,M}^{gen}.$$

Then [4, Theorem 6.7] tells us that

$$\sigma_G \cap Y_G(\mathbb{C}) = \emptyset.$$

This follows from the coefficients of Ψ_G being positive and $\text{Re}(\Xi_G(m, q)) > 0$ when $\alpha_i > 0$ and $(m, q) \in U_{F,M}^{gen}$, i.e., parameters have positive real parts. Then we have

$$[\sigma_G] \in \Gamma(U_{F,M}^{gen}, (mot_G)_B^\vee)$$

as desired.

Finally, for a general definition of a motivic Feynman amplitude, one needs to prove that the pull-back of the differential form $\pi_G^*(\omega_G(m, q))$ doesn't acquire any new poles along the exceptional divisors after blowing up. For a general criterion in terms of the degrees of divergence of motivic subgraphs see [4, §6.6]. We do not need this result in full generality, and will check for poles directly in section 5, when we first encounter motivic

subgraphs and having to blow-up. When this is satisfied $\pi_G^*(\omega_G(m, q))$ is a global section of $\Omega_{P_G \setminus Y_G/k_S}^{N_G-1}$ and defines a class

$$[\pi_G^*(\omega_G(m, q))] \in \Gamma(\text{Spec}(k_S), (mot_G)_{dR})$$

as required.

Definition 9. The *motivic Feynman amplitude* is the family of motivic periods:

$$I_G^m(m, q) = [mot_G, [\sigma_G], [\pi_G^*(\omega_G(m, q))]]^m \in \mathcal{P}_{\mathcal{H}(S)}^{m, U_{F,M}^{gen, gen}}$$

When we have

$$[\pi_G^*(\omega_G(m, q))] \in \Gamma(\text{Spec}(k_S), (mot'_G)_{dR}), \text{ and } [\sigma_G] \in \Gamma(U_{F,M}^{gen}, (mot'_G)_{\mathbb{B}}^{\vee}),$$

for example when $N_G \geq (h_G + 1)\frac{d}{2}$, the motivic Feynman amplitude can be considered as a motivic period of the motive mot'_G . In this case we denote it by the same symbol:

$$I_G^m(m, q) = [mot'_G, [\sigma_G], [\pi_G^*(\omega_G(m, q))]]^m \in \mathcal{P}_{\mathcal{H}(S)}^{m, U_{F,M}^{gen, gen}}.$$

In what follows we will make it clear if we are working with mot_G or mot'_G .

When the motive of a Feynman graph is mixed Tate (see 2.4.1), as will be the case in the rest of this paper, we define the *de Rham Feynman amplitude* as:

$$I_G^{dr}(m, q) = [mot_G, \epsilon, [\pi_G^*(\omega_G(m, q))]]^{dr} \in \mathcal{P}_{\mathcal{H}(S)}^{m, U_{F,M}^{gen, gen}},$$

where $\epsilon : \omega_{dR}^{gen}(mot_G)_{dR} \rightarrow \omega_{dR}^{gen}W_0(mot_G)_{dR}$. This map is simply the projection to the weight 0 part. Note that the weight filtration on the de Rham realization is split since the motive is Mixed Tate.

3. Motives of one-loop graphs

We specialize the previous discussion to the case of one-loop cycle graphs with N_G internal edges, and one external edge attached to each vertex, in $d = 4$ dimensions. In this case, the second Symanzik polynomial Ξ_G is homogeneous quadratic in the α_i , and the first Symanzik polynomial Ψ_G is linear in the α_i . The integral of interest is:

$$(19) \quad I_G(m, q) = \int_{\sigma} \frac{\Psi_G^{N_G-4}}{\Xi_G^{N_G-2}} \Omega_G.$$

3.1. Four and more internal edges

Let $N_G \geq 4$, and all masses and momenta non-vanishing, i.e., $F = M = N_G$. Note that in this case the polar locus of the integrand does not include the first Symanzik polynomial, therefore we can work with the restricted motive mot'_G .

In the case of non-vanishing masses and momenta, there are no motic subgraphs (see definition 6) of a one-loop graph. In particular there is no need to blow up, and denoting by Q the vanishing locus of Ξ_G , and following the previous section we have:

$$mot'_G = (H_{\text{dR}}^{N_G-1}(\mathbb{P}^{N_G-1} \setminus Q, \Delta \setminus Q \cap \Delta)_{/S}, H_{\text{dR}}^{N_G-1}(\mathbb{P}^{N_G-1} \setminus Q, \Delta \setminus Q \cap \Delta)_{/S}, c),$$

and

$$[\omega_G] \in \Gamma(\text{Spec}(k_S), (mot'_G)_{\text{dR}}), \quad [\sigma_G] \in \Gamma(U_{F,M}^{gen}, ((mot'_G)_B)^\vee).$$

The associated motivic Feynman amplitude is:

Definition 10.

$$(20) \quad I_G^m(m, q) = [mot'_G, [\sigma_G], [\omega_G]]^m \in \mathcal{P}_{\mathcal{H}(S)}^{m, U_{F,M}^{gen}.gen}$$

3.2. Semi-simplification of the motive

Lemma 1. *Let $Q \subset \mathbb{P}^n$ be a smooth quadric hypersurface. Then*

$$(21) \quad H^q(\mathbb{P}^n \setminus Q, \mathbb{Q}) \cong \begin{cases} \mathbb{Q}(-m), & \text{if } q = n = 2m - 1 \\ \mathbb{Q}(0), & \text{if } q = 0 \\ 0, & \text{otherwise} \end{cases}$$

Proof. Recall that $H^{2q}(Q, \mathbb{Q}) \cong \mathbb{Q}(-m)$ for a d -dimensional quadric Q , and $1 \leq q \leq d$ if d odd. If $d = 2l$ is even then $H^d(Q, \mathbb{Q}) \cong \mathbb{Q}(-l) \oplus \mathbb{Q}(-l)$. Let $n = 2m - 1$ and consider the Gysin long exact sequence:

$$\dots H^n(\mathbb{P}^n) \rightarrow H^n(\mathbb{P}^n \setminus Q) \xrightarrow{\text{Res}} H^{n-1}(Q)(-1) \xrightarrow{G} H^{n+1}(\mathbb{P}^n) \rightarrow \dots$$

Note that $H^{n-1}(Q)(-1) \cong \mathbb{Q}(-m) \oplus \mathbb{Q}(-m)$ and that the Gysin morphism G is surjective since $\mathbb{P}^n \setminus Q$ is a closed subset of dimension n of the affine space $\mathbb{P}^N \setminus H$, where H is a hyperplane, and the closed embedding is given by the Veronese embedding $i : \mathbb{P}^n \rightarrow \mathbb{P}^N$, where $N = \binom{n+2}{n} - 1$. Since

$H^n(\mathbb{P}^n) = 0$ we have that $H^n(\mathbb{P}^n \setminus Q) \cong \mathbb{Q}(-m)$. From the same long exact sequence we can see that $H^{n+k}(\mathbb{P}^n \setminus Q) = 0$ when $0 \leq k \leq n$ because in that case $H^{n+k}(Q)(-1) \cong H^{n+k}(\mathbb{P}^n)$, and $H^{n+k}(\mathbb{P}^n) = 0$ if k even and $H^{n+k-1}(Q) = 0$ if k is odd. By Poincaré duality we get the remaining cohomology. The result is proved analogously for $n = 2m$ even. \square

Recall that the rank of a quadric $Q = V(f) \subset \mathbb{P}^n$, where f is a homogeneous polynomial of degree 2, is the rank of the matrix C where $f = \vec{\alpha}C\vec{\alpha}^t$, and α is the row vector $(\alpha_1, \dots, \alpha_{n+1})$. Let $\Delta_i = V(\alpha_i)$, $\Delta_I = \cap_{i \in I} \Delta_i$, and $\Delta_\emptyset = \mathbb{P}^n$. The rank of $Q = V(\Xi_G(m, q)) \subset \mathbb{P}^{N_G-1}$, for $N_G \geq 6$, in $d = 4$ space-time dimensions is at most 6, and it is exactly 6 for generic values of masses and momenta [8, Lemma 6.3], i.e., the locus where the rank is strictly less than 6 is in the Landau variety L_G . When $N_G \leq 5$ the quadric Q is generically smooth. In order to determine the weight graded pieces of mot'_G , we first prove the following result, which applies to each fiber of mot'_G over $S(\mathbb{C})$.

Proposition 1. *Let $n \geq 3$, and $\Delta \subset \mathbb{P}^n$ be the standard simplex. Let $Q \subset \mathbb{P}^n$ be a quadric of rank $\min(6, n + 1)$ with Q and Δ in general position. Then the weight graded pieces of the motive $H = H^n(\mathbb{P}^n \setminus Q, \Delta \setminus Q \cap \Delta)$ are:*

$$(22) \quad gr_0^W H = \mathbb{Q}(0), \quad gr_2^W H = \bigoplus_{\binom{n+1}{n-1}} \mathbb{Q}(-1), \quad gr_4^W H = \bigoplus_{\binom{n+1}{n-3}} \mathbb{Q}(-2).$$

When $n \leq 4$ all the other weight-graded pieces, except for these three, vanish. When $n \geq 5$ we also have:

$$(23) \quad gr_6^W H = \bigoplus_{\binom{n+1}{n-5} - \binom{n+1}{n-6}} \mathbb{Q}(-3),$$

and all the others vanish.

Proof. We apply the relative cohomology spectral sequence

$$E_1^{p,q} = \bigoplus_{|I|=p} H^q(\Delta_I \setminus (Q \cap \Delta_I)) \Rightarrow H^{p+q}(\mathbb{P}^n \setminus Q, \Delta \setminus Q \cap \Delta).$$

To compute $W_4 H$ we note that our assumption on the rank of Q implies that $Q \cap \Delta_I$ is smooth when Δ_I is a face of Δ of dimension ≤ 5 . Therefore, by the previous lemma, the first row $E_1^{\bullet,1}$ is zero except for the entry

$$E_1^{n-1,1} \cong \bigoplus_{|I|=n-1} H^1(\Delta_I \setminus Q \cap \Delta_I) \cong \bigoplus_{\binom{n+1}{n-1}} \mathbb{Q}(-1).$$

Hence $gr_2^W H \cong \bigoplus_{\binom{n+1}{n-1}} \mathbb{Q}(-1)$ as required. Again, by the previous lemma, row $E_1^{\bullet,2}$ is all zero, while the row $E_1^{\bullet,3}$ is zero except for

$$E_1^{n-3,3} \cong \bigoplus_{|I|=n-3} H^3(\Delta_I \setminus Q \cap \Delta_I) \cong \bigoplus_{\binom{n+1}{n-3}} \mathbb{Q}(-2).$$

It remains to show the claim for the graded weight 6 piece of H , as well as that there are no higher weight graded pieces. By the assumption on the rank of Q , if $n \geq 6$ then $Q \subset \mathbb{P}^n$ is a generalized cone over a smooth quadric $Q_0 \subset \mathbb{P}^5$. Hence, $\mathbb{P}^n \setminus Q$ is a fiber bundle with fibers \mathbb{A}^{n-5} over $\mathbb{P}^5 \setminus Q_0$. By the Leray spectral sequence we get

$$(24) \quad H^i(\mathbb{P}^n \setminus Q, \mathbb{Q}) \cong H^i(\mathbb{P}^5 \setminus Q_0, \mathbb{Q}) \text{ for all } i \geq 0$$

Therefore we can apply the previous lemma again to conclude that for all $q \geq 6$ and $p \geq 0$ the $E_1^{p,q}$ vanish. Hence there are no weight graded pieces higher than 6. We also get $E_1^{n-4,5} = 0$ and

$$E_2^{n-5,5} \cong gr_6^W H \cong \text{coker} \left(E_1^{n-6,5} \rightarrow E_1^{n-5,5} \right) \cong \bigoplus_{\binom{n+1}{n-5} - \binom{n+1}{n-6}} \mathbb{Q}(-3).$$

Finally, the bottom row $E_1^{\bullet,0}$ computes the cohomology of the simplicial complex Δ which is homologically equivalent to a sphere, giving us the graded weight 0 part of the proposition. □

We must be careful when pulling back this result to S . It is not the case that, for S defined as above, $gr^W mot'_G$ is isomorphic to

$$\mathbb{Q}(0)_{/S} \bigoplus \mathbb{Q}(-1)_{/S}^{m_1} \bigoplus \mathbb{Q}(-2)_{/S}^{m_2} \bigoplus \mathbb{Q}(-3)_{/S}^{m_3},$$

for m_1, m_2, m_3 defined in the previous Proposition, and $\mathbb{Q}(-k)_{/S}$, constant variation of Tate Hodge structure defined in 2.4.1. The reason is that the cohomology of a smooth quadric in even dimensions has rank 2 in middle degree, and the monodromy might interchange the two classes. However, if we pass to the covering of S which includes the square roots of the determinant of the matrix associated to the quadric Q , as well as the square roots of the determinants of all the quadrics $Q|_{\Delta_I}$, where Δ_I is a face of Δ as before, we do indeed get that $gr^W mot'_G$ is a constant variation. This explains the prefactors $P_{j,k}$ in Theorem 1. We assume that this has been done and denote the new base by S as well, by abuse of notation.

