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On a class of non-simply connected Calabi-Yau 3-folds with positive Euler characteristic*

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In this work we obtain a class of non-simply connected Calabi-Yau 3-folds with positive Euler characteristic as the quotient of projective small resolutions of singular Schoen 3-folds under the free action of finite groups. A Schoen 3-fold is a fiber product $X = B_1 \times_{\mathbb{P}^1} B_2$ of two relatively minimal rational elliptic surfaces with section $\beta_i : B_i \to \mathbb{P}^1$, $i = 1, 2$. Schoen has shown that if X is smooth, then X is a simply connected Calabi-Yau 3-fold, and if the only singularities of X are on $I_r \times I_s$ type fibers with $r > 1$ and $s > 1$, then there exists a projective small resolution $\hat{X}$ of X, and $\hat{X}$ is a simply connected Calabi-Yau 3-fold [7]. If G is a finite group which acts freely on a smooth Schoen 3-fold X, then the quotient X/G is a non-simply connected Calabi-Yau 3-fold with fundamental group G, and all such group actions have been classified by Bouchard and Donagi [2]. Bouchard and Donagi have proposed the open problem of classifying all finite groups G which act freely on projective small resolutions $\hat{X}$ of singular Schoen 3-folds X. In this case the quotient $\hat{X}/G$ is again a Calabi-Yau 3-fold with fundamental group G. In this paper we first classify the finite groups G which act freely on singular Schoen 3-folds X where the only singularities of X are on $I_r \times I_s$ type fibers with $r > 1$ and $s > 1$ and the elements of G act on X as an automorphism $\tau_1 \times \tau_2$ where each τ_i is an automorphism of the elliptic surface B_i. A projective small resolution $\hat{X}$ of X is obtained by blowing up some components of the $I_r \times I_s$ fibers on X. We determine which of the free actions on the singular 3-fold X lift to free actions on the Calabi-Yau 3-fold $\hat{X}$. For the non-simply connected Calabi-Yau 3-folds $\hat{X}/G$ obtained with this construction, the distinct fundamental groups are $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, and $\mathbb{Z}_n$ for $n = 6, 5, 4, 3, 2$. These are the same groups obtained by Bouchard and Donagi by working on free actions on smooth Schoen 3-folds. While the Euler characteristic of each X/G obtained by Bouchard and Donagi is 0, the Euler characteristics of all non-simply connected Calabi-Yau 3-folds $\hat{X}/G$ we obtain in

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this paper are positive and they range in: 64, 54, 48, 40 and $2k$ for $2 \leq k \leq 18$. The given Euler characteristic values do not all occur for each of the listed fundamental groups. The classification of finite groups which act freely on singular Schoen 3-folds X whose singularities are on $I_r \times I_s$ type fibers with $r > 1$ and $s > 1$, the classification of such group actions which lift to free actions on projective small resolutions $\hat{X}$ of X, and the fundamental groups and Euler characteristic values of the non-simply connected Calabi-Yau 3-folds $\hat{X}/G$ are displayed in several tables. The study of the group actions on X which induce a non-trivial action on the base curve $\mathbb{P}^1$ and which induce a trivial action on $\mathbb{P}^1$ is carried out separately.

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1. Introduction

A Schoen 3-fold X is a fiber product $X = B_1 \times_{\mathbb{P}^1} B_2 = \{(a,b) \in B_1 \times B_2 | \beta_1(a) = \beta_2(b)\}$ of two relatively minimal rational elliptic surfaces with section $\beta_i : B_i \to \mathbb{P}^1$, $i = 1, 2$. The fiber product X is a smooth 3-fold if and only if $S = S_1 \cap S_2 = \emptyset$ where $S_i = \{p \in \mathbb{P}^1 | \beta_i^{-1}(p)$ is a singular fiber of $B_i\}$ for $i = 1, 2$. Schoen has shown that if X is smooth, then X is a simply connected Calabi-Yau 3-fold [7]. Bouchard and Donagi have studied finite abelian groups G which act on smooth fiber products $X = B_1 \times_{\mathbb{P}^1} B_2$ freely (without fixed points) where each element $g \in G$ acts on X as a product $\tau_1 \times \tau_2 \in Aut(B_1) \times Aut(B_2)$ of automorphisms τ_1 and τ_2 of the relatively minimal rational elliptic surfaces with section B_1 and B_2, respectively. Explicitly $\tau_1 \times \tau_2 : X \to X$ is defined by $(a,b) \mapsto (\tau_1(a), \tau_2(b))$ where τ_1 and τ_2 have the same induced action on the base curve $\mathbb{P}^1$ so that the map $\tau_1 \times \tau_2$ is well-defined as a map on the fiber product X. For such a group action on the smooth fiber product X which is a simply connected Calabi-Yau 3-fold, the quotient 3-fold X/G under the group action is a non-simply connected Calabi-Yau 3-fold with fundamental group G as stated in [2] by using Beauville's argument in [1]. Bouchard and Donagi have constructed and classified all finite abelian groups G which act freely on smooth Schoen 3-folds X and which induce cyclic group actions on the base curve $\mathbb{P}^1$, and they obtained a class of non-simply connected Calabi-Yau 3-folds as the quotient

3-folds under the group action [2]. Using a classification of the automorphism groups of relatively minimal rational elliptic surfaces with section developed in [3, 4], the author has proved in [5] that if a finite group G acts freely on a smooth Schoen 3-fold X as a subgroup of $Aut(B_1) \times Aut(B_2)$ (so that elements of G are of the form $\tau_1 \times \tau_2$ as above), then the action of G on X induces a cyclic action on the base curve $\mathbb{P}^1$, hence the list of non-simply connected Calabi-Yau 3-folds obtained as quotients of smooth Schoen 3-folds given in [2] is a complete list (no other fixed point free finite group action on smooth Schoen 3-folds exists).

For singular Schoen 3-folds X, Schoen has shown that if all singularities of X are ordinary double points, then under certain conditions X has a projective small resolution which is a simply connected Calabi-Yau 3-fold (see §2 and Lemma 3.1 in [7]). The second open problem proposed in §8 of [2] asks the classification of finite groups which act freely on such desingularizations of singular Schoen 3-folds in order to construct a possibly larger family of non-simply connected Calabi-Yau 3-folds than the family they obtained in [2]. In this paper we solve this open problem for the singular Schoen 3-folds X whose singularities are on $I_r \times I_s$ type fibers with $r > 1$ and $s > 1$. For such 3-folds X, a projective small resolution $\hat{X}$ of X is obtained by successively blowing up some components of the $I_r \times I_s$ fibers of X in order to resolve the singularities by small resolutions (see §1 and Lemma 3.1 in [7]). Note that in Lemma 3.1 in [7], it is also shown that if X is a singular Schoen 3-fold with only ordinary double point singularities and $\beta_1 = \beta_2$ (the two elliptic surfaces of the fiber product are identical), then a projective small resolution of X exists and is obtained by blowing up the diagonal Δ in X. Such singular Schoen 3-folds X where $\beta_1 = \beta_2$ are not considered in this paper.

In Theorem 13 in §5.2, we first classify the finite groups G which act freely on singular Schoen 3-folds X whose singularities are on $I_r \times I_s$ type fibers with $r > 1$ and $s > 1$ such that the induced action on the base curve $\mathbb{P}^1$ is non-trivial and the elements of G act on X as products of automorphisms of the two elliptic surfaces B_1 and B_2. In §6 we determine which of these actions lift to free actions on projective small resolutions $\hat{X}$ of X which are obtained by successively blowing up some components of the $I_r \times I_s$ fibers of X. The same classification and lifting task for the free actions on X where the induced action on $\mathbb{P}^1$ is trivial is accomplished in §6.3. The main result of the paper is stated in Theorem 18 in §7. For the non-simply connected Calabi-Yau 3-folds $\hat{X}/G$ obtained as the quotient spaces under these free actions, the distinct fundamental groups G are $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_n$ for $n = 6, 5, 4, 3, 2$. All of these 3-folds $\hat{X}/G$ have positive

Euler characteristics and the distinct χ values we obtain are 64, 54, 48, 40 and $2k$ for $2 \leq k \leq 18$. The Euler characteristic values achieved for each fundamental group G are displayed in Table 13. Comparing the results we obtain in this paper to the results of Bouchard and Donagi in [2] where free actions on smooth Schoen 3-folds have been classified, the same finite groups G act freely on smooth Schoen 3-folds X and on small projective resolutions $\hat{X}$ of the singular Schoen 3-folds studied in this paper. While the Euler characteristic of the 3-folds X/G obtained in [2] is 0, the Euler characteristics of all 3-folds $\hat{X}/G$ we obtain in this paper are positive.

As mentioned in the remarks 4.3.1 and 6.4 in [2], in their analysis Bouchard and Donagi have considered suitable σ-pairs (B, α) and suitable pairs (B, τ) where the fiber f_∞ over $\infty \in \mathbb{P}^1$ of both the elliptic surface B and the quotient surface $\hat{B}$ is smooth. They have excluded the analysis of the suitable σ-pairs and suitable pairs where the fiber f_∞ is a singular fiber of type I_r for some $r \geq 1$. These excluded cases form a codimension 1 family for the moduli space of elliptic surfaces corresponding to the action of a specific group G. In this paper, we make use of these suitable σ-pairs and suitable pairs excluded in [2] (which we call suitable σ-pairs and suitable pairs of *special type*) in order to construct free actions on the singular Schoen 3-folds under study. We also list the free actions on smooth Schoen 3-folds where the action on at least one of B_1 and B_2 is of *special type* in §5.1.

2. Free actions on desingularizations of singular Schoen 3-folds by small resolutions

Let $X = B_1 \times_{\mathbb{P}^1} B_2 = \{(a,b) \in B_1 \times B_2 | \beta_1(a) = \beta_2(b)\}$ be the fiber product of two relatively minimal rational elliptic surfaces $\beta_i : B_i \to \mathbb{P}^1$, $i = 1, 2$ with section. Let $S_i = \{p \in \mathbb{P}^1 | \beta_i^{-1}(p)$ is a singular fiber of $B_i\}$ for $i = 1, 2$. X is a smooth 3-fold if and only if $S = \emptyset$ where $S = S_1 \cap S_2$. A double point q on a hypersurface 3-fold Y is called an ordinary double point (or a node) if the projectivized tangent cone of Y at q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The fiber product X has only ordinary double point singularities if and only if $S \neq \emptyset$ and for every $p \in S$, $\beta_1^{-1}(p)$ and $\beta_2^{-1}(p)$ are singular fibers of type I_r for $r > 0$ using Kodaira's notation for singular fibers of elliptic surfaces. If we define the projection map $\beta : X \to \mathbb{P}^1$ by $\beta((a,b)) = \beta_1(a)$ (which also equals $\beta_2(b)$ by the definition of X), then X is singular with only ordinary double point singularities if and only if $S \neq \emptyset$ and $\beta^{-1}(p)$ is of the form $I_r \times I_s$ for some $r > 0$ and $s > 0$ for each $p \in S$. For such a singular fiber product X, the ordinary double points are exactly the points $(a,b) \in X$

such that a is a singular point of $\beta_1^{-1}(p) = I_r$ and b is a singular point of $\beta_2^{-1}(p) = I_s$ for each $p \in S$. There are rs ordinary double points in $\beta^{-1}(p)$ if $\beta^{-1}(p) = I_r \times I_s$. The singularity of X at each ordinary double point can be resolved by a local operation called small resolution. In general if $q \in Y$ is an ordinary double point on a singular 3-fold Y, a small resolution of Y at q is a 3-fold $\hat{Y}$ with a map $\pi : \hat{Y} \to Y$ such that the exceptional locus at q is $E_q = \pi^{-1}(q) = \mathbb{P}^1$ and π restricts to an isomorphism $\hat{Y} - E_q \to Y - p$ and $\hat{Y}$ has no singularities on E_q. Note that there are two small resolutions of Y at q which are non-isomorphic over Y. Successively applying the small resolution operation at all singularities of X, which are assumed to be ordinary double points, we can obtain a desingularization of X by small resolutions which we denote by $\hat{X}$ and which we call a small resolution (or minimal resolution) of X. The reader is referred to §2 in [7] for the construction of the small resolution and general facts about small resolutions. Schoen has shown in [7] (§2 and §3) that if X has only ordinary double points and if either (i) for each $p \in S$ neither $\beta_1^{-1}(p)$ nor $\beta_2^{-1}(p)$ is irreducible (singular fiber of type I_1), or (ii) $\beta_1 = \beta_2$ (the two elliptic surfaces of the fiber product are identical), then there exists a small resolution $\hat{X}$ of X which is a projective, simply connected Calabi-Yau 3-fold. The projective 3-fold $\hat{X}$ is obtained by blowing up the diagonal Δ of the fiber product X in Case (ii), and by successively blowing up some divisors of the form $\theta_i \times \Gamma_j$ in Case (i) where θ_i and Γ_j are components of the I_r and I_s fibers of B_1 and B_2 for the $I_r \times I_s$ fibers of X which contain the singularities of X. For such a 3-fold $\hat{X}$, if G is a finite group which acts freely (without fixed points for each non-identity element of G) on $\hat{X}$, then the quotient 3-fold $\hat{X}/G$ is a non-simply connected Calabi-Yau 3-fold with fundamental group G due to the same reason as explained in §1.2 of [2] using Beauville's argument in [1].

Bouchard and Donagi have proposed the open problem (see §8 in [2]) of classifying finite groups G which act freely on projective small resolutions $\hat{X}$ of singular Schoen 3-folds X with only ordinary double point singularities where X satisfies the condition (i) or (ii) stated above. In this paper we solve this open problem for the 3-folds X satisfying the condition (i) by studying finite groups G acting on X where the action of G lifts to a free action on $\hat{X}$.

Lemma 1. *If the action of a finite group G on X lifts to a free action on a small resolution $\hat{X}$, than the action of G on X is also free.*

Proof. Otherwise, if a non-identity $g \in G$ has a fixed point $q \in X$, then if q is not a singular point of X, q will be a fixed point of g in the lifted action of G. If q is a singular point of X, then in the lifted action on $\hat{X}$, g maps the exceptional locus $E_q = \mathbb{P}^1$ over q to itself, hence g has a fixed point on E_q. □

By this lemma, the problem reduces to studying finite groups G which act freely on the singular Schoen 3-folds X which satisfy the conditions stated above so that there is a small resolution $\hat{X}$ which is a projective, simply connected Calabi-Yau 3-fold. Once such group actions are classified, we need to determine which of these actions lift to free actions on the small resolution $\hat{X}$.

Remark. The criterion we develop in Lemma 14 in §6 for lifting the action of X to a projective small resolution $\hat{X}$ only applies in the case that X satisfies the condition (i) (the only singularities of X are on $I_r \times I_s$ type fibers with $r > 1$ and $s > 1$). In this paper we exclude the 3-folds X satisfying the condition (ii) from the discussion.

3. Free actions on singular Schoen 3-folds

3.1. Preliminaries on automorphism groups of rational elliptic surfaces

Before discussing the group actions on singular Schoen 3-folds, we give a brief summary of the general facts on the automorphism group of a relatively minimal rational elliptic surface B with section. The reader can consult [3, 4] and [2] for details. Let $\sigma \subset B$ be a section of the relatively minimal rational elliptic surface $\beta : B \to \mathbb{P}^1$ with section. All sections of B form a group $MW(B)$ called the Mordell-Weil group of B with σ as the identity of the group (we will call σ the zero section of B). The group operation on $MW(B)$ is performing the group law on each smooth fiber F which is an elliptic curve with identity $\sigma \cap F$. Mordell-Weil group $MW(B)$ naturally embeds in the automorphism group $Aut(B)$ as the automorphisms acting on B as translation by a section. More precisely, if $\epsilon \in MW(B)$ is a section, then the translation by ϵ is the automorphism t_ϵ which acts on each smooth fiber F which is an elliptic curve by $x \mapsto (\epsilon \cap F) + x$ where $+$ denotes the group operation on the elliptic curve F. This action extends to all of B as an automorphism. We will identify ϵ and t_ϵ, hence we will identify the Mordell-Weil group $MW(B)$ with its isomorphic copy in the automorphism group $Aut(B)$ of the surface B throughout the paper. Oguiso and Shioda have classified the Mordell-Weil groups and Mordell-Weil lattices of relatively minimal rational elliptic surfaces with section in [6].

If we define the subgroup $Aut_\sigma(B)$ of $Aut(B)$ by

$$Aut_\sigma(B) = \{\tau \in Aut(B) | \tau(\sigma) = \sigma\}$$

(the subgroup of automorphisms of B which preserve the zero section as a set), then we can define a group epimorphism

$$\psi : Aut(B) \to Aut_\sigma(B)$$
$$\tau \mapsto t_{-\tau(\sigma)} \circ \tau$$

where $\ker(\psi) = MW(B)$. Hence $Aut(B)$ is given as the semi-direct product

$$Aut(B) = MW(B) \rtimes Aut_\sigma(B).$$

For any $\alpha \in Aut_\sigma(B)$ and $t_\epsilon \in MW(B)$ we have $\alpha \circ t_\epsilon = t_{\alpha(\epsilon)} \circ \alpha$.

Since the canonical class of B is $K_B = -F$ where F is the fiber class of the rational elliptic surface B with section and the linear system of F is a pencil, every automorphism of B maps fibers to fibers. Thus, we get a group homomorphism

$$\phi : Aut(B) \to Aut(\mathbb{P}^1)$$
$$\tau \mapsto \tau_{\mathbb{P}^1}$$

such that $\beta \circ \tau = \tau_{\mathbb{P}^1} \circ \beta$. We denote the image of ϕ by $Aut_B(\mathbb{P}^1) := \phi(Aut(B))$ (the group of induced automorphisms on $\mathbb{P}^1$).

3.2. Conditions for free action on singular Schoen 3-folds

Let $X = B_1 \times_{\mathbb{P}^1} B_2$ be a singular Schoen 3-fold ($S = S_1 \cap S_2 \neq \emptyset$) where the only singularities of X are on $I_r \times I_s$ type fibers with $r > 1$ and $s > 1$ (for each $p \in S$ we have $\beta^{-1}(p) = I_r \times I_s$ for some $r > 1$ and $s > 1$). Such a singular Schoen 3-fold X has a projective small resolution $\hat{X}$ which is a simply connected Calabi-Yau 3-fold (see §2). We have reduced the problem to the classification of finite groups G acting on X freely (without fixed points). We consider the group G as a subgroup of $Aut(B_1) \times Aut(B_2)$ so that each element of G is of the form $\tau_1 \times \tau_2$ where $\tau_i \in Aut(B_i)$ ($i = 1, 2$). Since every finite order automorphism of a relatively minimal rational elliptic surface with section has a fixed point (see the appendix in [5] for a proof and some comments), the group homomorphisms $\pi_i : G \to Aut(B_i)$, $\tau_1 \times \tau_2 \mapsto \tau_i$ ($i = 1, 2$) are injective if G acts freely on X (that is, non-identity elements of G have no fixed points). Thus, G is isomorphic to the subgroup $\pi_i(G)$ of $Aut(B_i)$ for $i = 1, 2$. This reduces the analysis to working in the automorphism groups of the two rational elliptic surfaces B_1 and B_2.

The group G acts freely on X if and only if for every $\tau_1 \times \tau_2 \in G$ the cyclic group $\langle \tau_1 \times \tau_2 \rangle$ acts freely on X. Bouchard and Donagi have worked out the conditions for the free action of $\langle \tau_1 \times \tau_2 \rangle$ on a smooth Schoen 3-fold X in [2]. We generalize and modify these conditions to the case of singular X as follows (Note that the lemmas below are valid for any singular Schoen 3-fold X).

Let $n_i = ord(\tau_i)$, $\alpha_i = \psi(\tau_i) \in Aut_\sigma(B_i)$, $m_i = ord(\alpha_i)$ and $\bar{m}_i = ord(\phi(\tau_i))$ for $i = 1, 2$ where ord denotes the order of an element in a group. Let $d_i = n_i/m_i$ and $k_i = m_i/\bar{m}_i$ for $i = 1, 2$. In order to have a well-defined map $\tau_1 \times \tau_2$ on the fiber product X, we need $\phi(\tau_1) = \phi(\tau_2)$ which gives $\bar{m}_1 = \bar{m}_2$.

Lemma 2 (Generalization of Lemma 3.5 in [2])**.** *If $\langle \tau_1 \times \tau_2 \rangle$ acts freely on X, then $n_1 = n_2$ and $k_1 = k_2 = 1$.*

The proof is exactly the same as the proof given in [2]. Note that the proof uses the fact that every finite order automorphism of B_i has a fixed point, and if $k_i > 1$, then $\tau_i^{\bar{m}_i}$ has a fixed curve intersecting each smooth fiber (Lemma 3.3 in [2]), hence the fixed locus of $\tau_i^{\bar{m}_i}$ intersects all fibers of B_i.

As a result we obtain $m_1 = m_2 = \bar{m}_1 = \bar{m}_2$ and $n_1 = n_2$ if $\langle \tau_1 \times \tau_2 \rangle$ acts freely on X. In the below discussion in this section we assume that these equations hold for τ_1 and τ_2. From now on we will drop the subscript and write n, m and d in place of n_i, m_i and d_i, respectively. We have $n = dm$ where d is the order of τ_i^m which is translation by a torsion section on B_i $(i = 1, 2)$.

For $m = 1$ case τ_i is in the torsion subgroup of $MW(B_i)$ (τ_i is translation by a torsion section of B_i). Any nonzero torsion section is disjoint from the zero section σ (see Proposition 2.1 in [2]), hence translation by a nonzero torsion section does not have fixed points on smooth fibers. If X is a smooth fiber product, this implies that for $m = 1$ case $\langle \tau_1 \times \tau_2 \rangle$ acts freely on X since $S = \emptyset$ (singular fibers of B_1 and B_2 are not paired in the fiber product). But if X is singular as we are examining in this discussion, then $\langle \tau_1 \times \tau_2 \rangle$ may not act freely on X in the $m = 1$ case. This is one of the main differences between the smooth X and singular X cases.

