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Rational 2-functions are abelian

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We show that the coefficients of rational 2-functions are contained in an abelian number field. More precisely, we show that the poles of such functions are poles of order one and given by roots of unity and rational residue.

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1. Introduction

Fermat's and Euler's congruences are well-known in number theory and are rich of remarkable consequences. In the following we will give a short survey of these congruences. We start with the famous

Theorem 1.1 (Euler)**.** *The congruence*

$$a^{p^r} \equiv a^{p^{r-1}} \mod p^r \tag{1.1}$$

holds for all integers $a \in \mathbb{Z}$, all primes p, and all natural numbers $r \in \mathbb{N}$.

A sequence $(a_k)_{k\in\mathbb{N}}$ of rational numbers is called an *Euler sequence* (or *Gauss sequence* as in [4]) for the prime p, if a_k is a p-adic integer for all $k \in \mathbb{N}$ and

$$a_{mp^r} \equiv a_{mp^{r-1}} \mod p^r \tag{1.2}$$

for all integers $r \geq 1$ and $m \geq 1$. A survey of these congruences has been given in [9] and [12].

Beukers coined the term *supercongruence*: A supercongruence (with respect to a prime p) refers to a sequence $(a_n)_{n\in\mathbb{N}} \in \mathbb{Z}_p^{\mathbb{N}}$ that satisfies congruences of the type

$$a_{mp^r} \equiv a_{mp^{r-1}} \mod p^{sr}, \tag{1.3}$$

for all $m, r \in \mathbb{N}$ and a fixed $s \in \mathbb{N}$, $s > 1$ (cf. [6]). Such supercongruences are given by the Jacobsthal-Kazandzidis congruence (cf. [5]), Apéry numbers (cf. [2], [3]), generalized Domb numbers (cf. [10]) and Almkvist-Zudilin numbers (cf. [1], [7]) to name a few. Note that all the above mentioned supercongruences are valid for $s = 3$ with respect to $p \geq 5$.

Let K be an algebraic number field and $\mathcal{O}$ its ring of algebraic integers. We consider a generalization of supercongruences to sequences of algebraic integers in K. More precisely, for $s \in \mathbb{N}$, an s-sequence is a sequence $(a_n) \in K^{\mathbb{N}}$, such that for any unramified prime ideal $\mathfrak{p} \in \mathcal{O}$ lying above the prime $p \in \mathbb{Z}$, $a_n \in \mathcal{O}_{\mathfrak{p}}$, and for all $m, r \in \mathbb{N}$,

$$\mathrm{Frob}_{\mathfrak{p}}(a_{p^{r-1}m}) - a_{p^r m} \equiv 0 \mod \mathfrak{p}^{sr}\mathcal{O}_{\mathfrak{p}},$$

where $\mathcal{O}_{\mathfrak{p}}$ is the ring of $\mathfrak{p}$-adic integers and $\mathrm{Frob}_{\mathfrak{p}}$ is the canonical lift of the standard Frobenius element of $\mathfrak{p}$ in the Galois group of the local field extension $(\mathcal{O}/\mathfrak{p})|(\mathbb{Z}/p)$. The generating function $V(z)$ of an s-sequence then integrates to what is referred to as an s-function in [11]. More precisely, the s-sequence $a \in K^{\mathbb{N}}$ corresponds to the s-function $\textstyle\int^s V(z)$ given by the (formal) power series

$$\textstyle\int^s V(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} z^n \in zK[\![z]\!],$$

Interestingly, 2-functions (where $s = 2$) have their geometric origin in super symmetry. As stated in [11], see Thm. 22 therein, 2-functions appear as the non-singular part of the superpotential function (without the constant term) with algebraic coefficients. In other words, algebraic cycles on Calabi-Yau three-folds provide a source of 2-functions that are analytic and furthermore satisfy a differential equation with algebraic coefficients. It is therefore expected that understanding the numerical interpretation of open Gromov-Witten/BPS theory relative to Lagrangian submanifolds mirror to algebraic cycles highly depend on delivering some (natural) basis of the class of 2-functions with algebraic coefficients. It is therefore of main interest to characterize a submodule of s-functions by suitable algebraic or analytic properties, and a class of distinguished generators for this submodule. The contribution of the present work to this problem is to give a characterization of a 2-function $\int^2 V(z)$, where V represents a rational function. We have

