

KP integrability of triple Hodge integrals. I. From Givental group to hierarchy symmetries

ALEXANDER ALEXANDROV

In this paper, we investigate a relation between the Givental group of rank one and the Heisenberg-Virasoro symmetry group of the KP hierarchy. We prove, that only a two-parameter family of the Givental operators can be identified with elements of the Heisenberg-Virasoro symmetry group. This family describes triple Hodge integrals satisfying the Calabi-Yau condition. Using the identification of the elements of two groups we prove that the generating function of triple Hodge integrals satisfying the Calabi-Yau condition and its Θ -version are tau-functions of the KP hierarchy. This generalizes the result of Kazarian on KP integrability in the case of linear Hodge integrals.

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1. Introduction

Since the works of 't Hooft [35] and Brézin, Itzykson, Parisi, and Zuber [12], it is known that the matrix models provide an efficient tool for the investigation of enumerative geometry problems. Recent remarkable progress in enumerative geometry/matrix model theory, in particular, is related to the Chekhov-Eynard-Orantin topological recursion [15, 25] and Givental decomposition [31, 32]. Both describe a non-trivial extension of the matrix model theory, which ranges far beyond the scope of the classical matrix integrals. Both topological recursion and Givental decomposition are naturally described in terms of summation over graphs, thus formally can be represented by infinite-dimensional Gaussian matrix integrals with integrands made of the Kontsevich-Witten tau-function and its generalizations.

While a relationship between these two methods is known [16, 21], their relation to other classical ingredients of the matrix model theory is not always clear. Usually, matrix models satisfy two different families of differential (or differential-difference) equations. One family, the *linear* equations, is given by the Virasoro or, more generally, W-constraints. For known examples, recursive solution of these equations naturally leads to topological recursion and Givental-type decomposition, see, e.g., [1–5, 22]. Another family of equations is given by the *bilinear* Hirota equations of the KP/Toda type integrable hierarchies. While the Virasoro constraints constitute an inherent part of the Givental construction, the situation with integrability is much less clear. Many examples of the enumerative geometry generating functions, which can be described by topological recursion/Givental decomposition (TR/GD), are certain tau-functions of integrable hierarchies, however, a general relation between TR/GD and integrability is not known (see, however, [24, 28–30]).

The main goal of this paper is to clarify the role of KP/Toda type integrability in the general scheme of TR/GD. It is important, in particular, because some elements of the integrable systems can be naturally identified with the main components of the TR/GD. For example, the quantum spectral curve, playing a central role in TR/GD construction, can be identified with a specific Kac-Schwarz operator, while the wave function, annihilated by this operator, is given by the Baker-Akhiezer function [6–11].

In this paper, we address a question of the relation between the Givental group and the symmetry groups of the integrable hierarchies. These are

different groups, acting on different spaces. However, sometimes it is possible to identify the elements of two groups up to a linear change of variables. An example of such identification was constructed in the work of Liu and Wang [46]. They showed that the Givental operator associated with the generating function of linear Hodge integrals is equal to an element of the Heisenberg-Virasoro subgroup of $GL(\infty)$ up to a linear change of variables.

Therefore, a question arises: are there any other elements of the Givental group, which can be identified with the elements of the symmetry group of the integrable hierarchy? In this paper, we give a complete answer to this question for the rank one case of the Givental construction on one side and the KP integrable hierarchy on another. By *rank* here we mean the rank of the Frobenius manifold = the number of the KdV tau-functions in the Givental formula. The first important result of this paper, Theorem 1, states that the infinite-dimensional Givental group contains only a two-dimensional family of such operators.

On the enumerative geometry side, this two-dimensional family is known to describe a very interesting generating function, namely, a generating function of triple Hodge integrals satisfying the Calabi-Yau condition. Hence, from the identification of the Givental and Heisenberg-Virasoro group elements it follows, that after a certain linear change of variables the generating function of triple Hodge integrals satisfying the Calabi-Yau condition becomes a tau-function of the KP hierarchy. Theorem 2, which describes this KP tau-function, is the second important result of this paper. It generalizes the result of Kazarian for linear Hodge integrals [36]. Theorem 3 extends this KP integrability to the case with included Norbery's Θ -classes.

We restrict ourselves to the simplest rank one case. However, we expect that the results of this paper can be immediately generalized to the higher rank Givental groups. This should lead to new interesting examples of integrable generating functions of the cohomological field theory. In the companion paper [11] we investigate a family of the tau-functions of the KP hierarchy, that generalizes the two-parametric family constructed in this paper and conjecturally describes interesting enumerative geometry invariants in the r -spin case. This family can be described by a deformation of the generalized Kontsevich model. The current project, in particular, was motivated by [42], where a question about KP integrability of non-linear Hodge integrals and their description in terms of generalized Kontsevich model was raised.

It is known that enumerative geometry generating functions constitute only a small class of tau-functions of integrable hierarchies. We hope that

the identification between the elements of the Givental group and symmetry groups of integrable hierarchies will help us to classify all such tau-functions.

The present paper is organized as follows. In Section 2 we consider rank one Givental operators, which after a linear change of variables can be identified with the elements of the Heisenberg-Virasoro symmetry group of the KP hierarchy. In Section 3 we identify this family with the generating functions of triple Hodge integrals, which proves the KP integrability of the latter.

2. Two groups of symmetries

In this section we consider two different infinite-dimensional groups acting on the bosonic Fock spaces. First of them, the Givental group of the quantized symplectic transformations is a group of symmetries of semi-simple cohomological field theory. Second is a Heisenberg-Virasoro subgroup of the $GL(\infty)$ symmetry group of the KP hierarchy. We restrict the action of both groups to the space of functions of odd times. We prove, that on this space only a two-parametric family of the Givental operators can be identified, up to a linear change of variables, with the elements of the Heisenberg-Virasoro group.

2.1. Givental group

Let us consider the Givental quantization scheme for the rank one case. We basically follow the presentation of [19, 32, 40]. However, we consider a different normalization of the variables T_k , with an additional factor of \hbar . This change of normalization corresponds to the transition from the genus expansion of the generating function

$$(1) \quad \exp \left(\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \hbar^{2g-2} \mathcal{F}_{g,n} \right),$$

where g is the genus, n is the number of the marked points, to the topological expansion

$$(2) \quad \exp \left(\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \hbar^{2g-2+n} \mathcal{F}_{g,n} \right),$$

because the latter is more convenient for the KP hierarchy description.

Let $H = \mathbb{C}[[z, z^{-1}]]$ be the space of formal Laurent series in an indeterminate z . We introduce the symplectic form on this space

$$(3) \quad \Omega(f, g) = \frac{1}{2\pi i} \oint f(-z)g(z)dz.$$

For the natural decomposition

$$(4) \quad H = H_+ \oplus H_-,$$

where $H_+ = \text{span}_{\mathbb{C}}\{1, z, z^2, \dots\}$ and $H_- = \text{span}_{\mathbb{C}}\{z^{-1}, z^{-2}, z^{-3}, \dots\}$, we introduce the Darboux coordinate system $\{p_k, q_m\}$ with

$$(5) \quad \mathcal{J}(z) := \sum_{k=0}^{\infty} \left(q_k z^k + (-z)^{-k-1} p_k \right).$$

After Givental, for any infinitesimal symplectic transformation A on H (that is, a transformation such that $\Omega(Af, g) + \Omega(f, Ag) = 0$) we consider

$$(6) \quad H_A := \frac{1}{2} \Omega(A\mathcal{J}, \mathcal{J}).$$

This defines a Lie algebra isomorphism:

$$(7) \quad H_{[A,B]} = \{H_A, H_B\},$$

where the Poisson bracket is given by

$$(8) \quad \{H_A, H_B\} := \sum_{i=0}^{\infty} \left(\frac{\partial H_A}{\partial p_i} \frac{\partial H_B}{\partial q_i} - \frac{\partial H_B}{\partial p_i} \frac{\partial H_A}{\partial q_i} \right).$$

Using the standard Weyl quantization we quantize these operators to order ≤ 2 linear differential operators

$$(9) \quad \widehat{A} = \widehat{H}_A := \frac{1}{2} {}^* \Omega(A\widehat{\mathcal{J}}, \widehat{\mathcal{J}}) {}^*,$$

where

$$(10) \quad \widehat{\mathcal{J}}(z) = \sum_{k=0}^{\infty} \left(\widehat{T}_k z^k + (-z)^{-k-1} \frac{\partial}{\partial T_k} \right),$$

and $* \dots *$ denotes the standard bosonic normal ordering, which puts all $\frac{\partial}{\partial T_k}$ to the right of all T_k . The so-called *dilaton shift* is given by $\tilde{T}_k = T_k - \hbar^{-1} \delta_{k,1}$. This gives us a central extension of the original algebra, with the commutator

$$(11) \quad \left[\widehat{A}, \widehat{B} \right] = \widehat{[A, B]} + \mathcal{C}(H_A, H_B),$$

where the so-called 2-cocycle term satisfies

$$(12) \quad \mathcal{C}(p_i p_j, q_k q_m) = -\mathcal{C}(q_k q_m, p_i p_j) = \delta_{i,k} \delta_{j,m} + \delta_{i,m} \delta_{j,k}$$

and vanishes for all other elements H_A . For the finite symplectic transformations we define

$$(13) \quad \widehat{e^A} := e^{\widehat{A}}.$$

Below we consider only the symplectic transformations of the form $R(z) = 1 + R_1 z + R_2 z^2 + \dots \in 1 + z\mathbb{C}[[z]]$. They satisfy the symplectic condition

$$(14) \quad R(z)R(-z) = 1$$

if and only if $\log R(z)$ is odd function of z . Let us denote

$$(15) \quad \widehat{W}_k := \widehat{z^{2k-1}}, \quad k \in \mathbb{Z},$$

then from (9) we have [32, 40]

$$(16) \quad \widehat{W}_k = - \sum_m \tilde{T}_m \frac{\partial}{\partial T_{m+2k-1}} + \frac{1}{2} \sum_{m=0}^{-2k} (-1)^{l+1} \tilde{T}_m \tilde{T}_{-2k-m} + \frac{1}{2} \sum_{m=0}^{2k-2} (-1)^m \frac{\partial^2}{\partial T_m \partial T_{2k-m-2}}.$$

These operators satisfy the commutation relations

$$(17) \quad \left[\widehat{W}_k, \widehat{W}_m \right] = -\frac{2k-1}{2} \delta_{k+m,1}.$$

After quantization, using (16) we obtain

$$(18) \quad \widehat{R} := \exp \left(\widehat{\log R(z)} \right).$$

Below we call the group of such operators the *Givental group*.