For the $m > 1$ case, $\phi(\tau_1) = \phi(\tau_2)$ is an order m automorphism of the base curve $\mathbb{P}^1$. After a change of coordinates in $\mathbb{P}^1$, the fixed points of $\phi(\tau_1) = \phi(\tau_2)$ are 0 and ∞, and the map is given by $z \mapsto \omega_m z$ where ω_m is a primitive mth root of unity. With this convention:

Lemma 3 (Generalization of Lemma 3.6 in [2])**.** *Suppose that $m > 1$. If $\langle \tau_1 \times \tau_2 \rangle$ acts freely on X, then $\langle \tau_1 \rangle$ acts freely on the fiber $f_{1\infty}$ and $\langle \tau_2 \rangle$ acts freely on the fiber f_{20}, or vice versa ($\langle \tau_1 \rangle$ acts freely on f_{10} and $\langle \tau_2 \rangle$ acts freely on $f_{2\infty}$). Here f_{ij} denotes the fiber of the elliptic surface B_i over the point $j \in \mathbb{P}^1$.*

The converse is true in the $m > 1$ and $d = 1$ (n=m) case. For the $m > 1$ and $d > 1$ case, $\langle \tau_1 \rangle$ acts freely on $f_{1\infty}$ and $\langle \tau_2 \rangle$ acts freely on f_{20} (or vice versa, interchanging 0 and ∞) and $\langle \tau_1^m \times \tau_2^m \rangle$ acts freely on X implies that $\langle \tau_1 \times \tau_2 \rangle$ acts freely on X.

Note that in the generalization of Lemma 3.6 in [2] to the singular X case, the last sentence in the original version is dropped since when X is singular f_{10} and f_{20} both can be singular fibers (similarly for $f_{1\infty}$ and $f_{2\infty}$). The statement is still an if and only if statement for the $d = 1$ case, but for the $d > 1$ case we need an extra condition for the converse. This is due to the main difference between the smooth X and singular X cases when $m = 1$ as explained above. Here for $\langle \tau_1^m \times \tau_2^m \rangle$ the action on $\mathbb{P}^1$ is trivial ($ord(\psi(\tau_i^m)) = ord(\phi(\tau_i^m)) = 1$) which means that the elements in this cyclic group are of the form $t_{\epsilon_1} \times t_{\epsilon_2}$ for torsion sections ϵ_i, hence the free action on X is not guaranteed when X is singular. It must be checked. With these remarks, the proof of the generalized lemma follows the same argument as given in the proof of the original lemma in [2].

Lemma 4 (Lemma 3.7 in [2])**.** *If $m > 1$ and $d > 1$ and $\langle \tau_1 \rangle$ acts freely on $f_{1\infty}$, then f_{10} is a singular fiber of B_1.*

This lemma is valid no matter X is smooth or singular.

We can define a suitable pair for the singular X case in a similar way as it is defined for the smooth X case in [2]:

Definition 5. Let $\tau \in Aut(B)$ with integers $(n, m, \bar{m}, d, k)$. We say that (B, τ) is a suitable pair if $k = 1$ (which means $m = \bar{m}$) and one of the following three conditions holds: (1) $m = 1$ (which means $\tau \in MW_{tors}(B)$). (2) $m > 1$, $d = 1$ and $\langle \tau \rangle$ acts freely on the fiber f_∞ of B. (3) $m > 1$, $d > 1$ and $\langle \tau \rangle$ acts freely on f_∞ (in this case f_0 is a singular fiber due to Lemma 4).

Note that the requirement that f_∞ is a smooth fiber in condition (3) in the original definition in [2] is dropped in this new definition for the singular X case since this requirement is due to the fact that when $m > 1$ and $d > 1$, f_{10} and $f_{2\infty}$ are singular fibers (assuming $\langle \tau_1 \rangle$ acts freely on $f_{1\infty}$ and $\langle \tau_2 \rangle$ acts freely on f_{20}) by Lemma 4 and X is smooth. But, if X is singular then $f_{1\infty}$ and f_{20} need not be smooth fibers.

In the singular X case, if $\langle \tau_1 \times \tau_2 \rangle$ acts freely on X, then (B_1, τ_1) and (B_2, τ_2) are both suitable pairs (Note that 0 and ∞ should be interchanged for (B_2, τ_2) in order to call it a suitable pair according to the above definition). Conversely, using two suitable pairs (B_1, τ_1) and (B_2, τ_2) with common m and n values ($m_1 = m_2$ and $n_1 = n_2$), one can first make a change of coordinates on the base curve $\mathbb{P}^1$ of B_2 interchanging 0 and ∞, then construct the fiber product X. There exists u relatively prime to m such that $\phi(\tau_1) = \phi(\tau_2^u)$, hence $\tau_1 \times \tau_2^u$ is an automorphism of X. To conclude that $\langle \tau_1 \times \tau_2^u \rangle$ acts freely on X, due to Lemma 3 one needs to check that the non-identity elements of the form $t_{\epsilon_1} \times t_{\epsilon_2}$ in this cyclic group have no fixed points on X (this needs to be checked if $m = 1$ or if $m > 1$ and $d > 1$).

Note that in order for an automorphism $t_{\epsilon_1} \times t_{\epsilon_2}$ not to have a fixed point on an $I_r \times I_s$ fiber of X, it should be checked that ϵ_1 does not intersect the neutral component (the component intersecting the zero section) of I_r, or ϵ_2 does not intersect the neutral component of I_s. Here ϵ_i is a torsion section of B_i. This can be checked using the height pairing on the Mordell-Weil lattice and its explicit formula given in [8] (see also §2 in [2]).

Lemma 6 (Generalization of Lemma 3.10 in [2])**.** *Let (B, τ) be a suitable pair with $m > 1$ (conditions (2) and (3) in the definition). Then f_∞ is either smooth or singular of type I_{nr} for some integer $r > 0$.*

This lemma follows from the fact that for an elliptic surface the only singular fiber type which admits a free action of $\mathbb{Z}_n$ is I_{nr}. Here condition 3 is also included in the generalized lemma since in the singular X case f_∞ need not be a smooth fiber even if $m > 1$ and $d > 1$, contrary to the smooth X case.

Lemma 7 (Proposition 3.11 in [2])**.** *Let (B, τ) be a suitable pair with $m > 1$ (conditions (2) and (3) in the definition). Then $\alpha = \psi(\tau) \in Aut_\sigma(B)$ fixes the neutral component (the component intersecting the zero section σ) of f_∞ pointwise.*

The proof of the lemma is the same as the proof given in [2].

For a suitable pair (B, τ), the pair (B, α) where $\alpha = \psi(\tau) \in Aut_\sigma(B)$ is a suitable σ-pair which is given by the same definition as in [2]:

Definition 8. A pair (B, α) where B is a relatively minimal rational elliptic surface with section and $\alpha \in Aut_\sigma(B)$ is called a suitable σ-pair if $m = \bar{m} > 1$, the fiber f_∞ of B is either smooth or singular of type I_{mr} for some integer $r > 0$, and α fixes the neutral component of f_∞ pointwise.

Here $m = ord(\alpha) = ord(\psi(\alpha))$ (note that $\alpha = \psi(\alpha)$ since $\alpha \in Aut_\sigma(B)$) and $\bar{m} = ord(\phi(\alpha))$. The fiber f_∞ of B is the fiber over $\infty \in \mathbb{P}^1$ which is one of the two fixed points of $\phi(\alpha) \in Aut(\mathbb{P}^1)$. The neutral component of f_∞ is the component which intersects the zero section σ of B.

In §4 of [2], Bouchard and Donagi have classified all suitable σ-pairs. They have shown that any suitable σ-pair (B, α) is obtained by a pull-back of another relatively minimal rational elliptic surface $\hat{B}$ with section via the map $g_m : \mathbb{P}^1 \to \mathbb{P}^1$, $z \mapsto z^m$ of degree m on the base curve $\mathbb{P}^1$ where $\hat{f}_0$ of $\hat{B}$ is one of the singular fiber types as shown in Table 3 in [2] and $\hat{f}_\infty$ of $\hat{B}$ is either a smooth fiber or a singular fiber of type I_r. Here, $m = ord(\alpha)$ and the classification given by Bouchard and Donagi shows that m can be $2, 3, 4, 5$ or 6. At this point, Bouchard and Donagi make a simplification in their analysis by considering only the suitable σ-pairs with f_∞ of B a smooth fiber (hence $\hat{f}_\infty$ of $\hat{B}$ is also a smooth fiber) in the remaining of their work [2]. It is noted in §4.3.1 in [2] that suitable σ-pairs (B, α) with f_∞ of type I_{mr} are specializations of a one parameter family of suitable σ-pairs with smooth f_∞. Bouchard and Donagi then classified all suitable pairs (B, τ) corresponding to suitable σ-pairs (B, α) with smooth f_∞.

In this paper, we will make explicit use of the classification of the suitable σ-pairs given by Bouchard and Donagi, and also classify suitable pairs (B, τ) with f_∞ of type I_{mr} and include them in our analysis in order to obtain free actions on singular Schoen 3-folds. Before proceeding to construct such suitable pairs, we first show in the next section that for a finite group G acting freely on a singular Schoen 3-fold X, the induced action on the base curve $\mathbb{P}^1$ is cyclic.

3.3. The induced action on the base curve $\mathbb{P}^1$

Let G be a finite group which acts freely on a singular Schoen 3-fold X where the elements of G act on X as the map $\tau_1 \times \tau_2$ for $\tau_i \in Aut(B_i)$, $i = 1$, 2. Then, for each such element of G the cyclic group $\langle \tau_1 \times \tau_2 \rangle$ acts freely on X since it is a subgroup of G whose action on X is free. Therefore, each element $\tau_1 \times \tau_2$ of G satisfies the conditions given in the lemmas in §3.2. The projections $\pi_i : G \to Aut(B_i)$, $\tau_1 \times \tau_2 \mapsto \tau_i$ for $i = 1$, 2 are injective homomorphisms, hence G is isomorphic to each of $G_i := \pi_i(G) \subset Aut(B_i)$. If we define $\tilde{\phi} : G \to Aut(\mathbb{P}^1)$ by $\tilde{\phi} := \phi \circ \pi_1 = \phi \circ \pi_2$ (second equality holds since $\tau_1 \times \tau_2 \in G$ implies that $\phi(\tau_1) = \phi(\tau_2)$ so that $\tau_1 \times \tau_2$ is a well-defined map on the fiber product X), then $\tilde{\phi}(G) \subset Aut(\mathbb{P}^1)$ is the subgroup of induced automorphisms on $\mathbb{P}^1$ by elements of G. We call the action of $\tilde{\phi}(G)$ on $\mathbb{P}^1$ the induced action of G. In the paper [5], the author has proved that the induced action of G is a cyclic group if G is a finite group which acts freely on a smooth Schoen 3-fold. We now show that the induced action is again cyclic when G acts freely on a singular Schoen 3-fold X.

Lemma 9 (Lemma 1 in [5]). *If G is a finite group which acts freely on a singular Schoen 3-fold $X = B_1 \times_{\mathbb{P}^1} B_2$ such that elements of G are of the form $\tau_1 \times \tau_2$ where $\tau_i \in Aut(B_i)$, then for each $i = 1, 2$, the restriction of the homomorphism $\phi : Aut(B_i) \to Aut_{B_i}(\mathbb{P}^1)$ to the subgroup $\psi(G_i) \subset Aut_\sigma(B_i)$*

$$\phi|_{\psi(G_i)} : \psi(G_i) \to \tilde{\phi}(G)$$

is an isomorphism. Hence

$$(\phi|_{Aut_\sigma(B_i)})^{-1}(\tilde{\phi}(G)) = \psi(G_i) \times CM(B_i) \cong \tilde{\phi}(G) \times CM(B_i)$$

as a subgroup of $Aut_\sigma(B_i)$ where $CM(B_i)$ is the complex multiplication subgroup of $Aut_\sigma(B_i)$.

The proof of this lemma is the same as the proof given in [5] once we note that $m_1 = m_2 = \bar{m}_1 = \bar{m}_2$ is valid in the singular X case as stated in Lemma 2 and the discussion following it.

The proof of Theorem 2 in [5] which states that $\tilde{\phi}(G)$ is cyclic is a case by case analysis which depends on Lemma 1 in [5], the fact that for a suitable pair (B, τ) where $m > 1$ the fiber f_∞ is either smooth or singular of type I_{mr}, and the fact that for a suitable σ-pair (B, α) the neutral component of f_∞ is fixed pointwise by α. All of these are valid in the singular X case. Therefore, the same proof given in [5] works for the following result:

Lemma 10. *Let G be a finite group which acts freely on a singular Schoen 3-fold X as in Lemma 9. Then $\tilde{\phi}(G)$ is a cyclic group.*

4. The lists

4.1. The list of suitable σ-pairs (B, α)

As we noted in §3.2, Bouchard and Donagi have shown that a suitable σ-pair (B, α) is constructed by pulling back a relatively minimal rational elliptic surface $\hat{B}$ with section via the map $g_m : \mathbb{P}^1 \to \mathbb{P}^1$, $z \mapsto z^m$ where the fiber $\hat{f}_0$ of $\hat{B}$ over $0 \in \mathbb{P}^1$ is as shown in Table 3 in [2] (note that $m = ord(\alpha)$). Bouchard and Donagi have given a list of such elliptic surfaces $\hat{B}$ and the corresponding pullback surfaces B where both $\hat{f}_\infty$ and f_∞ are smooth fibers (fibers of $\hat{B}$ and B over the point $\infty \in \mathbb{P}^1$) in Table 4 in [2]. Instead of choosing the point $\infty \in \mathbb{P}^1$ so that $\hat{f}_\infty$ is a smooth fiber, one can choose $\infty \in \mathbb{P}^1$ such that the fiber $\hat{f}_\infty$ of $\hat{B}$ over that point is a singular fiber of type I_r. Then the pullback via the map g_m produces a surface B where the fiber f_∞ of B is of type I_{mr}. We give a list of suitable σ-pairs (B, α) with f_∞ a singular fiber of type I_{mr} in Table 1. In this table we use the same surfaces $\hat{B}$ as in Table 4 in [2], but we choose $\hat{f}_\infty$ of $\hat{B}$ as a singular fiber of type I_r so that the pullback surface B via the map g_m has f_∞ a singular fiber of type I_{mr}. In Table 1 we indicate the fibers over 0 and ∞, the root lattice of singular fibers, the Mordell-Weil lattice, and the torsion subgroup of the Mordell-Weil group of both of the surfaces $\hat{B}$ and B together with the generic configuration of singular fibers on these surfaces. We consulted the list given in [6] in order to provide the Mordell-Weil lattices and the torsion subgroups of the Mordell-Weil lattices in our Table 1. We note two mistakes in the list in [6] as indicated in pages 26 and 43 in [2]: For an elliptic surface B whose root lattice of singular fibers is $T = D_4 \oplus A_2$ the Mordell-Weil lattice is $\frac{1}{6}\left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$, and for a surface with $T = A_7 \oplus A_1$ the torsion subgroup of the Mordell-Weil group is $\mathbb{Z}_4$.

Such suitable σ-pairs where f_∞ is not smooth have been excluded from the analysis given in [2] as mentioned in §4.3.1 and Remark 6.4 in [2]. In this paper we will use such suitable σ-pairs in order to construct singular Schoen 3-folds X admitting a free action of a finite group G. As a second goal of this paper, we will construct the smooth Schoen 3-folds X with a free action of a finite group G using these suitable σ-pairs which were not considered in the paper [2].

We will call suitable σ-pairs (B, α) with smooth f_∞ *general type* suitable σ-pairs and the pairs with singular f_∞ of type I_{mr} *special type* suitable σ-pairs due to the fact that these special type pairs are obtained as a specialization of the general type pairs forming a codimension 1 family in the family of surfaces B admitting such automorphisms α (see §4.3.1 in [2] for details).

4.2. The list of $G \subset Aut(B)$ consisting of suitable automorphisms

For the elliptic surfaces B which admit an automorphism $\alpha \in Aut_\sigma(B)$ with order $m > 1$ where (B, α) is a suitable σ-pair such that f_∞ is a singular fiber of type I_{mr}, we will determine the subgroups G of $Aut(B)$ such that (B, τ) is a suitable pair and $\psi(\tau) \in \langle\alpha\rangle$ for each $\tau \in G$. Note that such a pair (B, α) is one of the cases listed in Table 1, and if G is a finite group which acts freely on a singular Schoen 3-fold $X = B_1 \times_{\mathbb{P}^1} B_2$, then the induced action $\bar{\phi}(G)$ on the base curve $\mathbb{P}^1$ is a cyclic group which is isomorphic to $\psi(G_i)$ for each $i = 1, 2$ (see §3.3), hence if $\psi(G_i) = \langle\alpha_i\rangle$ we must have $\psi(\tau) \in \langle\alpha_i\rangle$ for each $\tau \in G_i = \pi_i(G) \cong G$ for each $i = 1, 2$. For such a group G_i, $\ker(\psi) \subset G_i$ consists of translations by torsion sections (note that $\psi : G_i \to \langle\alpha_i\rangle$). As a

Table 1: Table of suitable σ-pairs (B,α) with f_∞ a singular fiber of type I_{mr} (special type suitable σ-pairs)

	$\hat{B}$						B						
m	$\hat{f}_0$	$\hat{T}$	Fibers	$\hat{f}_\infty$	MW_{lat}	MW_{tors}	f_0	T	Fibers	f_∞	MW_{lat}	MW_{tors}	Case
6	II^*	E_8	$II^*I_1^2$	I_1	0	0	I_0	A_5	$I_6I_1^6$	I_6	$A_2^*\oplus A_1^*$	0	1a
5	II^*	E_8	$II^*I_1^2$	I_1	0	0	II	A_4	$II\,I_5I_1^5$	I_5	A_4^*	0	2a
4	II^*	E_8	$II^*I_1^2$	I_1	0	0	IV	$A_3\oplus A_2$	$IV\,I_4I_1^4$	I_4	$\frac{1}{12}\left[\begin{smallmatrix}7&1&2\\1&7&2\\2&2&4\end{smallmatrix}\right]$	0	3a
	III^*	E_7	$III^*I_1^3$	I_1	A_1^*	0	I_0	A_3	$I_4I_1^8$	I_4	D_5^*	0	4a
		$E_7\oplus A_1$	$III^*I_2I_1$	I_1	0	$\mathbb{Z}_2$		$A_3\oplus A_1^{\oplus 4}$	$I_4I_2^4$	I_4	$\langle 1/4\rangle$	$\mathbb{Z}_2\times\mathbb{Z}_2$	5a
				I_2				A_7	$I_8I_1^4$	I_8	A_1^*	$\mathbb{Z}_2$	5b
3	II^*	E_8	$II^*I_1^2$	I_1	0	0	I_0^*	$D_4\oplus A_2$	$I_0^*I_3I_1^3$	I_3	$\frac{1}{6}\left[\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right]$	0	6a
	III^*	E_7	$III^*I_1^3$	I_1	A_1^*	0	III	$A_2\oplus A_1$	$III\,I_3I_1^6$	I_3	A_5^*	0	7a
		$E_7\oplus A_1$	$III^*I_2I_1$	I_1	0	$\mathbb{Z}_2$		$A_2\oplus A_1^{\oplus 4}$	$III\,I_3I_2^3$	I_3	$\frac{1}{6}\left[\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right]$	$\mathbb{Z}_2$	8a
				I_2				$A_5\oplus A_1$	$III\,I_6I_1^3$	I_6	A_2^*	$\mathbb{Z}_2$	8b
	IV^*	E_6	$IV^*I_1^4$	I_1	A_2^*	0	I_0	A_2	$I_3I_1^9$	I_3	E_6^*	0	9a
		$E_6\oplus A_1$	$IV^*I_2I_1^2$	I_1	$\langle 1/6\rangle$	0		$A_2\oplus A_1^{\oplus 3}$	$I_3I_2^3I_1^3$	I_3	$A_1^*\oplus\frac{1}{6}\left[\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right]$	0	10a
				I_2				A_5	$I_6I_1^6$	I_6	$A_2^*\oplus A_1^*$	0	10b
		$E_6\oplus A_2$	$IV^*I_3I_1$	I_1	0	$\mathbb{Z}_3$		$A_2^{\oplus 4}$	I_3^4	I_3	0	$\mathbb{Z}_3\times\mathbb{Z}_3$	11a
				I_3				A_8	$I_9I_1^3$	I_9	0	$\mathbb{Z}_3$	11b
2	II^*	E_8	$II^*I_1^2$	I_1	0	0	IV^*	$E_6\oplus A_1$	$IV^*I_2I_1^2$	I_2	$\langle 1/6\rangle$	0	12a
	III^*	E_7	$III^*I_1^3$	I_1	A_1^*	0	I_0^*	$D_4\oplus A_1$	$I_0^*I_2I_1^4$	I_2	$(A_1^*)^{\oplus 3}$	0	13a
		$E_7\oplus A_1$	$III^*I_2I_1$	I_1	0	$\mathbb{Z}_2$		$D_4\oplus A_1^{\oplus 3}$	$I_0^*I_2^3$	I_2	A_1^*	$\mathbb{Z}_2\times\mathbb{Z}_2$	14a
				I_2				$D_4\oplus A_3$	$I_0^*I_4I_1^2$	I_4	$\langle 1/4\rangle$	$\mathbb{Z}_2$	14b
	IV^*	E_6	$IV^*I_1^4$	I_1	A_2^*	0	IV	$A_2\oplus A_1$	$IV\,I_2I_1^6$	I_2	A_5^*	0	15a
		$E_6\oplus A_1$	$IV^*I_2I_1^2$	I_1	$\langle 1/6\rangle$	0		$A_2\oplus A_1^{\oplus 3}$	$IV\,I_2^3I_1^2$	I_2	$A_1^*\oplus\frac{1}{6}\left[\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right]$	0	16a
				I_2				$A_3\oplus A_2$	$IV\,I_4I_1^4$	I_4	$\frac{1}{12}\left[\begin{smallmatrix}7&1&2\\1&7&2\\2&2&4\end{smallmatrix}\right]$	0	16b
		$E_6\oplus A_2$	$IV^*I_3I_1$	I_1	0	$\mathbb{Z}_3$		$A_2^{\oplus 3}\oplus A_1$	$IV\,I_3^2I_2$	I_2	$\langle 1/6\rangle$	$\mathbb{Z}_3$	17a
				I_3				$A_5\oplus A_2$	$IV\,I_6I_1^2$	I_6	A_1^*	$\mathbb{Z}_3$	17b