3.3. Reduction to four edges

The main tool we use to reduce to the one-loop graph with 4 internal edges (Figure 1) is the following:

Lemma 2. *Let $n = N_G - 1 \geq 4$. Let $\Delta_i \subset \mathbb{P}^n$ be the face of Δ where $\alpha_i = 0$. Then the Feynman differential form $\omega_G(m, q)$ is exact, and we can find an $(n - 1)$ -form ω_{n-1} which is a global section of the sheaf $\Omega_{\mathbb{P}^n \setminus Q/k_S}^{n-1}$, along with constants $a_j \in k_S$, such that $d\omega_{n-1} = \omega_G$ and $\omega_{n-1}|_{\Delta_j} = a_j \omega_{G/e_j}$.*

Proof. See [8, Lemma 8.1] □

Proposition 2. *For generic values of masses and momenta and in $d = 4$ dimensions, the motivic Feynman amplitude of any one-loop graph G with $N_G \geq 5$ edges is a k_S -linear combination of motivic periods of graph motives of 4-edge quotient graphs of G .*

Proof. We start with the motivic period $[mot'_G, [\sigma_G], [\omega_G]]^m$, and we would like to show that there exist constants $a_I \in k_S$, for $I \subset \{1, \dots, N_G\}$, such that we can write

$$(25) \quad [mot'_G, [\sigma_G], [\omega_G]]^m = \sum_{|I|=N_G-4} a_I \cdot [mot'_{G/e_I}, [\sigma_{G/e_I}], [\omega_{G/e_I}]]^m$$

or, more compactly:

$$(26) \quad I_G^m = \sum_{|I|=N_G-4} a_I \cdot I_{G/e_I}^m$$

Restricting the family $\mathbb{P}^{N_G-1} \setminus Q$, and $\Delta \setminus Q \cap \Delta$ to the fiber at the generic point, we apply the previous lemma as follows. Because $\mathbb{P}^{N_G-1} \setminus Q$ is affine and $\Delta \setminus Q \cap \Delta$ is a simple normal crossing divisor, $(mot'_G)_{dR}$ is computed by the cohomology of the total complex of the double complex

$$(27) \quad \Gamma(\mathbb{P}^n \setminus Q, \Omega_{\mathbb{P}^n \setminus Q}^\bullet) \rightarrow \bigoplus_{|I|=1} \Gamma(\Delta'_I, \Omega_{\Delta'_I}^\bullet) \rightarrow \dots \rightarrow \bigoplus_{|I|=n} \Gamma(\Delta'_I, \Omega_{\Delta'_I}^\bullet)$$

where $\Delta'_I = \Delta_I \setminus Q|_{\Delta_I}$, and $\Omega_{\Delta'_I}^\bullet$ is the direct image under $\Delta'_I \hookrightarrow \mathbb{P}^n \setminus Q$ of the sheaf of algebraic differential forms on Δ'_I and vanishes outside of Δ'_I . The horizontal morphisms are restrictions, i.e. pullback along the inclusion of faces $\Delta_I \hookrightarrow \Delta_I \setminus i_j$ with the sign $(-1)^k$ where i_j is the k th element of I .

The differential of the total complex of (27) is defined by:

$$(28) \quad d_{Tot}^n = \sum_{n=p+q} d^{p,q} + (-1)^p d_2^{p,q}$$

where d is the exterior derivative and d_2 is the restriction. Recall that we have specialized to the generic point, which we drop from the notation. Therefore the coefficients of the algebraic differential forms above lie in the field k_S . This double complex is obtained by specializing (10) to a point and using the fact that we are working with affine schemes. Therefore, the class of the Feynman form $[\omega_G] \in (mot'_G)_{dR}$ can be represented by an element

$$(\omega_G, 0, \dots, 0) \in \bigoplus_{N_G-1=p+q} C^{p,q},$$

where

$$C^{p,q} = \bigoplus_{|I|=p} \Gamma \left(\Delta_I \setminus Q \cap \Delta_I, \Omega_{\Delta_I \setminus Q \cap \Delta_I}^q \right).$$

By Lemma 2 we can construct an $(N_G - 2)$ -form

$$\omega' := (\omega, 0, \dots, 0) \in \bigoplus_{N_G-2=p+q} C^{p,q}, \quad \text{where } \omega \in \Gamma \left(\mathbb{P}^{N_G-1} \setminus Q, \Omega_{\mathbb{P}^{N_G-1} \setminus Q}^{N_G-2} \right),$$

and an $(N_G - 1)$ -form

$$(29) \quad \begin{aligned} \omega'' &:= (0, \omega|_{\Delta_1}, \omega|_{\Delta_2}, \dots, \omega|_{\Delta_{N_G}}, 0, \dots, 0) = \\ &= (0, a_1 \cdot \omega_{G/e_1}, a_2 \cdot \omega_{G/e_2}, \dots, a_{N_G} \cdot \omega_{G/e_{N_G}}, 0, \dots, 0) \end{aligned}$$

where $\omega'' \in \bigoplus_{N_G-1=p+q} C^{p,q}$, such that $d_{Tot}(\omega') = \omega + \omega''$. Now we can write $[\omega'']$ as a linear combination, over k_S , of classes of $(N_G - 2)$ -forms in $(mot_{G/e_j})_{dR}$ for $1 \leq j \leq N_G$.

The de Rham component of a face map (Definition 1):

$$(30) \quad \Phi_{i,dR} : \Gamma(\text{Spec}(k_S), (mot'_{G/e_i})_{dR}) \rightarrow \Gamma(\text{Spec}(k_S), (mot'_G)_{dR}),$$

for all $1 \leq i \leq N_G$ sends

$$[(a_i \cdot \omega_{G/e_i}, 0, \dots, 0)] \mapsto [(0, \dots, 0, a_i \cdot \omega_{G/e_i}, 0, \dots, 0)]$$

Summing over all the faces we get a morphism

$$(31) \quad \Phi_{\text{dR}} : \bigoplus_i \Gamma(\text{Spec}(k_S), (mot'_{G/e_i})_{\text{dR}}) \rightarrow \Gamma(\text{Spec}(k_S), (mot'_G)_{\text{dR}}) \\ \left(a_1 \cdot [\omega_{G/e_1}], \dots, a_{N_G} \cdot [\omega_{G/e_{N_G}}] \right) \mapsto [\omega_G]$$

where we write $[\omega_{G/e_i}]$ for the class of the element $(\omega_{G/e_i}, 0, \dots, 0)$.

The Betti components of face maps (1) are given by restriction to the faces of $\Delta \setminus \Delta \cap Q$, and again we sum over all the faces to get:

$$(32) \quad \Phi_B^\vee : \Gamma(U_{F,M}^{gen}, (mot'_G)_B^\vee) \rightarrow \bigoplus_i \Gamma(U_{F,M}^{gen}, (mot'_{G/e_i})_B^\vee) \\ [\sigma_G] \mapsto \left([\sigma_{G/e_1}], \dots, [\sigma_{G/e_{N_G}}] \right)$$

which shows the equivalence of motivic periods:

$$[mot_G, [\sigma_G], [\omega_G]]^m = \sum_{1 \leq i \leq N_G} a_i \cdot [mot_{G/e_i}, [\sigma_{G/e_i}], [\omega_{G/e_i}]]^m$$

Iterating this same process we can then write each of the $(N_G - 2)$ -forms on the right hand side above as a k_S -linear combination of $(N_G - 3)$ -forms of graphs obtained by contracting two edges of G , up to a form that is exact in relative de Rham cohomology. This can be repeated until we get to a k_S -linear combination of 3-forms which belong to the de Rham realizations of motives of graphs with 4 edges obtained by contracting the $N_G - 4$ edges of the original graph. Gathering all the coefficients a_i at each stage into a_I for each set of $N_G - 4$ contracted edges I , we obtain the proof of the lemma. \square

Applying the period map to (26) we recover an old result of Nickel [25]:

Corollary 1. *For generic values of masses and momenta the Feynman integral of any one-loop graph G in $d = 4$ dimensions is a k_S -linear combination of periods of graph motives of 4-edge quotient graphs of G .*

Remark 2. Note that using face maps to show equivalences of families of motivic periods in the previous proposition is the “motivic lift” of repeatedly applying Stokes’ theorem on the corresponding integrals under the period map.

Remark 3. Since we have reduced to motives of graphs with 4 edges in the previous proposition, Proposition 1 implies that all motivic Feynman

amplitudes of one-loop graphs with generic masses and momenta are k_S -linear combinations of families of periods of motives that have the following weight graded pieces

$$\text{gr}^W(\text{mot}_G) = \mathbb{Q}(-2)_{/s} \oplus \mathbb{Q}(-1)_{/s}^{\oplus 6} \oplus \mathbb{Q}(0)_{/s}$$

Griffiths transversality is used in [8, §9] to show that, up to a constant of integration, periods of highest weight of such motives are always k_S -linear combinations of dilogarithms.

3.4. Fewer than four edges

3.4.1. The triangle graph We will also consider the graph with $N_G = 3$ edges, henceforth *the triangle graph*. In the next figure we have the triangle graph when $F = M = N_G$, i.e., all masses and momenta are non-zero. Recall that we denote the internal edges with non-vanishing masses with double lines.

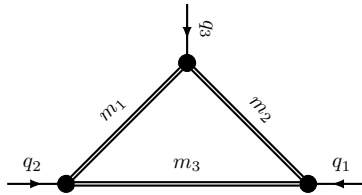


Figure 3: The triangle graph with generic masses and momenta.

The first and second Symanzik polynomials of this graph are:

$$(33) \quad \begin{aligned} \Psi_G &= \alpha_1 + \alpha_2 + \alpha_3, \text{ and} \\ \Xi_G &= q_1^2 \alpha_2 \alpha_3 + q_2^2 \alpha_1 \alpha_3 + q_3^2 \alpha_1 \alpha_2 + (m_1^2 \alpha_1 + m_2^2 \alpha_2 + m_3^2 \alpha_3) \Psi_G. \end{aligned}$$

Note that the differential form corresponding to this graph ω_G (19) has the first Symanzik polynomial in the denominator. Because of this we must consider the full graph motive. Let $L = V(\Phi_G)$, and $Q = V(\Xi_G(m, q))$. Then, the motivic Feynman amplitude is a family of periods associated to the object:

$$\text{mot}_G := H^2(\mathbb{P}^2 \setminus (Q \cup L), \Delta \setminus (Q \cup L) \cap \Delta)_{/s}$$

which we refer to as *the motive of the triangle graph*. We will study the triangle graph and the Galois coaction on the associated motivic Feynman

amplitude in section 5. We will also look at cases of the triangle graph where internal masses vanish.

3.4.2. The bubble graph Finally, we will be considering the one loop graph with $N_G = 2$ edges, henceforth *the bubble graph*. The first case we will look at is $F = M = N_G$, when the masses and the momentum are non-vanishing.

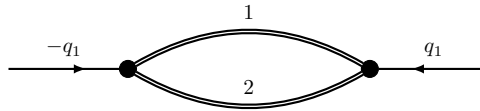


Figure 4: The bubble graph with generic masses and momenta.

The first and second Symanzik polynomials of this graph are:

$$(34) \quad \begin{aligned} \Psi_G &= \alpha_1 + \alpha_2, \text{ and} \\ \Xi_G &= q_1^2 \alpha_1 \alpha_2 + (m_1^2 \alpha_1 + m_2^2 \alpha_2) \Psi_G. \end{aligned}$$

The motive of the bubble graph is

$$(35) \quad \text{mot}_G := H^1(\mathbb{P}^1 \setminus (Q \cup L), \Delta \setminus (Q \cup L) \cap \Delta)_{/S},$$

where $L = V(\Phi_G)$, and $Q = V(\Xi_G(m, q))$. It can easily be seen from the relative cohomology long exact sequence

$$\dots \rightarrow H^0(\Delta')_{/S} \rightarrow H^1(\mathbb{P}^1 \setminus (Q \cup L), \Delta')_{/S} \rightarrow H^1(\mathbb{P}^1 \setminus (Q \cup L))_{/S} \rightarrow \dots,$$

where $\Delta' = \Delta \setminus (Q \cup L) \cap \Delta$, that it has rank 3.