The Givental group acts of the space of cohomological field theories (CohFT) with flat unit. If we relax the flat unit condition, we can include translations of the times, see an example in Section 3.2. However, translations are natural symmetries of the KP hierarchy, so this enrichment is trivial from the point of view of the identification of operators of two groups.

Operators \widehat{R} can be factorized, namely, one can factor out the part, corresponding to the linear change of variables and the translation of variables. This factorization, except for simple extraction of the translation operator, is given by Proposition 7.3 in [32]. Here we briefly remind the reader this relation. Let us introduce a formal series in two variables

$$(19) \quad V^R(z, w) := \frac{1 - R(-w)R(-z)}{w + z}.$$

The matrix of its coefficients

$$(20) \quad V^R(z, w) = \sum_{k,l=0}^{\infty} V_{kl}^R w^k z^l$$

is symmetric, $V_{kl}^R = V_{lk}^R$, and its entries are polynomials in R_k . Let the linear change of the shifted variables from \mathbf{T} to \mathbf{T}^R be given by

$$(21) \quad \sum_{k=0}^{\infty} \widetilde{T}_k^R z^k := R(-z) \sum_{k=0}^{\infty} \widetilde{T}_k z^k.$$

This transformation generates the change of the dilaton shift

$$(22) \quad \sum_{k=2}^{\infty} \delta_k z^k := z(1 - R(-z)).$$

Then, for any element of the Givental group (18) and any series $Z(\mathbf{T})$ we have [32]

Lemma 2.1 (Givental).

$$(23) \quad \widehat{R} \cdot Z(\mathbf{T}) = e^{\frac{1}{2} \sum_{i,j=0}^{\infty} V_{ij}^R \frac{\partial^2}{\partial T_i \partial T_j}} e^{\hbar^{-1} \sum_{k=2}^{\infty} \delta_k \frac{\partial}{\partial T_k}} \cdot Z(\mathbf{T}) \Big|_{T_k \mapsto T_k^R}.$$

In this expression one acts by the operator on the function $Z(\mathbf{T})$, and then in the result substitutes T_k with T_k^R .

2.2. Heisenberg-Virasoro subgroup of $GL(\infty)$

In this section we briefly describe the Heisenberg-Virasoro subgroup of $GL(\infty)$ symmetry group of KP hierarchy, for more details see the companion paper [11] and references therein. Let

$$(24) \quad \widehat{J}(z) := \sum_{k \in \mathbb{Z}} \frac{\widehat{J}_k}{z^{k+1}},$$

where

$$(25) \quad \widehat{J}_k = \begin{cases} \frac{\partial}{\partial t_k} & \text{for } k > 0, \\ 0 & \text{for } k = 0, \\ -kt_{-k} & \text{for } k < 0. \end{cases}$$

We consider the generating function of the Virasoro operators

$$(26) \quad {}^* \widehat{J}(z)^2 {}^* = 2 \sum_{k \in \mathbb{Z}} \frac{\widehat{L}_k}{z^{k+2}}$$

or

$$(27) \quad \widehat{L}_m = \frac{1}{2} \sum_{a+b=-m} abt_a t_b + \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k+m}} + \frac{1}{2} \sum_{a+b=m} \frac{\partial^2}{\partial t_a \partial t_b},$$

where, depending on the sign of m , only the first or the last summation appears. These formulas are the KP analogs of (10) and (9). The Heisenberg-Virasoro group \mathcal{V} is generated by the operators \widehat{J}_k , \widehat{L}_k and a unit. Let us consider a Virasoro subgroup of \mathcal{V} given by the operators of the form

$$(28) \quad \widehat{V} = \exp \left(\sum_{\mathbb{Z}_{>0}} a_k \widehat{L}_k \right) \in \mathcal{V}.$$

With any such operator one can associate the operator $e^{\sum_{\mathbb{Z}_{>0}} a_k \mathbf{1}_k}$, where

$$(29) \quad \mathbf{1}_m = -z^m \left(z \frac{\partial}{\partial z} + \frac{m+1}{2} \right)$$

are the generators of the Witt subalgebra of the algebra of diffeomorphisms on the circle.

For any set $a_k, k \in \mathbb{Z}_{>0}$, consider the series

$$(30) \quad f(z) := e^{\sum_{\mathbb{Z}_{>0}} a_k \mathbf{1}_k} z e^{-\sum_{\mathbb{Z}_{>0}} a_k \mathbf{1}_k} \in z + z\mathbb{C}[[z]].$$

Then

$$(31) \quad h(z) := e^{-\sum_{\mathbb{Z}_{>0}} a_k \mathbf{1}_k} z e^{\sum_{\mathbb{Z}_{>0}} a_k \mathbf{1}_k} \in z + z\mathbb{C}[[z]]$$

is an inverse series, $f(h(z)) = h(f(z)) = z$. There is a one-to-one correspondence between the space of series (30) and the subgroup (28) of the Virasoro group. From the commutation relations of the Heisenberg-Virasoro algebra,

$$(32) \quad \begin{aligned} [\widehat{J}_k, \widehat{J}_m] &= k\delta_{k,-m}, \\ [\widehat{L}_k, \widehat{J}_m] &= -m\widehat{J}_{k+m}, \\ [\widehat{L}_k, \widehat{L}_m] &= (k-m)\widehat{L}_{k+m} + \frac{1}{12}\delta_{k,-m}(k^3 - k) \end{aligned}$$

it follows that

$$(33) \quad \widehat{V} \widehat{J}(z) \widehat{V}^{-1} = h'(z) \widehat{J}(h(z)).$$

Let \widehat{V}_0 be a part of the element of the upper-triangular subgroup (28) that describes the linear change of variables,

$$(34) \quad \widehat{V}_0 := \exp \left(\sum_{k=1}^{\infty} a_k \sum_{m=1}^{\infty} m t_m \frac{\partial}{\partial t_{k+m}} \right).$$

With the help of the Campbell-Baker-Hausdorff formula operator \widehat{V} can be factorized

$$(35) \quad \widehat{V} = \widehat{V}_0 \exp \left(\frac{1}{2} \sum_{k,m=1}^{\infty} v_{km} \frac{\partial^2}{\partial t_k \partial t_m} \right)$$

for some $v_{ij} \in \mathbb{C}[[a_1, a_2, a_3, \dots]]$. Below we will describe the space of all v_{ij} which can be obtained in this way.

Using the commutation relations of the Virasoro algebra we can always factor out a term $\exp(a_1 \widehat{L}_1)$,

$$(36) \quad \exp \left(\sum_{\mathbb{Z}_{>0}} a_k \widehat{L}_k \right) = \exp(a_1 \widehat{L}_1) \exp \left(\sum_{\mathbb{Z}_{>1}} \tilde{a}_k \widehat{L}_k \right)$$

for some new \tilde{a}_k . Operator $\widehat{L}_1 = \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k+1}}$ is a first order differential operator, so $\exp(a_1 \widehat{L}_1)$ gives only a linear change of variables. As we are going to identify the Givental operators with the elements of the Heisenberg-Virasoro group up to the linear change of variables, there is a certain arbitrariness in the choice of a_1 . We call the transformation, given by $\exp(a_1 \widehat{L}_1)$, the *gauge transformation*, and the choice of a_1 will be called the choice of *gauge*. For the series (30) the gauge transformation is given by

$$(37) \quad f(z) \mapsto f(z/(1 + a_1 z)).$$

2.3. Quadratic part of the Virasoro group operators

To find the matrix v_{km} in (35) it is enough to act by both sides of (35) on the function

$$(38) \quad E := \exp \left(\sum_{k=1}^{\infty} kt_k q_k \right),$$

where \mathbf{q} are auxiliary variables, independent on \mathbf{t} . Let us show that the coefficients v_{nm} in (35) are the so-called Grunsky coefficients of function $h(z)$. Consider

$$(39) \quad v(\eta_1, \eta_2) := \sum_{k,m=1}^{\infty} v_{km} \eta_1^k \eta_2^m.$$

Lemma 2.2.

$$(40) \quad v(\eta_1, \eta_2) = \log \left(\frac{h(\eta_1) - h(\eta_2)}{\eta_1 - \eta_2} \right).$$

Remark 2.1. *This formula was known to the experts a long time ago, and can be proven via the Campbell-Baker-Hausdorff formula. Here we provide its proof for completeness.*

Proof. On one hand, the action of the right hand side of (35) on E yields

$$(41) \quad \widehat{V} \cdot E = \widehat{V}_0 \cdot \exp \left(\sum_{k=1}^{\infty} kt_k q_k + \frac{1}{2} \sum_{k,m=1}^{\infty} km v_{km} q_k q_m \right).$$

On the other hand

$$\begin{aligned}
 \widehat{V} \cdot E &= \exp \left(\sum_{k=1}^{\infty} k \left(\widehat{V} t_k \widehat{V}^{-1} \right) q_k \right) \cdot 1 \\
 (42) \qquad &= \exp \left(\sum_{k=1}^{\infty} \left(k t_k \tilde{q}_k + \tilde{q}_{-k} \frac{\partial}{\partial t_k} \right) \right) \cdot 1,
 \end{aligned}$$

where from (33) we have

$$(43) \qquad \tilde{q}_k := \frac{1}{2\pi i} \oint_{\infty} \left(\sum_{m=1}^{\infty} \frac{q_m}{f(z)^m} \right) \frac{dz}{z^{k+1}}, \quad k \in \mathbb{Z}.$$

Hence

$$\begin{aligned}
 \widehat{V} \cdot E &= \exp \left(\sum_{k=1}^{\infty} \left(k t_k \tilde{q}_k + \frac{1}{2} k \tilde{q}_k \tilde{q}_{-k} \right) \right) \\
 (44) \qquad &= \widehat{V}_0 \cdot \exp \left(\sum_{k=1}^{\infty} \left(k t_k q_k + \frac{1}{2} k \tilde{q}_k \tilde{q}_{-k} \right) \right).
 \end{aligned}$$