Table 1: *Continued*

	$\hat{B}$						B						
m	$\hat{f}_0$	$\hat{T}$	Fibers	$\hat{f}_\infty$	MW_{lat}	MW_{tors}	f_0	T	Fibers	f_∞	MW_{lat}	MW_{tors}	Case
2	I_4^*	D_8	$I_4^* I_1^2$	I_1	0	$\mathbb{Z}_2$	I_8	$A_7 \oplus A_1$	$I_8 I_2 I_1^2$	I_2	0	$\mathbb{Z}_4$	18a
	I_3^*	D_7	$I_3^* I_1^3$	I_1	$\langle 1/4 \rangle$	0	I_6	$A_5 \oplus A_1$	$I_6 I_2 I_1^4$	I_2	$A_1^* \oplus \langle 1/6 \rangle$	0	19a
	I_2^*	D_6	$I_2^* I_1^4$	I_1	$(A_1^*)^{\oplus 2}$	0	I_4	$A_3 \oplus A_1$	$I_4 I_2 I_1^6$	I_2	$A_3^* \oplus A_1^*$	0	20a
		$D_6 \oplus A_1$	$I_2^* I_2 I_1^2$	I_1	A_1^*	$\mathbb{Z}_2$		$A_3 \oplus A_1^{\oplus 3}$	$I_4 I_2^3 I_1^2$	I_2	$A_1^* \oplus \langle 1/4 \rangle$	$\mathbb{Z}_2$	21a
				I_2				$A_3^{\oplus 2}$	$I_4^2 I_1^4$	I_4	$(A_1^*)^{\oplus 2}$	$\mathbb{Z}_2$	21b
		$D_6 \oplus A_1^{\oplus 2}$	$I_2^* I_2^2$	I_2	0	$\mathbb{Z}_2 \times \mathbb{Z}_2$		$A_3^{\oplus 2} \oplus A_1^{\oplus 2}$	$I_4^2 I_2^2$	I_4	0	$\mathbb{Z}_4 \times \mathbb{Z}_2$	22a
	I_1^*	D_5	$I_1^* I_1^5$	I_1	A_3^*	0	I_2	$A_1^{\oplus 2}$	$I_2^2 I_1^8$	I_2	D_6^*	0	23a
		$D_5 \oplus A_1$	$I_1^* I_2 I_1^3$	I_1	$A_1^* \oplus \langle 1/4 \rangle$	0		$A_1^{\oplus 4}$	$I_2^4 I_1^4$	I_2	$(A_1^*)^{\oplus 4}$	0	24a
				I_2				$A_3 \oplus A_1$	$I_4 I_2 I_1^6$	I_4	$A_3^* \oplus A_1^*$	0	24b
		$D_5 \oplus A_2$	$I_1^* I_3 I_1^2$	I_1	$\langle 1/12 \rangle$	0		$A_2^{\oplus 2} \oplus A_1^{\oplus 2}$	$I_3^2 I_2^2 I_1^2$	I_2	$(\langle 1/6 \rangle)^{\oplus 2}$	0	25a
				I_3				$A_5 \oplus A_1$	$I_6 I_2 I_1^4$	I_6	$A_1^* \oplus \langle 1/6 \rangle$	0	25b
		$D_5 \oplus A_1^{\oplus 2}$	$I_1^* I_2^2 I_1$	I_1	$\langle 1/4 \rangle$	$\mathbb{Z}_2$		$A_1^{\oplus 6}$	I_2^6	I_2	$(A_1^*)^{\oplus 2}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	26a
				I_2				$A_3 \oplus A_1^{\oplus 3}$	$I_4 I_2^3 I_1^2$	I_4	$A_1^* \oplus \langle 1/4 \rangle$	$\mathbb{Z}_2$	26b
		$D_5 \oplus A_3$	$I_1^* I_4 I_1$	I_1	0	$\mathbb{Z}_4$		$A_3^{\oplus 2} \oplus A_1^{\oplus 2}$	$I_4^2 I_2^2$	I_2	0	$\mathbb{Z}_4 \times \mathbb{Z}_2$	27a
				I_4				$A_7 \oplus A_1$	$I_8 I_2 I_1^2$	I_8	0	$\mathbb{Z}_4$	27b
	I_0^*	D_4	$I_0^* I_1^6$	I_1	D_4^*	0	I_0	A_1	$I_2 I_1^{10}$	I_2	E_7^*	0	28a
		$D_4 \oplus A_1$	$I_0^* I_2 I_1^4$	I_1	$(A_1^*)^{\oplus 3}$	0		$A_1^{\oplus 3}$	$I_2^3 I_1^6$	I_2	$D_4^* \oplus A_1^*$	0	29a
				I_2				A_3	$I_4 I_1^8$	I_4	D_5^*	0	29b
		$D_4 \oplus A_2$	$I_0^* I_3 I_1^3$	I_1	$\frac{1}{6}\left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$	0		$A_2^{\oplus 2} \oplus A_1$	$I_3^2 I_2 I_1^4$	I_2	$A_2^* \oplus \langle 1/6 \rangle$	0	30a
				I_3				A_5	$I_6 I_1^6$	I_6	$A_2^* \oplus A_1^*$	0	30b
		$D_4 \oplus A_1^{\oplus 2}$	$I_0^* I_2^2 I_1^2$	I_1	$(A_1^*)^{\oplus 2}$	$\mathbb{Z}_2$		$A_1^{\oplus 5}$	$I_2^5 I_1^2$	I_2	$(A_1^*)^{\oplus 3}$	$\mathbb{Z}_2$	31a
				I_2				$A_3 \oplus A_1^{\oplus 2}$	$I_4 I_2^2 I_1^4$	I_4	A_3^*	$\mathbb{Z}_2$	31b
		$D_4 \oplus A_1^{\oplus 3}$	$I_0^* I_2^3$	I_2	A_1^*	$\mathbb{Z}_2 \times \mathbb{Z}_2$		$A_3 \oplus A_1^{\oplus 4}$	$I_4 I_2^4$	I_4	$\langle 1/4 \rangle$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	32a
		$D_4 \oplus A_3$	$I_0^* I_4 I_1^2$	I_1	$\langle 1/4 \rangle$	$\mathbb{Z}_2$		$A_3^{\oplus 2} \oplus A_1$	$I_4^2 I_2 I_1^2$	I_2	A_1^*	$\mathbb{Z}_4$	33a
				I_4				A_7	$I_8 I_1^4$	I_8	A_1^*	$\mathbb{Z}_2$	33b

result, G_i is generated by an automorphism τ such that $\psi(\tau) = \alpha_i$ together with possibly some translations by torsion sections of B_i.

Lemma 11. *For each relatively minimal rational elliptic surface B with section which admits an automorphism $\alpha \in Aut_\sigma(B)$ of order $m > 1$ such that (B, α) is a suitable σ-pair where f_∞ is a singular fiber of type I_{mr} (that is, for each of the cases listed in Table 1), the groups $G \subset Aut(B)$ where (B, τ) is a suitable pair and $\psi(\tau) \in \langle\alpha\rangle$ for each $\tau \in G$ are as listed in Table 2. In this table we list $m = ord(\alpha)$, $d = ord(\tau^m) = ord(\tau)/m$ for elements $\tau \in G$ such that $\psi(\tau) = \alpha$ and the dimension* dim *of the moduli space of the elliptic surfaces B admitting such a group $G \subset Aut(B)$.*

Remark. For each case the dimension of the moduli space of the rational elliptic surfaces B which admit an automorphism α of order m such that (B, α) is a suitable σ-pair with f_∞ a singular fiber of type I_{mr} is one less than the dimension of the moduli space of such surfaces where (B, α) is a suitable σ-pair with smooth f_∞ as explained in §4.3.1 in [2]. With this note the dimensions dim in Table 2 are calculated as in Proposition 6.2 in [2]. The dim values for each group G in Table 2 are less than the dim values for the same group G listed in Tables 8 and 9 in [2]. As a result the family of the Schoen 3-folds $X = B_1 \times_{\mathbb{P}^1} B_2$ admitting a free action of a finite group G with non-trivial action on $\mathbb{P}^1$ where at least one of the actions on B_1 and B_2 is special type has smaller dimension than such a family where both actions on B_1 and B_2 are general type.

Proof. We follow the same technique as in the paper [2] to list all such groups G for each of the cases in Table 1. For the suitable σ-pair in each of these cases, we determine whether *allowed sections* $\epsilon \in MW(B)$ exist such that (B, τ) is a suitable pair where $\tau = t_\epsilon \circ \alpha$.

For $\alpha \in Aut_\sigma(B)$ of order m and $\epsilon \in MW(B)$, the automorphism $\tau = t_\epsilon \circ \alpha$ has finite order n if and only if $\mathcal{P}_m(\epsilon)$ is a torsion section of order $d = n/m$, or equivalently $\Phi_m(\epsilon) = 0$ where $\mathcal{P}_m(\epsilon) = \alpha^{m-1}(\epsilon) + \alpha^{m-2}(\epsilon) + \cdots + \alpha(\epsilon) + \epsilon$ and $\Phi_m(\epsilon) = \langle \mathcal{P}_m(\epsilon), \mathcal{P}_m(\epsilon)\rangle$ (see Lemma 3.4 in [2]). Here $\langle -, -\rangle$ denotes the height pairing in the Mordell-Weil group of B (see [8] or §2.3 in [2]), and $+$ denotes the group operation in $MW(B)$. This fact simply follows from the relation $\alpha \circ t_\epsilon = t_{\alpha(\epsilon)} \circ \alpha$ for $\epsilon \in MW(B)$ and $\alpha \in Aut_\sigma(B)$ (see §3.1) and $ord(\alpha) = m$.

It is shown in §5.1 of [2] that if (B, α) is a suitable σ-pair with $ord(\alpha) = m$ where B is obtained as the pullback of an elliptic surface $\hat{B}$ via the map $z \mapsto z^m$ on $\mathbb{P}^1$, then $\ker(\Phi_m) = [MW(B)^\alpha]^\perp$ in $MW(B)$ (Lemma 5.1 in [2]) where $MW(B)^\alpha$ is the sections ϵ of B such that $\alpha(\epsilon) = \epsilon$ (sections preserved

Table 2: Finite groups $G \subset Aut(B)$ which consist of suitable automorphisms where $\psi(G) = \langle\alpha\rangle = \mathbb{Z}_m$ ($m > 1$) and f_∞ is of type I_{mr} ($r > 0$), *special type actions*

♯	m	d	dim	Sing. fibers	T	f_0	f_∞	Case
				$G = \mathbb{Z}_3 \times \mathbb{Z}_3$				
1	3	1	0	I_3^4	A_2^4	I_0	I_3	11a
				$G = \mathbb{Z}_4 \times \mathbb{Z}_2$				
2	4	1	0	$I_4 I_2^4$	$A_3 \oplus A_1^{\oplus 4}$	I_0	I_4	5a
3	2	2	0	$I_4^2 I_2^2$	$A_3^{\oplus 2} \oplus A_1^{\oplus 2}$	I_4	I_4	22a
				$G = \mathbb{Z}_6$				
4	6	1	0	$I_6 I_1^6$	A_5	I_0	I_6	1a
5	3	1 or 2	0	$III\, I_6 I_1^3$	$A_5 \oplus A_1$	III	I_6	8b
6	2	1 or 3	0	$IV\, I_6 I_1^2$	$A_5 \oplus A_2$	IV	I_6	17b
				$G = \mathbb{Z}_5$				
7	5	1	0	$II\, I_5 I_1^5$	A_4	II	I_5	2a
				$G = \mathbb{Z}_4$				
8	4	1	1	$I_4 I_1^8$	A_3	I_0	I_4	4a
9	4	1	0	$IV\, I_4 I_1^4$	$A_3 \oplus A_2$	IV	I_4	3a
10	4	1	0	$I_8 I_1^4$	A_7	I_0	I_8	5b
11	2	2	1	$I_4 I_2^3 I_1^2$	$A_3 \oplus A_1^{\oplus 3}$	I_2	I_4	26b
12	2	2	0	$I_0^* I_4 I_1^2$	$D_4 \oplus A_3$	I_0^*	I_4	14b
13	2	2	0	$I_8 I_2 I_1^2$	$A_7 \oplus A_1$	I_2	I_8	27b
				$G = \mathbb{Z}_3$				
14	3	1	2	$I_3 I_1^9$	A_2	I_0	I_3	9a
15	3	1	1	$III\, I_3 I_1^6$	$A_2 \oplus A_1$	III	I_3	7a
16	3	1	1	$I_3 I_2^3 I_1^3$	$A_2 \oplus A_1^{\oplus 3}$	I_0	I_3	10a
17	3	1	1	$I_6 I_1^6$	A_5	I_0	I_6	10b
18	3	1	0	$I_0^* I_3 I_1^3$	$D_4 \oplus A_2$	I_0^*	I_3	6a
19	3	1	0	$III\, I_3 I_2^3$	$A_2 \oplus A_1^{\oplus 4}$	III	I_3	8a
20	3	1	0	$I_9 I_1^3$	A_8	I_0	I_9	11b
				$G = \mathbb{Z}_2 \times \mathbb{Z}_2$				
21	2	1	2	$I_2^5 I_1^2$	$A_1^{\oplus 5}$	I_0	I_2	31a
22	2	1	1	$I_4 I_2^3 I_1^2$	$A_3 \oplus A_1^{\oplus 3}$	I_4	I_2	21a
23	2	1	1	I_2^6	$A_1^{\oplus 6}$	I_2	I_2	26a
24	2	1	1	$I_4 I_2^4$	$A_3 \oplus A_1^{\oplus 4}$	I_0	I_4	32a
25	2	1	1	$I_4^2 I_2 I_1^2$	$A_3^{\oplus 2} \oplus A_1$	I_0	I_2	33a
26	2	1	0	$I_0^* I_2^3$	$D_4 \oplus A_1^{\oplus 3}$	I_0^*	I_2	14a
27	2	1	0	$I_8 I_2 I_1^2$	$A_7 \oplus A_1$	I_8	I_2	18a
28	2	1	0	$I_4^2 I_2^2$	$A_3^{\oplus 2} \oplus A_1^{\oplus 2}$	I_4	I_4	22a
29	2	1	0	$I_4^2 I_2^2$	$A_3^{\oplus 2} \oplus A_1^{\oplus 2}$	I_2	I_2	27a
				$G = \mathbb{Z}_2$				
30	2	1	4	$I_2 I_1^{10}$	A_1	I_0	I_2	28a

Table 2: *Continued*

♯	m	d	dim	Sing. fibers	T	f_0	f_∞	Case
31	2	1	3	$I_2^2 I_1^8$	$A_1^{\oplus 2}$	I_2	I_2	23a
32	2	1	3	$I_2^3 I_1^6$	$A_1^{\oplus 3}$	I_0	I_2	29a
33	2	1	3	$I_4 I_1^8$	A_3	I_0	I_4	29b
34	2	1	2	$IV I_2 I_1^6$	$A_2 \oplus A_1$	IV	I_2	15a
35	2	1	2	$I_4 I_2 I_1^6$	$A_3 \oplus A_1$	I_4	I_2	20a
36	2	1	2	$I_2^4 I_1^4$	$A_1^{\oplus 4}$	I_2	I_2	24a
37	2	1	2	$I_4 I_2 I_1^6$	$A_3 \oplus A_1$	I_2	I_4	24b
38	2	1	2	$I_3^2 I_2 I_1^4$	$A_2^{\oplus 2} \oplus A_1$	I_0	I_2	30a
39	2	1	2	$I_6 I_1^6$	A_5	I_0	I_6	30b
40	2	1	2	$I_4 I_2^2 I_1^4$	$A_3 \oplus A_1^{\oplus 2}$	I_0	I_4	31b
41	2	1	1	$I_0^* I_2 I_1^4$	$D_4 \oplus A_1$	I_0^*	I_2	13a
42	2	1	1	$IV I_2^3 I_1^2$	$A_2 \oplus A_1^{\oplus 3}$	IV	I_2	16a
43	2	1	1	$IV I_4 I_1^4$	$A_3 \oplus A_2$	IV	I_4	16b
44	2	1	1	$I_6 I_2 I_1^4$	$A_5 \oplus A_1$	I_6	I_2	19a
45	2	1	1	$I_4^2 I_1^4$	$A_3^{\oplus 2}$	I_4	I_4	21b
46	2	1	1	$I_3^2 I_2^2 I_1^2$	$A_2^{\oplus 2} \oplus A_1^{\oplus 2}$	I_2	I_2	25a
47	2	1	1	$I_6 I_2 I_1^4$	$A_5 \oplus A_1$	I_2	I_6	25b
48	2	1	1	$I_4 I_2^3 I_1^2$	$A_3 \oplus A_1^{\oplus 3}$	I_2	I_4	26b
49	2	1	1	$I_8 I_1^4$	A_7	I_0	I_8	33b
50	2	1	0	$IV^* I_2 I_1^2$	$E_6 \oplus A_1$	IV^*	I_2	12a
51	2	1	0	$I_0^* I_4 I_1^2$	$D_4 \oplus A_3$	I_0^*	I_4	14b
52	2	1	0	$IV I_3^2 I_2$	$A_2^{\oplus 3} \oplus A_1$	IV	I_2	17a
53	2	1	0	$I_8 I_2 I_1^2$	$A_7 \oplus A_1$	I_2	I_8	27b

by α as a set). The symbol $\perp$ denotes the orthogonal complement with respect to the height pairing $\langle -,- \rangle$ in the Mordell-Weil group. The Gram matrix of $MW(B)^\alpha$ is given by $m\hat{MW}$ where $\hat{MW}$ is the Gram matrix of the Mordell-Weil lattice of $\hat{B}$ (see the comment after Lemma 5.1 in [2]). This makes it possible to calculate $\ker(\Phi_m)$ for each case listed in Table 1.

Let (B,α) be a suitable σ-pair with $ord(\alpha) = m > 1$ and let $\epsilon \in \ker \Phi_m$ such that $\mathcal{P}_m(\epsilon)$ is a torsion section of order d. In this case $\tau = t_\epsilon \circ \alpha$ has order $n = md$ and we have (B,τ) is a suitable pair if and only if $\langle \tau \rangle$ acts on f_∞ freely. Let f_∞ be a singular fiber of type I_s and label the components of f_∞ as $\theta_0, \theta_1, ..., \theta_{s-1}$ where θ_0 is the neutral component (the component which intersects the zero section σ) and θ_i intersects each of $\theta_{i\pm 1}$ with multiplicity 1. If a section η intersects f_∞ at the component θ_v, then the automorphism t_η maps each component θ_i to the component θ_{i+v} (subindices are considered modulo s). With this notation, since α fixes the neutral component θ_0 of f_∞ pointwise, we can conclude that $\langle \tau \rangle$ acts on f_∞ freely if and only if ϵ intersects a component $\theta_{sj/n}$ where j is relatively prime to n (in other words

ϵ intersects a component of order n of f_∞ which is of type I_s). In particular n divides s if (B, τ) is a suitable pair.

Using these facts and the explicit formula of the height pairing in terms of the intersection numbers of the sections given in Theorem 8.6 in [8], we can determine the existence of the allowed sections $\epsilon \in MW(B)$ for each case in Table 1 such that (B, τ) is a suitable pair where $\tau = t_\epsilon \circ \alpha$. We can determine whether $d > 1$ is possible or not for each case. In each case, we can also determine if there are one or more torsion sections η of B such that τ and the automorphism(s) t_η together generate a subgroup $G \subset Aut(B)$ such that (B, γ) is a suitable pair and $\psi(\gamma) \in \langle\alpha\rangle$ for each $\gamma \in G$. This analysis is carried out case by case through Table 1. We explicitly discuss some cases below. The analysis of each case is done in a way similar to the cases discussed below. The results are listed in Table 2.

Note that since $\alpha \in Aut_\sigma(B)$ it maps the zero of each smooth fiber (which is an elliptic curve) of B to the zero of a smooth fiber, hence α restricts to elliptic curve isomorphisms between the smooth fibers of B. As a result the map $\epsilon \mapsto \alpha(\epsilon)$ is an automorphism of the group $MW(B)$. In particular, α maps torsion sections to torsion sections.

We first consider the cases in Table 1 where $MW_{tors}(B) = 0$. In these cases $d = 1$ for any suitable pair (B, τ) since there is no torsion section on B. We have $G = \mathbb{Z}_m$ once we show the existence of allowed sections (if a suitable pair exists $n = m$ and there is no torsion section to generate a larger group together with τ).

• *Cases 1a, 2a, 3a, 6a and 12a:* In all of these cases $MW(\hat{B}) = 0$, hence $\ker \Phi_m = MW(B)$. In Case 3a, there exists $\epsilon \in \ker \Phi_4$ with $\langle\epsilon, \epsilon\rangle = 7/12$. Using the explicit formulation of the height pairing given in [8], we obtain $\langle\epsilon, \epsilon\rangle = 2 + 2\epsilon\sigma - \Sigma contr_\nu = 7/12$. In this formula the contribution at the singular fiber IV of B is either 0 or 2/3 and the contribution at the fiber I_4 is 0, 3/4 or 1. We can conclude that ϵ is disjoint from the zero section σ and the contribution at IV is 2/3 and the contribution at I_4 is 3/4. Thus, ϵ intersects I_4 at the component θ_1 or θ_3. As a result, ϵ is an allowed section. A similar argument gives the existence of allowed sections in the Cases 1a, 2a, 6a and 12a if we consider $\epsilon \in \ker \Phi_m = MW(B)$ where $\langle\epsilon, \epsilon\rangle = 7/6$, 6/5, 1/3 and 1/6, respectively.