In order to relate the bubble graph with the motivic coaction on one loop graphs with more than 2 edges, we are going to consider the motivic Feynman amplitude of this graph in $d = 2$ dimensions, in which case we have the integrand

$$\theta_G^1 = \frac{\alpha_2 d\alpha_1 - \alpha_1 d\alpha_2}{\Xi_G}.$$

It defines a class over the generic point of the de Rham realization of the restricted motive of the bubble graph:

$$\text{mot}'_G := H^1(\mathbb{P}^1 \setminus Q, \Delta \setminus Q \cap \Delta)_{/S}$$

Denote by

$$[p_0p_1|p_2p_3] = \frac{(p_2 - p_0)(p_3 - p_1)}{(p_2 - p_1)(p_3 - p_0)}$$

the cross-ratio of 4 points on \mathbb{P}^1 . Then the period corresponding to the motivic Feynman amplitude of the bubble graph in $d = 2$ dimensions is:

$$(36) \quad \begin{aligned} \text{per} \left([mot_G, [\sigma_G], [\theta_G^1]]^m \right) &= \frac{1}{\sqrt{4|\det C|}} \log([p_0p_1|u_0u_1]) = \\ &= \frac{1}{x - y} \log\left(\frac{y}{x}\right), \end{aligned}$$

where $\{u_0, u_1\} := V(\Xi_G) \subset \mathbb{P}^1$, C is the matrix of the quadratic form Ξ_G , x, y are the coordinates of u_0, u_1 respectively in the chart where $\alpha_2 \neq 0$, and $p_0 = [0 : 1], p_1 = [1 : 0]$. We know from cohomology computations that $[mot_G, [\sigma_G], [\theta_G^1]]^m$ is a motivic logarithm (up to an explicit prefactor), and since these are determined by their periods we have that

$$[mot_G, [\sigma_G], [\theta_G^1]]^m = \frac{1}{\sqrt{4|\det C|}} \log^m([p_0p_1|u_0u_1]).$$

In the case of the bubble graph with one vanishing mass one of the two points $Q \subset \mathbb{P}^1$ coincides with one of the points $\Delta = \{[1 : 0], [0 : 1]\}$. Without

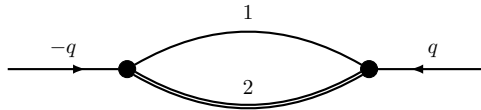


Figure 5: The bubble graph with one vanishing mass.

loss of generality let $u_0 \in \Delta$. In that case the relevant motive will be

$$(37) \quad mot''_G = H^1(\mathbb{P}^1 \setminus (u_1 \cup L), \Delta \setminus (u_1 \cup L) \cap \Delta)_{/S}$$

It has rank 2. The motivic period we will be interested in, which is associated to this motive, is obtained by pairing the interval $(0, \infty)$ with a generator of $(mot''_G)_{dR}$ represented by the form

$$(38) \quad \theta_G^2 = \frac{(x - y)(\alpha_2 d\alpha_1 - \alpha_1 d\alpha_2)}{(\alpha_2 - x\alpha_1)(\alpha_2 - y\alpha_1)}$$

where x, y are coordinates of u_1 and L respectively in the coordinate chart of \mathbb{P}^1 where $\alpha_2 \neq 0$. The associated period is:

$$\text{per} \left([mot''_G, [\sigma_G], [\theta_G^2]]^m \right) = \log \left(\frac{y}{x} \right).$$

Similarly to the previous case we see that $[mot''_G, [\sigma_G], [\theta_G^2]]^m$ is a motivic logarithm.

4. Coaction on the 4-edge graph: proof of Theorem 1

4.1. The motivic side of the coaction

The first and second Symanzik polynomials of the one-loop graph with 4 edges G are:

$$\begin{aligned} \Psi_G &= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \\ (39) \quad \Xi_G &= (q_2 + q_3)^2 \alpha_1 \alpha_3 + (q_1 + q_2)^2 \alpha_2 \alpha_4 + q_1^2 \alpha_1 \alpha_4 + q_2^2 \alpha_1 \alpha_2 + \\ &\quad + q_3^2 \alpha_2 \alpha_3 + q_4^2 \alpha_3 \alpha_4 + (m_1^2 \alpha_1 + m_2^2 \alpha_2 + m_3^2 \alpha_3 + m_4^2 \alpha_4) \Psi_G. \end{aligned}$$

Recall that the general formula for the coaction is:

$$\Delta [mot'_G, [\sigma_G], [\omega_G]]^m = \sum_j [mot'_G, [\sigma_G], [\omega_j]]^m \otimes [mot'_G, [\omega_j]^\vee, [\omega_G]]^{\text{dr}}$$

where $\{[\omega_j]\}$ is a basis of $(mot'_G)_{\text{dR}}$, and $\{[\omega_j]^\vee\}$ is the dual basis. The first sheet of the relative cohomology spectral sequence

$$E_1^{p,q} = \bigoplus_{|I|=p} H^q(\Delta_I \setminus (Q \cap \Delta_I)) \Rightarrow H^{p+q}(\mathbb{P}^n \setminus Q, \Delta \setminus Q \cap \Delta)$$

applied to any fiber of $(mot'_G)_B$ over $t \in S(\mathbb{C})$, by Lemma 1, reads:

$$\begin{aligned} E^{0,3} \cong \mathbb{Q}(-2) &\longrightarrow E^{1,3} \cong 0 \longrightarrow E^{2,3} \cong 0 \longrightarrow E^{3,3} \cong 0 \\ E^{0,2} \cong 0 &\longrightarrow E^{1,2} \cong 0 \longrightarrow E^{2,2} \cong 0 \longrightarrow E^{3,2} \cong 0 \\ E^{0,1} \cong 0 &\longrightarrow E^{1,1} \cong 0 \longrightarrow E^{2,1} \cong \mathbb{Q}(-1)^{\oplus 6} \longrightarrow E^{3,1} \cong 0 \\ E^{0,0} \cong \mathbb{Q}(0) &\longrightarrow E^{1,0} \cong \mathbb{Q}(0)^{\oplus 4} \longrightarrow E^{2,0} \cong \mathbb{Q}(0)^{\oplus 6} \longrightarrow E^{3,0} \cong \mathbb{Q}(0)^{\oplus 4}. \end{aligned}$$

We can see that the morphism

$$\bigoplus_{|I|=2} H^1(\Delta_I \setminus Q \cap \Delta_I, \Delta^I)_{/S} \rightarrow W_2 H^3(\mathbb{P}^3 \setminus Q, \Delta \setminus Q \cap \Delta),$$

given by the sum of face maps, is surjective by computing the relative cohomology spectral sequence for a fiber of $H^1(\Delta_I \setminus Q \cap \Delta_I, \Delta^I)_{/S}$, where $|I| = 2$, and noticing that the morphism of spectral sequences induced by the sum of face maps is surjective. Now we look at the de Rham and Betti realization of this sum of the face maps.

Consider the bubble graph obtained by contracting edges e_i and e_j . In the next figure we have chosen to contract e_2 and e_3 . The restricted motive

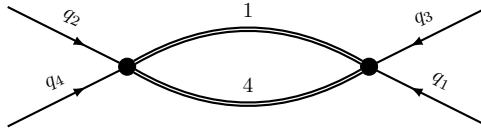


Figure 6: Bubble graph obtained by contracting edges e_2, e_3 of the four-edge graph.

of the bubble graph (35) obtained by contracting edges e_i and e_j is

$$mot'_{G/\{e_i, e_j\}} = H^1(\mathbb{P}^1 \setminus Q_{G/\{e_i, e_j\}}, \Delta \setminus Q_{G/\{e_i, e_j\}} \cap \Delta)_{/S},$$

where $Q_{G/\{e_i, e_j\}}$ is the vanishing locus of the second Symanzik polynomial of the bubble graph. Observe that $\Xi_{G/\{e_i, e_j\}} = \Xi_G|_{\{\alpha_i=0, \alpha_j=0\}}$, hence

$$H^1(\Delta_I \setminus Q \cap \Delta_I, \Delta^I)_{/S} \cong mot'_{G/\{e_i, e_j\}},$$

where $I = \{i, j\}$. We can then choose a basis of the fiber over the generic point of $(mot_G)_{dR}$ such that it contains the class of the Feynman form $[\omega_G]$ of G , which is of weight 4, and the 6 weight 2 classes denoted by $[\omega_G^{ij}]$, which we define to be the images of $[\theta_{G/\{e_i, e_j\}}^1]$ under the de Rham component of the face map (see Definition 1):

$$(40) \quad \begin{aligned} \Phi_{ij, dR} : \Gamma(\text{Spec}(k_S), (mot'_{G/\{e_i, e_j\}})_{dR}) &\rightarrow \Gamma(\text{Spec}(k_S), (mot'_G)_{dR}) \\ [\theta_{G/\{e_i, e_j\}}^1] &\mapsto [\omega_G^{ij}] \end{aligned}$$

for all $1 \leq i < j \leq 4$. The restriction to the faces of the domain of integration

$$\Phi_B^\vee : [\sigma_G] \mapsto [\sigma_{G/\{e_i, e_j\}}]$$

is the Betti component of the face map, which gives an equivalence of motivic periods:

$$\left[\text{mot}'_G, [\sigma_G], [\omega_G^{ij}] \right]^m = \left[\text{mot}'_{G/\{e_i, e_j\}}, [\sigma_{G/\{e_i, e_j\}}], [\theta_{G/\{e_i, e_j\}}^1] \right]^m.$$

We have

$$\left[\text{mot}'_{G/\{e_i, e_j\}}, [\sigma_{G/\{e_i, e_j\}}], [\theta_{G/\{e_i, e_j\}}^1] \right]^m = I_{G/\{e_i, e_j\}}^m \left(\theta_{G/\{e_i, e_j\}}^1 \right)$$

where $I_{G/\{e_i, e_j\}}^m \left(\theta_{G/\{e_i, e_j\}}^1 \right)$ is the motivic Feynman amplitude of the bubble graph in $d = 2$ dimensions. We have proved

Proposition 3. *Let G be a one loop graph with $N_G = 4$. Then the motivic Galois coaction on its motivic Feynman amplitude in $d = 4$ dimensions with non-vanishing masses and momenta is:*

$$(41) \quad \Delta I_G^m = \sum_{1 \leq i < j \leq 4} I_{G/\{e_i, e_j\}}^m \left(\theta_{G/\{e_i, e_j\}}^1 \right) \otimes \left[\text{mot}'_G, [\omega_G^{ij}]^\vee, [\omega_G] \right]^{\text{dr}} + I_G^m \otimes (\mathbb{L}^{\text{dr}})^2 + 1 \otimes I_G^{\text{dr}}.$$

Corollary 2. *We can rewrite the coaction in terms of motivic logarithms as*

$$\Delta I_G^m = I_G^m \otimes (\mathbb{L}^{\text{dr}})^2 + 1 \otimes I_G^{\text{dr}} + \sum_{1 \leq i < j \leq 4} \frac{1}{\sqrt{4|\det D_{i,j}|}} \log^m \left([p_0 p_1 | u_{\{i,j\}}^0 u_{\{i,j\}}^1] \right) \otimes \left[\text{mot}'_G, [\omega_G^{ij}]^\vee, [\omega_G] \right]^{\text{dr}}$$

where $D_{i,j}$ is the matrix of the quadratic form $\Xi_{G/\{e_i, e_j\}}$.

Proof. We have

$$\text{mot}'_{G/\{e_i, e_j\}} \cong H^1(\mathbb{P}^1 \setminus \{u_{\{i,j\}}^0, u_{\{i,j\}}^1, \{p_0, p_1\}\})/S$$

where

$$(42) \quad \begin{aligned} \{u_{\{i,j\}}^0, u_{\{i,j\}}^1\} &= Q_{G/\{e_i, e_j\}} \cap \Delta_{\{i,j\}} \\ \{p_0, p_1\} &= \{[1 : 0], [0 : 1]\} \end{aligned}$$

and the form $\theta_{G/\{e_i, e_j\}}^1$ has simple poles at $\{u_{\{i,j\}}^0, u_{\{i,j\}}^1\}$. The result follows. □

Remark 4. One can combine the previous proposition with Proposition 2 to obtain the coaction on all one-loop graphs with $N_G \geq 4$ edges in $d = 4$ dimensions with non-vanishing masses and momenta.

4.2. de Rham side of the coaction

In order to complete the proof of Theorem 1 we need to determine the de Rham side of the coaction, i.e. to interpret the objects

$$\left[\text{mot}'_G, \left[\omega_G^{ij} \right]^\vee, [\omega_G] \right]^{\text{dr}}$$

4.2.1. Connection and the de Rham periods The connection allows us to differentiate a section $[\omega]$ of $H_{\text{dR}}^n(X, D)/_S$ with respect to parameters, which in the case of Feynman amplitudes are the masses and momenta. Consider a section $[\omega]$ of \mathcal{V}_{dR} on an open affine $U \subset S$ with coordinate q . We can compose the connection with the contraction by a vector field $\frac{\partial}{\partial q}$ to obtain a map

$$\mathcal{V}_{\text{dR}} \xrightarrow{\nabla} \mathcal{V}_{\text{dR}} \otimes \Omega_{S/k}^1 \xrightarrow{\frac{\partial}{\partial q}} \mathcal{V}_{\text{dR}}$$

which sends $[\omega]$ to its first derivative with respect to q , denoted $\nabla_q(\omega)$. Since \mathcal{V}_{dR} has finite rank we will get a relation between

$$[\omega], \nabla_q([\omega]), \nabla_q^2([\omega]), \dots, \nabla_q^k([\omega])$$

for some finite k . This is the Picard-Fuchs equation satisfied by ω and therefore by $\text{per}([\mathcal{V}, [\sigma], [\omega]]^{\text{m}})$, for some cycle of integration $\sigma \in (\omega_{\mathbb{B}}^X(\mathcal{V}))^\vee$, defined over some simply connected $X \subset S(\mathbb{C})$.