Comparing this expression with (41) we conclude

$$(45) \qquad \sum_{k,m=1}^{\infty} k m v_{km} q_k q_m = \sum_{k=1}^{\infty} k \tilde{q}_k \tilde{q}_{-k}.$$

Therefore,

$$\begin{aligned}
 v_{km} &= \frac{1}{km} \sum_{j=1}^{\infty} \frac{j}{(2\pi i)^2} \oint_{\infty} \frac{dz}{z^{j+1} f(z)^k} \oint_{\infty} \frac{w^{j-1} dw}{f(w)^m} \\
 (46) \qquad &= -\frac{1}{km} \sum_{j=1}^{\infty} \frac{1}{j} \frac{1}{(2\pi i)^2} \oint_{\infty} \frac{dz^{-j}}{f(z)^k} \oint_{\infty} \frac{dw^j}{f(w)^m} \\
 &= \frac{1}{(2\pi i)^2} \oint_{\infty} \frac{dz}{z^{k+1}} \oint_{\infty} \frac{dw}{w^{m+1}} \log \left(1 - \frac{h(w)}{h(z)} \right) \\
 &= \frac{1}{(2\pi i)^2} \oint_{\infty} \frac{dz}{z^{k+1}} \oint_{\infty} \frac{dw}{w^{m+1}} \log \left(\frac{h(z) - h(w)}{z - w} \right).
 \end{aligned}$$

Here in the transition from the second to the third line we use the change of variables from z and w to $f(z)$ and $f(w)$. To derive the last line we use

the positivity of k and m , and the function there is chosen in such way that it possess formal Taylor series expansion

$$(47) \quad \log \left(\frac{h(z) - h(w)}{z - w} \right) \in \mathbb{C}[[z, w]].$$

This completes the proof. \square

2.4. Equivalence of Givental and Virasoro group elements

Let us compare the subgroup (28) of the Virasoro group and Givental group, given by Lemma 2.1. First, we identify

$$(48) \quad T_k \equiv (2k + 1)!! t_{2k+1}.$$

Second, we consider the action of two groups on the Fock space of functions of odd variables $\mathbf{t}^o = \{t_1, t_3, t_5, \dots\}$. On this space for the element of the Virasoro group \widehat{V} one can neglect all terms with even k or m in (35) and all terms with even values of $k + m$ in (34). Therefore, on this space the action of two operators will coincide, up to linear change and translation of variables, if and only if

$$(49) \quad V_{km}^R = (2k + 1)!!(2m + 1)!! v_{2k+12m+1}$$

for all $k, m \in \mathbb{Z}_{\geq 0}$.

Remark 2.2. *There is no restriction for the elements v_{km} when k or m is even. However, we will see that the constraint (49) is already very restrictive.*

Remark 2.3. *Arguments of this section are closely related to the standard manipulations in the context of TR/GD, often referred to as the Laplace transform, see, e.g., [21, 23, 26].*

Let

$$(50) \quad x(z) := \frac{f(z)^2}{2} = \frac{z^2}{2} + z^3 \mathbb{C}[[z]],$$

where $f(z)$ describes the Virasoro group element by (30) and

$$(51) \quad N(z) := \frac{z}{x'(z)} = 1 + \sum_{k \in \mathbb{Z}_{>1}} n_k z^k.$$

We fix the gauge by $a_1 = 0$, which immediately leads to $n_1 = 0$. Then we have

Lemma 2.3. *From (49) it follows that $n_k = 0$ for $k > 3$.*

Proof. From (19) and Lemma 2.2 we see that the relation (49) is equivalent to

$$(52) \quad \frac{1 - R(-w)R(-z)}{w + z} = \frac{1}{2\pi(zw)^{3/2}} \int_{\gamma^2} d\eta_1 d\eta_2 \eta_1 \eta_2 e^{-\frac{\eta_1^2}{2z} - \frac{\eta_2^2}{2w}} \log \left(\frac{h(\eta_1) - h(\eta_2)}{\eta_1 - \eta_2} \right).$$

Here γ is a local steepest descent contour in the vicinity of $\eta_1 = \eta_2 = 0$, and we consider the asymptotic expansion of the integral at small values of $|z|$ and $|w|$. Let $x_k = x(\eta_k)$. The right hand side of (52) after the change of variables of integration $\eta_k \mapsto f(\eta_k)$ is equal to

$$(53) \quad \begin{aligned} & \frac{1}{2\pi(zw)^{3/2}} \int_{\gamma^2} dx_1 dx_1 \log \left(\frac{\eta_1 - \eta_2}{f(\eta_1) - f(\eta_2)} \right) e^{-\frac{x_1}{z} - \frac{x_2}{w}} \\ &= \frac{(-1)}{2\pi(z+w)\sqrt{zw}} \int_{\gamma^2} dx_1 dx_2 \log \left(\frac{\eta_1 - \eta_2}{f(\eta_1) - f(\eta_2)} \right) \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) e^{-\frac{x_1}{z} - \frac{x_2}{w}} \\ &= \frac{1}{2\pi(z+w)\sqrt{zw}} \int_{\gamma^2} dx_1 dx_2 e^{-\frac{x_1}{z} - \frac{x_2}{w}} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \log \left(\frac{\eta_1 - \eta_2}{f(\eta_1) - f(\eta_2)} \right), \end{aligned}$$

where the last equality follows from the integration by parts. From the identity

$$(54) \quad \frac{\partial x_1}{\partial \eta_1} \frac{\partial x_2}{\partial \eta_2} \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \log \left(\frac{\eta_1 - \eta_2}{f(\eta_1) - f(\eta_2)} \right) = f'(\eta_1) f'(\eta_2) - \frac{\frac{\partial x_1}{\partial \eta_1} - \frac{\partial x_2}{\partial \eta_2}}{\eta_1 - \eta_2}$$

it follows that the last line of (53) is equal to

$$(55) \quad \frac{1}{2\pi(z+w)\sqrt{zw}} \int_{\gamma^2} \left(d\sqrt{2x_1} d\sqrt{2x_2} - \frac{dx_1 d\eta_2 - d\eta_1 dx_2}{\eta_1 - \eta_2} \right) e^{-\frac{x_1}{z} - \frac{x_2}{w}}.$$

Comparing it with (52), we see that the latter is equivalent to

$$(56) \quad R(-w)R(-z) = \frac{1}{2\pi\sqrt{zw}} \int_{\gamma^2} \frac{dx_1 d\eta_2 - d\eta_1 dx_1}{\eta_1 - \eta_2} e^{-\frac{x_1}{z} - \frac{x_2}{w}}.$$

The left hand side factorizes, so should the right hand side. The factors can be easily found if one puts $w = 0$,

$$(57) \quad R(-z) = \frac{1}{\sqrt{2\pi z}} \int_{\gamma} \frac{dx(\eta)}{\eta} e^{-\frac{x(\eta)}{z}}.$$

Hence $x(\eta)$ should satisfy the equation

$$(58) \quad \frac{1}{2\pi\sqrt{zw}} \int_{\gamma^2} \left(\frac{dx_1 d\eta_2 - d\eta_1 dx_2}{\eta_1 - \eta_2} - \frac{dx_1 dx_2}{\eta_1 \eta_2} \right) e^{-\frac{x_1}{z} - \frac{x_2}{w}} = 0.$$

Let us solve this equation for $x(\eta)$. First, we rewrite the left hand side as

$$(59) \quad \begin{aligned} & \frac{1}{2\pi\sqrt{zw}} \int_{\gamma^2} \frac{d\eta_1 d\eta_2}{N(\eta_1)N(\eta_2)} \left(\frac{\eta_1 N(\eta_2) - \eta_2 N(\eta_1)}{\eta_1 - \eta_2} - 1 \right) e^{-\frac{x_1}{z} - \frac{x_2}{w}} \\ &= -\frac{1}{2\pi\sqrt{zw}} \int_{\gamma^2} d\eta_1 d\eta_2 \eta_1 \eta_2 \sum_{k=2}^{\infty} n_k \sum_{j+m=k-2} h(\eta_1)^j h(\eta_2)^m e^{-\frac{\eta_1^2}{2z} - \frac{\eta_2^2}{2w}}, \end{aligned}$$

where we apply the change of integration variables $\eta_k \mapsto h(\eta_k)$ and integrate by parts.

For any $g(\eta) \in \mathbb{C}[[\eta]]$ let

$$(60) \quad [g(\eta)]_{\eta} := \frac{1}{2} (g(\eta) - g(-\eta))$$

be its odd part, and for a series of two variables $g(\eta_1, \eta_2) \in \mathbb{C}[[\eta_1, \eta_2]]$

$$(61) \quad [g(\eta_1, \eta_2)]_{\eta_1, \eta_2} := \frac{1}{4} (g(\eta_1, \eta_2) - g(-\eta_1, \eta_2) - g(\eta_1, -\eta_2) + g(-\eta_1, -\eta_2)).$$

Then (58) is equivalent to

$$(62) \quad \left[\sum_{k=2}^{\infty} n_k \sum_{j+m=k-2} h(\eta_1)^j h(\eta_2)^m \right]_{\eta_1, \eta_2} = 0,$$

so we need to prove that it implies $n_k = 0$ for $k > 3$. It is obvious that it implies $n_4 = 0$. Let

$$(63) \quad M_j := \left[\sum_{k=5}^{\infty} n_k h(\eta_1)^{k-j-2} \right]_{\eta_1}.$$

It is easy to find the first terms of the series $h(x)$:

$$(64) \quad h(\eta) = \eta + \frac{n_2}{4}\eta^3 + \frac{n_3}{5}\eta^4 + \frac{5n_2^2}{96}\eta^5 + O(\eta^6).$$

Then

$$(65) \quad \left[\sum_{k=5}^{\infty} n_k \sum_{j+m=k-2} h(\eta_1)^j h(\eta_2)^m \right]_{\eta_1, \eta_2} = M_1 \eta_2^1 + (M_3 + \frac{n_2}{4} M_1) \eta_2^3 + O(\eta_2^5).$$

Hence, from (62) it follows that

$$(66) \quad M_1 = 0, \quad M_3 = 0.$$

Let us prove by induction that these two equations have the only solution $n_k = 0$ for $k > 4$. Assume that $n_k = 0$ for all $k = 4, 5, \dots, m$. If m is odd, then from the equation $M_1 = 0$ it follows that $n_{m+1} = 0$, so we can assume that m is even, $m = 2k_0$. Assume that $n_{2k_0+1} \neq 0$ for some $k_0 \geq 2$. There are two possibilities:

- Let $n_3 = 0$. Then

$$(67) \quad h(\eta) + h(-\eta) = 2 \frac{n_{2k_0+1}}{2k_0+3} \eta^{2k_0+2} + O(\eta^{2k_0+2})$$

and from the equation $M_1 = 0$ it follows that $n_{2k} = 0$ for all $k \in \{3, \dots, 2k_0\}$.