• *Cases 10a, 10b, 19a, 25a, 25b, 30a and 30b:* In all of these cases $MW(B)^\alpha$ whose Gram matrix is $m\hat{MW}$ is a direct summand of $MW_{lat}(B)$ (note that $\hat{MW}$ is the Gram matrix of $MW_{lat}(\hat{B})$). Thus, $\ker \Phi_m = [MW(B)^\alpha]^\perp$ in $MW(B)$ is easily computed. In Case 10a, we have $m = 3$, $\ker \Phi_3 = [3\langle 1/6\rangle]^\perp = [A_1^*]^\perp$ in $MW(B)$ which equals $\frac{1}{6}\left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$. There exists $\epsilon \in \ker \Phi_3$ such that $\langle\epsilon, \epsilon\rangle = 1/3$. Using the explicit formula of the height pairing we can conclude that the contribution at I_3 fiber in this formula is $2/3$, hence ϵ intersects $f_\infty = I_3$ at the component θ_1 or θ_2 (an order 3 component). Thus, ϵ is an allowed section. Similarly in Cases 10b, 19a, 25a, 25b, 30a and 30b, we can show the existence of $\epsilon \in \ker \Phi_m$ with $\langle \epsilon, \epsilon \rangle = 2/3$, $1/6$, $1/6$, $1/2$, $1/6$ and $1/2$, respectively. In all of these cases we can show that ϵ intersects f_∞ at an order m component, hence ϵ is an allowed section. This follows directly using the explicit formula of the height pairing except for Case 25a for which the following analysis yields the result: In Case 25a, $\ker \Phi_2 = [MW(B)^\alpha]^\perp = [\langle 1/6 \rangle]^\perp$ in $MW(B)$ which equals $\langle 1/6 \rangle$. There are $\epsilon \in \ker \Phi_2$ and $\delta \in MW(B)^\alpha$ such that $\langle \epsilon, \epsilon \rangle = \langle \delta, \delta \rangle = 1/6$ and $\langle \epsilon, \delta \rangle = 0$. The first two equalities imply that both ϵ and δ are disjoint from the zero section σ and each of them intersect two of the I_3 fibers (or the two fibers corresponding to $A_2^{\oplus 2}$ in the root lattice T) at non-neutral components and only one of the two I_2 fibers (f_0 and f_∞) at a non-neutral component. As a result of the facts $\alpha(\delta) = \delta$, α has two fixed points on the neutral component Γ_0 of $f_0 = I_2$ one of which is $\Gamma_0 \cap \sigma$, and δ is disjoint from σ, we can conclude that if δ intersects Γ_0, then both δ and $-\delta$ intersect Γ_0 at the second fixed point of α on Γ_0. If δ and $-\delta$ intersect, $\langle \delta, -\delta \rangle = -1/6$ is a contradiction (this can be seen using the explicit formula of the height pairing). Therefore, δ intersects f_0 at the non-neutral component Γ_1 of f_0. The equality $\langle \epsilon, \delta \rangle = 0$ then implies that ϵ intersects $f_\infty = I_2$ at the component θ_1 (a degree 2 component), hence ϵ is an allowed section.

• *Cases 13a, 16a, 16b, 20a, 24a and 24b:* In Case 13a we have $\ker \Phi_2 = [\langle 1 \rangle]^\perp$ in $(A_1^*)^{\oplus 3}$ which gives $\ker \Phi_2 = \langle 1 \rangle \oplus A_1^*$ (we can embed $\langle 1 \rangle$ as the subspace generated by $e_1 + e_2$ in $(A_1^*)^{\oplus 3}$ whose orthogonal complement is then generated by $e_1 - e_2$ and e_3 where e_i are the standard basis vectors). Thus, there is $\epsilon \in \ker \Phi_2$ with height $1/2$, i.e., $\langle \epsilon, \epsilon \rangle = 1/2$. This section ϵ intersects $f_\infty = I_2$ at the order 2 component θ_1, hence ϵ is an allowed section. In Cases 16a and 16b, we have $\ker \Phi_2 = [\langle 1/3 \rangle]^\perp$ in $MW(B)$ and we can show the existence of $\epsilon \in \ker \Phi_2$ with height $1/2$ and 1 in these two cases, respectively. This section ϵ is an allowed section in each case since ϵ intersects f_∞ at an order 2 component. In Cases 20a, 24a and 24b, using simple linear algebra we can show that $\ker \Phi_2 = \langle 1 \rangle \oplus A_1^*$, $\langle 1 \rangle \oplus A_1^*$ and $\langle 1 \rangle \oplus \langle 1 \rangle$, respectively. The section $\epsilon \in \ker \Phi_2$ with height $1/2$, $1/2$ and 1 in these three cases are allowed sections. This directly follows for Cases 20a and 24b using the explicit formula of the height pairing (ϵ intersects f_∞ at the order 2 component). For Case 24a we need the following argument: There exists $\delta \in MW(B)^\alpha$ with height $1/2$. We can show that δ intersects $f_0 = I_2$ at the non-neutral component Γ_1 by using the same discussion we gave for Case 25a above. The equations $\langle \epsilon, \epsilon \rangle = \langle \delta, \delta \rangle = 1/2$ and $\langle \epsilon, \delta \rangle = 0$ imply that ϵ intersects $f_\infty = I_2$ at θ_1, hence ϵ is an allowed section.

- *Case 29a:* There exists $\epsilon \in \ker \Phi_2 = [\langle 1 \rangle^{\oplus 3}]^\perp$ in $D_4^* \oplus A_1^*$ with height 1/2. This section ϵ is an allowed section since it intersects $f_\infty = I_2$ at the order 2 component θ_1.
- *Case 29b:* There exists $\epsilon \in \ker \Phi_2 = [\langle 1 \rangle^{\oplus 3}]^\perp$ in D_5^* with height 1. This section ϵ is an allowed section since it intersects $f_\infty = I_4$ at the order 2 component θ_2.
- *Case 4a:* There exists $\epsilon \in \ker \Phi_4 = [\langle 2 \rangle]^\perp$ in D_5^* with height 5/4. This section ϵ intersects $f_\infty = I_4$ at an order 4 component θ_1 or θ_3, hence ϵ is an allowed section.
- *Cases 7a and 9a:* We have $\ker \Phi_3$ is given by $[\langle 3/2 \rangle]^\perp$ in A_5^* and $[3A_2^*]^\perp$ in E_6^* in these two cases, respectively. We can show the existence of $\epsilon \in \ker \Phi_3$ with height 4/3 in both cases. This ϵ intersects $f_\infty = I_3$ at an order 3 component θ_1 or θ_2, hence ϵ is an allowed section.
- *Case 15a:* We have $\ker \Phi_2 = [2A_2^*]^\perp$ in A_5^* and we can show the existence of $\epsilon \in \ker \Phi_2$ with height 3/2. This section ϵ intersects $f_\infty = I_2$ at the order 2 component θ_1, hence ϵ is an allowed section.
- *Case 28a:* We have $\ker \Phi_2 = [2D_4^*]^\perp = [D_4]^\perp$ in E_7^*. Using the result $[2A_3^*]^\perp$ in E_7^* equals D_4 which is proved in Lemma 5.2 in [2] as Case 23, we can conclude that $\ker \Phi_2 = [D_4]^\perp$ in E_7^* contains $2A_3^*$ as a sublattice. Thus, there exists $\epsilon \in \ker \Phi_2$ with height 3/2. This section ϵ is an allowed section since it intersects $f_\infty = I_2$ at θ_1.
- *Case 23a:* We have $\ker \Phi_2 = [2A_3^*]^\perp$ in D_6^* is the lattice given by the Gram matrix $\frac{1}{2}\left[\begin{smallmatrix} 4 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 3 \end{smallmatrix}\right]$. Thus, there exists ϵ_1 and ϵ_2 in $\ker \Phi_2$ such that $\langle \epsilon_1, \epsilon_1 \rangle = 2$, $\langle \epsilon_2, \epsilon_2 \rangle = 3/2$ and $\langle \epsilon_1, \epsilon_2 \rangle = 1$. First two equalities imply that both ϵ_1 and ϵ_2 are disjoint from the zero section σ, ϵ_1 intersects the neutral components of f_0 and f_∞, and ϵ_2 intersects exactly one of f_0 and f_∞ at the neutral component. If ϵ_2 intersects $f_\infty = I_2$ at the component θ_1, then ϵ_2 is an allowed section and we are done. If ϵ_2 intersects $f_\infty = I_2$ at the neutral component θ_0, then we get a contradiction as follows: The equality $\langle \epsilon_1, \epsilon_2 \rangle = 1$ implies that the intersection number $\epsilon_1 \epsilon_2$ is 0. Since $\epsilon_1, \epsilon_2 \in \ker \Phi_2$ we have $\epsilon_i + \alpha(\epsilon_i) = 0$ for each $i = 1, 2$ since $MW_{tors}(B) = 0$. Since α fixes the neutral component θ_0 (which is $\mathbb{P}^1$) pointwise and both ϵ_i are disjoint from σ, we can conclude that in order to have $\epsilon_i + \alpha(\epsilon_i) = 0$, both ϵ_i must intersect $\theta_0 = \mathbb{P}^1$ at -1 (after choosing coordinates on θ_0 such that $\sigma \cap \theta_0$ is 1 and the two intersection points of θ_0 and θ_1 are 0 and ∞. Note here that 0 and ∞ are fixed by t_ϵ if ϵ intersects θ_0, and if $\epsilon \cap \theta_0 = \omega$, then t_ϵ acts on $\theta_0 = \mathbb{P}^1$ by $z \mapsto \omega z$). This implies $\epsilon_1 \epsilon_2 \geq 1$ contradicting $\epsilon_1 \epsilon_2 = 0$.

In the rest of the proof we consider the cases in Table 1 where $MW(B)$ has a non-trivial torsion subgroup. After determining $\ker \Phi_m$, we determine whether there are allowed sections $\epsilon \in \ker \Phi_m$ where $\mathcal{P}_m(\epsilon)$ is a torsion section of order $d > 1$ or not. Here d divides the order of $MW_{tors}(B)$ and since we require that $\langle t_\epsilon \circ \alpha \rangle$ acts freely on f_∞ which is a singular fiber of type I_s, dm divides s. For some allowed sections in $\ker \Phi_m$ we may have $d = 1$ while $d > 1$ for other allowed sections. This way we can obtain different groups G for the same case in Table 1. Finally, we determine whether such an automorphism $t_\epsilon \circ \alpha$ where ϵ is an allowed section and translations by other torsion sections together generate a larger group G (order of G is greater than md) consisting of suitable automorphisms ($\tau \in G$ implies (B, τ) is a suitable pair). Note that by Lemma 4, $d > 1$ is not possible if f_0 is a smooth fiber (which is denoted as I_0).

• *Case 5a:* We have $\ker \Phi_4 = [0 \oplus \mathbb{Z}_2]^\perp$ in $MW(B)$ which equals $MW(B) = \langle 1/4 \rangle \oplus (\mathbb{Z}_2 \times \mathbb{Z}_2)$. There is $\epsilon \in \ker \Phi_4$ with height $1/4$ and this section ϵ intersects $f_\infty = I_4$ at an order 4 component θ_1 or θ_3, hence it is an allowed section. Since f_0 is I_0 (a smooth fiber), we have $d = 1$ by Lemma 4. Let $\tau = t_\epsilon \circ \alpha$, then (B, τ) is a suitable pair with $m = 4$ and $d = 1$. Is it possible to form a group G generated by τ and some translations by torsion sections such that every element of G is a suitable automorphism? $MW_{tor}(B) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Let η_1, η_2 and $\eta_1 + \eta_2$ be the order 2 sections of B. Note that α maps torsion sections to torsion sections. Without loss of generality, η_1 and η_2 intersect $f_\infty = I_4$ at the component θ_2 and two of the four I_2 fibers at non-neutral components. The section $\eta_1 + \eta_2$ intersects $f_\infty = I_4$ at the neutral component θ_0 and all four I_2 fibers at non-neutral components. G cannot contain t_{η_1} or t_{η_2} since otherwise $t_{\eta_i} \circ \tau^2 \in G$, but it is not a suitable automorphism for $i = 1, 2$ (they have fixed points on f_∞, their action on f_∞ is not free). We can form the group $G = \langle \tau, t_{\eta_1 + \eta_2} \rangle = \mathbb{Z}_4 \times \mathbb{Z}_2$ such that all elements of G are suitable automorphisms. Note that in Table 2 we do not list the subgroups of $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ for Case 5a as separate items, only $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ is displayed in this table.

• *Case 5b:* In this case $\ker \Phi_4 = A_1^* \oplus \mathbb{Z}_2$. Since $f_0 = I_0$, $d > 1$ is not possible. For $\epsilon \in \ker \Phi_4$ with height $1/2$, ϵ intersects $f_\infty = I_8$ at an order 4 component θ_2 or θ_6, hence $\tau = t_\epsilon \circ \alpha$ is a suitable automorphism with $m = 4$, $d = 1$. The order 2 section η of B intersects f_∞ at the component θ_4. G cannot contain t_η, otherwise $t_\eta \circ \tau^2 \in G$ has a fixed point on f_∞, hence $t_\eta \circ \tau^2$ is not a suitable automorphism. Note that $t_\eta \circ \tau^2 = t_\gamma \circ \alpha^2$ where $\gamma = \eta + \epsilon + \alpha(\epsilon)$ intersects $f_\infty = I_8$ at the neutral component θ_0. This is why $t_\eta \circ \tau^2$ has a fixed point on f_∞ (note that α fixes the neutral component of f_∞ pointwise). As a result, we have $G = \langle \tau \rangle = \mathbb{Z}_4$.

• *Case 8a:* There exists $\epsilon \in \ker \Phi_3$ with height $1/3$ and ϵ is an allowed section. $d = 2$ is not possible since $\mathbb{Z}_6$ cannot act freely on $f_\infty = I_3$. If η is the order 2 section of B, G contains t_η and a suitable automorphism τ with $m = 3$ and $d = 1$ implies $t_\eta \circ \tau \in G$ is a suitable automorphism with $m = 3$ and $d = 2$, a contradiction. Thus, $t_\eta \notin G$, and $G = \mathbb{Z}_3$.

• *Case 8b:* There exists $\epsilon \in \ker \Phi_3$ with height $2/3$ and ϵ intersects $f_\infty = I_6$ at an order 3 component θ_2 or θ_4. $\tau = t_\epsilon \circ \alpha$ is a suitable automorphism with $m = 3$ and $d = 1$. If η is the order 2 section of B, η intersects $f_\infty = I_6$ at the component θ_3. Thus $t_\eta \circ \tau = t_{\eta+\epsilon} \circ \alpha$ is a suitable automorphism with $m = 3$ and $d = 2$ (its action on $f_\infty = I_6$ is free since $\eta + \epsilon$ intersects I_6 at an order 6 component θ_1 or θ_5). We have $G = \langle \tau, t_\eta \rangle = \mathbb{Z}_6$ in this case.

• *Case 11a:* We have $\ker \Phi_3 = \mathbb{Z}_3 \times \mathbb{Z}_3$. If we denote the four I_3 fibers of B as S_1, S_2, S_3 and $R = f_\infty$, then without loss of generality α maps S_i to S_{i+1} (considering the indices modulo 3) and maps R to itself. We can denote the components of these I_3 fibers by $S_{i,j}$ with $i = 1, 2, 3$ and $j = 0, 1, 2$ where $S_{i,0}$ is the neutral component of S_i and α maps $S_{i,j}$ to $S_{i+1,j}$. We can similarly denote the components of $f_\infty = R$ by R_j with $j = 0, 1, 2$ and α maps R_j to itself. One can show that the components intersected by each of the eight order 3 sections are given by $(1, 2, 0, 1)$, $(0, 1, 2, 1)$, $(2, 0, 1, 1)$, $(2, 1, 0, 2)$, $(0, 2, 1, 2)$, $(1, 0, 2, 2)$, $(1, 1, 1, 0)$ and $(2, 2, 2, 0)$ (here (a, b, c, d) means that the section intersects the components $S_{1,a}$, $S_{2,b}$, $S_{3,c}$ and R_d, and if necessary we can interchange 1 and 2 in the labeling of R_j in order to obtain these 8 tuples). Let η_1 be the section corresponding to the tuple $(1, 1, 1, 0)$. The order 3 sections which are preserved by α are η_1 and $\eta_1 + \eta_1$. If η_2 is any of the first six sections in the given list, then $\tau = t_{\eta_2} \circ \alpha$ is a suitable automorphism with $m = 3$ and $d = 1$. It is not possible to have a suitable automorphism with $d > 1$ in this case since $f_0 = I_0$. Considering $\tau \in G$, we want to determine whether G can contain a translation by a torsion section or not. If the translation by any of the first six torsion sections in the given list is in G, then there is $t_\gamma \in G$ where γ and η_2 intersect two distinct non-neutral components of $f_\infty = R$, hence $t_\gamma \circ \tau = t_{\gamma+\eta_2} \circ \alpha \in G$ is not a suitable automorphism (there are fixed points on R since $\gamma + \eta_2$ intersects the neutral component R_0). Thus, G may contain t_{η_1} and $t_{\eta_1+\eta_1}$, but not the translations by other six torsion sections. It can be checked that every element in $G = \langle \tau, t_{\eta_1} \rangle = \mathbb{Z}_3 \times \mathbb{Z}_3$ is a suitable automorphism.

• *Case 11b:* We have $\ker \Phi_3 = \mathbb{Z}_3 = \langle \eta \rangle$ where η intersects $f_\infty = I_9$ at θ_3 or θ_6. Thus $\tau = t_\eta \circ \alpha$ is a suitable automorphism with $m = 3$ and $d = 1$. It is not possible to have $d > 1$ in this case since $f_0 = I_0$. Considering $\tau \in G$ where τ is any suitable automorphism with $m = 3$ and $d = 1$, G cannot contain t_η (and $t_{\eta+\eta}$) since otherwise $t_\eta \circ \tau$ or $t_{\eta+\eta} \circ \tau$ which is in G has a fixed point on f_∞ and is not a suitable automorphism. Thus, $G = \langle\tau\rangle = \mathbb{Z}_3$ in this case.

• *Case 14a:* We have $\ker\Phi_2 = A_1^* \oplus (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and $MW_{tors}(B) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Let η_1, η_2 and $\eta_1 + \eta_2$ be the order 2 sections of B. Denote $f_0 = I_0^*$ by Q, $f_\infty = I_2$ by R and the other two I_2 fibers of B by S_i, $i = 1, 2$. Let R_j and $S_{i,j}$ where $j = 0, 1$ be the components of R, S_1 and S_2 such that $j = 0$ corresponds to the neutral component. Denote the components of $Q = I_0^*$ by Q_j, $0 \leq j \leq 4$ such that Q_0 is the neutral component and Q_4 is the component with multiplicity 2. The automorphism α maps $S_{i,j}$ to $S_{i+1,j}$ (considering i modulo 2), Q_1 to Q_2 (see the discussion under *Case* 14 on page 34 in [2]) and maps the other components to themselves. Using the formula of the height pairing and the facts that $\langle\eta, \epsilon\rangle = 0$ for any torsion section η, and α maps torsion sections to torsion sections, we can determine the components each torsion section intersects as follows. Without loss of generality, η_1 intersects the components $(Q_1, S_{1,1}, R_1)$, η_2 intersects $(Q_2, S_{2,1}, R_1)$ and $\eta_1 + \eta_2$ intersects $(Q_3, S_{1,1}, S_{2,1})$ (we only listed the non-neutral components intersected). We get $\alpha(\eta_1) = \eta_2$ and $\alpha(\eta_1 + \eta_2) = \eta_1 + \eta_2$. Let now $\epsilon \in \ker\Phi_2$ with height 1/2. Then, either ϵ intersects all three I_2 fibers at non-neutral components, or it intersects $Q = I_0^*$ at a non-neutral component and only one of the three I_2 fibers at a non-neutral component. In the former case we have $\mathcal{P}_2(\epsilon) = \epsilon + \alpha(\epsilon) = 0$ and $\tau = t_\epsilon \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 1$. In the latter case, if ϵ intersects $f_\infty = R = I_2$ at R_1, then again τ is a suitable automorphism with $m = 2$ and $d = 1$. If ϵ intersects $S_{1,1}$ or $S_{2,1}$, then $\mathcal{P}_2(\epsilon)$ which is a torsion section is $\eta_1 + \eta_2$ and we have $\mathcal{P}_2(\epsilon + \eta_i) = 0$ for any $i = 1, 2$. Thus $\tau_i = t_{\epsilon+\eta_i} \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 1$. In any case, there is a suitable automorphism $\tilde{\tau}$ with $m = 2$ and $d = 1$. Note that there is no suitable automorphism with $d > 1$ in this case since $\mathbb{Z}_4$ does not act freely on $f_\infty = I_2$. Considering $\tilde{\tau} \in G$, can G contain a translation by a torsion section? Since $t_{\eta_i} \circ \tilde{\tau}$ has fixed points on f_∞, G cannot contain t_{η_i}. Every element of $G = \langle t_{\eta_1+\eta_2}, \tilde{\tau}\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$ is a suitable automorphism. We list G as $\mathbb{Z}_2 \times \mathbb{Z}_2$ in this case.