We use the Gauss–Manin connection on the vector bundle

$$(\text{mot}'_G)_{\text{dR}} = H_{\text{dR}}^3(\mathbb{P}^3 \setminus Q, \Delta \setminus Q \cap \Delta)_{/S}$$

This vector bundle sits in a long exact sequence of vector bundles:

$$\dots \rightarrow H_{\text{dR}}^2(\Delta \setminus Q \cap \Delta)_{/S} \rightarrow H_{\text{dR}}^3(\mathbb{P}^3 \setminus Q, \Delta \setminus Q \cap \Delta)_{/S} \rightarrow H_{\text{dR}}^3(\mathbb{P}^3 \setminus Q)_{/S} \rightarrow \dots,$$

where $H_{\text{dR}}^k(\Delta \setminus Q \cap \Delta)_{/S}$ are objects of $\mathcal{H}(S)$ which are obtained by truncating the complexes (9) and (10) on the left so that the non-zero components are

in $|J| \geq n - 1$, and we denote the section of $H^3_{\text{dR}}(\mathbb{P}^3 \setminus Q)_{/S}$ over the generic point which is the image of $[\omega_G]$ by the same symbol. Since $H^3_{\text{dR}}(\mathbb{P}^3 \setminus Q)_{/S}$ is a vector bundle with connection of rank 1 we know that the Feynman integrand satisfies a relation $\nabla_{q_1}([\omega_G]) + B(m, q)[\omega_G] = 0$, where ∇_{q_1} is the Gauss–Manin connection on $H^3_{\text{dR}}(\mathbb{P}^3 \setminus Q)_{/S}$ composed with contraction by the vector field $\partial/\partial q_1$, and $B(m, q) \in k_S$. This relation lifts to a relation of sections of $H^3_{\text{dR}}(\mathbb{P}^3 \setminus Q, \Delta \setminus Q \cap \Delta)_{/S}$. We compute it explicitly in the following.

Lemma 3. *Let q_1 be one of the momentum parameters of a one-loop four-edge Feynman integral with non-vanishing masses and momenta. Then*

$$[\omega_G] \in \Gamma(\text{Spec}(k_S), (\text{mot}'_G)_{\text{dR}})$$

satisfies a relation of Picard–Fuchs type

$$(43) \quad \nabla_{q_1}([\omega_G]) + B(m, q)[\omega_G] = [d\beta]$$

where β is a section of $\Omega^2_{\mathbb{P}^3 \setminus Q/\text{Spec}(k_S)}$, and $B(m, q) \in k_S$. Both β and $B(m, q)$ can be computed explicitly.

Proof. We use the general description of rational forms on \mathbb{P}^n with poles along a hypersurface [20]. A 2-form on \mathbb{P}^3 with a pole along Ξ_G of order 2 is of the form

$$(44) \quad \beta = \frac{\sum_{i < j} (-1)^{i+j} \alpha_j A_i - \alpha_i A_j}{\Xi_G^2} d\alpha_1 \wedge \dots \wedge \widehat{d\alpha_i} \wedge \dots \wedge \widehat{d\alpha_j} \wedge \dots \wedge d\alpha_4$$

for some linear polynomials A_i in the variables α_i . One then computes the exterior derivative

$$d\beta = -2 \frac{\sum_i A_i \frac{\partial \Xi_G}{\partial \alpha_i}}{\Xi_G^3} \Omega_G + \frac{\sum_i \frac{\partial A_i}{\partial \alpha_i}}{\Xi_G^2} \Omega_G$$

In [20, Proposition 4.6] Griffiths makes a quite general observation that whenever we have a rational form $\frac{A}{F^k} \Omega$ such that $A \in J(F)$, where $J(F)$ is the Jacobian ideal generated by partial derivatives of F , we can reduce the order of the pole up to an exact form. To compute $\nabla_{q_1}([\omega_G])$ one differentiates ω_G with respect to q_1 and writes the resulting form in terms of a basis of sections of $H^3(\mathbb{P}^3 \setminus Q)_{/S}$, which we have chosen to be $[\omega_G]$. We can use

the Groebner basis of $J(\Xi_G)$ to find the A_i 's such that we can reduce the pole of $\frac{\partial}{\partial q_1}(\omega_G)$

$$\begin{aligned}
 \frac{\partial}{\partial q_1}(\omega_G) &= \frac{-2(2(q_1 + q_2)\alpha_2\alpha_4 + 2q_1\alpha_1\alpha_4)}{\Xi_G^3} \Omega_G = \\
 (45) \qquad \qquad &= \frac{\sum_i A_i \frac{\partial \Xi_G}{\partial \alpha_i}}{\Xi_G^3} \Omega_G = \frac{1}{2} \frac{\sum_i \frac{\partial A_i}{\partial \alpha_i}}{\Xi_G^2} \Omega_G - \frac{1}{2} d\beta
 \end{aligned}$$

Defining

$$(46) \qquad \qquad B(m, q) := \frac{1}{2} \sum_i \frac{\partial A_i}{\partial \alpha_i}$$

which indeed depends only on masses and momenta since A_i are necessarily linear, gives us the stated relation. \square

Proposition 4. *Let $C, D_{j,k}$ be the matrices associated to the quadratic forms $\Xi_G, \Xi_{G/\{e_j, e_k\}}$ respectively, and $U = C^{-1}$. Let*

$$f_{ij}(m, q) = \frac{\sqrt{(U)_{i,j}^2 - (U)_{i,i}(U)_{j,j} - (U)_{i,j}}}{\sqrt{(U)_{i,j}^2 - (U)_{i,i}(U)_{j,j} + (U)_{i,j}}},$$

as in Theorem 1, and $P = \frac{\sqrt{|\det D_{j,k}|}}{8\sqrt{|\det C|}}$. Then

$$(47) \qquad \qquad \left[\text{mot}'_G, \left[\omega_G^{ij} \right]^\vee, [\omega_G] \right]^{\text{dr}} = P \log^{\text{dr}}(f_{ij}) \mathbb{L}^{\text{dr}}.$$

Proof. Let G and β be as in the previous lemma. We first show that $[d\beta]$, viewed as a section of $H_{\text{dR}}^3(\mathbb{P}^3 \setminus Q, \Delta \setminus Q \cap \Delta)_{/S}$ over the generic point, can be written as a sum of images under the face maps (Definition 1)

$$\Phi_{ij,dR} : \Gamma(\text{Spec}(k_S), (\text{mot}'_{G/\{e_i, e_j\}})_{\text{dR}}) \rightarrow \Gamma(\text{Spec}(k_S), (\text{mot}'_G)_{\text{dR}}).$$

We will work over the generic point of the space of kinematics throughout this proof, and we drop the explicit reference to $\text{Spec}(k_S)$ to ease the notation.

Because $\mathbb{P}^3 \setminus Q$ is affine $[d\beta]$ is represented by the cocycle of the total complex of the double complex:

$$(d\beta, 0, \dots, 0) \in \bigoplus_{3=p+q} \bigoplus_{|I|=p} \Gamma(\Delta_I \setminus Q \cap \Delta_I, \Omega_{\Delta_I \setminus Q \cap \Delta_I}^q)$$

Then

$$(48) \quad d_{Tot}^2(\beta, 0, \dots, 0) = (d\beta, 0, \dots, 0) + (0, \beta|_{\Delta_1}, \beta|_{\Delta_2}, \beta|_{\Delta_3}, \beta|_{\Delta_4}, 0, \dots, 0)$$

where d_{Tot}^2 is the differential of the total complex defined in (28). As sections of $(mot'_G)_{dR}$ over the generic point, we have

$$(49) \quad [d\beta] = -[(0, \beta|_{\Delta_1}, \beta|_{\Delta_2}, \beta|_{\Delta_3}, \beta|_{\Delta_4}, 0, \dots, 0)]$$

Note that each $\beta|_{\Delta_i} \in \Gamma(\Delta_i \setminus Q \cap \Delta_i, \Omega_{\Delta_i \setminus Q \cap \Delta_i}^2)$, but since $H^2(\Delta_i \setminus Q) \cong H^2(\mathbb{P}^2 \setminus Q) \cong 0$ by Lemma 1, there must exist forms $\beta_i \in \Gamma(\Delta_i \setminus Q \cap \Delta_i, \Omega_{\Delta_i \setminus Q \cap \Delta_i}^1)$ such that $d\beta_i = \beta|_{\Delta_i}$. Using this fact we see that we can write for each β_i

$$(50) \quad d_{Tot}^1(0, \dots, 0, \beta_i, \dots, 0) = (0, \dots, \beta|_{\Delta_i}, \dots, 0) - (0, \dots, 0, r_j^i(\beta_i), r_k^i(\beta_i), r_l^i(\beta_i), 0, \dots, 0)$$

where r_j^i is the restriction of differential forms on Δ_i to forms on Δ_j with appropriate signs. Combining (49) and (50) for each i we get:

$$(51) \quad [d\beta] = - \left[\sum_{0 \leq i < j \leq 3} (0, \dots, r_j^i(\beta_i) + r_i^j(\beta_j), 0, \dots, 0) \right].$$

Note that in the previous equation we get two contributions for each 1-face $\Delta_{\{i,j\}}$. One from first restricting β to the 2-face Δ_i , taking the primitive of the restriction, then in turn restricting that primitive to the 1-face $\Delta_{\{i,j\}}$, and the other by the same procedure, but starting by restricting to Δ_j . If we write

$$(52) \quad \beta_{ij} = (r_j^i(\beta_i) + r_i^j(\beta_j), 0, 0) \in \bigoplus_{3=p+q} \bigoplus_{\substack{|I|=p \\ i,j \subset I}} \Gamma(\Delta_I \setminus Q \cap \Delta_I, \Omega_{\Delta_I \setminus Q \cap \Delta_I}^q)$$

where the latter is the double complex which computes

$$H^1(\Delta_{\{i,j\}} \setminus Q \cap \Delta_{\{i,j\}}, \Delta_{\{k,l\}} \cap \Delta_{\{i,j\}}),$$

for i, j, k, l pairwise distinct. Therefore we proved

$$[d\beta] = - \sum_{1 \leq i < j \leq 4} \Phi_{ij}([\beta_{ij}]).$$

Since $[\beta_{ij}]$ is a section of $(\text{mot}'_{G/\{e_i, e_j\}})_{dR}$ over the generic point, we can write

$$[\beta_{ij}] = a_{ij}(m, q)[\theta_{G/\{e_i, e_j\}}^1],$$

where $a_{ij}(m, q) \in k_S$. For each $1 \leq i < j \leq 4$ we have equalities of families of de Rham periods

$$\begin{aligned} & \left[\text{mot}'_G, [\omega_G^{ij}]^\vee, \nabla_{q_1}([\omega_G]) \right]^{\text{dr}} = \\ & = \left[\text{mot}'_G, [\omega_G^{ij}]^\vee, [B(m, q)\omega_G - \frac{1}{2}d\beta] \right]^{\text{dr}} = \\ (53) \quad & = \left[\text{mot}'_G, [\omega_G^{ij}]^\vee, \left([B(m, q)\omega_G] + \frac{1}{2} \sum_{1 \leq i < j \leq 4} \Phi_{ij, dR}([\beta_{ij}]) \right) \right]^{\text{dr}} = \\ & = B(m, q) \left[\text{mot}'_G, [\omega_G^{ij}]^\vee, [\omega_G] \right]^{\text{dr}} + \\ & + \frac{a_{ij}(m, q)}{2} \left[\text{mot}'_{G/\{e_i, e_j\}}, [\theta_{G/\{e_i, e_j\}}^1]^\vee, [\theta_{G/\{e_i, e_j\}}^1] \right]^{\text{dr}} = \\ & = B(m, q) \left[\text{mot}'_{G/\{e_i, e_j\}}, [\omega_G^{ij}]^\vee, [\omega_G] \right]^{\text{dr}} + \frac{a_{ij}(m, q)}{2} \mathbb{L}^{\text{dr}} \end{aligned}$$

where the second to last equality holds because $\Phi_{ij, dR}([\theta_{G/\{e_i, e_j\}}^1]) = [\omega_G^{ij}]$.

On the other hand one can check that

$$\nabla_x(P \log^{\text{dr}}(x) \mathbb{L}^{\text{dr}}) = \frac{\partial}{\partial x}(P) \log^{\text{dr}}(x) \mathbb{L}^{\text{dr}} + P \frac{1}{x} \mathbb{L}^{\text{dr}}.$$

Comparing this expression with (53), we see that $a_{ij}(m, q) = 2P \frac{1}{f_{ij}} \frac{\partial f_{ij}}{\partial q_1}$, and we obtain the result up to a constant:

$$\left[\text{mot}'_G, [\omega_G^{ij}]^\vee, [\omega_G] \right]^{\text{dr}} = P \log^{\text{dr}}(c f_{ij})$$

The constant c can be determined to be 1 by specializing to a convenient point in the space of kinematics. □

Remark 5. Note that an equivalence of de Rham periods induced by a face map $\Phi_{ij,dR}$ in the previous proposition is a version of Stokes’ theorem for de Rham periods.