Then

$$(68) \quad \begin{aligned} M_1 &= \left(\frac{2k_0 - 2}{2k_0 + 3} n_{2k_0+1}^2 + n_{4k_0+2} \right) \eta_1^{4k_0-1} + O(\eta_1^{4k_0+1}), \\ M_3 &= \left(\frac{2k_0 - 4}{2k_0 + 3} n_{2k_0+1}^2 + n_{4k_0+2} \right) \eta_1^{4k_0-3} + O(\eta_1^{4k_0-1}). \end{aligned}$$

Hence from (66) we have $n_{2k_0+1} = 0$, which contradicts the assumption.

- Let $n_3 \neq 0$. Then from the first equation in (66) we immediately see that $n_{2k_0+2} = 0$, and

$$(69) \quad \begin{aligned} M_1 &= \left(\frac{2k_0 - 2}{5} n_3 n_{2k_0+1} + n_{2k_0+4} \right) \eta_1^{2k_0+1} + O(\eta_1^{2k_0+3}), \\ M_3 &= \left(\frac{2k_0 - 4}{5} n_3 n_{2k_0+1} + n_{2k_0+4} \right) \eta_1^{2k_0-1} + O(\eta_1^{2k_0+2}). \end{aligned}$$

Hence from (66) we have $n_{2k_0+1} = 0$, which contradicts the assumption. This completes the proof. \square

Therefore, to identify the rank one Givental operators with the elements of the Heisenberg-Virasoro group we can restrict ourself to a two-dimensional family of the Virasoro operators (28), identification of the translation part will be considered later. Let us choose a more convenient parametrization of this family. For this purpose we use the gauge transformation (37). Let us denote by $\widehat{V}_{n_1, n_2, n_3}$ the element of the Virasoro subgroup (28) with function $f(z)$ given by

$$(70) \quad \frac{f(z)^2}{2} = \int_0^z \frac{\eta d\eta}{1 + n_1\eta + n_2\eta^2 + n_3\eta^3}.$$

Lemma 2.4.

$$(71) \quad \widehat{V}_{n_1, n_2, 0} = e^{\frac{n_1}{3}\widehat{L}_1} \widehat{V}_{0, n_2 - n_1^2/3, -n_1n_2/3 + 2n_1^3/27}.$$

Proof. Because of the one-to-one correspondence between the group (28) and the space of formal series $z + z^2\mathbb{C}[[z]]$, the statement of the lemma follows from the identity

$$(72) \quad \int_0^{\frac{z}{1+n_1z/3}} \frac{\eta d\eta}{1 + (n_2 - n_1^2/3)\eta^2 + (2n_1^3/27 - n_1n_2/3)\eta^3} = \int_0^z \frac{\eta d\eta}{1 + n_1\eta + n_2\eta^2}$$

and the transformation rule (37). \square

Its easy to see, that for any \tilde{n}_2 and \tilde{n}_3 the equations

$$(73) \quad \tilde{n}_2 = n_2 - n_1^2/3, \quad \tilde{n}_3 = -n_1n_2/3 + 2n_1^3/27$$

have solutions. Hence, we can work with the 2-parameter subspace of the Virasoro group, described by

$$(74) \quad \frac{f(z)^2}{2} = \int_0^z \frac{\eta d\eta}{1 + n_1\eta + n_2\eta^2}.$$

Let us parametrize it by p and q

$$(75) \quad 1 + n_1z + n_2z^2 = (1 + \sqrt{p+qz})(1 + qz/\sqrt{p+q}),$$

or, equivalently,

$$(76) \quad q = n_2, \quad p = \frac{n_1^2 - 2n_2 \pm \sqrt{n_1^4 - 4n_1^2n_2}}{2}.$$

We assume that $p + q \neq 0$, then x is defined by

$$(77) \quad dx = f(z)df(z) = \frac{zdz}{(1 + \sqrt{p+qz})(1 + qz/\sqrt{p+q})}.$$

In this parametrization, if $p \neq 0$ and $q \neq 0$ we have

$$(78) \quad x(z) = \frac{p+q}{pq} \log\left(1 + \frac{qz}{\sqrt{p+q}}\right) - \frac{1}{p} \log(1 + \sqrt{p+qz}).$$

For $q = 0$ it degenerates to

$$(79) \quad x(z) = \frac{z}{\sqrt{p}} - \frac{1}{p} \log(1 + \sqrt{p}z),$$

and for $p = 0$ it degenerates to

$$(80) \quad x(z) = \frac{1}{q} \log(1 + \sqrt{q}z) - \frac{1}{\sqrt{q}} \frac{z}{1 + \sqrt{q}z}.$$

Let us denote

$$(81) \quad I(z) := \frac{1}{\sqrt{2\pi z}} \int_{\gamma} \frac{dx(\eta)}{\eta} e^{-\frac{x(\eta)}{z}}.$$

It appears that if the factorization constraint (56) is satisfied, then corresponding $R(z)$ satisfies the symplectic condition. Namely, let us consider a two-parametric family of functions satisfying this condition,

$$(82) \quad R_{q,p}(z) = \exp\left(-\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} \left(p^{2k-1} + q^{2k-1} - \left(\frac{pq}{p+q}\right)^{2k-1}\right) z^{2k-1}\right).$$

Here B_{2k} are the Bernoulli numbers

$$(83) \quad \frac{xe^x}{e^x - 1} = 1 + \frac{x}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}x^{2k}}{(2k)!}.$$

Lemma 2.5.

$$(84) \quad I(z) = R_{q,p}(-z).$$

Proof. We need to consider three different cases.

1) Let $p \neq 0$ and $q \neq 0$. Then $x(z)$ is given by (78). From the integration by parts we have

$$(85) \quad \int_{\gamma} dx(\eta) e^{-\frac{x(\eta)}{z}} = 0,$$

hence

$$(86) \quad \begin{aligned} I(z) &= \frac{1}{\sqrt{2\pi z}} \int_{\gamma} dx(\eta) \left(\frac{1}{\eta} - q\sqrt{p+q} \right) e^{-\frac{x(\eta)}{z}} \\ &= \frac{1}{\sqrt{2\pi z}} \int_{\gamma} \frac{d\eta}{(1 + \sqrt{p+q}\eta)^{1-\frac{1}{pz}} (1 + q\eta/\sqrt{p+q})^{\frac{p+q}{pqz}}}. \end{aligned}$$

Let us make a change of integration variable $\eta \mapsto \frac{p}{q\sqrt{p+q}}\eta - \frac{1}{\sqrt{p+q}}$, then

$$(87) \quad I(z) = \frac{1}{\sqrt{2\pi z(p+q)}} \frac{(p+q)^{\frac{1}{pz} + \frac{1}{qz}}}{q^{\frac{1}{pz}} p^{\frac{1}{qz}}} \int_{\tilde{\gamma}} \frac{\eta^{\frac{1}{pz}-1} d\eta}{(1+\eta)^{\frac{1}{pz} + \frac{1}{qz}}}.$$

Here $\tilde{\gamma}$ is the new contour, corresponding to the saddle point at $\eta = q/p$. We can deform the integration contour in such a way that the asymptotic expansion does not change, which allows us to identify it with the asymptotic expansion of the beta-function

$$(88) \quad B(a, b) = \int_0^{\infty} \frac{\eta^{a-1} dx}{(1+\eta)^{a+b}}$$

for $a = 1/pz$ and $b = 1/qz$. From Stirling's expansion

$$(89) \quad \Gamma(z^{-1}) = \sqrt{2\pi} e^{-z} z^{z-1/2} e^{\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} z^{2k-1}}$$

one has the asymptotic expansion of the beta-function

$$(90) \quad \begin{aligned} &B((qz)^{-1}, (pz)^{-1}) \\ &= \frac{\sqrt{2\pi z(p+q)} q^{\frac{1}{pz}} p^{\frac{1}{qz}}}{(p+q)^{\frac{1}{pz} + \frac{1}{qz}}} e^{\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} \left((pz)^{2k-1} + (qz)^{2k-1} - \left(\frac{pq}{p+q} z \right)^{2k-1} \right)}. \end{aligned}$$

The statement of the lemma follows from the comparison of this expansion with (87).

2) Let $q = 0$. Then $x(z)$ is given by (79), and

$$(91) \quad I(z) = \frac{1}{\sqrt{2\pi z}} \int_{\gamma} \frac{d\eta}{(1 + \sqrt{p\eta})^{1 - \frac{1}{pz}}} e^{-\frac{\eta}{z\sqrt{p}}}.$$

After a change of the integration variable, $\eta \mapsto z\sqrt{p}\eta - \frac{1}{\sqrt{p}}$, we get

$$(92) \quad I(z) = \frac{1}{\sqrt{2\pi zp}} e^{\frac{1}{zp}} (zp)^{\frac{1}{zp}} \int_{\tilde{\gamma}} d\eta \eta^{\frac{1}{pz} - 1} e^{-\eta}.$$

The asymptotic expansion of the integral coincides with the asymptotic expansion of the gamma-function

$$(93) \quad \Gamma(z^{-1}) = \int_0^{\infty} d\eta \eta^{\frac{1}{z} - 1} e^{-\eta},$$

and the statement of lemma follows from the Stirling expansion (89).