Theorem 1.2. *Let $V \in zK[\![z]\!]$, $V(z) \neq 0$, be the generating function of a 2-sequence $(a_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}}$, representing the rational function $F(z) \in K(z)$ as its Maclaurin expansion. Then, there is root of unity ζ primitive of order N and rational coefficients $A_i \in \mathbb{Q}$ for $i = 1, ..., N$ such that*

$$F(z) = \sum_{i=1}^{N} \frac{A_i \zeta^i z}{1 - \zeta^i z}. \tag{1.4}$$

In particular, the coefficients a_n of $V(z)$ are periodic and have the form

$$a_n = \sum_{i=1}^{N} A_i \zeta^{in}. \tag{1.5}$$

The first reduction in the proof of Theorem 1.2 is given by Theorem 3.1, a statement due to Minton (cf. [9]). It states, that the generating functions of Euler sequences are given by sums of logarithmic derivatives of polynomials with integral coefficients.

2. Preliminaries

In this section, we recall basic notations in number fields and introduce s-sequences. For further reference see also [11]. Throughout this paper, the natural numbers will be meant to be the set of all positive integers, $\mathbb{N} = \{1, 2, ...\}$, while $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If X is a set, then $X^{\mathbb{N}}$ denotes the set of all sequences indexed by the natural numbers, $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$. For a ring R let $R[\![z]\!]$ denote the ring of formal power series in the variable z with coefficients in R.

Let K be a fixed algebraic number field and assume K to be normal over $\mathbb{Q}$. Denote by $\mathcal{O}$ the ring of integers of K. Let D be the discriminant of $K|\mathbb{Q}$. We say that a prime $p \in \mathbb{Z}$ is unramified in $K|\mathbb{Q}$ if all prime ideals $\mathfrak{p} \mid p\mathcal{O}$ are unramified. Note that an unramified prime p is characterized by the property that $p \nmid D$. For any prime ideal $\mathfrak{p}$, $\mathcal{O}_{\mathfrak{p}}$ denotes the ring of $\mathfrak{p}$-adic integers. Then $\mathcal{O}_{\mathfrak{p}}$ is an integral domain and its field of fractions $K_{\mathfrak{p}} = \mathrm{Quot}(\mathcal{O}_{\mathfrak{p}})$ is the $\mathfrak{p}$-adic completion of K.

For $\mathfrak{p} \mid (p)$, the *Frobenius element* $\mathrm{Fr}_{\mathfrak{p}}$ at $\mathfrak{p}$ is the unique element satisfying the following two conditions: $\mathrm{Fr}_{\mathfrak{p}}$ is an element in the decomposition group $D(\mathfrak{p}) \subset \mathrm{Gal}(K/\mathbb{Q})$ of $\mathfrak{p}$ and for all $x \in \mathcal{O}$, $\mathrm{Fr}_{\mathfrak{p}}(x) \equiv x^p \mod \mathfrak{p}$. By Hensel's Lemma, $\mathrm{Fr}_{\mathfrak{p}}$ can be lifted to $\mathcal{O}_{\mathfrak{p}}$ and then extended to an automorphism $\mathrm{Frob}_{\mathfrak{p}} \colon K_{\mathfrak{p}} \to K_{\mathfrak{p}}$. By declaring $\mathrm{Frob}_{\mathfrak{p}}(z) = z$, $\mathrm{Frob}_{\mathfrak{p}}$ can be (linearly) extended to an endomorphism $\mathrm{Frob}_{\mathfrak{p}} \colon K_{\mathfrak{p}}[\![z]\!] \to K_{\mathfrak{p}}[\![z]\!]$.