• *Case 14b:* We have $\ker\Phi_2 = \langle 1/4\rangle \oplus \mathbb{Z}_2$. If η is the order 2 section, η intersects $f_\infty = I_4$ at the component θ_2 and $\tau = t_\eta \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 1$. If such an automorphism τ is in G, then $t_\eta \notin G$ since $t_\eta \circ \tau \in G$ has a fixed point on f_∞. Thus, $G = \mathbb{Z}_2 = \langle\tau\rangle$ in the case that $d = 1$ for τ. In this case $d = 2$ also occurs. Let $\epsilon \in \ker\Phi_2$ with height 1/4. Such a section ϵ intersects $f_\infty = I_4$ at θ_1 or θ_3, and we get $\mathcal{P}_2(\epsilon) = \epsilon + \alpha(\epsilon)$, which is a torsion section, equals η. Thus, $\tilde{\tau} = t_\epsilon \circ \alpha$ is a suitable automorphism with $m = d = 2$ and $G = \langle\tilde{\tau}\rangle = \mathbb{Z}_4$ in this case (note that G already contains $t_\eta = (\tilde{\tau})^2$).

• *Case 17a:* We have $\ker \Phi_2 = \langle 1/6 \rangle \oplus \mathbb{Z}_3$. Let η be an order 3 section of B, then η intersects $f_\infty = I_2$ at θ_0. Let $\epsilon \in \ker \Phi_2$ with height 1/6, then ϵ intersects f_∞ at θ_1. We may have $\mathcal{P}_2(\epsilon) = \epsilon + \alpha(\epsilon)$ as any of 0, η or $\eta + \eta$. In any case one of $\mathcal{P}_2(\epsilon)$, $\mathcal{P}_2(\epsilon + \eta)$ or $\mathcal{P}_2(\epsilon + \eta + \eta)$ is 0 (note that $\alpha(\eta) = \eta$), hence we get an allowed section for a suitable automorphism with $m = 2$ and $d = 1$. It is not possible to have a suitable automorphism with $m = 2$ and $d = 3$ since $f_\infty = I_2$ does not admit a free action of $\mathbb{Z}_6$. Considering $\tau \in G$ with $m = 2$ and $d = 1$, if we also have $t_\eta \in G$, then $t_\eta \circ \tau \in G$ has $m = 2$ and $d = 3$, which is not possible. Therefore, $G = \mathbb{Z}_2$ in this case.

• *Case 17b:* We have $\ker \Phi_2 = A_1^* \oplus \mathbb{Z}_3$. Let η be an order 3 section of B, then η intersects $f_\infty = I_6$ at an order 3 component θ_2 or θ_4 and it intersects $f_0 = IV$ at a non-neutral component. Let $\epsilon \in \ker \Phi_2$ with height 1/2, then either it intersects $f_\infty = I_6$ at θ_3 and $f_0 = IV$ at the neutral component, or it intersects f_∞ at θ_1 or θ_5 and $f_0 = IV$ at a non-neutral component. In the former case, $\tau = t_\epsilon \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 1$. We have $G = \langle t_\eta, \tau \rangle = \mathbb{Z}_6$ where $t_\eta \circ \tau$ is a suitable automorphism with $m = 2$ and $d = 3$. In the latter case, $\tau = t_\epsilon \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 3$, hence $G = \langle \tau \rangle = \mathbb{Z}_6$ (here G already contains t_η and $t_{\eta+\eta}$). In any case $G = \mathbb{Z}_6$ and G contains elements with both $d = 1$ and $d = 3$.

• *Case 18a:* We have $\ker \Phi_2 = \mathbb{Z}_4 = \langle \eta \rangle$ where the order 4 section η intersects $f_0 = I_8$ at an order 4 component Γ_2 or Γ_6 and intersects $f_\infty = I_2$ at the component θ_1. The automorphism α maps Γ_i to Γ_{8-i} (see the discussion under *Case* 18 on page 36 in [2]). We get $\mathcal{P}_2(\eta) = \eta + \alpha(\eta) = 0$, hence $\tau = t_\eta \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 1$. It is not possible to have a suitable automorphism with $m = d = 2$ since $f_\infty = I_2$ does not admit a free action of $\mathbb{Z}_4$. Considering $\tau \in G$, t_η or $t_{\eta+\eta+\eta}$ cannot be in G since $t_\eta \circ \tau$ and $t_{3\eta} \circ \tau$ have fixed points on f_∞. But every element of $G = \langle t_{\eta+\eta}, \tau \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$ is a suitable automorphism.

• *Case 21a:* We have $\ker \Phi_2 = [\langle 1 \rangle \oplus \mathbb{Z}_2]^\perp$ in $A_1^* \oplus \langle 1/4 \rangle \oplus \mathbb{Z}_2$ which equals $A_1^* \oplus \mathbb{Z}_2$. Let η be the order 2 section of B, then $\alpha(\eta) = \eta$ and η intersects $f_0 = I_4$ at the component Γ_2, $f_\infty = I_2$ at the component θ_0 and the other two I_2 fibers at non-neutral components. Let $\epsilon \in \ker \Phi_2$ with height 1/2, then either ϵ intersects f_0 at Γ_0 and the three I_2 fibers at non-neutral components (hence intersects $f_\infty = I_2$ at θ_1), or ϵ intersects f_0 at Γ_2 and one of the three I_2 fibers at a non-neutral component. In any case $\mathcal{P}_2(\epsilon) = \epsilon + \alpha(\epsilon) = 0$ (the zero section σ) since $\mathcal{P}_2(\epsilon) \in MW_{tors}(B) = \langle \eta \rangle$ and $\mathcal{P}_2(\epsilon)$ intersects f_0 at the component Γ_0 whereas η intersects f_0 at Γ_2. In the former case, $\tau = t_\epsilon \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 1$. In the latter case, $\mathcal{P}_2(\epsilon) = 0$ implies that ϵ intersects $f_\infty = I_2$ at θ_1 (otherwise, if ϵ intersects one of the other I_2 fibers at a non-neutral component, then $\mathcal{P}_2(\epsilon)$ intersects these two I_2 fibers at non-neutral components, which contradicts $\mathcal{P}_2(\epsilon)=0$). Thus, in this latter case $\tau=t_\epsilon\circ\alpha$ is again a suitable automorphism with $m=2$ and $d=1$. In any case $G=\langle\tau,t_\eta\rangle=\mathbb{Z}_2\times\mathbb{Z}_2$ consists of suitable automorphisms.

• *Case 21b:* We have $\ker\Phi_2=[\langle 1\rangle\oplus\mathbb{Z}_2]^\perp$ in $(A_1^*)^{\oplus 2}\oplus\mathbb{Z}_2$ which equals $\langle 1\rangle\oplus\mathbb{Z}_2$. Let τ be the order 2 section of B, then η intersects $f_0=I_4$ at the component Γ_2 and $f_\infty=I_4$ at the component θ_2. Let $\epsilon\in\ker\Phi_2$ with height 1, then either ϵ intersects f_0 at Γ_2 or ϵ intersects f_∞ at θ_2. In the latter case $\tau=t_\epsilon\circ\alpha$ is a suitable automorphism with $m=2$ and $d=1$, and in the former case $\tau=t_{\eta+\epsilon}\circ\alpha$ is such a suitable automorphism. It is not possible to have a suitable automorphism with $m=d=2$ since there is no section $\gamma\in\ker\Phi_2=\langle 1\rangle\oplus\mathbb{Z}_2$ which intersects $f_\infty=I_4$ at θ_1 or θ_3 (since $\langle\gamma,\gamma\rangle$ is an integer) so that $\mathcal{P}_2(\gamma)=\eta$. Considering $\tau\in G$ is a suitable automorphism with $m=2$ and $d=1$, we cannot have $t_\eta\in G$ since $t_\eta\circ\tau\in G$ has a fixed point on f_∞. Thus, $G=\langle\tau\rangle=\mathbb{Z}_2$ in this case.

• *Case 22a:* We have $\ker\Phi_2=\mathbb{Z}_4\times\mathbb{Z}_2$. If we denote the fibers f_0 and f_∞ by Q and R, the two I_2 fibers by S_1 and S_2, and their components by Q_j, R_j, $S_{i,k}$ where $i=1,2$, $j=0,1,2,3$ and $k=0,1$ such that $j=0$ and $k=0$ correspond to neutral components, then α maps Q_j to Q_{4-j} (see the discussion under *Case* 22 on page 36 in [2]), $S_{i,k}$ to $S_{i+1,k}$ and maps R_j to itself. We can describe the torsion sections of B by giving the intersection tuples (a,b,c,d) of each as follows where the intersection tuple (a,b,c,d) denotes that the torsion section intersects the components Q_a, $S_{1,b}$, $S_{2,c}$ and R_d. The three order 2 sections of B are described by the intersection tuples as $\eta_1:(0,1,1,2)$, $\eta_2:(2,1,1,0)$ and $\eta_1+\eta_2:(2,0,0,2)$. The order 4 sections of B are described as $\omega_1:(1,1,0,1)$, $\omega_2:(3,0,1,1)$, $\omega_3:(3,1,0,3)$ and $\omega_4:(1,0,1,3)$. The order two automorphism α maps ω_1 to ω_2, ω_3 to ω_4 and maps each order 2 section to itself. We have $2\omega_i=\eta_1+\eta_2$, $-\omega_1=\omega_3$ and $-\omega_2=\omega_4$. Using this notation, the only suitable automorphisms with $m=2$ and $d=1$ are $t_{\eta_1}\circ\alpha$ and $t_{\eta_1+\eta_2}\circ\alpha$ (note that $t_{\eta_2}\circ\alpha$ is not suitable since if has a fixed point on f_∞ as η_2 intersects f_∞ at the neutral component R_0). If one of $t_{\eta_1}\circ\alpha$ or $t_{\eta_1+\eta_2}\circ\alpha$ is in G then C cannot contain any translation by a non-trivial torsion section except for t_{η_2} since otherwise G contains α or $t_{\eta_2}\circ\alpha$ which are not suitable automorphisms. In this case (for $m=2$, $d=1$) we have $G=\langle t_{\eta_2},t_{\eta_1+\eta_2}\circ\alpha\rangle=\langle t_{\eta_2},t_{\eta_1}\circ\alpha\rangle=\{1,t_{\eta_2},t_{\eta_1}\circ\alpha,t_{\eta_1+\eta_2}\circ\alpha\}=\mathbb{Z}_2\times\mathbb{Z}_2$. We have $\mathcal{P}_2(\omega_i)=\omega_i+\alpha(\omega_i)=\eta_1$ and ω_i intersects $f_\infty=R$ at an order 4 component R_1 or R_3, hence each of the automorphisms $t_{\omega_i}\circ\alpha$ is a suitable automorphism with $m=d=2$. Considering $t_{\omega_i}\circ\alpha\in G$, none of t_{ω_j} $(j=1,..,4)$ is in G since otherwise we have $\alpha\in G$, but α is not a suitable automorphism. In this case (for $m=d=2$) $G=\langle t_{\eta_2},t_{\omega_1}\circ\alpha\rangle=\mathbb{Z}_4\times\mathbb{Z}_2$

which contains all $t_{\omega_i} \circ \alpha$ for $i = 1, 2, 3, 4$ and all t_η where η is an order 2 section of B.

• *Case 26a:* We have $MW(B)^\alpha = A_1^* \oplus \mathbb{Z}_2$ and $\ker \Phi_2 = A_1^* \oplus (\mathbb{Z}_2 \times \mathbb{Z}_2)$, hence there exists $\delta \in MW(B)^\alpha$ and $\epsilon \in \ker \Phi_2$ which both have height $1/2$ such that $\langle \delta, \epsilon \rangle = 0$. We first determine the components of the six I_2 fibers intersected by each torsion section in $MW_{tors}(B) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Let η_1, η_2 and $\eta_1 + \eta_2$ be the order 2 sections of B. If we denote f_0 and f_∞ by Q and R and the other four I_2 fibers by S_i $(i = 1, 2, 3, 4)$ and the components of these singular fibers by Q_j, R_j, $S_{i,j}$ where $j = 0, 1$, $i = 1, 2, 3, 4$ such that $j = 0$ corresponds to neutral components, then without loss of generality α maps $S_{1,j}$ to $S_{2,j}$, $S_{3,j}$ to $S_{4,j}$ and maps each of the other components to itself. Each of the order 2 sections intersects exactly four of the I_2 fibers at non-neutral components. Since α maps each torsion section to a torsion section, we have two possibilities for the intersection tuples of the torsion sections. If the tuple (a, b, c, d, e, f) means that the given section intersects the components Q_a, $S_{1,b}, \ldots, S_{4,e}$ and R_f, then the first possibility is that the intersection tuples of the torsion sections are given by $\eta_1 : (1, 1, 1, 0, 0, 1)$, $\eta_2 : (1, 0, 0, 1, 1, 1)$ and $\eta_1 + \eta_2 : (0, 1, 1, 1, 1, 0)$. For the second possibility, we have $\eta_1 : (1, 1, 0, 1, 0, 1)$, $\eta_2 : (1, 0, 1, 0, 1, 1)$ and $\eta_1 + \eta_2 : (0, 1, 1, 1, 1, 0)$. The first possibility does not occur as the following argument shows: For δ with height $1/2$ such that $\alpha(\delta) = \delta$ whose existence is shown above, δ has height $1/2$ implies δ is disjoint from the zero section and it intersects exactly three of the I_2 fibers at non-neutral components. The same holds for the section $-\delta$. Assume that δ and $-\delta$ intersect $f_0 = Q$ at Q_0. Since α has 2 fixed points on Q_0 one of which is $\sigma \cap Q$ and $\alpha(\delta) = \delta$, we have δ and $-\delta$ both intersect Q_0 at the second fixed point of α on Q_0 (note that δ and $-\delta$ are disjoint from σ). The fact that δ and $-\delta$ intersect gives a contradiction when we consider $\langle \delta, -\delta \rangle = -1/2$. Thus, δ intersects $f_0 = Q$ at the component Q_1. Since $\alpha(\delta) = \delta$, intersection tuple of δ must be one of $(1, 1, 1, 0, 0, 0)$ or $(1, 0, 0, 1, 1, 0)$. The equalities $\langle \delta, \eta_1 \rangle = \langle \delta, \eta_2 \rangle = \langle \delta, \eta_1 + \eta_2 \rangle = 0$ imply that δ and each of the three order 2 sections have exactly two common I_2 fibers where they both intersect the non-neutral component. This is not the case in the first possibility given above. Thus, the second possibility occurs. As a result, we have $\alpha(\eta_1) = \eta_2$ and $\alpha(\eta_1 + \eta_2) = \eta_1 + \eta_2$. Let $\epsilon \in \ker \Phi_2$ with height $1/2$, then ϵ intersects three of the I_2 fibers at non-neutral components. The equalities $\langle \epsilon, \eta_1 \rangle = \langle \epsilon, \eta_2 \rangle = \langle \epsilon, \eta_1 + \eta_2 \rangle = 0$ imply that ϵ and each of the three order 2 sections have exactly two common I_2 fibers where they both intersect the non-neutral components. $\langle \delta, \epsilon \rangle = 0$ implies δ and ϵ have 0 or 2 common I_2 fibers where they both intersect the non-neutral component. These conditions and $\mathcal{P}_2(\epsilon) = 0$ or $\eta_1 + \eta_2$ reduce the possibilities for the intersection tuple of ϵ to the following four cases: $(0,0,0,1,1,1)$, $(0,1,1,0,0,1)$, $(1,1,0,0,1,0)$ and $(1,0,1,1,0,0)$. For the first two cases, $\tau = t_\epsilon \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 1$. For the last two cases, $\tau = t_{\epsilon+\eta_1} \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 1$. In any case we obtain the existence of $\tau \in G$ with $m = 2$ and $d = 1$. It is not possible to have a suitable automorphism with $m = d = 2$ since $f_\infty = I_2$. Considering $\tau \in G$, t_{η_1} or t_{η_2} cannot be in G since $t_{\eta_i} \circ \tau$ has a fixed point on f_∞. But, $G = \langle \tau, t_{\eta_1+\eta_2} \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$ consists of suitable automorphisms.

• *Case 26b:* We have $\ker \Phi_2 = \langle 1/4 \rangle \oplus \mathbb{Z}_2$. Let η be the order 2 section of B, then $\alpha(\eta) = \eta$ and η intersects f_∞ at the order 2 component θ_2 and the two I_2 fibers different from f_0 at the non-neutral components. $\tau = t_\eta \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 1$. Considering such a $\tau \in G$ with $m = 2$ and $d = 1$, G cannot contain t_η since $t_\eta \circ \tau$ has a fixed point on f_∞. Thus, for $m = 2$ and $d = 1$ case $G = \langle \tau \rangle = \mathbb{Z}_2$. We also have $m = d = 2$ case. Let $\epsilon \in \ker \Phi_2$ with height $1/4$, then ϵ intersects f_∞ at an order 4 component θ_1 or θ_3 and $\mathcal{P}_2(\epsilon) = \eta$. Thus, $\tilde{\tau} = t_\epsilon \circ \alpha$ is a suitable automorphism with $m = d = 2$. We get $G = \langle \tilde{\tau} \rangle = \mathbb{Z}_4$ (note that G already contains t_η).

• *Case 27a:* We have $\ker \Phi_2 = \mathbb{Z}_4 \times \mathbb{Z}_2$. If we denote $f_0 = I_2$ and $f_\infty = I_2$ by Q and R, the two I_4 fibers by S_1 and S_2, and the components of these singular fibers by Q_j, R_j and $S_{i,k}$ where $i = 1, 2$, $j = 0, 1$ and $k = 0, 1, 2, 3$ such that $j = 0$ and $k = 0$ correspond to neutral components, then without loss of generality α maps $S_{i,k}$ to $S_{i+1,k}$ and α maps each of the other components to itself. The intersection tuple (a, b, c, d) for a section means that this section intersects the components Q_a, $S_{1,b}$, $S_{2,c}$ and R_d. The order 2 sections of B are described by the intersection tuples as $\eta_1 : (1, 2, 0, 1)$, $\eta_2 : (1, 0, 2, 1)$ and $\eta_1 + \eta_2 : (0, 2, 2, 0)$. The order 4 sections of B are described by the intersection tuples as $\omega_1 : (1, 1, 1, 0)$, $\omega_2 : (1, 3, 3, 0)$, $\omega_3 : (0, 3, 1, 1)$ and $\omega_4 : (0, 1, 3, 1)$. Here we can explain why we have a section with intersection tuple $(1, 1, 1, 0)$ instead of $(0, 1, 1, 1)$ as follows: A torsion section ϵ which intersects $S_{1,1}$ and $S_{2,1}$ intersects either Q_1 or R_1, and satisfies $\alpha(\epsilon) = \epsilon$. Assume that ϵ intersects Q_0 and R_1. Choosing coordinates on $Q_0 = \mathbb{P}^1$ such that $\sigma \cap Q_0$ is 1 and the two intersection points of Q_0 and Q_1 are 0 and ∞, α maps z to $1/z$ since α interchanges 0 and ∞ (α fixes $1 \in Q_0$ and does not fix Q_0 pointwise, see the proof of Lemma 4.6 in [2]) and preserves the zero section σ. Since $\alpha(\epsilon) = \epsilon$, we get $\epsilon \cap Q_0 = -1$ (note that non-zero torsion sections are disjoint from the zero section σ, see Proposition 2.1 in [2]). This implies that t_ϵ maps z to $-z$ on Q_0 and $t_{\epsilon+\epsilon}$ is identity on Q_0, hence $(\epsilon+\epsilon) \cap Q_0 = 1$. But, $\epsilon + \epsilon$ is a non-trivial torsion section (it intersects $S_{1,2}$ and $S_{2,2}$), hence it is disjoint from the zero section σ, which contradicts $(\epsilon+\epsilon)\cap Q_0 = 1 \in \sigma$. Therefore, ϵ has intersection tuple $(1,1,1,0)$ instead of $(0,1,1,1)$. Using the above descriptions of the torsion sections of B in terms of intersection tuples, we can now conclude that $\tau = t_{\omega_3} \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 1$. Every element of $G = \langle t_{\eta_1+\eta_2}, \tau\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$ is a suitable automorphism. There is no suitable automorphism with $m = d = 2$ since $f_\infty = I_2$ does not admit a free action of $\mathbb{Z}_4$.