3) Let $p = 0$. This case can be reduced to the previous one. Namely, for $p = 0$ we have

$$(94) \quad I(z) = \frac{1}{\sqrt{2\pi z}} \int_{\gamma} \frac{d\eta}{(1 + \sqrt{q\eta})^{2 + \frac{1}{qz}}} e^{\frac{1}{\sqrt{qz}} \frac{\eta}{1 + \sqrt{q\eta}}}$$

which reduces to (79) after the change of integration variable $\eta \mapsto -\eta/(1 + \sqrt{q\eta})$ and identification of q and p if we apply (85). This completes the proof. \square

Below we assume that coefficients \mathbf{a} are fixed by (30), where the function $f(z)$ is given by (77). Let us consider the operator

$$(95) \quad D_{q,p} = -e^{\sum_{k \in \mathbb{Z}_{>0}} a_k \mathbf{1}_k} \frac{1}{z} \frac{\partial}{\partial z} e^{-\sum_{k \in \mathbb{Z}_{>0}} a_k \mathbf{1}_k} = -\frac{\partial}{\partial x},$$

or, equivalently,

$$(96) \quad D_{q,p} = -\frac{(1 + \sqrt{p + qz})(1 + qz/\sqrt{p + q})}{z} \frac{\partial}{\partial z}.$$

Then the functions

$$(97) \quad \phi_k(z) := D_{q,p}^k \cdot \frac{1}{z}$$

define a change of variables if we associate kt_k with z^{-k} and $T_k^{q,p}$ with $\phi_k(z)$. This change of variables can be described by the recursion

$$(98) \quad \begin{aligned} T_0^{q,p}(\mathbf{t}) &= t_1, \\ T_k^{q,p}(\mathbf{t}) &= \left(q\widehat{L}_0 + \frac{2q+p}{\sqrt{p+q}}\widehat{L}_{-1} + \left(\widehat{L}_{-2} - \frac{t_1^2}{2} \right) \right) \cdot T_{k-1}^{q,p}. \end{aligned}$$

Let us prove that the linear change of variables (98) gives a transformation, required for identification of the Givental and Heisenberg-Virasoro operators.

Lemma 2.6.

$$(99) \quad T_k^R(\mathbf{T}_{q,p}(\mathbf{t})) = \widehat{V}_0 \cdot (2k+1)!!t_{2k+1}, \quad k \in \mathbb{Z}_{\geq 0}.$$

Proof. Let us prove an equivalent relation

$$(100) \quad \widehat{V}_0^{-1} \cdot \sum_{k=0}^{\infty} T_k^R(\mathbf{T}_{q,p}(\mathbf{t}))w^k = \sum_{k=0}^{\infty} (2k+1)!!t_{2k+1}w^k.$$

Let us associate kt_k with z^{-k} . Then $\sum_{k=0}^{\infty} T_k^R w^k$ is associated with

$$(101) \quad R(-w) \sum_{k=0}^{\infty} w^k \left(-\frac{1}{z} \frac{\partial}{\partial z} \right)^k \cdot \frac{1}{z},$$

and, by (95), the sum $\sum_{m=0}^{\infty} T_m^R(\mathbf{T}_{q,p}(\mathbf{t}))w^m$ is associated with

$$(102) \quad R(-w) \sum_{m=0}^{\infty} w^m D_{q,p}^m \cdot \frac{1}{z}.$$

Let us omit the factor $R(-w)$ for a moment and act on (102) by $e^{-\sum_{z>0} a_k 1_k}$:

$$(103) \quad \begin{aligned} & \left(e^{-\sum_{z>0} a_k 1_k} \sum_{m=0}^{\infty} w^m D_{q,p}^m \cdot \frac{1}{z} \right)_{-} \\ &= \left(\sum_{m=0}^{\infty} w^m \left(-\frac{1}{z} \frac{\partial}{\partial z} \right)^m e^{-\sum_{z>0} a_k 1_k} \cdot \frac{1}{z} \right)_{-} \\ &= \left(\sum_{m=0}^{\infty} w^m \left(-\frac{1}{z} \frac{\partial}{\partial z} \right)^m \frac{1}{h(z)} \right)_{-} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k,m=0}^{\infty} \frac{1}{2\pi i} \frac{w^m}{z^{2k+1}} \oint \frac{d\eta}{h(\eta)} \left(\frac{\partial}{\partial \eta} \frac{1}{\eta} \right)^m \eta^{2k} \\
 &= \sum_{k,m=0}^{\infty} \frac{(2k-1)!!}{2\pi i} \frac{w^{k+m}}{z^{2k+1}} \oint d\eta \frac{(-1)^m (2m-1)!!}{h(\eta) \eta^{2m}} \\
 &= \sum_{k=0}^{\infty} (2k-1)!! \frac{w^k}{z^{2k+1}} \frac{1}{\sqrt{-2\pi z}} \int_{\gamma} dx \frac{x}{h(x)} e^{\frac{x^2}{2z}} \\
 &= R(w) \sum_{k=0}^{\infty} (2k-1)!! \frac{w^k}{z^{2k+1}},
 \end{aligned}$$

where for any series $(\sum_{k \in \mathbb{Z}} b_k z^k)_- := \sum_{k \in \mathbb{Z}_{<0}} b_k z^k$. Hence,

$$(104) \quad \left(e^{-\sum_{z>0} a_k 1_k} \cdot R(-w) \sum_{m=0}^{\infty} w^m D_{q,p}^m \cdot \frac{1}{z} \right)_- = \sum_{k=0}^{\infty} (2k-1)!! \frac{w^k}{z^{2k+1}},$$

and two linear transformations coincide. □

Let us also introduce

$$(105) \quad v_k := [z^k] \int_0^z (f(\eta) - y(\eta)) dx(\eta),$$

where

$$(106) \quad y(z) = \int^z \frac{dx(\eta)}{\eta}.$$

For $p \neq 0$ and $q \neq 0$

$$(107) \quad y(z) = \frac{\sqrt{p+q}}{p} \left(\log(1 + \sqrt{p+q}z) - \log\left(1 + \frac{qz}{\sqrt{p+q}}\right) \right).$$

For $q = 0$ it reduces to

$$(108) \quad y(z) = \frac{1}{\sqrt{p}} \log(1 + \sqrt{p}z)$$

and for $p = 0$ it degenerates to

$$(109) \quad y(z) = \frac{z}{1 + \sqrt{q}z}.$$

It is easy to see that $v_k = 0$ for $k < 4$.

Having in mind that the coefficients v_k are given by (105), let us remind the reader the main ingredients of the following theorem, given by (27), (98), (30), (77), and (82), namely the Virasoro operators

$$(110) \quad \widehat{L}_m = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+m}} + \frac{1}{2} \sum_{a+b=m} \frac{\partial^2}{\partial t_a \partial t_b}, \quad m \geq 0,$$

the linear change of variables

$$(111) \quad \begin{aligned} T_0^{q,p}(\mathbf{t}) &= t_1, \\ T_k^{q,p}(\mathbf{t}) &= \left(q \widehat{L}_0 + \frac{2q+p}{\sqrt{p+q}} \widehat{L}_{-1} + \left(\widehat{L}_{-2} - \frac{t_1^2}{2} \right) \right) \cdot T_{k-1}^{q,p}. \end{aligned}$$

and the coefficients a_k are given implicitly through the formal series

$$(112) \quad f(z) := e^{\sum_{k \in \mathbb{Z}_{>0}} a_k \mathbf{1}_k} z e^{-\sum_{k \in \mathbb{Z}_{>0}} a_k \mathbf{1}_k} \in z + z\mathbb{C}[[z]],$$

which is defined by

$$(113) \quad f(z)df(z) = \frac{zdz}{(1 + \sqrt{p+qz})(1 + qz/\sqrt{p+q})},$$

and a formal series

$$(114) \quad R_{q,p}(z) = \exp \left(- \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} \left(p^{2k-1} + q^{2k-1} - \left(\frac{pq}{p+q} \right)^{2k-1} \right) z^{2k-1} \right),$$

which defines an element of the Givental group. Then

Theorem 1. *Rank one Givental operator coincides up to a linear change of variables with the element of the Heisenberg-Virasoro group of symmetries of KP hierarchy if and only if $R(z)$ belongs to a family (114). For this family*

$$(115) \quad \widehat{R}_{q,p} \cdot Z(\mathbf{t}^o) \Big|_{\mathbf{T}=\mathbf{T}^{q,p}(\mathbf{t})} = e^{\hbar^{-1} \sum_{k=4} v_k \frac{\partial}{\partial t_k}} e^{\sum_{k \in \mathbb{Z}_{>0}} a_k \widehat{L}_k} \cdot Z(\mathbf{t}^o)$$

for any function $Z(\mathbf{t}^o)$.

Proof. We have already identified the quadratic parts of two operators and linear changes of variables in Lemmas 2.5 and 2.6. It remains to identify the

translations of the variables. Let us show that for any $Z(\mathbf{t}^o)$

$$(116) \quad \widehat{V}_0^{-1} \left(\sum_{k=4}^{\infty} v_k \frac{\partial}{\partial t_k} \right) \widehat{V}_0 \cdot Z(\mathbf{t}^o) = \sum_{k=2}^{\infty} \delta_k \frac{\partial}{\partial T_k} Z(\mathbf{t}^o),$$

where δ_k 's are given by (22). By definition,

$$(117) \quad e^{-\sum_{z>0} a_k 1_k} \sum_{k=4}^{\infty} v_k z^k e^{\sum_{z>0} a_k 1_k} = \int_0^{h(z)} (f(\eta) - y(\eta)) dx(\eta) \\ = \int_0^z (\eta - y(h(\eta))) \eta d\eta.$$

Using integration by parts we have

$$(118) \quad \frac{1}{\sqrt{2\pi z}} \int_{\gamma} d\eta (\eta - y(h(\eta))) \eta e^{-\frac{\eta^2}{2z}} = z - \frac{z}{\sqrt{2\pi z}} \int_{\gamma} dy(\eta) e^{-\frac{x(\eta)}{z}} \\ = z(1 - R(-z)).$$

This completes the proof. □

Let us stress that this theorem essentially describes the equality of two operators, therefore the formula is true for any function $Z(\mathbf{t}^o)$ for which both sides make sense, not only for the solutions of the KdV hierarchy.

Remark 2.4. *The identification of the Givental and Heisenberg-Virasoro operators (115) for $p = 0$ was established in [46].*

It is natural to compare the higher rank Givental group with the symmetry group of the multicomponent KP hierarchy. However, for higher rank Givental groups there are other promising possibilities, and one can try to identify the Givental operators with the operators from the symmetry groups of other solitonic integrable hierarchies. In particular, for rank two, one can consider 2D Toda lattice and extended Toda lattice. It would be interesting to find a relation between our analysis and the vertex operator analysis of the bilinear Hirota equations considered in [29, 30] for the ADE type singularities.