4.2.2. Residues and the de Rham projection Since $[\omega_G^{ij}]$ and $[\omega_G]$ are sections over the generic point of $H_{dR}^3(\mathbb{P}^3 \setminus Q, (\Delta_i \cup \Delta_j) \setminus Q \cap (\Delta_i \cup \Delta_j))_{/S}$, and we are interested in the de Rham period

$$\left[\text{mot}'_G, [\omega_G^{ij}]^\vee, [\omega_G] \right]^{\text{dr}},$$

which only depends on $\Delta_i \cup \Delta_j \subset \Delta$, we can restrict our attention to the fiber over the generic point of

$$H^3(\mathbb{P}^3 \setminus Q, (\Delta_i \cup \Delta_j) \setminus Q \cap (\Delta_i \cup \Delta_j))_{/S}.$$

Consider the residue map

$$H^3(\mathbb{P}^3 \setminus Q, (\Delta_i \cup \Delta_j) \setminus Q \cap (\Delta_i \cup \Delta_j)) \xrightarrow{\text{Res}} H^2(Q, (\Delta_i \cup \Delta_j) \cap Q)(-1),$$

where we have restricted to the fiber over the generic point. Denote the former by H'_G , and the latter by H_Q . Consider the motivic period

$$[H'_G, [\text{Tube}(\sigma_{Lune}), [\omega_G]]^m] = [H_Q, [\sigma_{Lune}, \text{Res}([\omega_G])]^m]$$

where σ_{Lune} is the spherical lune cut out by the two hyperplanes Δ_i and Δ_j on $Q(\mathbb{C})$, and Tube denotes the tubular neighbourhood homomorphism — it is the transpose of the residue morphism in the Betti realization. The period of $[H_Q, [\sigma_{Lune}, \text{Res}(\omega_G)]^m]$ can be computed in spherical coordinates (see also [5, 5.3.3]). Since we are working with a smooth quadric we can reduce to the case of a sphere $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \alpha_4^2$, where one checks that the corresponding period is $\frac{1}{2i}\theta_{ij}$, where θ_{ij} is the angle between Δ_i and Δ_j .

To show the relation of this computation with the de Rham period we are interested in, one should show that

$$\text{Res}^\vee \left([\omega_Q^{ij}] \right) = [\omega_G^{ij}]^\vee,$$

where Res^\vee is the dual of the residue homomorphism, and ω_Q^{ij} is a form on Q with simple poles at $Q \cap \Delta_i \cap \Delta_j$. The *de Rham projection*, denoted π_{dR} , is a certain natural morphism associating to an effective motivic period of

a separated motive a de Rham period of the same motive (see [6, §4]), and we have

$$\pi_{\text{dR}} \left(\left[H_Q, [\omega_Q^{ij}]^\vee, [\omega_G] \right]^{\text{dR}} \right) = [H_Q, [\sigma_{Lune}], \text{Res}(\omega_G)]^{\text{m}}.$$

If the previous statement about the transpose of the residue homomorphism is true, we have an equivalence of de Rham periods

$$\left[H_Q, [\omega_Q^{ij}]^\vee, \text{Res}(\omega_G) \right]^{\text{dR}} = \left[\text{mot}'_G, [\omega_G^{ij}]^\vee, [\omega_G] \right]^{\text{dR}},$$

and have therefore associated a motivic period to the de Rham period we are interested in.

4.2.3. Comparing two computations of the coaction We are going to apply the motivic coaction to an explicit expression for the Feynman integral of the 1-loop 4-edge graph in 4 dimensions in terms of dilogarithms, as it appears in the physics literature [26, Proposition 8.], and check that it matches the coaction computed in Theorem 1. As a consequence, we will see that the coaction is both shorter and more symmetrical than the full expression of the Feynman integral in terms of dilogarithms as found in [26]. For a classical reference for a geometric approach to one-loop integrals and dilogarithms see also [13].

The expression given in [26] for I_G is as follows: let C be the 4×4 matrix associated to the quadratic form Ξ_G , and $U = C^{-1}$. Let

$$\begin{aligned}
 \nu^0(r, s, t) &:= \arctan \left(\frac{C_{t,4} U_{r,4} \sqrt{|\det U|}}{U_{r,s} U_{r,4} - U_{r,r} U_{s,4}} \right) \\
 \nu^1(r, s, t) &:= \arctan \left(\frac{C_{t,4} U_{r,4} \sqrt{|\det U|}}{(U_{r,s} U_{r,4} - U_{r,r} U_{s,4}) \sqrt{1 - C_{4,4} U_{4,4}}} \right) \\
 \nu^2(r, s, t) &:= \arctan \left(\frac{U_{r,4} \sqrt{U_{r,r} U_{s,s} - U_{r,s}^2}}{U_{r,s} U_{r,4} - U_{r,r} U_{s,4}} \right) \\
 \nu^3(r, s, t) &:= \arctan \left(\frac{U_{r,4}}{\sqrt{U_{r,r} U_{4,4} - U_{r,4}^2}} \right)
 \end{aligned}
 \tag{54}$$

then the family of periods, depending on masses and momenta, of interest is:

$$\begin{aligned}
 I_G = \frac{1}{16\sqrt{|\det C|}} \sum_{\{r,s,t\} \in S_3} & (2 \operatorname{Im} \operatorname{Li}_2(\exp(2\nu^0(r, s, t))) + \\
 (55) \qquad \qquad \qquad & + \sum_{l=1}^3 (-1)^l [\operatorname{Im} \operatorname{Li}_2(\exp(2\nu^0(r, s, t) + 2\nu^l(r, s, t))] \\
 & + \operatorname{Im} \operatorname{Li}_2(\exp(2\nu^0(r, s, t) - 2\nu^l(r, s, t)))]).
 \end{aligned}$$

Note that the above expression consists of a linear combination of 42 dilogarithms. We replace the dilogarithms with their motivic versions, and apply the coaction to each motivic dilogarithm

$$\Delta \operatorname{Li}_2^{\mathfrak{m}}(x) = \operatorname{Li}_2^{\mathfrak{m}}(x) \otimes \mathbb{L}^{\partial\mathfrak{r}} + \operatorname{Li}_1^{\mathfrak{m}}(x) \otimes \log^{\partial\mathfrak{r}}(x) \mathbb{L}^{\partial\mathfrak{r}} + 1 \otimes \operatorname{Li}_2^{\partial\mathfrak{r}}(x).$$

Taking care to note that the arguments are all on the unit circle, we get a linear combination of 42 terms of the form

$$\begin{aligned}
 \Delta \operatorname{Im}(\operatorname{Li}_2^{\mathfrak{m}}(z)) &= \Delta \frac{1}{2i} (\operatorname{Li}_2^{\mathfrak{m}}(z) - \operatorname{Li}_2^{\mathfrak{m}}(\bar{z})) = \\
 &= \frac{1}{2i} ((\operatorname{Li}_2^{\mathfrak{m}}(z) - \operatorname{Li}_2^{\mathfrak{m}}(\bar{z})) \otimes (\mathbb{L}^{\partial\mathfrak{r}})^2 + \operatorname{Li}_1^{\mathfrak{m}}(z) \otimes \log^{\partial\mathfrak{r}}(z) - \\
 (56) \qquad \qquad \qquad &- \operatorname{Li}_1^{\mathfrak{m}}\left(\frac{1}{z}\right) \otimes \log^{\partial\mathfrak{r}}\left(\frac{1}{z}\right) + 1 \otimes (\operatorname{Li}_2^{\partial\mathfrak{r}}(z) - \operatorname{Li}_2^{\partial\mathfrak{r}}(\bar{z}))) = \\
 &= \frac{1}{2i} ((\operatorname{Li}_2^{\mathfrak{m}}(z) - \operatorname{Li}_2^{\mathfrak{m}}(\bar{z})) \otimes (\mathbb{L}^{\partial\mathfrak{r}})^2 + \\
 &+ \log^{\mathfrak{m}}\left(\frac{(1-z)^2}{z}\right) \otimes \log^{\partial\mathfrak{r}}(z) + 1 \otimes (\operatorname{Li}_2^{\partial\mathfrak{r}}(z) - \operatorname{Li}_2^{\partial\mathfrak{r}}(\bar{z})))
 \end{aligned}$$

Notice that in Theorem 1 we have 6 terms of the form $\log^{\mathfrak{m}}(f_i) \otimes \log^{\partial\mathfrak{r}}(g_i)$ (up to prefactors), while in the coaction (56) on (55) we have 42 such terms.

All the computations below were done in Maple. In order to check that the two expressions are equivalent, we take the 6 motivic logarithms from Theorem 1 and the 42 motivic logarithms from the coaction on (55), and we find a basis for these 48 functions which includes the 6 motivic logarithms in Theorem 1. One can do this by applying the LLL algorithm to a matrix whose entries are the evaluations of the (q_1 -derivative) of the 48 motivic logarithms at sufficiently many points in the space of generic kinematics. The basis found contains 27 motivic logarithms. One then checks that the relations found indeed hold on the level of the logarithms themselves and

expresses the non-basis elements in terms of the basis. Plugging this back into the original expression for the coaction on I_G^m in (55) and collecting the de Rham logarithms with each of the 27 basis motivic logarithms, one then repeats the procedure of finding a basis for the de Rham logarithms one is left with on the right hand side of the tensor product. The basis on the de Rham side contains 20 logarithms. Expressing the non-basis de Rham logarithms in terms of these 20 and plugging this back into the previous expression with the motivic side reduced to 27 terms one observes that everything cancels out but 6 terms:

$$\sum_{1 \leq j < k \leq 4} 2 \log^m \left(\left[p_0 p_1 | u_{\{j,k\}}^0 u_{\{j,k\}}^1 \right] \right) \otimes \log^{\text{dr}} (f_{j,k}),$$

where

$$f_{j,k} = \frac{\sqrt{(U)_{j,k}^2 - (U)_{j,j}(U)_{k,k} - (U)_{j,k}}}{\sqrt{(U)_{j,k}^2 - (U)_{j,j}(U)_{k,k} + (U)_{j,k}}}.$$

After taking into account prefactors — checking that

$$\frac{\partial}{\partial q_1} \left(\frac{1}{16 \sqrt{|\det C|}} \right) = B(m, q),$$

where $B(m, q)$ is computed as in Lemma 3 — this expression matches the one in Theorem 1, as expected.

5. The triangle graph: motives and coaction

In this section we study the motives and coaction of the triangle graph, both with non-vanishing and vanishing masses.

5.1. The triangle graph with non-vanishing masses

5.1.1. Motive of the triangle graph with non-vanishing masses In the case of the triangle graph we must work with the full graph motive because there is a linear part as well as the quadric in the polar locus of the integrand for $d = 4$ dimensions. We start with the case when $F = M = N_G = 3$, i.e., all masses and momenta are non-vanishing. In this case the motive of interest is:

$$\text{mot}_G = H^2(\mathbb{P}^2 \setminus (Q \cup L), \Delta \setminus (Q \cup L) \cap \Delta)_{/S}$$

We will need the following lemma to compute the semi-simplification of mot_G . We will make use of the following spectral sequence, called the *Gysin*

spectral sequence [16]. Let D be a simple normal crossing divisor in X , and let $D_I = D_{i_1} \cap \dots \cap D_{i_k}$ where D_i are the irreducible components of D , and $I = \{i_1, \dots, i_k\}$, finally let $D_\emptyset = X$. Then there is the spectral sequence

$$(57) \quad E_1^{-p,q} = \bigoplus_{|I|=p} H^{q-2p}(D_I)(-p) \Rightarrow H^{-p+q}(X \setminus D)$$

where the $d_1^{-p,q} : E_1^{-p,q} \rightarrow E_1^{-p+1,q}$ is the alternating sum of Gysin homomorphisms

$$H^{q-2p}(D_I)(-1) \rightarrow H^{q-2p+2}(D_{I \setminus \{i\}})$$

for each $i \in I$, multiplied by the appropriate sign.