- *Case 27b:* We have $\ker \Phi_2 = \mathbb{Z}_4 = \langle \eta\rangle$. The order 4 section η intersects $f_0 = I_2$ at the non-neutral component and $f_\infty = I_8$ at an order 4 component θ_2 or θ_6. We have $\alpha(\eta) = \eta$, $\mathcal{P}_2(\eta) = \mathcal{P}_2(3\eta) = 2\eta$ which is the order 2 section of B, hence $\tau = t_\eta \circ \alpha$ (and also $\tilde{\tau} = t_{3\eta} \circ \alpha$) is a suitable automorphism with $m = d = 2$. We get $G = \langle\tau\rangle = \langle\tilde{\tau}\rangle = \mathbb{Z}_4$. Considering $\tau \in G$, t_η or $t_{3\eta}$ cannot be in G (otherwise there are elements in G which have fixed points on f_∞) and $\tau^2 = t_{2\eta}$ is already in G. $\bar{\tau} = t_{2\eta} \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 1$ since $\mathcal{P}_2(2\eta) = 0$. Considering $\bar{\tau} \in G$, none of t_γ where γ is a torsion section can be in G (otherwise G has elements which have fixed points on f_∞). Thus, for $d = 1$ case we have $G = \langle\bar{\tau}\rangle = \mathbb{Z}_2$.
- *Case 31a:* We have $\ker \Phi_2 = A_1^* \oplus \mathbb{Z}_2$ and $MW(B)^\alpha = \langle 1\rangle^{\oplus 2} \oplus \mathbb{Z}_2$. Let $\epsilon \in \ker \Phi_2$ with height $1/2$ and $\delta \in MW(B)^\alpha$ with height 1. Let η be the order 2 section of B. If we denote $f_\infty = I_2$ by R, the other four I_2 fibers by S_i, $i = 1,2,3,4$, and the components of these singular fibers by R_j and $S_{i,j}$, $j = 0,1$ such that $j = 0$ corresponds to the neutral components, then without loss of generality α maps $S_{1,j}$ to $S_{2,j}$, $S_{3,j}$ to $S_{4,j}$ and maps R_j to itself for any $j = 0,1$. The order 2 section η intersects four of the five I_2 fibers at non-neutral components, and since $\alpha(\eta) = \eta$, η intersects $f_\infty = I_2$ at the neutral component R_0. ϵ intersects three of the five I_2 fibers at non-neutral components. Since $\mathcal{P}_2(\epsilon) = \eta$ or σ (zero section), ϵ intersects R_1. δ has height 1 and $\alpha(\delta) = \delta$ implies that the non-neutral components δ intersects are either $(S_{1,1}, S_{2,1})$ or $(S_{3,1}, S_{4,1})$. Using the explicit formula of the height pairing in $\langle\epsilon, \delta\rangle = 0$, we can conclude that there are 0 or 2 common I_2 fibers where both ϵ and δ intersect these fibers at non-neutral components. As a result, ϵ intersects the components $(S_{1,1}, S_{2,1}, R_1)$ or $(S_{3,1}, S_{4,1}, R_1)$, and in any case $\mathcal{P}_2(\epsilon) = \sigma$. Thus, $\tau = t_\epsilon \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 1$. There is no suitable automorphism with $m = d = 2$ since $f_\infty = I_2$. Every element of $G = \langle t_\eta, \tau\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$ is a suitable automorphism.
- *Case 31b:* Let η be the order 2 section of B, then η intersects $f_\infty = I_4$ at the order 2 component θ_2 and $\mathcal{P}_2(\eta) = 0$. Thus $\tau = t_\eta \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 1$. Since $f_0 = I_0$ (a smooth fiber), it is not possible to have any suitable automorphism with $m = d = 2$ (see

Lemma 4). Considering $\tilde{\tau} \in G$ with $m = 2$ and $d = 1$, G cannot contain t_η since $t_\eta \circ \tilde{\tau}$ has a fixed point on f_∞. Therefore, $G = \langle \tau \rangle = \mathbb{Z}_2$ in this case.

• *Case 32a:* We have $\ker \Phi_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$. Two of the three order 2 sections, η_1 and η_2, intersect $f_\infty = I_4$ at the order 2 component θ_2, and $\eta_1 + \eta_2$ intersects f_∞ at θ_0. Note that α maps each torsion section to itself since B is obtained by pull-back of the elliptic surface $\hat{B}$ via the map g_2 (see §4 in [2]) and $MW_{tors}(\hat{B}) = \mathbb{Z}_2 \times \mathbb{Z}_2$. We have $\tau = t_{\eta_i} \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 1$ for each $i = 1, 2$. Since $f_0 = I_0$, there is no suitable automorphism with $m = d = 2$. We have $G = \langle t_{\eta_1+\eta_2}, t_{\eta_1} \circ \alpha \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$ in this case.

• *Case 33a:* We have $\ker \Phi_2 = \mathbb{Z}_4$. Let η be an order 4 section of B. If we denote $f_\infty = I_2$ by R, the two I_4 fibers by S_1 and S_2, and the components of these singular fibers by R_0, R_1, $S_{i,j}$ where $i = 1, 2$ and $j = 0, 1, 2, 3$ where subindex 0 corresponds to neutral components, then without loss of generality α maps $S_{1,j}$ to $S_{2,j}$ for all j, and α maps R_i to itself for all i. η intersects $f_\infty = I_2 = R$ at the non-neutral component R_1, $S_1 = I_4$ at $S_{1,1}$ or $S_{1,3}$ and S_2 at $S_{2,1}$ or $S_{2,3}$. We first show that η intersects $(S_{1,1}, S_{2,3})$ or $(S_{1,3}, S_{2,1})$ by considering $\delta \in MW(B)^\alpha$ with height 1/2. $\alpha(\delta) = \delta$ and $\langle \delta, \delta \rangle = 1/2$ implies that δ intersects $(S_{1,1}, S_{2,1})$ or $(S_{1,3}, S_{2,3})$. In any case $\langle \eta, \delta \rangle = 0$ implies the contributions at the two I_4 fibers in the explicit formula of the height pairing must add up to an integer, hence η intersects $(S_{1,1}, S_{2,3})$ or $(S_{1,3}, S_{2,1})$ as claimed. Thus, $\alpha(\eta) \neq \eta$, hence $\alpha(\eta) = 3\eta$. We have $\mathcal{P}_2(\eta) = \eta + \alpha(\eta) = 0$. Since η intersects $f_\infty = I_2$ at R_1, $\tau = t_\eta \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 1$. Since $f_0 = I_0$, a suitable automorphism with $m = d = 2$ is not possible (see Lemma 4). We get $G = \langle t_{2\eta}, \tau \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$ in this case.

• *Case 33b:* We have $\ker \Phi_2 = \mathbb{Z}_2$. If η is the order 2 section of B, then η intersects $f_\infty = I_8$ at the order 2 component θ_4 and $\mathcal{P}_2(\eta) = 0$. Thus, $\tau = t_\eta \circ \alpha$ is a suitable automorphism with $m = 2$ and $d = 1$. Considering $\tau \in G$, C cannot contain t_η since $t_\eta \circ \tau$ has a fixed point on f_∞. Thus, we have $G = \langle \tau \rangle = \mathbb{Z}_2$ in this case. Note that since $f_0 = I_0$, there is no suitable automorphism with $m = d = 2$. □

5. Construction of free actions on Schoen 3-folds with non-trivial induced action on $\mathbb{P}^1$

In this section we use the information obtained in the previous section and construct the smooth or singular Schoen 3-folds $X = B_1 \times_{\mathbb{P}^1} B_2$ which admit a free action of a finite group G with non-trivial induced action on the base curve $\mathbb{P}^1$ ($m > 1$ case) where for the singular 3-folds X we require that the singularities of X are on $I_r \times I_s$ type fibers with $r > 1$ and $s > 1$ so that the small resolution of X is a projective Calabi-Yau 3-fold. In the paper [2], Bouchard and Donagi have produced the list of groups G which act on a rational elliptic surface B with section such that (B, τ) is a suitable pair for each $\tau \in G$, $\psi(G)$ is a cyclic subgroup of $Aut_\sigma(B)$, and f_∞ is a smooth fiber of B in Table 8 and Table 9 (pages 47–49 in [2]). For easy reference in the proceeding discussion, we include a copy of this list in Table 3 in this paper. We will call these group actions *general type* group actions. In Table 2 in the previous section we listed such group actions where f_∞ of B is a singular fiber of type I_{mr} ($m > 1$ and $r \geq 1$) (see Lemma 11), and we will call these group actions *special type* group actions. Bouchard and Donagi have classified the free actions on smooth Schoen 3-folds $X = B_1 \times_{\mathbb{P}^1} B_2$ where both actions on B_1 and B_2 are *general type*. We now construct smooth or singular Schoen 3-folds X which admit a free action by a finite group G with non-trivial induced action on $\mathbb{P}^1$ (where in the smooth X case at least one of the actions on B_1 and B_2 is *special type*). Note that for such an action, the induced action is cyclic, hence $\psi(G) \subset Aut_\sigma(B_i)$ is cyclic (see §3.3). Furthermore, for each $\tau_1 \times \tau_2 \in G$, each (B_i, τ_i) is a suitable pair, hence G is isomorphic to a subgroup of $Aut(B_i)$ consisting of suitable automorphisms (see §3.2). As a result, in order to construct such group actions, we choose B_1 and B_2 from Table 3 (Tables 8 and 9 in [2]) or Table 2 (where at least one is chosen from Table 2 if X is smooth) which admit the action of the same group G. That is, if B_i is listed under the group G_i in these tables, then G is a subgroup of G_i for each $i = 1, 2$. We also require that the induced actions of G on the base curve $\mathbb{P}^1$ are isomorphic for B_1 and B_2. We form the fiber product $X = B_1 \times_{\mathbb{P}^1} B_2$ after changing the coordinates in the base curve $\mathbb{P}^1$ of B_2 by interchanging 0 and ∞ and then identifying the two base curves by an automorphism $z \mapsto \mu z$ of $\mathbb{P}^1$ which fixes 0 and ∞. After this change of coordinates, the fiber of X over $0 \in \mathbb{P}^1$ is $f_{10} \times f_{20}$ and the fiber of X over $\infty \in \mathbb{P}^1$ is $f_{1\infty} \times f_{2\infty}$ (Here the f_0 and f_∞ fibers of B_2 are denoted by $f_{2\infty}$ and f_{20}, respectively, due to the change of coordinates on $\mathbb{P}^1$ while f_0 and f_∞ fibers of B_1 are denoted by f_{10} and $f_{1\infty}$, respectively). We determine if there exists an isomorphism $\Delta : G \to G$ such that τ and $\Delta(\tau)$ have the same m and d values and $\phi(\tau) = \phi(\Delta(\tau))$ for all $\tau \in G$ (This is a simple task since G is abelian with at most two generators). Here we consider the domain of Δ as $G \subset Aut(B_1)$ and the image of Δ as $G \subset Aut(B_2)$. By definition, (B_1, τ) and $(B_2, \Delta(\tau))$ are suitable pairs for each $\tau \in G$. For every $\tau \in G$ with $m > 1$, $\langle\tau\rangle$ acts freely on $f_{1\infty}$ and $\langle\Delta(\tau)\rangle$ acts freely on f_{20}. The isomorphic copy $\{(\tau, \Delta(\tau)) \in G \times G | \tau \in G\} \subset Aut(B_1) \times Aut(B_2)$ of G acts on X where $(\tau, \Delta(\tau))$ is the automorphism $\tau \times \Delta(\tau)$ of X using our notation in the previous sections. If X is smooth, then the action of G on X is free by definition of suitable pairs (see §3 in [2]). If X is singular, to conclude that this action of G on X is free, we only need to check that the non-identity automorphisms of the form $t_{\epsilon_1} \times t_{\epsilon_2}$ in this group $G \subset Aut(B_1) \times Aut(B_2)$ have no fixed points on X (see Lemma 3). With these guidelines to construct free actions on Schoen 3-folds, we can prove the following two theorems by examining the cases in Table 2 and Table 3.

5.1. Free action on smooth Schoen 3-folds

Theorem 12. *The smooth Schoen* 3*-folds* $X = B_1 \times_{\mathbb{P}^1} B_2$ *which admit a free action by a finite group* G *such that the induced action on* $\mathbb{P}^1$ *is non-trivial and at least one of the actions of* G *on* B_1 *and* B_2 *is special type (*f_∞ *fiber is a singular fiber of type* I_{mr}*) are as listed in Table 4. In Table 4 the numbers in brackets refer to the case numbers in Table 3 (general type actions), and the numbers not in brackets in the columns* B_1 *and* B_2 *refer to the case numbers in Table 2 (special type actions).*

Proof. The general strategy of the proof is as explained above. We consider elliptic surfaces B_1 and B_2 admitting the action of the same group G where the action on $\mathbb{P}^1$ is non-trivial and there is an isomorphism $\Delta : G \to G$ as mentioned above. At least one of B_1 and B_2 is chosen from Table 2. To have a smooth fiber product X, at least one of f_{10} and f_{20}, and at least one of $f_{1\infty}$ and $f_{2\infty}$ must be a smooth fiber (fiber of type I_0), and the automorphism $z \mapsto \mu z$ of $\mathbb{P}^1$ mentioned in the above construction must be chosen such that the singular fibers of B_1 and B_2 are not paired over the same point in the fiber product $B_1 \times_{\mathbb{P}^1} B_2$. With these remarks, the proof is completed by a case by case analysis through the Tables 2 and 3. □

Remark. The suitable pairs (B, τ) with $m > 1$ where f_∞ is not a smooth fiber were excluded from the analysis in the paper [2]. Theorem 12 displays the results when such cases are included in the analysis. When Table 4 is examined, the same groups G (except for $\mathbb{Z}_5$) appear as in the results listed in Table 11 in [2]. For each group G in Table 4, the dimension of the moduli space of the Schoen 3-folds X admitting the free action by G is strictly smaller than the dimension of the corresponding moduli space obtained in Table 11 in [2].

5.2. Free action on singular Schoen 3-folds

Theorem 13. *All singular Schoen* 3*-folds* $X = B_1 \times_{\mathbb{P}^1} B_2$ *with singularities on* $I_r \times I_s$ *type fibers (*$r > 1$*,* $s > 1$*) which admit a free action by a finite group*

Table 3: Finite groups $G \subset Aut(B)$ which consist of suitable automorphisms where $\psi(G) = \langle\alpha\rangle = \mathbb{Z}_m$ ($m > 1$) and f_∞ is of type I_0 (smooth fiber), *general type actions*. (A copy of Tables 8 and 9 in [2] with less details)

G	♯	m	d	dim	Sing. fibers	T	f_0
$\mathbb{Z}_3 \times \mathbb{Z}_3$	[9]	3	1	1	$I_3^3 I_1^3$	$A_2^{\oplus 3}$	I_0
$\mathbb{Z}_4 \times \mathbb{Z}_2$	[10]	4	1	1	$I_2^4 I_1^4$	$A_1^{\oplus 4}$	I_0
	[11]	2	2 or 1	1	$I_4^2 I_2 I_1^2$	$A_3^{\oplus 2} \oplus A_1$	I_2
	[12]	2	2	1	$I_4 I_2^4$	$A_3 \oplus A_1^{\oplus 4}$	I_4
$\mathbb{Z}_6$	[13]	6	1	1	I_1^{12}	0	I_0
	[14]	3	2 or 1	1	$III\ I_2^3 I_1^3$	$A_1^{\oplus 4}$	III
	[15]	2	3 or 1	1	$IV\ I_3^2 I_1^2$	$A_2^{\oplus 3}$	IV
$\mathbb{Z}_5$	[16]	5	1	1	$II\ I_1^{10}$	0	II
$\mathbb{Z}_4$	[17]	4	1	2	I_1^{12}	0	I_0
	[18]	4	1	1	$IV\ I_1^8$	A_2	IV
	[19]	2	2	2	$I_2^5 I_1^2$	$A_1^{\oplus 5}$	I_2
	[20]	2	2	1	$I_0^* I_2^2 I_1^2$	$D_4 \oplus A_1^{\oplus 2}$	I_0^*
$\mathbb{Z}_2 \times \mathbb{Z}_2$	[21]	2	1	3	$I_2^4 I_1^4$	$A_1^{\oplus 4}$	I_0
	[22]	2	1	2	I_2^6	$A_1^{\oplus 6}$	I_0
	[23]	2	1	2	$I_4^2 I_1^4$	$A_3^{\oplus 2}$	I_0
	[24]	2	1	2	$I_2^5 I_1^2$	$A_1^{\oplus 5}$	I_2
	[25]	2	1	2	$I_4 I_2^2 I_1^4$	$A_3 \oplus A_1^{\oplus 2}$	I_4
	[26]	2	1	1	$I_4 I_2^4$	$A_3 \oplus A_1^{\oplus 4}$	I_4
	[27]	2	1	1	$I_8 I_1^4$	A_7	I_8
	[28]	2	1	1	$I_0^* I_2^2 I_1^2$	$D_4 \oplus A_1^{\oplus 2}$	I_0^*
	[29]	2	1	0	$I_0^* I_0^*$	$D_4^{\oplus 2}$	I_0
$\mathbb{Z}_3$	[30]	3	1	3	I_1^{12}	0	I_0
	[31]	3	1	2	$I_2^3 I_1^6$	$A_1^{\oplus 3}$	I_0
	[32]	3	1	2	$III\ I_1^9$	A_1	III
	[33]	3	1	1	$I_0^* I_1^6$	D_4	I_0^*
$\mathbb{Z}_2$	[34]	2	1	5	I_1^{12}	0	I_0
	[35]	2	1	4	$I_2^2 I_1^8$	$A_1^{\oplus 2}$	I_0
	[36]	2	1	4	$I_2 I_1^{10}$	A_1	I_2
	[37]	2	1	3	$I_3^2 I_1^6$	$A_2^{\oplus 2}$	I_0
	[38]	2	1	3	$I_2^3 I_1^6$	$A_1^{\oplus 3}$	I_2
	[39]	2	1	3	$I_4 I_1^8$	A_3	I_4
	[40]	2	1	3	$IV\ I_1^8$	A_2	IV
	[41]	2	1	2	$I_3^2 I_2 I_1^4$	$A_2^{\oplus 2} \oplus A_1$	I_2
	[42]	2	1	2	$I_6 I_1^6$	A_5	I_6
	[43]	2	1	2	$IV\ I_2^2 I_1^4$	$A_2 \oplus A_1^{\oplus 2}$	IV
	[44]	2	1	2	$I_0^* I_1^6$	D_4	I_0^*
	[45]	2	1	1	$IV^* I_1^4$	E_6	IV^*

Table 4: Groups G which act on a smooth Schoen 3-fold $X = B_1 \times_{\mathbb{P}^1} B_2$ freely where the induced action on $\mathbb{P}^1$ is non-trivial and at least one of the actions on B_1 and B_2 is special type

G	B_1	B_2
$\mathbb{Z}_3 \times \mathbb{Z}_3$	1	[9],1.
$\mathbb{Z}_4 \times \mathbb{Z}_2$	2	[10],2.
$\mathbb{Z}_6$	4	[13],4.
$\mathbb{Z}_4$	2,8,9,10.	[10],[17].
	2,8,10.	2,8,10.
$\mathbb{Z}_3$	1,4,5,14–20.	[9],[13],[30],[31].
	1,4,14,16,17,20.	1,4,14,16,17,20.
$\mathbb{Z}_2 \times \mathbb{Z}_2$	2,21–29.	[10],[21],[22],[23],[29].
	2,21,24,25.	2,21,24,25.
$\mathbb{Z}_2$	2,4,6,8–10,21–53.	[10],[13],[17],[21]–[23], [29],[34],[35],[37].
	2,4,8,10,21,24,25, 30,32,33,38–40,49.	2,4,8,10,21,24,25, 30,32,33,38–40,49.

Table 5: Finite groups G which act freely on singular Schoen 3-folds $X = B_1 \times_{\mathbb{P}^1} B_2$ with non-trivial induced action on $\mathbb{P}^1$ such that the singularities of X are on $I_r \times I_s$ type fibers over points different from 0 and $\infty \in \mathbb{P}^1$

G	B_1	B_2
$\mathbb{Z}_3 \times \mathbb{Z}_3$	1,[9].	1,[9].
$\mathbb{Z}_4 \times \mathbb{Z}_2$	2,[10].	2,[10].
	[11]	[11]
	[12]	[12]
$\mathbb{Z}_6$	[14]	[14]
	[15]	[15]
$\mathbb{Z}_4$	2,[10].	2,[10].
	[11],[12],[19],[20].	[11],[12],[19],[20].
$\mathbb{Z}_3$	1,16.	1,16.
	[9],[14],[31].	[9],[14],[31].
	1,16,19.	[9],[31].
$\mathbb{Z}_2 \times \mathbb{Z}_2$	2,21,24,25.	2,21,24,25.
	[10],[11],[21]–[26],[28].	[10],[11],[21]–[26],[28].
	2,21–26,28,29.	[10],[21],[22],[23].
$\mathbb{Z}_2$	2,21,24,25,32,38,40.	2,21,24,25,32,38,40.
	2,21–26,28,29,32,36, 38,40,42,46,48,52.	[10],[21],[22],[23],[35],[37].
	[10],[11],[15],[21]–[26], [28],[35],[37],[38],[41],[43].	[10],[11],[15],[21]–[26], [28],[35],[37],[38],[41],[43].

Table 6: Finite groups G which act freely on singular Schoen 3-folds $X = B_1 \times_{\mathbb{P}^1} B_2$ with non-trivial induced action on $\mathbb{P}^1$ such that the singularities of X are on $I_r \times I_s$ type fibers over 0 and/or $\infty \in \mathbb{P}^1$

G	B_1	B_2
$\mathbb{Z}_4 \times \mathbb{Z}_2$	3	3,[12].
$\mathbb{Z}_4$	3,11,13.	3,11,13.
	3,11,12,13.	[11],[12],[19].
$\mathbb{Z}_2 \times \mathbb{Z}_2$	2,21,22,24,25,27,28.	22,27,28.
	2,21–29.	[25],[26],[27].
$\mathbb{Z}_2$	2,4,8,10,21–25, 27–33,35–40,44–49,53.	22,23,27–29,31, 35–37,44–48,53.
	2,4,6,8–10,21–53.	[11],[24]–[27],[36], [38],[39],[41],[42].

Table 7: Finite groups G which act freely on singular Schoen 3-folds $X = B_1 \times_{\mathbb{P}^1} B_2$ with non-trivial induced action on $\mathbb{P}^1$ such that the singularities of X are on $I_r \times I_s$ type fibers over 0 and/or ∞, and some other points $p \in \mathbb{P}^1$

G	B_1	B_2
$\mathbb{Z}_4 \times \mathbb{Z}_2$	3	3,[12].
$\mathbb{Z}_4$	3,11.	3,11,[11],[12],[19].
$\mathbb{Z}_2 \times \mathbb{Z}_2$	2,21,22,24,25,28.	22,28.
	2,21–26,28,29.	[25],[26].
$\mathbb{Z}_2$	2,21–25,28,29,32,36,38,40,46,48.	22,23,28,29,36,46,48.
	2,21–26,28,29,32,36,38,40,42,46,48,52.	[11],[24]–[26],[38],[41].

G such that the induced action on $\mathbb{P}^1$ is non-trivial are listed in Tables 5, 6 and 7.