Remark 2.5. *For any semisimple cohomological field theory one can derive the Virasoro constraints by conjugation of the constraints for the KdV tau-functions with the Givental group element. However, for the cases, when after a linear change of variables the generating function coincides with a tau-function of integrable hierarchy, these constraints gain additional properties. Namely, after a linear change of variables, they belong to the symmetry*

algebra of the hierarchy. Virasoro type constrains and their conjugation by the Givental operators can provide alternative way for analysis of the integrability and other interesting properties.

3. Hodge integrals

3.1. Triple Hodge integrals

Denote by $\mathcal{M}_{g,n}$ the moduli space of all compact Riemann surfaces of genus g with n distinct marked points. Deligne and Mumford [18] defined a natural compactification $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ via stable curves with possible nodal singularities. The moduli space $\overline{\mathcal{M}}_{g,n}$ is a non-singular complex orbifold of dimension $3g - 3 + n$. It is empty unless the stability condition

$$(119) \quad 2g - 2 + n > 0$$

is satisfied. We refer the reader to [18, 34] for the basic theory.

In his seminal paper [49], Witten initiated new directions in the study of $\overline{\mathcal{M}}_{g,n}$. For each marking index i consider the cotangent line bundle $\mathbb{L}_i \rightarrow \overline{\mathcal{M}}_{g,n}$, whose fiber over a point $[\Sigma, z_1, \dots, z_n] \in \overline{\mathcal{M}}_{g,n}$ is the complex cotangent space $T_{z_i}^* \Sigma$ of Σ at z_i . Let $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ denote the first Chern class of \mathbb{L}_i . We consider the intersection numbers

$$(120) \quad \langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_n^{a_n}.$$

The integral on the right-hand side of (120) vanishes unless the stability condition (119) is satisfied, all a_i are non-negative integers, and the dimension constraint $3g - 3 + n = \sum a_i$ holds true. Let $T_i, i \geq 0$, be formal variables and let

$$(121) \quad \tau_{KW} := \exp \left(\sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \hbar^{2g-2+n} F_{g,n} \right),$$

where

$$(122) \quad F_{g,n} := \sum_{a_1, \dots, a_n \geq 0} \langle \tau_{a_1} \tau_{a_2} \cdots \tau_{a_n} \rangle_g \frac{\prod T_{a_i}}{n!}.$$

Witten’s conjecture [49], proved by Kontsevich [38], states that the partition function τ_{KW} becomes a tau-function of the KdV hierarchy after the change of variables $T_n = (2n + 1)!! t_{2n+1}$. Integrability immediately follows

[37] from Kontsevich’s matrix integral representation, for more details see the companion paper [11].

There are different interesting deformations of this tau-function. Let us consider the Hodge bundle \mathbb{E} , a rank g vector bundle over $\mathcal{M}_{g,n}$. Let $\text{ch}_k \in H^{2k}(\overline{\mathcal{M}}_{g,n})$ be the components of the Chern character of \mathbb{E} . According to Mumford, the even components of the Chern character of \mathbb{E} identically vanish. Then we consider

$$(123) \quad \tilde{\mathcal{F}}_{g,n} = \sum_{a_1, \dots, a_n \geq 0} \frac{\prod T_{a_i}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} e^{\sum_{k=1}^{\infty} s_k \text{ch}_{2k-1}} \psi_1^{a_1} \psi_2^{a_2} \dots \psi_n^{a_n}.$$

We introduce a generating function

$$(124) \quad Z(\mathbf{T}; \mathbf{s}) = \exp \left(\sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \hbar^{2g-2+n} \tilde{\mathcal{F}}_{g,n} \right).$$

In general, this deformation does not preserve integrability, because it is easy to check that $Z(\mathbf{T}; \mathbf{s})$ is not a tau-function of the KdV hierarchy in the variables t_k anymore. Investigation of the integrable properties of $Z(\mathbf{t}; \mathbf{s})$ is one of the main goals of this paper.

From Mumford’s theorem [47] it follows that

$$(125) \quad Z(\mathbf{T}; \mathbf{s}) = \widehat{R}(\mathbf{s}) \cdot \tau_{KW}(\mathbf{T}),$$

where

$$(126) \quad \widehat{R}(\mathbf{s}) := \exp \left(\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} s_k \widehat{W}_k \right)$$

and operators \widehat{W}_k are given by (16).

For the Hodge bundle over $\overline{\mathcal{M}}_{g,n}$ let $\lambda_i = c_i(\mathbb{E})$. Then

$$(127) \quad \Lambda_g(u) = \sum_{i=0}^g u^i \lambda_i = e^{\sum_{m=1}^{\infty} (2m-2)! \text{ch}_{2m-1} u^{2m-1}}$$

satisfies $\Lambda_g(u)\Lambda_g(-u) = 1$. Particular specifications of the parameters \mathbf{s} correspond to various families of Hodge integrals. The generating function of the m -linear Hodge integrals

$$(128) \quad \int_{\overline{\mathcal{M}}_{g,n}} \Lambda_g(u_1)\Lambda_g(u_2) \dots \Lambda_g(u_m) \psi_1^{a_1} \psi_2^{a_2} \dots \psi_n^{a_n}$$

corresponds to the Miwa parametrization of \mathbf{s} variables

$$(129) \quad s_k = (2k - 2)! \sum_{j=1}^m u_j^{2k-1}.$$

In particular, the generating function of linear Hodge integrals corresponds to the case $s_k = (2k - 2)!u^{2k-1}$. From Givental’s theory it follows that a product of the Hodge classes $\Lambda_g(u)$ defines a CohFT with flat unit.

Remark 3.1. *We see that it is natural to consider a Miwa-like parametrization of the \mathbf{s} variables. It might be an indication of the existence of the integrable structure in these variables, at least for certain values of \mathbf{T} variables.*

Let us consider the case of cubic Hodge integrals with an additional Calabi-Yau condition

$$(130) \quad \frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} = 0.$$

It is convenient to use the parametrization

$$(131) \quad u_1 = -p, u_2 = -q, u_3 = \frac{pq}{p + q}.$$

Remark 3.2. *Cubic Hodge integrals satisfying the Calabi-Yau condition are related to the topological vertex, which plays an important role in the topological string models on 3-dimensional Calabi-Yau manifolds. In this context the parameters p and q are related to the local framings, for more detail see, e.g., [26] and references therein.*

Consider the generating function

$$(132) \quad Z_{q,p}(\mathbf{T}) = e^{\sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \hbar^{2g-2+n} \mathcal{F}_{g,n}},$$

where

$$(133) \quad \mathcal{F}_{g,n} = \sum_{a_1, \dots, a_n} \frac{\prod T_{a_i}}{n!} \int_{\mathcal{M}_{g,n}} \Lambda_g(-q) \Lambda_g(-p) \Lambda_g\left(\frac{pq}{p + q}\right) \psi_1^{a_1} \psi_2^{a_2} \dots \psi_n^{a_n}.$$

Linear Hodge integrals appear as a specification at $p = u^2, q = 0$ (or $q = u^2, p = 0$), because $\Lambda_g(0) = 1$. From Theorem 1 and equation (125) it immediately follows that

Theorem 2. *Generating function of triple Hodge integrals, satisfying the Calabi-Yau condition, in the variables (98) is a tau-function of the KP hierarchy,*

$$(134) \quad \tau_{q,p}(\mathbf{t}) = Z_{q,p}(\mathbf{T}^{q,p}(\mathbf{t})).$$

It is related to the Kontsevich-Witten tau-function by an element of the Heisenberg-Virasoro group

$$(135) \quad \tau_{q,p}(\mathbf{t}) = e^{\hbar^{-1} \sum_{k=4} v_k \frac{\partial}{\partial t_k}} e^{\sum_{k \in \mathbb{Z}_{>0}} a_k \widehat{L}_k} \cdot \tau_{KW}(\mathbf{t}).$$

For $p = 0$ the KP integrability was proved by Kazarian [36].

Remark 3.3. *After announcement of the main results of this project, Kramer [39] found an independent proof of KP integrability of triple Hodge integrals, satisfying the Calabi-Yau condition. This prove uses the Mariño-Vafa formula (see [50] for more details).*

One can reverse the logic and prove the Mariño-Vafa formula using our proof of Theorem 2. Indeed, the Mariño-Vafa formula describes the coefficients of expansion of two KP tau-functions, related by an element of the Heisenberg-Virasoro subgroup of the KP hierarchy symmetry group. Therefore, to prove the Mariño-Vafa formula it is enough to prove the relation between two tau-functions of the KP hierarchy, which can be done using the relations between corresponding points of the Sato Grassmannian and Kac-Schwarz algebras. The details will be given elsewhere.

Conjecture 3.1. *The case of triple Hodge integrals, satisfying the Calabi-Yau condition, is the most general case of generating functions of Hodge integrals (124), satisfying the KP hierarchy in the variables \mathbf{T} after a linear change of variables. A direct analog of this statement is also true for the Θ -Hodge integrals, see next section.*

Conjecture 3.2. *Theorem 2 has a direct generalization for the r -spin case.*

In a companion paper [11] we develop the methods of the KP hierarchy, suitable for the investigation of the family of the tau-functions, conjecturally describing a generalization of $\tau_{q,p}$ for the r -spin case.

Remark 3.4. *It would also be interesting to investigate KP type integrability of Hodge integrals in the context of Gromov-Witten theory [17, 27]. Particularly attracting is the Gromov-Witten theory of \mathbb{P}^1 , where the generating function is given by the tau-function of the extended Toda hierarchy.*

Hodge integrals, in particular, the representatives of the Calabi-Yau family for the specific values of the parameters q and p , are related to several other interesting integrable structures, including the Dubrovin-Zhang, Volterra or Hodge integrable hierarchies [14, 19, 20, 41, 48, 50]. Connection between them and the KP integrability, investigated in this paper, will be discussed elsewhere.

It is also interesting to consider the specific values of the parameters q and p , for which the tau-function $\tau_{q,p}$ describe k -reductions of the KP hierarchy.