Lemma 4. *Let $Q \subset \mathbb{P}^2$ be a smooth quadric, and $L \subset \mathbb{P}^2$ be a projective line. Then*

$$H^1(\mathbb{P}^2 \setminus (Q \cup L)) \cong \mathbb{Q}(-1) \quad \text{and} \quad H^2(\mathbb{P}^2 \setminus (Q \cup L)) \cong \mathbb{Q}(-2).$$

Proof. Next we need to compute $H^1(\mathbb{P}^2 \setminus (Q \cup L))$ and $H^2(\mathbb{P}^2 \setminus (Q \cup L))$, which we can do using the Gysin spectral sequence. Let us compute the relevant part of the first page. We will need the following elements:

$$(58) \quad \begin{aligned} E_1^{-2,4} &\cong H^0(Q \cap L)(-2) \cong \mathbb{Q}(-2)^{\oplus 2} \\ E_1^{-1,4} &\cong H^2(Q)(-1) \oplus H^2(L)(-1) \cong \mathbb{Q}(-2)^{\oplus 2} \\ E_1^{0,4} &\cong H^4(\mathbb{P}^2) \cong \mathbb{Q}(-2) \\ E_1^{-1,2} &\cong H^0(Q)(-1) \oplus H^0(L)(-1) \cong \mathbb{Q}(-2)^{\oplus 2} \\ E_1^{0,2} &\cong H^2(\mathbb{P}^2) \cong \mathbb{Q}(-1) \\ E_1^{0,0} &\cong H^0(\mathbb{P}^2) \cong \mathbb{Q}(0) \end{aligned}$$

From this we can see that the first page of the spectral sequence is

$$(59) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathbb{Q}(-2)^{\oplus 2} & \rightarrow & \mathbb{Q}(-2)^{\oplus 2} & \rightarrow & \mathbb{Q}(-2) \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Q}(-1)^{\oplus 2} & \rightarrow & \mathbb{Q}(-1) \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Q}(0) \end{array}$$

Let v_L and v_Q be generators of $H^0(L)(-1)$ and $H^0(Q)(-1)$ respectively. Then on the second row we have the Gysin morphisms

$$H^0(L)(-1) \xrightarrow{G} H^2(\mathbb{P}^2) \text{ and } H^0(Q)(-1) \xrightarrow{G} H^2(\mathbb{P}^2),$$

the images of which are the fundamental classes $G(v_L) = [L]$, $G(v_Q) = [Q]$, respectively. Moreover, $[L]$ generates $H^2(\mathbb{P}^2)$, and $[Q] = 2[L]$, since Q intersects L in two points, generically. Therefore, the morphism $\mathbb{Q}(-1)^{\oplus 2} \rightarrow \mathbb{Q}(-1)$ is surjective, and the kernel is generated by $(2v_L, -v_Q)$. Hence

$$gr_2^W H^2(\mathbb{P}^2 \setminus Q \cup L) \cong 0 \text{ and } gr_2^W H^1(\mathbb{P}^2 \setminus Q \cup L) \cong \mathbb{Q}(-1).$$

Finally, $gr_4^W H^2(\mathbb{P}^2 \setminus Q \cup L)$ is the kernel of the map $E_{-2,4}^1 \rightarrow E_{-1,4}^1$. The kernel of

$$H^0(Q \cap L)(-2) \rightarrow H^2(L)(-1)$$

is one-dimensional because $H^2(L \setminus Q \cap L) = 0$, and similarly the kernel of the other Gysin map

$$H^0(Q \cap L)(-2) \rightarrow H^2(Q)(-1)$$

is one-dimensional. Hence

$$gr_4^W H^2(\mathbb{P}^2 \setminus Q \cup L) \cong \mathbb{Q}(-2). \quad \square$$

Lemma 5. *The weight graded pieces of the one-loop triangle graph motive mot_G with non-vanishing masses and momenta are*

$$gr^W mot_G \cong \mathbb{Q}(-2)_{/S} \oplus \mathbb{Q}(-1)_{/S}^{\oplus 5} \oplus \mathbb{Q}(0)_{/S}$$

Proof. It is enough to prove the claim over any fibre $t \in S(\mathbb{C})$. Let

$$\Delta'_I = \Delta_I \setminus (Q \cup L) \cap \Delta_I.$$

Then the E_1 page of the relative cohomology spectral sequence is

$$\begin{aligned} & H^2(\mathbb{P}^2 \setminus (Q \cup L)) \rightarrow \bigoplus_{|I|=1} H^2(\Delta'_I) \rightarrow \bigoplus_{|I|=2} H^2(\Delta'_I) \\ (60) \quad & H^1(\mathbb{P}^2 \setminus (Q \cup L)) \rightarrow \bigoplus_{|I|=1} H^1(\Delta'_I) \rightarrow \bigoplus_{|I|=2} H^1(\Delta'_I) \\ & H^0(\mathbb{P}^2 \setminus (Q \cup L)) \rightarrow \bigoplus_{|I|=1} H^0(\Delta'_I) \rightarrow \bigoplus_{|I|=2} H^0(\Delta'_I) \end{aligned}$$

To compute this, we note that

$$H^1(\Delta_i \setminus (Q \cup L) \cap \Delta_i) \cong H^1(\mathbb{P}^1 \setminus \{u_{1_i}, u_{2_i}, l_{1_i}\}) \cong \mathbb{Q}(-1)^{\oplus 2},$$

where $u_{1_i}, u_{2_i}, l_{1_i}$ are the points of intersection of the quadric Q and the line L with the face $\Delta_i \hookrightarrow \Delta$. From the previous lemma we get the leftmost column, and note that $E_1^{1,2} = E_1^{2,1} = E_1^{1,2} = E_1^{2,2} = 0$. Taking cohomology of the rows we get the result. \square

5.1.2. Coaction on the triangle graph with non-vanishing masses and momenta Notice from the proofs of the previous two lemmas that all the motivic periods of the triangle graph motive in the case of non-vanishing masses, except the one of weight 4 and one of weight 0, are equivalent to motivic periods of the motives associated to the faces of Δ via the face maps, as in (4.1). Note that G/e_i , for $i \in \{1, 2, 3\}$, are bubble graphs with non-vanishing masses. In this case we consider the full motive of the bubble graph mot_{G/e_i} (35) for all i . The fiber over the generic point of $(gr_2^W mot_{G/e_i})_{dR}$ is of rank 2 and we can choose its basis to be the classes of the two forms

$$(61) \quad \begin{aligned} \theta_{G/e_i}^1 &= \frac{\alpha_k d\alpha_j - \alpha_j d\alpha_k}{\Xi_{G/e_i}}, \\ \theta_{G/e_i}^2 &= \frac{(x-y)(\alpha_k d\alpha_j - \alpha_j d\alpha_k)}{(\alpha_k - x\alpha_j)(\alpha_k - y\alpha_j)} \end{aligned}$$

where i, j, k are pairwise distinct and x, y are coordinates of points $u_1 \in Q \cap \Delta_i$ and $L|_{\Delta_i}$ respectively in the coordinate chart of Δ_i where $\alpha_j \neq 0$.

We can choose a basis of the fiber of $(mot_G)_{dR}$ over the generic point such that it contains the class of the Feynman form of G in 4 dimensions $[\omega_G]$, and the 5 weight 2 classes denoted by $[\omega_i^j]$, which we define to be the images of $[\theta_{G/e_i}^j]$ under the de Rham component of the face map (Definition 1)

$$(62) \quad \begin{aligned} \Phi_{i,dR} : (mot_{G/e_i})_{dR} &\rightarrow (mot_G)_{dR} \\ [\theta_{G/e_i}^j] &\mapsto [\omega_i^j] \end{aligned}$$

Note that there are 6 such classes $[\omega_i^j]$, two for each face Δ_i , but there is a relation between them, as can be seen in the proof of Lemma 5.

Proposition 5. *Let G be a triangle graph with generic masses and momenta. Then the motivic Galois coaction on the associated motivic Feynman amplitude is:*

(63)

$$\begin{aligned} \Delta I_G^m &= \sum_{i=2,3} \sum_{j=1,2} I_{G/e_i}^m \left(\theta_{G/e_i}^j \right) \otimes \left[\text{mot}_G, [\omega_i^j]^\vee, [\omega_G] \right]^{\text{dr}} \\ &+ I_{G/e_1}^m \left(\theta_{G/e_1}^1 \right) \otimes \left[\text{mot}_G, [\omega_1^1]^\vee, [\omega_G] \right]^{\text{dr}} + I_G^m \otimes (\mathbb{L}^{\text{dr}})^2 + 1 \otimes I_G^{\text{dr}} \end{aligned}$$

Remark 6. Both the motivic and the de Rham side of the coaction in the previous proposition can be expressed in terms of motivic and de Rham logarithms respectively, using the same techniques as in the previous section.

5.2. Triangle graph with vanishing masses

5.2.1. Motive When one of the masses m_i vanishes, i.e., when we have $F = 3$ and $M = 2$, the quadric Q passes through a point of Δ defined by $V(\alpha_j) \cap V(\alpha_k)$, where i, j, k are pairwise distinct. In the following figure, we have chosen $m_1 = 0$.

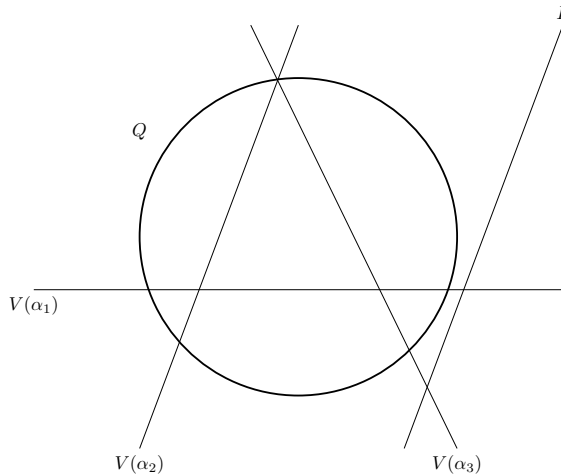


Figure 7: Geometry underlying the triangle graph with one vanishing mass.

This means that the poles of the integrand meet the boundary of the domain of integration, and $\Delta \cup Q \cup L$ is not simple normal crossing anymore, over each fiber of $K_{F,M}^{gen}$. Thus we cannot realize the relevant Feynman integral of the graph as a family of motivic periods of the motive

$H^2(\mathbb{P}^2 \setminus (Q \cup L), \Delta \setminus (Q \cup L) \cap \Delta)_{/S}$. To remedy the situation we must blow up along $V(\alpha_j) \cap V(\alpha_k) \subset \mathbb{P}^2$. The point $V(\alpha_j) \cap V(\alpha_k)$ corresponds to a motivic subgraph γ (see Definition 6) spanned by the edges $\{e_j, e_k\}$ of our triangle graph G .

Let $\pi_G : P^G \rightarrow \mathbb{P}^2$ for the blow-up of \mathbb{P}^2 at the point

$$V(\alpha_3) \cap V(\alpha_2) = [1 : 0 : 0].$$

Denote by \tilde{Q}, \tilde{L} the strict transforms of Q and L where these are given by the vanishing of

$$(64) \quad \begin{aligned} \Psi_G &= \alpha_1 + \alpha_2 + \alpha_3, \text{ and} \\ \Xi_G(m, q) &= q_1^2 \alpha_2 \alpha_3 + q_2^2 \alpha_1 \alpha_3 + q_3^2 \alpha_1 \alpha_2 + (m_2^2 \alpha_2 + m_3^2 \alpha_3) \Psi_G. \end{aligned}$$

Let $D = \pi_G^{-1}(\Delta)$ be the total transform of Δ , and denote by D_{-1} the exceptional divisor corresponding to the motivic subgraph γ (see Figure 8). By the results recalled in 2.5 we get

$$[\tilde{\sigma}_G] \in \Gamma(U_{2,3}^{gen}, (mot_G)_B^\vee)$$

where $(mot_G)_B$ is the Betti realization of the graph motive

$$mot_G = H^2(P^G \setminus \tilde{Q} \cup \tilde{L}, D \setminus (\tilde{Q} \cup \tilde{L}) \cap D)_{/S}.$$

Lemma 6. $\pi_G^*(\omega_G(m, q))$ does not have any poles along the exceptional divisor D_{-1} . Hence $\pi_G^*(\omega_G(m, q))$ is a global section of $\Omega_{P^G \setminus \tilde{Q} \cup \tilde{L}/k_S}^2$, and defines a class $[\pi_G^*(\omega_G(m, q))] \in (mot_G)_{dR}$.

Proof. We can consider the following affine charts of P^G : let $\mathbb{A}_{23,1}$ be the affine space with the coordinate ring $\mathcal{O}(\mathbb{A}_{23,1}) = \mathbb{Z}[\beta_1^{23,1}, \beta_2^{23,1}]$, where $\beta_1^{23,1} = \frac{\alpha_1}{\alpha_2}$, and $\beta_2^{23,1} = \alpha_2$, and $\alpha_3 = 1$. The exceptional divisor, in this affine chart, is given by $\beta_2^{23,1} = 0$. One analogously defines $\mathbb{A}_{23,2}$, by $\beta_1^{23,2} = \alpha_1$, and $\beta_2^{23,2} = \frac{\alpha_2}{\alpha_1}$, where $\alpha_3 = 1$. Two more affine spaces, given by $\alpha_1 = 1$ and $\alpha_2 = 1$, away from the exceptional divisor, complete a covering of P^G .