In [11], an *s-function with coefficients in K* (for $s \in \mathbb{N}$) is defined to be a formal power series $\widetilde{V} \in zK[\![z]\!]$ such that for every unramified prime ideal $\mathfrak{p} \subset \mathcal{O}$ dividing $p \in \mathbb{Z}$, we have

$$\frac{1}{p^s}\mathrm{Frob}_{\mathfrak{p}}\widetilde{V}(z^p) - \widetilde{V}(z) \in z\mathcal{O}_{\mathfrak{p}}[\![z]\!]. \tag{2.1}$$

A sequence $(a_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}}$ is said to satisfy the *local s-function property for p*, if $p \in \mathbb{Z}$ is unramified in $K|\mathbb{Q}$, and $a_n \in \mathcal{O}_p$ is a p-adic integer for all $n \in \mathbb{N}$, and

$$\mathrm{Frob}_{\mathfrak{p}}\left(a_{mp^{r-1}}\right) \equiv a_{mp^r} \mod p^{sr}\mathcal{O}_{\mathfrak{p}}, \tag{2.2}$$

for all $m, r \in \mathbb{N}$ and all prime ideals $\mathfrak{p} \mid (p)$. $(a_n)_{n\in\mathbb{N}}$ is called an *s-sequence* if it satisfies the local s-function property for all unramified primes p in $K|\mathbb{Q}$. By definition, it is evident that the coefficients of an s-sequence are contained in $\mathcal{O}\left[D^{-1}\right]$. The coefficients of s-functions are given by an s-sequence after applying s-fold derivation by the Euler operator $z\frac{\mathrm{d}}{\mathrm{d}z}$ (cf. [11, Lem. 4]). We denote by $\mathcal{S}^s(K|\mathbb{Q}) \subset z\mathcal{O}\left[D^{-1}\right][\![z]\!]$ the set of all generating functions of s-sequences with coefficients in K. Furthermore, the set $\mathcal{S}^s_{\mathrm{rat}}(K|\mathbb{Q})$ denote the subset in $\mathcal{S}^s(K|\mathbb{Q})$ of power series which represent rational functions.

3. A theorem due to Minton

The next Theorem 3.1 is a adaptation of Thm. 7.1 in [4], which on the other hand is a re-proven statement from [9]. It is the starting point for the proof of Theorem 1.2. The crucial insight is that a rational 1-function only admits poles of order 1. We give a proof for the sake of completeness. In the course of this, we follow the ideas given in [4].

Theorem 3.1 (compare with [4], [9])**.** *Let $V \in \mathcal{S}^1_{\mathrm{rat}}(K|\mathbb{Q})$ representing the rational function $F(z) \in K(z)$ as its Maclaurin expansion. Then there is an integer $r \in \mathbb{N}$, distinct algebraic numbers $\alpha_i \in \overline{\mathbb{Q}}^{\times}$, and $A_i \in \mathbb{Q}^{\times}$, for*

$i = 1, ..., r$, such that F can be written as

$$F(z) = \sum_{i=1}^{r} \frac{A_i \alpha_i z}{1 - \alpha_i z}.$$

Proof. Let F be given by the fraction of $P, Q \in K[z]$, $Q \not\equiv 0$, i.e. $F = \frac{P}{Q}$. We may assume that $Q(0) \neq 0$ and $P(0) = 0$. By [4, Prop. 3.5], we have $\deg(P) \leq \deg(Q)$. By adding a constant $C \in K$ to F it does not affect the 1-function condition but we may assume $\deg(P) < \deg(Q)$. Then, by the *Partial Fraction Decomposition* $\widetilde{F} = \frac{P}{Q} + C$ has the form

$$\widetilde{F} = \sum_{i=1}^{r} \sum_{j=1}^{m_i} \frac{A_{i,j}}{(1 - \alpha_i z)^j},$$

where the $\alpha_i \in \overline{\mathbb{Q}}^{\times}$, $i \in \{1, ..., r\}$ are distinct algebraic numbers, $m_i \in \mathbb{N}$ and $A_{i,j} \in \overline{\mathbb{Q}}$ for all $(i, j) \in \{1, ..., r\} \times \{1, ..., m_i\}$. Let p be a sufficiently large prime, unramified in $K|\mathbb{Q}$, such that $\alpha_i, \alpha_i - \alpha_j \in \mathcal{O}_{\mathfrak{p}}^{\times}$, and $p > m_i$ for all $i, j \in \{1, ..., r\}$. What we need to show is $m_i = 1$ for all $i \in \{1, ..., r\}$. We have

$$\frac{1}{(1 - \alpha_i z)^j} = \sum_{k=0}^{\infty} \binom{k + j - 1}{j - 1} \alpha_i^k z^k.$$