Corollary 3.1. *For $p = -2q$ the tau-function $\tau_{q,p}$ does not depend on even times,*

$$(136) \quad \frac{\partial}{\partial t_{2k}} \tau_{q,-2q}(\mathbf{t}) = 0, \quad k \in \mathbb{Z}_{>0},$$

that is $\tau_{q,-2q}(\mathbf{t})$ is a tau-function of the KdV hierarchy.

Proof. It follows from the change of variables (98). Indeed, since $2q + p = 0$ the operator, in (98) does not contain \widehat{L}_{-1} , and the operators $\widehat{L}_0 - \frac{t_1^2}{2}$ and \widehat{L}_{-2} preserve the parity of the indices of time variables. \square

It is easy to see that it is impossible to get higher k -reductions of the KP hierarchy (aka Gelfand-Dickey hierarchies) this way.

In [19] the relation between the generating function $Z_{q,-2q}$ and the discrete KdV was established. Our results show that there is a simpler relation between $Z_{q,-2q}$ and the ordinary KdV hierarchy, given by a linear change of variables. We expect that this relation can help us better understand the relation between the discrete and ordinary KdV hierarchies.

3.2. Triple Θ -Hodge integrals

From Theorem 1 it follows that the action of Givental operator $\widehat{R}_{q,p}$ on any KdV tau-function leads to the solution of the KP hierarchy after a linear change of variables. The most natural alternative to the KW tau-function here is the Brézin-Gross-Witten (BGW) tau-function. This KdV tau-function governs the intersection theory with the insertions of the fascinating Norbury's Θ -classes. We refer the reader to [16, 44, 45] for a detailed description. The role of the BGW tau-function as the universal building block of TR/GD was first observed by the author, Mironov and Morozov [3–5] in the context of matrix models.

Norbury’s Θ -classes are the cohomology classes, $\Theta_{g,n} \in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n})$, described in [44]. Consider the generating function of the intersection numbers of Θ -classes and ψ -classes

$$(137) \quad F^\Theta = \sum_{a_1, \dots, a_n \geq 0} \frac{\prod T_{a_i}}{n!} \int_{\mathcal{M}_{g,n}} \Theta_{g,n} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_n^{a_n}$$

then we have a direct analog of the Kontsevich-Witten tau-function [44]:

Theorem (Norbury). *Generating function*

$$(138) \quad \tau_\Theta = \exp \left(\sum_{g=0}^\infty \sum_{n=0}^\infty \hbar^{2g-2+n} F_{g,n}^\Theta \right)$$

becomes a tau-function of the KdV hierarchy after the change of variables $T_n = (2n + 1)!! t_{2n+1}$.

Norbury also proved, that τ_Θ is nothing but a tau-function of the BGW model, described by a unitary matrix integral [13, 33]

$$(139) \quad \tau_\Theta = \tau_{BGW}.$$

KdV integrability of the BGW model follows from the relation to the generalized Kontsevich model and was established by Mironov, Morozov and Semenov [43].

Insertion of Norbury’s Θ -classes can be naturally absorbed by the Givental formalism [45]. This allows us to relate integrable properties of generating functions of intersection numbers with and without Θ -classes. Consider the symplectic transformation $R(z)$, introduced in Section 2.1. For it we introduce the change of the dilaton shift associated to the BGW tau-function

$$(140) \quad \sum_{k=1}^\infty \delta_k^0 z^k := 1 - R(-z).$$

Note, that the right hand side of this equation is equal to the right hand side of (22), divided by z . For any symplectic transformation $R(z)$ consider the corresponding action of the Givental operator with new translation part

$$(141) \quad \widehat{R}^0 \cdot Z(\mathbf{T}) = e^{\frac{1}{2} \sum_{i,j=0}^\infty V_{ij}^R \frac{\partial^2}{\partial T_i \partial T_j}} e^{\hbar^{-1} \sum_{k=1}^\infty \delta_k^0 \frac{\partial}{\partial T_k}} Z(\mathbf{T}) \Big|_{T_k \mapsto T_k^R}.$$

The classes $\Theta_{g,n}$ define a degenerate CohFT. It was shown by Norbury that the multiplication of the CohFT by these classes corresponds to a simple modification of the Givental operators, given by (141). It is described by Proposition 3.9 of [45], let us formulate a version of this proposition for the rank 1 case. We stress that the original proposition of Norbury describes a relation between CohFT's, and we consider only its corollary, a relation between the partition functions of these CohFT's. For more details see [45].

The action of the Givental group on the trivial CohFT, associated with the Kontsevich-Witten tau-function, produces certain classes of cohomology $\Omega_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n})$. Let us consider it on the level of partition functions. For any element of the Givental group \widehat{R} one has

$$(142) \quad \widetilde{Z}^\Omega(\mathbf{T}) = \widehat{R} \cdot \tau_{KW},$$

where

$$(143) \quad \widetilde{Z}^\Omega(\mathbf{T}) = \exp \left(\sum_{g=0}^\infty \sum_{n=0}^\infty \hbar^{2g-2+n} \mathcal{F}_{g,n}^\Omega \right),$$

with

$$(144) \quad \mathcal{F}_{g,n}^\Omega = \sum_{a_1, \dots, a_n} \frac{\prod T_{a_i}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n} \psi_1^{a_1} \psi_2^{a_2} \dots \psi_l^{a_n}.$$

For Givental's theory it follows that the classes $\Omega_{g,n}$ exist and satisfy certain peculiar properties. In particular, for the rank one case which we consider in this paper, they always can be expressed in terms of the components of the Chern character of \mathbb{E} , see (123).

Then the insertion of the Θ -classes is described by

Proposition 3.2 (Norbury).

$$(145) \quad \widetilde{Z}^{\Theta\Omega}(\mathbf{T}) = \widehat{R}^0 \cdot \tau_\Theta,$$

where $\Theta\Omega$ denotes the classes $\Theta_{g,n}\Omega_{g,n}$.

Let us apply this proposition for triple Hodge integrals, that is, $\Omega_{g,n} = \Lambda_g(-q)\Lambda_g(-p)\Lambda_g(\frac{pq}{p+q})$. Let

$$(146) \quad Z_{q,p}^\Theta(\mathbf{T}) = \exp \left(\sum_{g=0}^\infty \sum_{n=0}^\infty \hbar^{2g-2+n} \mathcal{F}_{g,n}^\Theta \right),$$

where

$$(147) \quad \mathcal{F}_{g,n}^\Theta = \sum_{a_1, \dots, a_n} \frac{\prod T_{a_i}}{n!} \int_{\mathcal{M}_{g,n}} \Theta_{g,n} \Lambda_g(-q) \Lambda_g(-p) \Lambda_g\left(\frac{pq}{p+q}\right) \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_l^{a_n}.$$

From Proposition 3.2 it immediately follows that

Lemma 3.3.

$$(148) \quad Z_{q,p}^\Theta(\mathbf{T}) = \widehat{R}_{q,p}^0 \cdot \tau_\Theta.$$

Let us introduce the coefficients

$$(149) \quad \tilde{v}_k^0 = [z^k] \int_0^z (df(\eta) - dy(\eta)).$$

They are related to the translations (140),

$$(150) \quad \widehat{V}_0^{-1} \left(\sum_{k=2}^\infty \tilde{v}_k^0 \frac{\partial}{\partial t_k} \right) \widehat{V}_0 \cdot Z(\mathbf{t}^\circ) = \sum_{k=1}^\infty \delta_k^0 \frac{\partial}{\partial T_k} \cdot Z(\mathbf{t}^\circ).$$

Indeed

$$(151) \quad e^{-\sum_{z>0} a_k \mathbf{1}_k} \sum_{k=2}^\infty v_k^0 z^k e^{\sum_{z>0} a_k \mathbf{1}_k} = \int_0^{h(z)} (df(\eta) - dy(\eta)) \\ = \int_0^z (d\eta - dy(h(\eta))),$$

and from integration by parts

$$(152) \quad \frac{1}{\sqrt{2\pi z}} \int_\gamma (d\eta - dy(h(\eta))) e^{-\frac{\eta^2}{2z}} = 1 - \frac{1}{\sqrt{2\pi z}} \int_\gamma dy(\eta) e^{-\frac{x(\eta)}{z}} \\ = 1 - R(-z).$$

Then we have

Theorem 3. *Generating function of triple Θ -Hodge integrals, satisfying the Calabi-Yau condition, in the variables (98) is a tau-function of the KP hierarchy,*

$$(153) \quad \tau_{q,p}^\Theta(\mathbf{t}) = Z_{q,p}^\Theta(\mathbf{T}^{q,p}(\mathbf{t})).$$

It is related to the Brézin-Gross-Witten tau-function by an element of the Heisenberg-Virasoro group

$$(154) \quad \tau_{q,p}^\Theta(\mathbf{t}) = e^{\hbar^{-1} \sum_{k>0} \tilde{v}_k^0 \frac{\partial}{\partial t_k}} e^{\sum_{k \in \mathbb{Z}_{>0}} a_k \hat{L}_k} \cdot \tau_{BGW}(\mathbf{t}).$$

Conjecture 3.3. *Theorem 3 has a direct generalization for the r -spin case.*

Similar to Corollary 3.1 one has

Corollary 3.4. *For $p = -2q$ the tau-function $\tau_{q,p}$ does not depend on even times,*

$$(155) \quad \frac{\partial}{\partial t_{2k}} \tau_{q,-2q}^\Theta(\mathbf{t}) = 0, \quad k \in \mathbb{Z}_{>0},$$

that is $\tau_{q,-2q}^\Theta(\mathbf{t})$ is a tau-function of the KdV hierarchy.

The proof is completely analogous to the proof Corollary 3.1.

Generating functions $Z_{q,p}$, $Z_{q,p}^\Theta$ satisfy the Virasoro constraints, and tau-functions $\tau_{q,p}$, $\tau_{q,p}^\Theta$ satisfy the Heisenberg-Virasoro constraints. All these constraints can be obtained from the constraints for τ_{KW} and τ_{BGW} by conjugation with Givental (for Z 's) and Heisenberg-Virasoro (for τ 's) group elements. Moreover, all these generating functions possess simple cut-and-join description. These topics, as well as matrix integral description of $\tau_{q,p}^\Theta(\mathbf{t})$ and its generalization for the r -spin case, will be described elsewhere.