Let us see what $\pi_G^*(\omega_G(m, q))$ looks like in $\mathbb{A}_{23,1}$. The differential form Ω_G is pulled back to $\pi_G^*(\Omega_G) = d(\beta_1^{23,1} \beta_2^{23,1}) d\beta_2^{23,1} = \beta_2^{23,1} d\beta_1^{23,1} \beta_2^{23,1}$. We also have

$$(65) \quad \begin{aligned} \Psi_{23} &:= \pi_G^*(\Psi_G) = \beta_2^{23,1} d\beta_1^{23,1} + \beta_2^{23,1} + 1 \\ \Xi_{23} &:= \pi_G^*(\Xi_G) = \beta_2^{23,1} (q_1^2 + q_2^2 \beta_1^{23,1} + q_3^2 \beta_1^{23,1} \beta_2^{23,1} + m_2^2 \beta_2^{23,1} \Psi_{23}) \end{aligned}$$

and therefore

$$\pi_G^*(\omega_G(m, q)) = \left(\frac{\beta_2^{23,1} d\beta_1^{23,1} \beta_2^{23,1}}{\Psi_{23} \Xi_{23}} \right)$$

has no poles along $\beta_2^{23,1}$ since it cancels out. The result follows. □

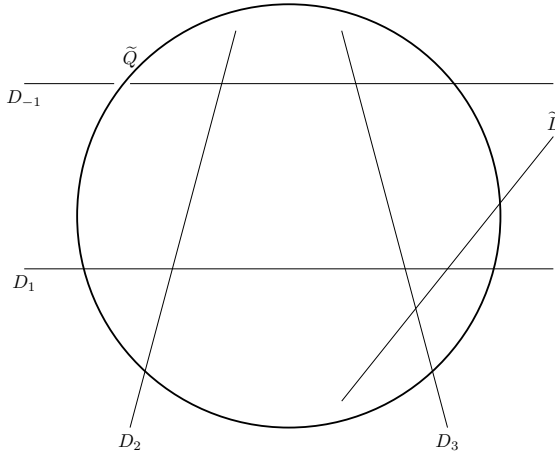


Figure 8: Geometry underlying the triangle graph with one vanishing mass after blowing up.

When either of the other two masses, m_2 or m_3 , vanishes, we have to blow up points $V(\alpha_1) \cap V(\alpha_3)$, and $V(\alpha_1) \cap V(\alpha_2)$ respectively. Denote the corresponding exceptional divisors D_{-2}, D_{-3} . The proof of the previous lemma proceeds analogously, for each of the exceptional divisors. Having done this we get the associated motivic Feynman amplitude:

$$[mot_G, [\sigma_G], [\pi^*(\omega_G)]]^m \in \mathcal{P}_{\mathcal{H}(S)}^{m, U_{k,3}^{gen}, gen},$$

where $1 \leq k < 3$ is the number of non-vanishing masses.

Lemma 7. *Let G be a 1-loop graph with 3 internal edges. Let $1 \leq v \leq 3$ be the number of vanishing masses. Then*

$$gr^W mot_G \cong \mathbb{Q}(-2)_{/S} \oplus \mathbb{Q}(-1)_{/S}^{\oplus 5-v} \oplus \mathbb{Q}(0)_{/S}$$

Proof. Let us see how one proceeds in the case of one vanishing mass. Without loss of generality let that mass be $m_1 = 0$.

Let $D'_I = D_I \setminus (\tilde{Q} \cup \tilde{L}) \cap D_I$. The first page of the relative cohomology spectral sequence for each fiber of $(mot_G)_B$ over any $t \in S(\mathbb{C})$ is

$$\begin{aligned} H^2(P^G \setminus (\tilde{Q} \cup \tilde{L})) &\rightarrow \bigoplus_{i \in \{-1,1,2,3\}} H^2(D'_i) \rightarrow \bigoplus_{i \in \{-1,1,2,3\}} H^2(D'_{ij}) \\ H^1(P^G \setminus (\tilde{Q} \cup \tilde{L})) &\rightarrow \bigoplus_{i \in \{-1,1,2,3\}} H^1(D'_i) \rightarrow \bigoplus_{i \in \{-1,1,2,3\}} H^1(D'_{ij}) \\ H^0(P^G \setminus (\tilde{Q} \cup \tilde{L})) &\rightarrow \bigoplus_{i \in \{-1,1,2,3\}} H^0(D'_i) \rightarrow \bigoplus_{i \in \{-1,1,2,3\}} H^0(D'_{ij}) \end{aligned}$$

We apply the spectral sequence (57) to compute

$$H^2(P^G \setminus (\tilde{Q} \cup \tilde{L})) \text{ and } H^1(P^G \setminus (\tilde{Q} \cup \tilde{L})).$$

Its first page is

$$\begin{aligned} &\dots \rightarrow H^2(\tilde{Q})(-1) \oplus H^2(\tilde{L})(-1) \rightarrow H^4(P^G) \\ &\dots \longrightarrow 0 \longrightarrow 0 \\ (66) \quad &\dots \rightarrow H^0(\tilde{Q})(-1) \oplus H^0(\tilde{L})(-1) \rightarrow H^2(P^G) \\ &\dots \longrightarrow 0 \longrightarrow 0 \\ &\dots \longrightarrow 0 \longrightarrow H^0(P^G) \end{aligned}$$

We have that $H^2(P^G)$ is generated by $[\tilde{L}], [D_{-1}]$, and the image of the Gysin morphism

$$H^0(\tilde{Q})(-1) \rightarrow H^2(P^G)$$

is $[Q] = 2[\tilde{L}] - [D_{-1}]$ because the intersection of $[Q] \cdot [D_{-1}] = -1$ and $[Q] \cdot [L] = 2$. The image of

$$H^0(\tilde{L})(-1) \rightarrow H^2(P^G)$$

is simply $[\tilde{L}]$, so the difference of these two morphisms, which is the map, $E^1_{-1,2} \rightarrow E^1_{0,2}$, is an isomorphism. Therefore

$$gr_2^W H^1(P^G \setminus (\tilde{Q} \cup \tilde{L})) = 0 \text{ and } gr_2^W H^2(P^G \setminus (\tilde{Q} \cup \tilde{L})) = 0.$$

In the top row nothing changes by blowing up, hence the computation is the same as in the Lemma 4. We get

$$H^2(P^G \setminus (\tilde{Q} \cup \tilde{L})) \cong \mathbb{Q}(-2) \text{ and } H^1(P^G \setminus (\tilde{Q} \cup \tilde{L})) \cong 0.$$

Next, we compute $H^1(D'_i)$ for $i \in \{-1, 1, 2, 3\}$. For the exceptional divisor we have $D_{-1} \cap (\tilde{Q} \cup \tilde{L}) = \{1 \text{ point}\}$, therefore

$$H^1(D_{-1} \setminus (\tilde{Q} \cup \tilde{L}) \cap D_{-1}) \cong H^1(\mathbb{A}^1) = 0.$$

For $D_1 = \widetilde{V(\alpha_1)}$, since it is away from the exceptional divisor, we have that $D_1 \cap (\tilde{Q} \cup \tilde{L}) \cong D_1 \cap (Q \cup L) = \{3 \text{ points}\}$, therefore

$$H^1(D_1 \setminus (\tilde{Q} \cup \tilde{L}) \cap D_1) \cong \mathbb{Q}(-1)^{\oplus 2}.$$

For the remaining two irreducible components $D_{3/2} \cap (\tilde{Q} \cup \tilde{L}) = \{2 \text{ points}\}$, therefore

$$H^1(D_{3/2} \setminus (\tilde{Q} \cup \tilde{L}) \cap D_{3/2}) \cong \mathbb{Q}(-1).$$

Putting this together into (5.2.1) we get

$$gr^W H^2(P^G \setminus (\tilde{Q} \cup \tilde{L}), D \setminus (\tilde{Q} \cup \tilde{L}) \cap D) \cong \mathbb{Q}(-2) \bigoplus \mathbb{Q}(-1)^{\oplus 4} \bigoplus \mathbb{Q}(0).$$

When another mass vanishes, say m_2 , in the spectral sequence (66) we get $H^2(P^G) \cong \mathbb{Q}(-1)^{\oplus 3}$, generated by $[\tilde{L}], [D_{-1}], [D_{-2}]$. The image of the Gysin morphism

$$H^0(\tilde{Q})(-1) \rightarrow H^2(P^G)$$

is generated by $2[\tilde{L}] - [D_{-1}] - [D_{-2}]$. Therefore the rank of the cokernel of the morphism $E^1_{-1,2} \rightarrow E^1_{0,2}$ is of rank 1, and the kernel remains trivial. Hence

$$gr^W_2 H^2(P^G \setminus (\tilde{Q} \cup \tilde{L})) \cong \mathbb{Q}(-1).$$

Everything else remains the same as in the previous case, therefore

$$H^2(P^G \setminus (\tilde{Q} \cup \tilde{L})) \cong \mathbb{Q}(-2) \oplus \mathbb{Q}(-1) \text{ and } H^1(P^G \setminus (\tilde{Q} \cup \tilde{L})) \cong 0.$$

We also have

$$\bigoplus_{i \in \{-2, -1, 1, 2, 3\}} H^1(D_i \setminus (\tilde{Q} \cup \tilde{L}) \cap D_i) \cong \mathbb{Q}(-1)^{\oplus 2}$$

where one $\mathbb{Q}(-1)$ comes from D_3 and D_1 each. Plugging these results into the spectral sequence (5.2.1) we get

$$\mathrm{gr}^W H^2(P^G \setminus (\tilde{Q} \cup \tilde{L}), D \setminus (\tilde{Q} \cup \tilde{L}) \cap D) \cong \mathbb{Q}(-2) \bigoplus \mathbb{Q}(-1)^{\oplus 3} \bigoplus \mathbb{Q}(0).$$

Finally, if all three masses vanish we calculate, analogously to the previous two cases,

$$H^2(P^G \setminus (\tilde{Q} \cup \tilde{L})) \cong \mathbb{Q}(-2) \oplus \mathbb{Q}(-1)^{\oplus 2} \text{ and } H^1(P^G \setminus (\tilde{Q} \cup \tilde{L})) \cong 0,$$

where P^G is the blow up of \mathbb{P}^2 at 3 points. This case differs from the previous ones only in $\mathrm{gr}_2^W H^2(P^G \setminus (\tilde{Q} \cup \tilde{L}))$, which is given by the cokernel of the morphism $E_{-1,2}^1 \rightarrow E_{0,2}^1 \cong H^2(P^G)$ in (66). The classes $[\tilde{L}], [D_{-1}], [D_{-2}], [D_{-3}]$ generate $H^2(P^G)$, and the image of the morphism $H^0(\tilde{Q})(-1) \rightarrow H^2(P^G)$ is generated by $2[\tilde{L}] - [D_{-1}] - [D_{-2}] - [D_{-3}]$. Hence the cokernel is of rank 2. Moreover

$$\bigoplus_{i \in \{-3, -2, -1, 1, 2, 3\}} H^1(D_i \setminus (\tilde{Q} \cup \tilde{L}) \cap D_i) \cong 0$$

because we remove 1 point from each of the three exceptional divisors, its intersection with \tilde{Q} , and one from each of the D_i , their intersection with \tilde{L} , which makes D into a hexagon of affine lines. Therefore, from (5.2.1) we get

$$\mathrm{gr}^W H^2(P^G \setminus (\tilde{Q} \cup \tilde{L}), D \setminus (\tilde{Q} \cup \tilde{L}) \cap D) \cong \mathbb{Q}(-2) \bigoplus \mathbb{Q}(-1)^{\oplus 2} \bigoplus \mathbb{Q}(0).$$

□

5.2.2. Coaction on the triangle graph with two or more vanishing masses

When one of the masses in the triangle graph vanishes, and other masses and momenta are generic, the proof of Lemma 7, and the same argument as in 5.1.2, imply that the coaction in this case is

$$\begin{aligned} \Delta I_G^m &= \left[\mathrm{mot}_{G/e_2}'' , [\sigma_{G/e_2}] , [\theta_{G/e_1}^2] \right]^m \otimes \left[\mathrm{mot}_G , [\omega_2^2]^\vee , [\omega_G] \right]^{\mathrm{dr}} + \\ &+ \left[\mathrm{mot}_{G/e_3}'' , [\sigma_{G/e_3}] , [\theta_{G/e_3}^2] \right]^m \otimes \left[\mathrm{mot}_G , [\omega_3^2]^\vee , [\omega_G] \right]^{\mathrm{dr}} + \\ (67) \quad &+ \sum_{j=1,2} I_{G/e_1}^m \left(\theta_{G/e_1}^j \right) \otimes \left[\mathrm{mot}_G , [\omega_1^j]^\vee , [\omega_G] \right]^{\mathrm{dr}} + \\ &+ I_G^m \otimes (\mathbb{L}^{\mathrm{dr}})^2 + 1 \otimes I_G^{\mathrm{dr}} \end{aligned}$$

where we have taken, without loss of generality, $m_1 = 0$, and mot''_{G/e_2} and mot''_{G/e_3} are motives of bubble graphs with one vanishing mass (37), while mot_{G/e_1} is the motive of a bubble graph with non-vanishing masses (35).