Therefore, if $\widetilde{V}(z) = V(z) + C$ is the Maclaurin series expansion of $\widetilde{F}$, the Cartier operator $\mathscr{C}_p$ applied to $\widetilde{V}$ is given by

$$\mathscr{C}_p \widetilde{V} = \sum_{k=0}^{\infty} \left[\sum_{i=1}^{r} \sum_{j=1}^{m_i} A_{i,j} \binom{pk + j - 1}{j - 1} \alpha_i^{pk} \right] z^k.$$

Since $p > m_i$, we find $\binom{pk+\nu}{\nu} \equiv 1 \mod p$ for all $0 \leq \nu < m_i$ (in particular, $\nu < p$) by the following calculation

$$\binom{pk + \nu}{\nu} = \prod_{\ell=1}^{\nu} \left(1 + \frac{pk}{\ell}\right) \equiv 1 \mod p.$$

Consequently,

$$\mathscr{C}_p \widetilde{V} \equiv \sum_{k=0}^{\infty} \left[\sum_{i=1}^{r} \sum_{j=1}^{m_i} A_{i,j} \alpha^{pk} \right] z^k = \sum_{i=1}^{r} \sum_{j=1}^{m_i} \frac{A_{i,j}}{1 - \alpha_i^p z} = \sum_{i=1}^{r} \frac{A_i}{1 - \alpha_i^p z} \mod p,$$

where $A_i = \sum_{j=1}^{m_i} A_{i,j}$. Hence, $\mathscr{C}_p\widetilde{V}$ represents a rational function with exclusively simple poles modulo p. Thus, the 1-function property $\mathscr{C}_p\widetilde{V} - \mathrm{Frob}_{\mathfrak{p}}\, \widetilde{V} \equiv 0 \mod p\mathcal{O}_p[\![z]\!]$ ensures that $\widetilde{F}$ has only simple poles as well. Therefore, we write from now on

$$\widetilde{F} = \sum_{i=1}^{r} \frac{A_i}{1-\alpha_i z},$$

where $A_i, \alpha_i \in \overline{\mathbb{Q}}^{\times}$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. Evaluating $\widetilde{F}$ at $z = 0$ we conclude that $C = \sum_{i=1}^{r} A_i$. Therefore,

$$F = \widetilde{F} - C = \sum_{i=1}^{r} \frac{A_i}{1-\alpha_i z} - \sum_{i=1}^{r} A_i = \sum_{i=1}^{r} \frac{A_i \alpha_i z}{1-\alpha_i z}.$$

In particular, we have $a_n = \sum_{i=1}^{r} A_i \alpha_i^n$ for all $n \in \mathbb{N}$. The local 1-function property for p then gives

$$0 \equiv \mathrm{Frob}_{\mathfrak{p}}(a_m) - a_{mp} = \sum_{i=1}^{r} (\mathrm{Frob}_{\mathfrak{p}}(A_i)\mathrm{Frob}_{\mathfrak{p}}(\alpha_i^m) - A_i \alpha_i^{mp}) \mod p\mathcal{O}_{\mathfrak{p}},$$

for all $m \in \mathbb{N}$. Since Frob_p is given by taking component-wise the p-th power modulo $\mathfrak{p}$ for all $\mathfrak{p} \mid (p)$, we conclude

$$0 \equiv \sum_{i=1}^{r} (A_i^p - A_i)\, \alpha_i^{mp} \mod p\mathcal{O}_{\mathfrak{p}},$$

for all $m \in \mathbb{N}$. The Vandermonde type $r \times r$ matrix $M = \left(\alpha_j^{ip}\right)_{i,j=1,\dots,r}$. is invertible modulo $p\mathcal{O}_p$. Indeed, its determinant is given by

$$\det(M) \equiv \Big(\prod_{i=1}^{r} \alpha_i^p\Big) \times \prod_{1 \leq i < j \leq r} (\alpha_j - \alpha_i)^p \mod p\mathcal{O}_{\mathfrak{p}}.$$