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References

- [1] A. S. Alexandrov, A. Mironov and A. Morozov, "Unified description of correlators in non-Gaussian phases of Hermitean matrix model," *Int. J. Mod. Phys. A* **21** (2006) 2481. [MR2234724](#)

- [2] A. S. Alexandrov, A. D. Mironov and A. Y. Morozov, “Partition functions of matrix models as the first special functions of string theory: Finite Hermitian one-matrix model,” *Theor. Math. Phys.* **142** (2005) 349. [MR2165901](#)
- [3] A. S. Alexandrov, A. Mironov and A. Morozov, “M-theory of matrix models,” *Theor. Math. Phys.* **150** (2007) 153. [MR2325922](#)
- [4] A. S. Alexandrov, A. Mironov and A. Morozov, “Instantons and merons in matrix models,” *Physica D* **235** (2007) 126. [MR2385123](#)
- [5] A. Alexandrov, A. Mironov and A. Morozov, “BGWM as Second Constituent of Complex Matrix Model,” *JHEP* **0912** (2009) 053. [MR2592997](#)
- [6] A. Alexandrov, “Enumerative Geometry, Tau-Functions and Heisenberg-Virasoro Algebra,” *Commun. Math. Phys.* **338** (2015) no. 1, 195. [MR3345375](#)
- [7] A. Alexandrov, “Cut-and-join description of generalized Brezin-Gross-Witten model,” *Adv. Theor. Math. Phys.* **22** (2018) 1347–1399. [MR3952352](#)
- [8] A. Alexandrov, “Open intersection numbers, Kontsevich-Penner model and cut-and-join operators,” *JHEP* **1508** (2015) 028. [MR3402137](#)
- [9] A. Alexandrov, D. Lewanski and S. Shadrin, “Ramifications of Hurwitz theory, KP integrability and quantum curves,” *JHEP* **1605** (2016) 124. [MR3521843](#)
- [10] A. Alexandrov, G. Chapuy, B. Eynard and J. Harnad, “Fermionic approach to weighted Hurwitz numbers and topological recursion,” *Commun. Math. Phys.* **360** (2018) no. 2, 777–826. [MR3800798](#)
- [11] A. Alexandrov, “KP integrability of triple Hodge integrals. II. Generalized Kontsevich matrix model,” *Anal. Math. Phys.* **11** (2021) no. 1, 24. [MR4195117](#)
- [12] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, “Planar Diagrams,” *Commun. Math. Phys.* **59** (1978) 35. [MR0471676](#)
- [13] E. Brezin and D. J. Gross, “The External Field Problem in the Large N Limit of QCD,” *Phys. Lett. B* **97** (1980) 120. [MR0593129](#)
- [14] A. Buryak, “Dubrovin-Zhang hierarchy for the Hodge integrals,” *Commun. Num. Theor. Phys.* **9** (2015) no. 2, 239. [MR3361294](#)

- [15] L. Chekhov and B. Eynard, “Hermitean matrix model free energy: Feynman graph technique for all genera,” *JHEP* **0603** (2006) 014. [MR2222762](#)
- [16] L. Chekhov and P. Norbury, “Topological recursion with hard edges,” *Int. J. Math.* **30** (2019) no. 03, 1950014. [MR3941980](#)
- [17] A. Chiodo and D. Zvonkine, “Twisted r-spin potential and Givental’s quantization,” *Adv. Theor. Math. Phys.* **13** (2009) no. 5, 1335. [MR2672465](#)
- [18] P. Deligne, D. Mumford, “The irreducibility of the space of curves of given genus,” *Publications mathématiques de l’I.H.É.S.* **36** (1969) 75–109. [MR0262240](#)
- [19] B. Dubrovin, S. Q. Liu, D. Yang and Y. Zhang, “Hodge–GUE Correspondence and the Discrete KdV Equation,” *Commun. Math. Phys.* **379** (2020) no. 2, 461–490. [MR4156215](#)
- [20] B. Dubrovin, S. Q. Liu, D. Yang and Y. Zhang, “Hodge integrals and tau-symmetric integrable hierarchies of Hamiltonian evolutionary PDEs,” *Advances in Mathematics* **293** (2016) 382–435. [MR3474326](#)
- [21] P. Dunin-Barkowski, N. Orantin, S. Shadrin and L. Spitz, “Identification of the Givental formula with the spectral curve topological recursion procedure,” *Commun. Math. Phys.* **328** (2014) 669. [MR3199996](#)
- [22] B. Eynard, “Topological expansion for the 1-Hermitian matrix model correlation functions,” *JHEP* **11** (2004) 031 [MR2118807](#)
- [23] B. Eynard, “Invariants of spectral curves and intersection theory of moduli spaces of complex curves,” *Commun. Num. Theor. Phys.* **08** (2014) 541. [MR3282995](#)
- [24] B. Eynard, “The Geometry of integrable systems. Tau functions and homology of Spectral curves. Perturbative definition” [[arXiv:1706.04938](#) [\[math-ph\]](#)].
- [25] B. Eynard and N. Orantin, “Invariants of algebraic curves and topological expansion,” *Commun. Num. Theor. Phys.* **1** (2007) 347. [MR2346575](#)
- [26] B. Eynard and N. Orantin, “Computation of Open Gromov-Witten Invariants for Toric Calabi-Yau 3-Folds by Topological Recursion, a Proof of the BKMP Conjecture,” *Commun. Math. Phys.* **337** (2015) no. 2, 483–567. [MR3339157](#)

- [27] C. Faber and R. Pandharipande, “Hodge integrals and Gromov-Witten theory,” *Inventiones mathematicae* **139** (2000) no. 1, 173. [MR1728879](#)
- [28] E. Feigin, J. van de Leur and S. Shadrin, “Givental symmetries of Frobenius manifolds and multi-component KP tau-functions,” *Adv. Math.* **224** (2010) no. 3, 1031. [MR2628802](#)
- [29] E. Frenkel, A. Givental and T. Milanov, “Soliton equations, vertex operators, and simple singularities,” *Funct. Anal. Other Math.* **3** (2010) no. 1, 47. [MR2734560](#)
- [30] A. Givental, and T. Milanov, “Simple singularities and integrable hierarchies.” *The breadth of symplectic and Poisson geometry.* Birkhäuser Boston, 2005, 173–201. [MR2103007](#)
- [31] A. Givental, “Semisimple Frobenius structures at higher genus,” *Int. Math. Res. Not.* **2001** (2001) no. 23, 1265. [MR1866444](#)
- [32] A. B. Givental, “Gromov-Witten invariants and quantization of quadratic hamiltonians,” *Moscow Math. J.* **1** (2001) no. 4, 551. [MR1901075](#)
- [33] D. J. Gross and E. Witten, “Possible Third Order Phase Transition in the Large N Lattice Gauge Theory,” *Phys. Rev. D* **21** (1980) 446.
- [34] J. Harris, I. Morrison. *Moduli of curves.* Graduate Texts in Mathematics, 187. Springer-Verlag, New York, 1998. [MR1631825](#)
- [35] G. 't Hooft, “A Planar Diagram Theory for Strong Interactions,” *Nucl. Phys. B* **72** (1974) 461.
- [36] M. Kazarian, “KP hierarchy for Hodge integrals,” *Adv. Math.* **221** (2009) no. 1, 1. [MR2509319](#)
- [37] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and A. Zabrodin, “Towards unified theory of 2-d gravity,” *Nucl. Phys. B* **380** (1992) 181. [MR1186584](#)
- [38] M. Kontsevich, “Intersection theory on the moduli space of curves and the matrix Airy function,” *Commun. Math. Phys.* **147** (1992) 1–23. [MR1171758](#)
- [39] R. Kramer, “Cycles of curves, cover counts, and central invariants,” PhD thesis, 2019.
- [40] Y.-P. Lee, “Invariance of tautological equations II: Gromov-Witten theory,” *Journal of the American Mathematical Society* **22** (2008) no. 2, 331. [MR2476776](#)

- [41] S. Q. Liu, D. Yang, Y. Zhang and C. Zhou, “The Hodge-FVH Correspondence,” *Journal für die reine und angewandte Mathematik* (2021), <https://doi.org/10.1515/crelle-2020-0051>. [MR4265184](#)
- [42] A. Mironov and A. Morozov, “Virasoro constraints for Kontsevich-Hurwitz partition function,” *JHEP* **02** (2009) 024. [MR2486405](#)
- [43] A. Mironov, A. Morozov and G. W. Semenoff, “Unitary matrix integrals in the framework of generalized Kontsevich model. 1. Brezin-Gross-Witten model,” *Int. J. Mod. Phys. A* **11** (1996) 5031. [MR1411777](#)
- [44] P. Norbury, “A new cohomology class on the moduli space of curves” [[arXiv:1712.03662](#)].
- [45] P. Norbury, “Gromov-Witten invariants of \mathbb{P}^1 coupled to a KdV tau function” [[arXiv:1812.04221](#)].
- [46] X. Liu and G. Wang, “Connecting the Kontsevich-Witten and Hodge Tau-functions by the $\widehat{GL}(\infty)$ Operators,” *Commun. Math. Phys.* **346** (2016) no. 1, 143. [MR3528419](#)
- [47] D. Mumford, “Towards an enumerative geometry of the moduli space of curves.” *Arithmetic and geometry*. Birkhäuser Boston, 1983, 271–328. [MR0717614](#)
- [48] K. Takasaki, “Cubic Hodge integrals and integrable hierarchies of Volterra type” [[arXiv:1909.13095](#) [[math-ph](#)]]. [MR1445352](#)
- [49] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*. *Surveys in differential geometry* (Cambridge, MA, 1990), 243–310, Lehigh Univ., Bethlehem, PA, 1991. [MR1144529](#)
- [50] J. Zhou, “Hodge Integrals and Integrable Hierarchies,” *Lett. Math. Phys.* **93** (2010) no. 1, 55–71. [MR2661523](#)

ALEXANDER ALEXANDROV
CENTER FOR GEOMETRY AND PHYSICS
INSTITUTE FOR BASIC SCIENCE (IBS)
POHANG 37673
KOREA
E-mail address: alexandrovsash@gmail.com

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