If we denote the two fixed points of the induced automorphisms in $\tilde{\phi}(G)$ by $0 \in \mathbb{P}^1$ and $\infty \in \mathbb{P}^1$, Table 5 lists the cases where the singularities of X are not on the fibers over 0 and ∞, Table 6 lists the cases where X has singularities only on the fibers over 0 or ∞, and Table 7 lists the cases where X has singularities on fibers over 0 or ∞ and also on other fibers.

Proof. We consider the elliptic surfaces B_1 and B_2 chosen from Tables 2 and 3 admitting the action of the same group G where the induced action on $\mathbb{P}^1$ is non-trivial, there is an isomorphism $\Delta : G \to G$ as mentioned in the general construction given in the beginning of §5, and the fiber product $X = B_1 \times_{\mathbb{P}^1} B_2$ is singular such that all singularities are on fibers of type $I_r \times I_s$ ($r > 1$, $s > 1$). We group the results in three categories according to whether the singularities of X occur on fibers over 0 and/or $\infty \in \mathbb{P}^1$, or over other points on $\mathbb{P}^1$, or both. For pairs (B_1, B_2) where the fiber product X does not have singularities over 0 and ∞, depending on the choice of the automorphism $z \mapsto \mu z$ of $\mathbb{P}^1$ mentioned in the general construction, X may have singularities over points different from 0 and ∞. We search for the pairs (B_1, B_2) for which the singularities of X are only on fibers of type $I_r \times I_s$ with $r > 1$ and $s > 1$. For the pairs (B_1, B_2) where X has singularities on such $I_r \times I_s$ fibers over 0 or ∞, we can choose the automorphism $z \mapsto \mu z$ of $\mathbb{P}^1$ such that X does not have other singularities.

These are the pairs to be considered for Table 6. But, we may also choose the automorphism $z \mapsto \mu z$ such that X has singularities on fibers of type $I_r \times I_s$ ($r > 1$ and $s > 1$) over points different from 0 and $\infty \in \mathbb{P}^1$, and these are the cases to be considered for Table 7. In all of these cases under consideration, to conclude that the action of G on the singular 3-fold X is free, we need to check that the non-identity automorphisms of type $t_\epsilon \times t_\delta$ of X in G do not have fixed points on X. Note that the singularities of the 3-folds X under consideration are only on fibers of type $I_r \times I_s$ ($r > 1$ and $s > 1$), and $t_\epsilon \times t_\delta$ has a fixed point on X iff ϵ intersects I_r and δ intersects I_s at neutral components on one of such $I_r \times I_s$ fibers of X. With these remarks, the proof is completed by a careful case by case analysis through Tables 2 and 3 (consulting the proof of Lemma 11 for a detailed description of the elements of G if necessary). The only subtle point worth noting is the following: For the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ where X has singularities over 0 or ∞ (Table 6), and possibly over other points on $\mathbb{P}^1$ (Table 7), the choice of B_1 or B_2 as Case 23 or 29 from Table 2, or Case [11] or [24] from Table 3 does not result in a free action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on X if the other surface (B_1 or B_2) is chosen from Table 2. Suppose without loss of generality that B_1 is one of these four cases and B_2 is chosen from Table 2 such that $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts on $B_1 \times_{\mathbb{P}^1} B_2$. In this case, there is one non-identity automorphism of type $t_\epsilon \times t_\delta$ and it has a fixed point on the fiber $f_{10} \times f_{20}$ (since ϵ intersects f_{10} and δ intersects f_{20} at neutral components. Note that here f_{20} is the fiber of B_2 which is denoted as f_∞ in Table 2 as explained in the general construction in the beginning of §5). □

6. Lifting the free action to a projective small resolution of *X*

In Theorem 13 in the previous section we classified all finite groups G which act freely on a singular Schoen 3-fold $X = B_1 \times_{\mathbb{P}^1} B_2$ such that the induced action on $\mathbb{P}^1$ is non-trivial and the singularities of X are on $I_r \times I_s$ fibers with $r > 1$ and $s > 1$. Such singular Schoen 3-folds X have projective small resolutions $\hat{X}$ which are simply connected Calabi-Yau 3-folds. In this section we determine which of the free group actions classified in Theorem 13 can be lifted to a free action on a desingularization $\hat{X}$ of X by small resolutions. Note that in the case where $\beta^{-1}(p)$ is a fiber of type $I_r \times I_s$ with $r > 1$ and $s > 1$ for all $p \in S$ (see §2), the small resolutions of X are obtained by blowing up X along a sequence of divisors of the form $\theta_i \times \Gamma_j$ where θ_i and Γ_j are components of the singular fibers I_r and I_s so that the resulting 3-fold is projective (see Lemma 3.1 in [7]). There are two non-isomorphic small resolutions over Y at an ordinary double point singularity on a 3-fold Y (see §1 in [7]). For an ordinary double point (a, b) on a fiber $I_r \times I_s$ of X where a is on the components θ_i and θ_{i+1} of the singular fiber I_r of B_1 and b is on the components Γ_j and Γ_{j+1} of the fiber I_s on B_2, the small resolution at (a, b) performed by blowing up the divisor $\theta_i \times \Gamma_j$ or blowing up the divisor $\theta_{i+1} \times \Gamma_{j+1}$ are isomorphic over X. Similarly the small resolution performed by blowing up the divisor $\theta_i \times \Gamma_{j+1}$ or blowing up the divisor $\theta_{i+1} \times \Gamma_j$ are isomorphic to each other over X. But, the small resolution obtained by blowing up $\theta_i \times \Gamma_j$ or blowing up $\theta_{i+1} \times \Gamma_j$ are non-isomorphic over X. Schoen has discussed the problem of lifting an involution $\tau_1 \times \tau_2$ which acts on a singular Schoen 3-fold $X = B_1 \times_{\mathbb{P}^1} B_2$ to an automorphism of the desingularization of X by small resolutions in §6 in [7]. Generalizing the argument given by Schoen, we obtain the following criterion for lifting an automorphism $\tau_1 \times \tau_2$ of X to an automorphism of the small resolution $\hat{X}$ of X.

Lemma 14. *Let $\langle \tau_1 \times \tau_2 \rangle$ act freely on a singular Schoen 3-fold $X = B_1 \times_{\mathbb{P}^1} B_2$ where the singularities of X are on fibers of type $I_r \times I_s$ such that $r > 1$ and $s > 1$. If the orders of τ_1 and τ_2 are n and there is a component $\theta_i \times \Gamma_j$ of an $I_r \times I_s$ fiber of X such that the blow up of the divisors $\theta_i \times \Gamma_j$, $\tau_1(\theta_i) \times \tau_2(\Gamma_j)$, $\tau_1^2(\theta_i) \times \tau_2^2(\Gamma_j)$, ..., $\tau_1^{n-1}(\theta_i) \times \tau_2^{n-1}(\Gamma_j)$ in any order results in isomorphic partial resolutions $\tilde{X}$ over X, then $\langle \tau_1 \times \tau_2 \rangle$ lifts to a free action on $\tilde{X}$.*

The action of $\langle \tau_1 \times \tau_2 \rangle$ on X lifts to a free action on a desingularization $\hat{X}$ of X by small resolutions if there is a sequence of partial resolutions $\tilde{X}_k$ of X ending with $\hat{X}$ such that $\tilde{X}_{k+1}$ is isomorphic to the blow up of $\tilde{X}_k$ at the proper transforms in $\tilde{X}_k$ of the divisors $\theta_{i_k} \times \Gamma_{j_k}$, $\tau_1(\theta_{i_k}) \times \tau_2(\Gamma_{j_k})$, ..., $\tau_1^{n-1}(\theta_{i_k}) \times \tau_2^{n-1}(\Gamma_{j_k})$ in any order for some component $\theta_{i_k} \times \Gamma_{j_k}$ of a fiber $I_r \times I_s$ of X (so that the action lifts to a free action on each partial resolution $\tilde{X}_{k+1}$ step by step).

Proof. Note that when the action on X is free, the lifted action on $\hat{X}$ is free since the lift of a non-trivial automorphism maps an exceptional $\mathbb{P}^1$ of the small resolution to a different exceptional $\mathbb{P}^1$ (if a singular point Q_1 is mapped to a singular point Q_2, the exceptional $\mathbb{P}^1$ over Q_1 is mapped to the exceptional $\mathbb{P}^1$ over Q_2 in the resolution).

For a 3-fold Y and an ordinary double point $Q \in Y$ there are two small resolutions of Y at Q which are non-isomorphic over Y. The projectivized tangent cone of Y at Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Let S_1 and S_2 be two surfaces on Y which contain Q. If the tangent planes of S_1 and S_2 at Q correspond to $\mathbb{P}^1$ in the same ruling of the projectivized tangent cone $\mathbb{P}^1 \times \mathbb{P}^1$, then the small resolution of Y at Q along the surfaces S_1 and S_2 are isomorphic over Y, and if they correspond to $\mathbb{P}^1$ in different rulings of $\mathbb{P}^1 \times \mathbb{P}^1$, the small resolutions at Q are not isomorphic over Y (see §1 in [7]). Small resolution is a local operation. If there is an isomorphism between a neighborhood U of an ordinary double point Q of a 3-fold Y and a neighborhood U' of an ordinary double point Q' of a 3-fold Y' which maps a surface S containing Q to a surface S' containing Q', then the isomorphism lifts to an isomorphism from the small resolution of U along S to the small resolution of U' along S'. But if the small resolution of U' is along another surface which yields a non-isomorphic small resolution over U', then the given isomorphism between U and U' does not lift to the small resolutions.

In this lemma, we consider small resolutions of a Schoen 3-fold X obtained by blowing up X along a sequence of divisors so that the resolution is projective. Assume that blowing up the divisors in the orbit of the divisor $\theta_i \times \Gamma_j$ under the group action in any order results in isomorphic partial resolutions of X. Let $\bar{X}$ be such a partial resolution of X. Consider any ordinary double point Q_1 of X on one of the given divisors $D_k = \tau_1^k(\theta_i) \times \tau_2^k(\Gamma_j)$. The small resolution at Q_1 is isomorphic over X to the small resolution at Q_1 obtained by blowing up D_k (even if the singularity at Q_1 is resolved by blowing up another divisor before D_k, the order of blow ups does not change the isomorphism class of the resolutions). Let $\tau_1 \times \tau_2$ map Q_1 to Q_2. Similarly the small resolution of X at Q_2 is isomorphic over X to the resolution obtained by blowing up D_{k+1}. Since $\tau_1 \times \tau_2$ maps D_k to D_{k+1}, by the above argument we can conclude that the birational isomorphism $\tau_1 \times \tau_2$ of $\bar{X}$ extends to a rational map mapping E_{Q_1} (the exceptional $\mathbb{P}^1$ of the small resolution over Q_1) to E_{Q_2}. This way, $\tau_1 \times \tau_2$ extends to an isomorphism of $\bar{X}$. This completes the proof of the first statement of the lemma. The second statement is a direct consequence of the first statement. □

The groups which act on a singular Schoen 3-fold listed in Theorem 13 are cyclic or abelian with two generators. In the former case, Lemma 14 directly applies in order to check the lifting to a free action on the resolution.

In the latter case, in order to conclude that the whole group action lifts to a free action on the resolution, it suffices to check that the action of each of the two generators separately lifts to a free action on the same resolution since the commutator of the lifted automorphisms is the lift of the identity, hence the lifted automorphisms commute.

6.1. Lifting the actions listed in Table 6

For the cases listed in Table 6, the only singularities of X are on the fibers over 0 or $\infty \in \mathbb{P}^1$, i.e., the fiber over 0 or ∞ (or both) is of the form $I_r \times I_s$ with $r > 1$ and $s > 1$. One of the I_r and I_s fibers is f_0 of the surfaces B_1 or B_2 and the other fiber is the f_∞ of the other surface (recall that in $X = B_1 \times_{\mathbb{P}^1} B_2$ the fiber f_0 and f_∞ of B_2 is denoted as $f_{2\infty}$ and f_{20} respectively as fibers over ∞ and 0 due to the change of coordinates on the base curve $\mathbb{P}^1$ of B_2 as explained in §5). To simplify the notation in the discussion, we will always consider the first fiber I_r in $I_r \times I_s$ fiber as the f_0 fiber of one of the surfaces B_1 and B_2, and the second fiber I_s as the f_∞ fiber of the other surface (that is, for the fiber $f_{1\infty} \times f_{2\infty}$ we will abuse the notation and denote it by $f_{2\infty} \times f_{1\infty}$ interchanging the positions of the fibers so that the one on the left is f_0 fiber of one of the surfaces). We will denote the components of I_r by θ_i, $0 \le i < r$ and the components of I_s by Γ_j, $0 \le j < s$.

• **Lifting the $\mathbb{Z}_4$ action:** Let $\tau = t_\epsilon \circ \alpha$ be a generator of $\mathbb{Z}_4$. In any of the cases, ϵ intersects f_∞ at an order 4 component. Hence, after renaming the components of $f_\infty = I_s$ if necessary, τ maps Γ_i to Γ_{i+1} if $f_\infty = I_4$ and maps Γ_i to Γ_{i+2} if $f_\infty = I_8$. For the cases $f_0 = I_2$, τ maps θ_i to θ_{i+1} (since ϵ intersects f_0 at θ_1). For the cases $f_0 = I_4$, τ interchanges θ_0 and θ_1, and interchanges θ_2 and θ_3. If $I_r \times I_s = I_2 \times I_8$, the orbit of the divisor $\theta_0 \times \Gamma_0$ under the $\mathbb{Z}_4$ action is $\theta_0 \times \Gamma_0 \to \theta_1 \times \Gamma_2 \to \theta_0 \times \Gamma_4 \to \theta_1 \times \Gamma_6$, and the action lifts to a free action on the resolution of X obtained by blowing up all of these divisors by Lemma 14 (Note that blowing up the given four divisors resolves all singularities on this $I_2 \times I_8$ fiber). Similarly, for $I_r \times I_s = I_2 \times I_4$ blowing up the divisors in the orbit $\theta_0 \times \Gamma_0 \to \theta_1 \times \Gamma_1 \to \theta_0 \times \Gamma_2 \to \theta_1 \times \Gamma_3$ resolves all singularities on $I_2 \times I_4$ fiber and the action lifts to this partial resolution by Lemma 14. For $I_r \times I_s = I_4 \times I_4$ we need to blow up the divisors in the two orbits $\theta_0 \times \Gamma_0 \to \theta_1 \times \Gamma_1 \to \theta_0 \times \Gamma_2 \to \theta_1 \times \Gamma_3$ and $\theta_2 \times \Gamma_0 \to \theta_3 \times \Gamma_1 \to \theta_2 \times \Gamma_2 \to \theta_3 \times \Gamma_3$. And lastly, for $I_r \times I_s = I_4 \times I_8$, we need to blow up the divisors in the orbits $\theta_0 \times \Gamma_0 \to \theta_1 \times \Gamma_2 \to \theta_0 \times \Gamma_4 \to \theta_1 \times \Gamma_6$ and $\theta_2 \times \Gamma_0 \to \theta_3 \times \Gamma_2 \to \theta_2 \times \Gamma_4 \to \theta_3 \times \Gamma_6$. In all cases the resulting partial resolutions are isomorphic no matter in which order the divisors are blown up in each orbit. The blow up of all the listed divisors resolves all singularities on the given $I_r \times I_s$ fiber and the action lifts to the partial resolutions obtained. Since in all cases the action lifts to the partial resolutions, the action lifts to the full resolution when singularities on both fibers over 0 and ∞ are resolved.

In the remaining cases, the arguments will be similar to the discussion in the above paragraph. We will only give the orbits of divisors that need to be blown up to resolve the singularities on the given $I_r \times I_s$ fiber so that the action lifts to the partial resolution for this fiber.

• **Lifting the $\mathbb{Z}_4 \times \mathbb{Z}_2$ action:** For this action we consider the cases 3×3 and $3 \times [12]$ from Table 6. In any of these cases, we have $I_r \times I_s = I_4 \times I_4$. We have shown above that $\mathbb{Z}_4$ action lifts to a free action on the full resolution of X if we blow up the following divisors in the two orbits for each $I_r \times I_s$ fiber: $\theta_0 \times \Gamma_0 \to \theta_1 \times \Gamma_1 \to \theta_0 \times \Gamma_2 \to \theta_1 \times \Gamma_3$ and $\theta_2 \times \Gamma_0 \to \theta_3 \times \Gamma_1 \to \theta_2 \times \Gamma_2 \to \theta_3 \times \Gamma_3$. The second generator of $\mathbb{Z}_4 \times \mathbb{Z}_2$ is an automorphism of the form t_η where η is an order 2 section of B_i. This generator t_η sends each component Γ_i of $f_\infty = I_4$ to itself or to Γ_{i+2} in Case 3 depending on whether η intersects f_∞ at the component Γ_0 or Γ_2. In both of the Cases 3 and [12], t_η acts on $f_0 = I_r = I_4$ by mapping θ_i to θ_{i+2}. The 8 divisors listed above are permuted under the action by $t_{\eta_1} \times t_{\eta_2}$ on X. Since the blow ups of the 8 listed divisors in any order result in isomorphic resolutions, the second generator of the $\mathbb{Z}_4 \times \mathbb{Z}_2$ also lifts to the same resolution $\hat{X}$. Therefore, the $\mathbb{Z}_4 \times \mathbb{Z}_2$ action lifts to a free action on $\hat{X}$.

• **Lifting the $\mathbb{Z}_2$ action:** For all cases in Table 6, r and s are both even whenever X has a fiber of the form $I_r \times I_s$ over 0 or $\infty \in \mathbb{P}^1$. Let $r = 2a$ and $s = 2b$. If τ is the order two automorphism of B_1 or B_2, then τ acts on $f_\infty = I_s = I_{2b}$ by mapping the component Γ_j to Γ_{j+b}. The action of τ on the components of $f_0 = I_r = I_{2a}$ is one of two types. Type 1 action is by mapping two components θ_c and θ_{c+a} to themselves for some c and by interchanging θ_{c+i} and θ_{c-i} for each $1 \leq i < a$ (the subindices are considered modulo $r = 2a$ for components θ_j of f_0). Type 2 action on $f_0 = I_{2a}$ is by interchanging the components θ_{c+i} and θ_{c-1-i} for each $0 \leq i < a$ for some c. Note that $\tau = t_\epsilon \circ \alpha$ and α maps θ_i to θ_{-i}, and t_ϵ maps θ_i to θ_{i+d} for each $0 \leq i < r = 2a$ where the section ϵ intersects the fiber f_0 at the component θ_d. Whether the action of τ on f_0 is of type 1 or type 2 depends on d being even or odd.

Let $I_r \times I_s = I_{2a} \times I_2$ where $a > 1$. If we resolve all singularities on this $I_r \times I_s$ fiber by blowing up the divisors in the sequence of $\mathbb{Z}_2$ orbits $\theta_{c+i} \times \Gamma_0 \to \theta_{c-i} \times \Gamma_1$ where $1 \leq i < a$ in the type 1 action case, then the $\mathbb{Z}_2$ action lifts to this partial resolution. Similarly, for type 2 action case, we need to blow up the divisors in the sequence of $\mathbb{Z}_2$ orbits given by $\theta_{c+i}\times\Gamma_0 \to \theta_{c-1-i}\times\Gamma_1$ for $0 \leq i < a$ so that the $\mathbb{Z}_2$ action lifts to this partial resolution. Note that the order of divisors chosen in any of these orbits does not change the resolution obtained.

Let now $I_r\times I_s = I_{2a}\times I_{2b}$ where $a \geq 1$ and $b > 1$. No matter the action on f_0 is of type 1 or type 2, the $\mathbb{Z}_2 = \langle\tau_1\times\tau_2\rangle$ action lifts to the partial resolution obtained by blowing up the divisors in the following sequence of $\mathbb{Z}_2$ orbits: $\theta_0\times\Gamma_0 \to \tau_1(\theta_0)\times\Gamma_b$, $\theta_2\times\Gamma_0 \to \tau_1(\theta_2)\times\Gamma_b, ..., \theta_{2a-2}\times\Gamma_0 \to \tau_1(\theta_{2a-2})\times\Gamma_b$, $\theta_0 \times \Gamma_1 \to \tau_1(\theta_0) \times \Gamma_{b+1}, ..., \theta_{2a-2} \times \Gamma_1 \to \tau_1(\theta_{2a-2}) \times \Gamma_{b+1}, ..., \theta_0 \times \Gamma_{b-2} \to \tau_1(\theta_0)\times\Gamma_{2b-2}, ..., \theta_{2a-2}\times\Gamma_{b-2} \to \tau_1(\theta_{2a-2})\times\Gamma_{2b-2}$. In short we blow up the divisors in the $\mathbb{Z}_2$ orbits containing the divisors $\theta_{2i} \times \Gamma_j$ where $0 \leq i < a$ and $0 \leq j \leq b-2$. The blow up of these divisors resolve all singularities on this $I_r \times I_s$ fiber.

Let $I_r \times I_s = I_2 \times I_2$. In all cases in Table 6 with $G = \mathbb{Z}_2$ and $f_0 = I_2$, the action of τ on $f_0 = I_2$ is by mapping θ_i to itself for $i = 0, 1$. Then the $\mathbb{Z}_2$ orbits we obtain are $\theta_0 \times \Gamma_0 \to \theta_0 \times \Gamma_1$ and $\theta_1 \times \Gamma_0 \to \theta_1 \times \Gamma_1$, and in any of these two orbits the blow ups of the divisors in different orders result in non-isomorphic partial resolutions. Blowing up any of these four divisors resolves all singularities on the $I_2 \times I_2$ fiber. If we blow up $\theta_0 \times \Gamma_0$ to resolve all singularities on $I_2 \times I_2$, then for an ordinary double point Q_1 on $I_2 \times I_2$ which is mapped to Q_2 by $\tau_1\times\tau_2$ this map extends locally to a map from the small resolution of X at Q_1 along $\theta_0\times\Gamma_0$ to the small resolution of X at Q_2 along $\theta_0 \times \Gamma_1$. But, the singularity at Q_2 has already been resolved by the blow up of $\theta_0 \times \Gamma_0$ which gives a small resolution at Q_2 non-isomorphic to the small resolution at Q_2 along $\theta_0 \times \Gamma_1$. This means that the map does not lift to the given partial resolution obtained by blowing up $\theta_0 \times \Gamma_0$. A similar argument works if we blow up a different divisor to resolve the singularities on $I_2 \times I_2$. Therefore, the $\mathbb{Z}_2$ action does not lift to the resolution of X if X contains a fiber of type $I_2 \times I_2$ over 0 or ∞.