By assumption, we obtain $\det(M) \in \mathcal{O}_{\mathfrak{p}}^{\times}$. Hence, $A_i^p \equiv A_i \mod p$ for all $i \in \{1, \dots, r\}$. From *Frobenius's Densitiy Theorem*, see for instance [8], it follows that $A_i \in \mathbb{Q}$ for all $i \in \mathbb{N}$. □

4. Proof of Theorem 1.2

In the present section we will give a proof of Theorem 1.2. Let $V \in \mathcal{S}^2_{\mathrm{rat}}(K|\mathbb{Q})$ and let a_n be the n-th coefficient of V. In particular, $V \in \mathcal{S}^1(K|\mathbb{Q})$ and by Theorem 3.1, there is an $r \in \mathbb{N}$, $A_i \in \mathbb{Q}^\times$ and distinct $\alpha_i \in \overline{\mathbb{Q}}^\times$ for $i \in \{1, ..., r\}$ such that a_n are given by the power sums $a_n = \sum_{i=1}^r A_i \alpha_i^n$ for all $n \in \mathbb{N}$. In the following, let us assume $\alpha_i \in K$, since we might otherwise substitute K by a normal closure of $K(\alpha_1, ..., \alpha_r)$. Let $p \in \mathbb{Z}$ be a prime, unramified in $K|\mathbb{Q}$ and let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}$ dividing p such that

(i) A_i, α_i, $\alpha_i - \alpha_j$ and all their $\mathrm{Frob}_\mathfrak{p}$-conjugates are $\mathfrak{p}$-adic units, and
(ii) $\max\{2, r\} < p$.

In particular, p is a generator of the maximal ideal in $\mathcal{O}_\mathfrak{p}$.

Lemma 4.1. *Let $p \geq 3$ and let $x \in \mathcal{O}_\mathfrak{p} \setminus \{0\}$. Then for every $n \geq 1$ there is a $\rho_n \in \mathcal{O}_\mathfrak{p}$ such that*

$$\frac{x^{p^n}}{\mathrm{Frob}_\mathfrak{p}\left(x^{p^{n-1}}\right)} = 1 + p^n \rho_n \tag{4.1}$$

and $\mathrm{ord}_p(\rho_n) = \mathrm{ord}_p(\rho_1)$ for all $n \in \mathbb{N}$.

Proof. For $n = 1$, by definition of $\mathrm{Frob}_\mathfrak{p}$ there is a $\rho_1 \in \mathcal{O}_\mathfrak{p}$ such that eq. (4.1) is satisfied. Let $n \in \mathbb{N}$ and suppose that eq. (4.1) is true for $k < n$. Then

$$\frac{x^{p^n}}{\mathrm{Frob}_\mathfrak{p}\left(x^{p^{n-1}}\right)} = \left(\frac{x^{p^{n-1}}}{\mathrm{Frob}_\mathfrak{p}\left(x^{p^{n-2}}\right)}\right)^p = \left(1 + p^{n-1}\rho_{n-1}\right)^p$$

$$= 1 + p^n \rho_{n-1}\left(1 + \sum_{k=2}^{p-1} \binom{p}{k} p^{(k-1)(n-1)-1} \rho_{n-1}^{k-1} + p^{(p-1)(n-1)-n} \rho_{n-1}^{p-1}\right).$$

Since $p > 2$, the sum in the brackets has p-order greater than 0. Therefore, $\rho_n = \rho_{n-1}(1 + \rho')$ with $\rho' \in p\mathcal{O}_\mathfrak{p}$. Hence, $\mathrm{ord}_p(\rho_{n-1}) = \mathrm{ord}_p(\rho_n)$ for all $n \in \mathbb{N}$. ∎

For $x \in \mathcal{O}_\mathfrak{p} \setminus \{0\}$ we then obtain for all $n \in \mathbb{N}$

$$\frac{x^{mp^n}}{\mathrm{Frob}_\mathfrak{p}\left(x^{mp^{n-1}}\right)} \equiv 1 + mp^n \rho_n \mod p^{2n}\mathcal{O}_\mathfrak{p}. \tag{4.2}$$