In the proof of Lemma 7 we saw that when 2 masses of the triangle graph vanish we have

$$gr^W H^2(P^G \setminus (\tilde{Q} \cup \tilde{L})) \cong \mathbb{Q}(-2) \oplus \mathbb{Q}(-1).$$

Similarly when all 3 masses vanish we get

$$gr^W H^2(P^G \setminus (\tilde{Q} \cup \tilde{L})) \cong \mathbb{Q}(-2) \oplus \mathbb{Q}(-1)^{\oplus 2}.$$

Note that for the motive of the triangle graph with generic non-vanishing masses and momenta all motivic periods of weight 2 are equivalent to motivic periods of the motives $H^1(\Delta_i \setminus (Q \cup L) \cap \Delta_i, \Delta_i \cap (\Delta_j \cup \Delta_k))$, where $1 \leq i \leq 3$, and i, j, k pairwise distinct, via the face maps. This is crucial in relating the motivic side of the coaction to the motivic periods associated to motives of quotient graphs. However, when two or more masses vanish, this is not the case any longer.

We can still compute the coaction, as we now show in the case of all three masses vanishing and three non-trivial external momenta. Consider the following hyperplanes:

$$F_1 := V(\alpha_2 + \alpha_3), \quad F_2 := V(\alpha_1 + \alpha_3)$$

and denote by F_i their strict transforms as well. Note that we still consider all schemes base changed to $K_{F,M}$. Let D_{-i} be the exceptional divisor over the point $V(\alpha_j) \cap V(\alpha_k)$ for $1 \leq i, j, k \leq 3$ pairwise distinct. Denote the following intersection points:

$$D_i \cap \tilde{L} = l_i, \quad D_{-i} \cap \tilde{Q} = u_i, \quad F_i \cap D_{-i} = g_i, \quad F_i \cap D_i = t_i, \quad \text{for } i = 1, 2$$

and

$$D_{-i} \cap D_j = p_0^i, \quad D_{-i} \cap D_k = p_1^i, \quad i = 1, 2, \quad i, j, k \text{ pairwise distinct}$$

as well as

$$\tilde{Q} \cap \tilde{L} = \{f_0, f_1\}, \quad D_i \cap \tilde{L} = d_i, \quad \text{for } i = 1, 2, 3.$$

Theorem 3. *Let G be the triangle graph with all internal masses vanishing, and non-trivial momenta. Then the coaction on the associated motivic Feynman amplitude is:*

$$\begin{aligned} \Delta I_G^m &= a_1 \log^m([g_1 u_1 | p_0^1 p_1^1]) \otimes b_1 \log^{\text{dr}}(f_0 f_1 | d_1 d_2) + \\ &+ a_2 \log^m([g_2 u_2 | p_0^2 p_1^2]) \otimes b_2 \log^{\text{dr}}(f_0 f_1 | d_1 d_2) + \\ &+ a_3 \log^m([g_1 u_1 | p_0^1 p_1^1]) \otimes b_3 \log^{\text{dr}}(f_0 f_1 | d_1 d_3) + \\ &+ a_4 \log^m([g_2 u_2 | p_0^2 p_1^2]) \otimes b_4 \log^{\text{dr}}(f_0 f_1 | d_1 d_3) + \\ &+ I_G^m \otimes (\mathbb{L}^{\text{dr}})^2 + 1 \otimes I_G^{\text{dr}} \end{aligned}$$

where a_i, b_i are undetermined constants in k_S . The cross-ratios evaluate to:

$$\log^m([g_i u_i | p_0^i p_1^i]) = \log^m \left(\frac{q_j^2}{q_k^2} \right), \quad i = 1, 2, \quad i, j, k \text{ pairwise distinct}$$

and

$$\log^{\text{dr}}([f_0 f_1 | d_1 d_2]) = \log^{\text{dr}} \left(\frac{q_3^2 - q_2^2 - q_1^2 - \sqrt{q_1^4 + q_2^4 + q_3^4 - 2q_1^2 q_3^2 - 2q_2^2 q_3^2}}{q_3^2 - q_2^2 - q_1^2 + \sqrt{q_1^4 + q_2^4 + q_3^4 - 2q_1^2 q_3^2 - 2q_2^2 q_3^2}} \right)$$

and

$$\begin{aligned} \log^{\text{dr}}([f_0 f_1 | d_1 d_3]) &= \log^{\text{dr}}([f_0 f_1 | d_1 d_2]) - \\ &+ \log^{\text{dr}} \left(\frac{q_3^2 + q_2^2 - q_1^2 - \sqrt{q_1^4 + q_2^4 + q_3^4 - 2q_1^2 q_3^2 - 2q_2^2 q_3^2}}{q_3^2 + q_2^2 - q_1^2 + \sqrt{q_1^4 + q_2^4 + q_3^4 - 2q_1^2 q_3^2 - 2q_2^2 q_3^2}} \right) \end{aligned}$$

Proof. From Lemma 7 and the definition of the coaction we have

$$\Delta I_G^m = I_G^m \otimes (\mathbb{L}^{\text{dr}})^2 + \sum_{i=1,2} [\text{mot}_G, [\sigma_G], e_i]^m \otimes [\text{mot}_G, e_i^\vee, \omega_G]^m + 1 \otimes I_G^{\text{dr}},$$

where e_i are weight 2 elements of a chosen basis of $(\text{mot}_G)_{\text{dR}}$. We restrict to the fiber over the generic point. To identify the de Rham side we consider the residue morphism along \tilde{L} :

$$\text{Res}_{\tilde{L}} : H^2(P^G \setminus (\tilde{Q} \cup \tilde{L}), D \setminus (\tilde{Q} \cup \tilde{L}) \cap D) \rightarrow H^1(\tilde{L} \setminus \tilde{L} \cap \tilde{Q}, \tilde{L} \cap D)(-1).$$

Note that

$$H^1(\tilde{L} \setminus \tilde{L} \cap \tilde{Q}, \tilde{L} \cap D)(-1) \cong H^1(\mathbb{P}^1 \setminus \{f_0, f_1\}, \{d_1, d_2, d_3\})(-1).$$

This morphism gives us an equivalence of de Rham periods, and by computing the motivic periods of the motive on the right hand side, and their de Rham projections (see 2.4.2), we get that each de Rham period in our coaction is a k_S -linear combination of $\log^{\text{dr}}([f_0 f_1 | d_1 d_2])$ and $\log^{\text{dr}}([f_0 f_1 | d_1 d_3])$.

To determine the motivic periods in the coaction consider the pullback morphism:

$$i^* : H^2(P^G \setminus (\tilde{Q} \cup \tilde{L}), D \setminus (\tilde{Q} \cup \tilde{L}) \cap D) \rightarrow H^2(P^G \setminus W, D \setminus W \cap D)$$

where $W = \tilde{Q} \cup \tilde{L} \cup F_1 \cup F_2$. Computing the Gysin spectral sequence (57) and the relative cohomology spectral sequence for the motive on the right we get that

$$H^2(P^G \setminus W) \cong \mathbb{Q}(-2)$$

and

$$H^1(D_j \setminus W \cap D_j) \cong \mathbb{Q}(-1) \quad \text{for } j = \{-1, -2\}$$

and

$$H^1(D_j \setminus W \cap D_j) \cong 0 \quad \text{for } j = \{1, 2\},$$

therefore, since the pullback preserves the weight, the class of $i^*(e_i)$ vanishes in $H^2(P^G \setminus W)$. We can therefore write it as the differential of a 1-form which in turn restricts to a k_S -linear combination of non-exact differential forms on $D_j \setminus W \cap D_j$ for $j = \{-2, -1\}$. We get an equivalence of classes of differential forms:

$$(68) \quad \begin{aligned} [(i^*(e_i), 0, \dots, 0)] &= a_{i1}[(0, \theta_1, 0, \dots, 0)] + \\ &+ a_{i2}[(0, 0, \theta_2, 0, \dots, 0)] \end{aligned}$$

where θ_l are forms of the form (38) with simple poles at the points u_l and g_l , for $l = 1, 2$. For $j = \{-2, -1\}$ and l, k, j pairwise distinct, let

$$E_j = H^1(D_j \setminus W \cap D_j, D_j \cap (D_k \cup D_l)).$$

We get an equivalence of motivic periods:

$$(69) \quad \begin{aligned} [H_G, [\sigma_G], e_i]^{\text{m}} &= a_{i1}[E_{-1}, [\sigma_G|_{D_{-1}}], [(0, \theta_1, 0, \dots, 0)]]^{\text{m}} + \\ &+ a_{i2}[E_{-2}, [\sigma_G|_{D_{-2}}], [(0, 0, \theta_2, \dots, 0)]]^{\text{m}} \end{aligned}$$

Now notice that $[E_{-l}, [\sigma_G|_{D_{-l}}], [(0, \dots, \theta_l, 0, \dots, 0)]]^{\text{m}} = \log^{\text{m}}(g_l u_l | p_0^l p_1^l)$, and collect all the constants into a_1, a_2 to obtain the result. It is a standard exercise to put coordinates on a blow up of \mathbb{P}^2 at a point (see proof of

Lemma 6), which in turn enables us to compute the intersection points and the cross ratios. \square

Remark 7. The constants a_1, a_2, b_1, b_2 depend on the choice of a basis of $(\text{mot}_G)_{\text{dR}}$. In order to determine them one would need to write down such a basis, and follow the recipe in the proof of the previous theorem.

Remark 8. There are two ways of relating the motivic periods which are the conjugates of the motivic Feynman amplitude of the graph in the previous theorem to motives of its subquotient graphs. One is to consider the *affine motive of a graph*, defined in [4, 5.4 and 8.5]. This involves removing a hyperplane for each motic subgraph of a graph, or equivalently for each exceptional divisor in the blow up, in order to make the faces of D affine, as was done in the proof of the previous theorem. In the example of the massless triangle the affine motive would be

$$H^2(P \setminus (\tilde{Q} \cup \tilde{L} \cup F_1 \cup F_2 \cup F_3), D \setminus (\tilde{Q} \cup \tilde{L} \cup F_1 \cup F_2 \cup F_3))$$

where F_1 and F_2 are as in the theorem above, and $F_3 := V(\alpha_1 + \alpha_2)$. Removing the third hyperplane is superfluous in this example because in that case we would get another motivic logarithm in the coaction, $\log^m\left(\frac{q_1^2}{q_2^2}\right)$, but there is an obvious relation with the two motivic logarithms in the theorem. These three motivic logarithms are periods of the affine motives of the subgraphs of the triangle graph obtained by removing one edge. However, in physics one rarely thinks of cycle-free graphs contributing logarithms.

Therefore, in order to produce a more satisfying graphical interpretation of the motivic side of the coaction in the preceding example, we should regularize the motivic periods of the bubble graphs which are obtained by contracting an edge of the triangle graph. We could then obtain the conjugates in the coaction as a linear combination of these regularized motivic Feynman periods of quotient graphs. Conjecture 1 in [4] predicts that after including regularized motivic Feynman periods the Galois conjugates would be motivic periods of subquotient graph motives.

From the point of view of physics, it would be preferable to use the method of dimensional regularization. This method involves allowing the number of space-time dimensions d to vary, and considering the Laurent expansion of the Feynman integral around a fixed point. In [1] the authors consider one-loop graphs in this setting, in which case the coefficients in the regularization parameters are multiple polylogarithms. They found that after applying the coaction for multiple polylogarithms term by term to the coefficients of this expansion, it might be possible to rearrange the

resulting expression into a linear combination of tensor products, where of the left hand side of the tensor product we find dimensionally regularized Feynman integrals associated to subquotient graphs, and on the right hand side there are Cutkosky cuts of the graphs. They check this up to a finite order in the regularization parameter for several examples, and conjectured that it should hold for all one-loop graphs. For the example of the triangle graph with vanishing internal masses which is the object of Theorem 2 see [1, §3.3]. An algebro-geometric approach to the interplay of dimensional regularization and the motivic coaction will be the topic of future work.

Another method of regularizing, familiar in algebraic geometry, is by using tangential base points. For an example of how this works for a triangle graph with one mass vanishing in $d = 2$, which is very close to Theorem 2, see [4, Appendix II].

Other interesting directions for further inquiry include applying the tools presented here to some non-polylogarithmic Feynman integrals which have already been studied from a motivic point of view [10, 11], as well as studying the situations in which the masses and momenta lie outside of $K_{F,G}^{gen}$ in Definition 5, but for which there is numerical evidence that shows the Galois coaction could still be closed on the space of motivic Feynman periods, such as the ones which arise in QED [30].

Acknowledgements

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 724638). The author owes special thanks to Francis Brown for his encouragement to study the motivic Galois theory of Feynman periods in general, and many discussions and comments on this paper in particular. The author also thanks Erik Panzer for discussions, and in particular for his help with a computation in section 3 which compares the computations in this paper with those in the physics literature. Thanks are also owed to Clément Dupont and Nils Matthes for helpful discussions.

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RECEIVED NOVEMBER 4, 2019

ACCEPTED OCTOBER 17, 2020