Set

$$\frac{\alpha_i^{p^n}}{\mathrm{Frob}_\mathfrak{p}\left(\alpha_i^{p^{n-1}}\right)} = 1 + p^n \rho_{i,n},$$

with $\rho_{i,n} \in \mathcal{O}_{\mathfrak{p}}$ for all $i = 1, ..., r$. The local 2-function property eq. (2.2) then gives

$$0 \equiv a_{mp^n} - \mathrm{Frob}_{\mathfrak{p}}\left(a_{mp^{n-1}}\right) = \sum_{i=1}^{r} A_i \left(\alpha_i^{mp^n} - \mathrm{Frob}_{\mathfrak{p}}\left(\alpha_i^{mp^{n-1}}\right)\right) \mod p^{2n}\mathcal{O}_{\mathfrak{p}}.$$

Applying Lemma 4.1 and eq. (4.2) then gives

$$0 \equiv mp^n \sum_{i=1}^{r} A_i \,\mathrm{Frob}_{\mathfrak{p}}\left(\alpha_i^{mp^{n-1}}\right) \rho_{i,n} \mod p^{2n}\mathcal{O}_{\mathfrak{p}}.$$

For $\gcd(m,p) = 1$, we may divide this equation by mp^n and obtain

$$0 \equiv \sum_{i=1}^{r} A_i \,\mathrm{Frob}_{\mathfrak{p}}\left(\alpha_i^{mp^{n-1}}\right) \rho_{i,n} \mod p^{n}\mathcal{O}_{\mathfrak{p}}.$$

For $m = 1, ..., r$, (in particular $p \nmid m$ by the assumptions on p) we find the system of linear equations

$$\begin{pmatrix} \mathrm{Frob}_{\mathfrak{p}}\, \alpha_1^{p^{n-1}} & \mathrm{Frob}_{\mathfrak{p}}\, \alpha_2^{p^{n-1}} & \cdots & \mathrm{Frob}_{\mathfrak{p}}\, \alpha_r^{p^{n-1}} \\ \mathrm{Frob}_{\mathfrak{p}}\, \alpha_1^{2p^{n-1}} & \mathrm{Frob}_{\mathfrak{p}}\, \alpha_2^{2p^{n-1}} & \cdots & \mathrm{Frob}_{\mathfrak{p}}\, \alpha_r^{2p^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Frob}_{\mathfrak{p}}\, \alpha_1^{rp^{n-1}} & \mathrm{Frob}_{\mathfrak{p}}\, \alpha_2^{rp^{n-1}} & \cdots & \mathrm{Frob}_{\mathfrak{p}}\, \alpha_r^{rp^{n-1}} \end{pmatrix} \begin{pmatrix} A_1\rho_{1,n} \\ A_2\rho_{2,n} \\ \vdots \\ A_r\rho_{r,n} \end{pmatrix} \equiv 0 \mod p^n.$$

The determinant of the Vandermonde type matrix is given by

$$\mathrm{Frob}_{\mathfrak{p}}\left(\prod_{i=1}^{r} \alpha_i^{p^{n-1}} \times \prod_{i<j}\left(\alpha_i^{p^{n-1}} - \alpha_j^{p^{n-1}}\right)\right), \tag{4.3}$$

which is a $\mathfrak{p}$-adic unit by assumption (1) on p. Indeed, modulo p, eq. (4.3) is congruent to

$$\mathrm{Frob}_{\mathfrak{p}}^{n}\left(\prod_{i=1}^{r} \alpha_i \times \prod_{i<j}(\alpha_i - \alpha_j)\right) \in \mathcal{O}_{\mathfrak{p}}^{\times}.$$

Therefore, $A_i\rho_{i,n} \equiv 0 \mod p^n\mathcal{O}_{\mathfrak{p}}$ and since $A_i \in \mathcal{O}_{\mathfrak{p}}^{\times}$, we obtain the limit $\lim_{n\to\infty} \rho_{i,n} = 0$ for all $i \in \{1, ..., r\}$. Due to Lemma 4.1 this implies $\rho_{i,1} = 0$ and equivalently, $\mathrm{Frob}_{\mathfrak{p}}(\alpha_i) = \alpha_i^p$. Assuming $\mathrm{Frob}_{\mathfrak{p}}^{\kappa} = \mathrm{id}$ for a suitable $\kappa \in \mathbb{N}$ implies $\alpha_i^{p^\kappa} = \alpha_i$, hence α_i is a root of unity for all $i \in \{1, ..., r\}$.

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