• **Lifting the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action:** In the singular Schoen 3-folds X listed in Table 6 for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, for the $I_r \times I_s$ fibers over 0 or $\infty \in \mathbb{P}^1$ we have $f_0 = I_r$ is either I_4 or I_8 and $f_\infty = I_s$ is either I_2 or I_4. $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ has two order 2 generators τ and t_η where $m = 2$ for τ and $m = 1$ for t_η. The action of τ on $f_\infty = I_s$ is by mapping Γ_i to $\Gamma_{i+s/2}$ and the action of t_η on I_s is by mapping Γ_i to itself for all i. The action of t_η on $f_0 = I_r$ is by mapping θ_i to $\theta_{i+r/2}$ for all i. In all the cases we consider, we can choose the order 2 generator τ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ such that its action on $f_0 = I_r$ is as follows (τ or the other automorphism with $m = 2$, namely $t_\eta \circ \tau$ has the given action on I_r): For $I_r = I_4$, θ_0 and θ_2 are interchanged, θ_1 and θ_3 are mapped to themselves, and for $I_r = I_8$ θ_1 and θ_5 are mapped to themselves and θ_{5+i} and θ_{5-i} are interchanged for $i = 1, 2, 3$.

Let $I_r \times I_s = I_4 \times I_4$. Blowing up the divisors in the following orbits of $\langle \tau_1 \times \tau_2 \rangle$ resolves all singularities on $I_r \times I_s$: $\theta_0 \times \Gamma_0 \to \theta_2 \times \Gamma_2$, $\theta_0 \times \Gamma_2 \to \theta_2 \times \Gamma_0$. The action of $t_{\eta_1} \times t_{\eta_2}$ permutes these divisors and blowing up all four divisors in any order results in the same partial resolution. Thus, $\mathbb{Z}_2 \times \mathbb{Z}_2$ action lifts to this partial resolution.

Using the same argument as above, we can lift the action to the partial resolution of singularities in $I_8 \times I_2$ and $I_8 \times I_4$ cases blowing up the following divisors (in any order): For $I_8 \times I_2$ blow up $\theta_0 \times \Gamma_0$, $\theta_2 \times \Gamma_1$, $\theta_4 \times \Gamma_0$, $\theta_6 \times \Gamma_1$, and for $I_8 \times I_4$ we blow up $\theta_0 \times \Gamma_0$, $\theta_2 \times \Gamma_2$, $\theta_0 \times \Gamma_2$, $\theta_2 \times \Gamma_0$, $\theta_4 \times \Gamma_0$, $\theta_6 \times \Gamma_2$, $\theta_4 \times \Gamma_2$, $\theta_6 \times \Gamma_0$.

For $I_r \times I_s = I_4 \times I_2$, in order to lift the action of $\langle \tau_1 \times \tau_2 \rangle$, the divisor blown up first must be one of $\theta_0 \times \Gamma_i$ and $\theta_2 \times \Gamma_i$, $i = 1, 2$ (Assume on the contrary that the divisor blown up first is $\theta_1 \times \Gamma_i$. $\tau_1 \times \tau_2$ maps $\theta_1 \times \Gamma_i$ to $\theta_1 \times \Gamma_{i+1}$, hence permutes the four ordinary double points on $\theta_1 \times \Gamma_i$. The small resolutions at these four points are along $\theta_1 \times \Gamma_i$, but in order to lift $\tau_1 \times \tau_2$ to the resolution, the small resolution at the image points should be along $\theta_1 \times \Gamma_{i+1}$ which is a non-isomorphic small resolution to the small resolution along $\theta_1 \times \Gamma_i$, contradiction. A similar argument works if the divisor blown up first is $\theta_3 \times \Gamma_i$). Assume that the divisor blown up first is $\theta_0 \times \Gamma_i$ (a similar argument works for $\theta_2 \times \Gamma_i$). If $\tau_1 \times \tau_2$ lifts to a partial resolution, then the small resolution of X at the four ordinary double points on the divisor $\theta_2 \times \Gamma_{i+1}$ (which are the images of the other 4 ordinary double points on $\theta_0 \times \Gamma_i$) are isomorphic to the small resolutions along $\theta_2 \times \Gamma_{i+1}$ since $\tau_1 \times \tau_2$ maps $\theta_0 \times \Gamma_i$ to $\theta_2 \times \Gamma_{i+1}$. By the same reasoning, if $t_{\eta_1} \times t_{\eta_2}$ also lifts to the same partial resolution of X, this time the small resolutions at the four ordinary double points on the divisor $\theta_2 \times \Gamma_{i+1}$ (or equivalently on $\theta_2 \times \Gamma_i$) are isomorphic to the small resolutions along $\theta_2 \times \Gamma_i$, which gives a contradiction since such small resolutions are not isomorphic to the small resolutions along $\theta_2 \times \Gamma_{i+1}$ at these four points. Therefore, if X contains an $I_4 \times I_2$ fiber over 0 or $\infty \in \mathbb{P}^1$, then the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action does not lift to a free action on its resolution.

We can combine the information obtained in this section in the following lemma:

Lemma 15. *For the cases listed in Table 6 the free action of $G = \mathbb{Z}_2$ on X does not lift to a free action on a projective small resolution of X if there exists an $I_2 \times I_2$ fiber over 0 or $\infty \in \mathbb{P}^1$. The free action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on X does not lift to a free action on a projective small resolution of X if there*

Table 8: The cases in Table 6 where the action does not lift

G	B_1	B_2
$\mathbb{Z}_2 \times \mathbb{Z}_2$	21–23,25–27,29.	[25],[26].
	22,28.	27
	21,22,25,27,28.	22
	21,22,25,27.	28
$\mathbb{Z}_2$	21–23,25–27,29–32,34–36, 38,41,42,44,46,50,52.	[11],[24],[36],[38],[41].
	21–23,25,27,29–32,35,36, 38,44,46.	23,29,31,36,37,46,47, 48,53.
	37,47,48,53.	23,29,31,36,46.
	23,29,31,36,37,46,47,48,53.	22,27,35,44.

exists an $I_4 \times I_2$ or $I_2 \times I_4$ fiber over 0 or $\infty \in \mathbb{P}^1$. In all other cases in Table 6 such a lifting exists. The cases in Table 6 where the action does not lift are listed in Table 8.

6.2. Lifting the action listed in Tables 5 and 7

The singular Schoen 3-folds X listed in Table 5 have singularities on fibers of type $I_r \times I_s$ ($r > 1$ and $s > 1$ throughout this section) over points $p \in \mathbb{P}^1$ where p is different from 0 and $\infty \in \mathbb{P}^1$. The singular Schoen 3-folds X listed in Table 7 have $I_r \times I_s$ fibers over 0 or $\infty \in \mathbb{P}^1$ as well as over other points $p \in \mathbb{P}^1$. For $X = B_1 \times_{\mathbb{P}^1} B_2$ in Table 7, the pair (B_1, B_2) also appears in Table 6. For such a pair, we identify the base curves $\mathbb{P}^1$ of the elliptic surfaces B_1 and B_2 by an automorphism of $\mathbb{P}^1$ (which interchanges 0 and ∞ as explained in §5) in order to form the fiber product X. If the only $I_r \times I_s$ fibers occur over 0 or ∞, then X is listed in Table 6. If the fiber product X also has $I_r \times I_s$ fibers over points $p \in \mathbb{P}^1$ different from 0 and ∞, then X is listed in Table 7. In the previous section, we checked whether the action lifts to a resolution of the 3-folds X in Table 6 or not. For a pair (B_1, B_2) in Table 7, the action lifts to a partial resolution of the singularities where all singularities over 0 and ∞ are resolved if the action lifts for the 3-fold X in Table 6 corresponding to the same pair (B_1, B_2) (it suffices to blow up the same divisors as indicated in §6.1). If the action does not lift for X in Table 6 corresponding to the pair (B_1, B_2), then the action does not lift for X in Table 7 due to the same reason as indicated in §6.1. In this section we check whether the action lifts when the singularities on $I_r \times I_s$ fibers over $p \in \mathbb{P}^1$ different from 0 and ∞ are resolved.

• **Lifting the $\mathbb{Z}_m$ action:** Let $G = \mathbb{Z}_m = \langle\tau\rangle$ where $m = ord(\psi(\tau)) = ord(\alpha)$, that is, $d = 1$ for the generator τ of G. For an $I_r \times I_s$ fiber of X over a point different from 0 or ∞, X has m distinct copies of $I_r \times I_s$ fibers permuted by the action of G. Choose divisors $D_i = \theta_{a_i} \times \Gamma_{b_i}$ on one of the $I_r \times I_s$ fibers such that blowing up all divisors D_i resolves the singularities on this $I_r \times I_s$ fiber. If K_i consists of the divisors in the orbit of D_i under the G action, then blowing up the divisors in K_i in any order will result in the same partial resolution (each divisor in K_i is on a distinct fiber), hence we can lift the G action to the partial resolution obtained by blowing up the divisors in K_i. Therefore, the G action lifts to the partial resolution where all singularities on the m copies of $I_r \times I_s$ are resolved if we complete the sequence of partial resolutions where in the ith step the divisors in K_i are blown up. This argument covers $\mathbb{Z}_2$, $\mathbb{Z}_3$ actions and $m = 4$ cases for the $\mathbb{Z}_4$ action in the Tables 5 and 7.

• **Lifting the $\mathbb{Z}_4$ action for $m = 2$ case:** The $I_r \times I_s$ fibers over points different from 0 and ∞ we need to consider for this action are $I_2 \times I_2$, $I_2 \times I_4$ and $I_4 \times I_4$. If $G = \mathbb{Z}_4 = \langle\tau\rangle$, then $\tau^2 = t_\eta$ for an order 2 section η. The action of t_η on an I_2 fiber is by mapping θ_i to θ_{i+1} if there is an I_2 fiber over a point different from 0 and ∞. Similarly, if there is an I_4 fiber over such a point, then the action of t_η is by mapping θ_i to θ_{i+2}. If X has an $I_4 \times I_4$ fiber over a point different from 0 and ∞, to lift the $\mathbb{Z}_4$ action to a partial resolution where the singularities of the two $I_4 \times I_4$ fibers in the same orbit are resolved, we need to blow up the divisors in the two orbits of $\theta_0 \times \Gamma_0$ and $\theta_0 \times \Gamma_2$. These orbits are $\theta_0 \times \Gamma_0 \to \tilde{\theta}_i \times \tilde{\Gamma}_j \to \theta_2 \times \Gamma_2 \to \tilde{\theta}_{i+2} \times \tilde{\Gamma}_{j+2}$ and $\theta_0 \times \Gamma_2 \to \tilde{\theta}_i \times \tilde{\Gamma}_{j+2} \to \theta_2 \times \Gamma_0 \to \tilde{\theta}_{i+2} \times \tilde{\Gamma}_j$. Note that the blow up of these divisors in any order results in the same resolution and all singularities on the two $I_4 \times I_4$ fibers are resolved (here $\tilde{\theta}_i$ and $\tilde{\Gamma}_j$ denote the components of the I_4 fibers in the second $I_4 \times I_4$ in the orbit). Similarly we can lift the $\mathbb{Z}_4$ action in the case of an $I_2 \times I_4$ or $I_2 \times I_2$ by blowing up the divisors in the orbit of $\theta_0 \times \Gamma_0$.

• **Lifting the $\mathbb{Z}_6$ action:** For the $[14] \times [14]$ case we have $m = 3$ and for a generator τ of $G = \mathbb{Z}_6$, $\tau^3 = t_\eta$ for an order 2 section η. The action of t_η on the I_2 fibers is by mapping θ_i to θ_{i+1}. For the $[15] \times [15]$ case we have $m = 2$ and $\tau^2 = t_\eta$ where η is an order 3 section. The action of t_η on the I_3 fibers is by mapping θ_i to θ_{i+1} or to θ_{i+2}. In both cases, the action lifts if we blow up the divisors in the orbit of $\theta_0 \times \Gamma_0$ (the same argument as given for $\mathbb{Z}_4$ action above works).

• **Lifting the $\mathbb{Z}_3 \times \mathbb{Z}_3$ action:** Without loss of generality, we can choose generators $\tau_1 \times \tau_2$ and $t_{\eta_1} \times t_{\eta_2}$ of $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ such that for an appropriate labeling of the components of the I_3 fibers the action of these generators on the components of the three $I_3 \times I_3$ fibers is as follows: $\tau_1 \times \tau_2$ orbits are $\theta_i \times \Gamma_j \to \tilde{\theta}_{i+2} \times \tilde{\Gamma}_{j+2} \to \bar{\theta}_{i+2} \times \bar{\Gamma}_{j+2}$ for $0 \leq i, j \leq 2$ (here subindex 0 corresponds to neutral components, θ_i, $\tilde{\theta}_i$ and $\bar{\theta}_i$ denote the components of I_3 fibers of B_1 over the first, second and third points on $\mathbb{P}^1$). The action of $t_{\eta_1} \times t_{\eta_2}$ on the first $I_3 \times I_3$ is by mapping $\theta_i \times \Gamma_j$ to $\theta_{i+1} \times \Gamma_{j+1}$ (similarly for the second and third $I_3 \times I_3$). With this notation, it can be checked that blowing up the 9 divisors in the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbit of $\theta_0 \times \Gamma_0$ in any order results in the same resolution and all singularities are resolved. The action lifts to this resolution.

• **Lifting the $\mathbb{Z}_4 \times \mathbb{Z}_2$ action:** For the cases 2×2, $2 \times [10]$ and $[10] \times [10]$, we have $m = 4$ and X has four $I_2 \times I_2$ fibers. The action of the order two generator $t_{\eta_1} \times t_{\eta_2}$ on the first $I_2 \times I_2$ is by mapping $\theta_i \times \Gamma_j$ to $\theta_{i+1} \times \Gamma_{j+1}$ and the action on the other 3 $I_2 \times I_2$ is similar. The $\mathbb{Z}_4 \times \mathbb{Z}_2$ action lifts to the resolution obtained by blowing up the divisors in the orbit of $\theta_0 \times \Gamma_0$.

For the three cases 3×3, $3 \times [12]$ and $[12] \times [12]$, we have $m = 2$, hence there is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup consisting of automorphisms of the form $t_{\eta_i} \times t_{\eta_j}$ where η_i and η_j are torsion sections of B_1 and B_2. If the action is free, then one of such automorphisms acts on the first $I_2 \times I_2$ fiber by mapping $\theta_i \times \Gamma_j$ to $\theta_i \times \Gamma_{j+1}$ for each $0 \leq i, j \leq 1$. The two orbits of the action of this order 2 automorphism are $\theta_0 \times \Gamma_0 \to \theta_0 \times \Gamma_1$ and $\theta_1 \times \Gamma_0 \to \theta_1 \times \Gamma_1$, and no matter which orbit is chosen, the blow ups of the divisors in the orbit in different orders result in non-isomorphic partial resolutions. The order 2 automorphism under question does not lift to any partial resolution resolving the singularities of this $I_2 \times I_2$ fiber (by the notation of §6.3, this automorphism has intersection numbers $(0, 1)$, hence it does not lift. See §6.3 for a detailed explanation). Therefore, the action in these three cases does not lift to a resolution of X.

For the case $[11] \times [11]$, we have $m = 2$ and one of the order four generators is of the form $t_{\eta_1} \times t_{\eta_2}$ whose action on the first $I_4 \times I_4$ fiber is by mapping $\theta_i \times \Gamma_j$ to $\theta_{i+1} \times \Gamma_{j+1}$ for each $0 \leq i, j \leq 3$ and the action is similar on the second $I_4 \times I_4$. $\mathbb{Z}_4 \times \mathbb{Z}_2$ is generated by $t_{\eta_1} \times t_{\eta_2}$ and $\tau_1 \times \tau_2$ where $m = 2$ and $d = 1$ for each of the automorphisms τ_1 and τ_2. It can be checked that the $\mathbb{Z}_4 \times \mathbb{Z}_2$ action lifts to the resolution obtained by blowing up the 8 divisors in the orbit of $\theta_0 \times \Gamma_0$ under this action. Note that the order in which these 8 divisors are blown up does not change the resolution obtained.

• **Lifting the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action:** For $I_r \times I_s = I_2 \times I_2$, if the action of $t_{\eta_1} \times t_{\eta_2}$ on the components of $I_2 \times I_2$ is by mapping $\theta_i \times \Gamma_j$ to $\theta_{i+1} \times \Gamma_{j+1}$, then the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action lifts to the partial resolution obtained by blowing up the 4 divisors in the orbit of $\theta_0 \times \Gamma_0$. Note that in the cases [22] and [26], t_η acts on some of the I_2 fibers by mapping θ_i to itself. If one of the I_2 fibers of

Table 9: Torsion subgroups G of Mordell-Weil groups of rational elliptic surfaces with section

G	T	Sing. fibers	G	T	Sing. fibers
$\mathbb{Z}_3 \times \mathbb{Z}_3$	$A_2^{\oplus 4}$	I_3^4	$\mathbb{Z}_2$	$A_1^{\oplus 4}$	$I_2^4 I_1^4$
$\mathbb{Z}_4 \times \mathbb{Z}_2$	$A_3^{\oplus 2} \oplus A_1^{\oplus 2}$	$I_4^2 I_2^2$		$A_3 \oplus A_1^{\oplus 2}$	$I_4 I_2^2 I_1^4$
$\mathbb{Z}_6$	$A_5 \oplus A_2 \oplus A_1$	$I_6 I_3 I_2 I_1$		$A_1^{\oplus 5}$	$I_2^5 I_1^2$
$\mathbb{Z}_5$	$A_4^{\oplus 2}$	$I_5^2 I_1^2$		$A_5 \oplus A_1$	$I_6 I_2 I_1^4$
$\mathbb{Z}_4$	$A_3^{\oplus 2} \oplus A_1$	$I_4^2 I_2 I_1^2$		$D_4 \oplus A_1^{\oplus 2}$	$I_0^* I_2^2 I_1^2$
	$A_7 \oplus A_1$	$I_8 I_2 I_1^2$		$A_3^{\oplus 2}$	$I_4^2 I_1^4$
	$D_5 \oplus A_3$	$I_1^* I_4 I_1$		$A_3 \oplus A_1^{\oplus 3}$	$I_4 I_2^3 I_1^2$
$\mathbb{Z}_3$	$A_2^{\oplus 3}$	$I_3^3 I_1^3$		$A_2 \oplus A_1^{\oplus 4}$	$I_3 I_2^4 I_1$
	$A_5 \oplus A_2$	$I_6 I_3 I_1^3$		A_7	$I_8 I_1^4$
	$A_2^{\oplus 3} \oplus A_1$	$I_3^3 I_2 I_1$		$D_6 \oplus A_1$	$I_2^* I_2 I_1^2$
	A_8	$I_9 I_1^3$		$D_5 \oplus A_1^{\oplus 2}$	$I_1^* I_2^2 I_1$
	$E_6 \oplus A_2$	$IV^* I_3 I_1$		$A_5 \oplus A_1^{\oplus 2}$	$I_6 I_2^2 I_1^2$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$A_1^{\oplus 6}$	I_2^6		$D_4 \oplus A_3$	$I_0^* I_4 I_1^2$
	$D_4 \oplus A_1^{\oplus 3}$	$I_0^* I_2^3$		$A_3 \oplus A_2 \oplus A_1^{\oplus 2}$	$I_4 I_3 I_2^2 I_1$
	$A_3 \oplus A_1^{\oplus 4}$	$I_4 I_2^4$		D_8	$I_4^* I_1^2$
	$D_6 \oplus A_1^{\oplus 2}$	$I_2^* I_2^2$		$E_7 \oplus A_1$	$III^* I_2 I_1$
	$D_4^{\oplus 2}$	$I_0^* I_0^*$			

$I_2 \times I_2$ has such an action by t_η (if $t_{\eta_1} \times t_{\eta_2}$ has intersection numbers $(0,1)$ or $(1,0)$ in the notation of §6.3), then the action does not lift (see §6.3 for a detailed explanation). In all cases where $I_r \times I_s = I_2 \times I_4$, the action lifts to the partial resolution obtained by blowing up the divisors in the orbit of $\theta_0 \times \Gamma_0$. In all cases where $I_r \times I_s = I_4 \times I_4$, the action lifts to the partial resolution if the divisors in the orbits of $\theta_0 \times \Gamma_0$ and $\theta_0 \times \Gamma_2$ are blown up.

Combining the information obtained in this section we have proved the following lemma:

Lemma 16. *For the cases listed in Table 7, if the pair* (B_1, B_2) *corresponding to a group* G *also appears in Table 8, then the free action of* G *on* X *does not lift to a free action on a projective small resolution of* X*. For the cases listed in Table 5 and Table 7 the free action of* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *on* X *does not lift to a projective small resolution of* X *in the cases where* B_1 *or* B_2 *is one of the cases [22] and [26] and* X *has an* $I_2 \times I_2$ *fiber on which the action of the non-trivial automorphism of the form* $t_{\eta_1} \times t_{\eta_2}$ *is not by mapping* $\theta_i \times \Gamma_j$ *to* $\theta_{i+1} \times \Gamma_{j+1}$*. The* $\mathbb{Z}_4 \times \mathbb{Z}_2$ *action does not lift in the cases* 3×3, $3 \times [12]$ *and* $[12] \times [12]$*. In all other cases in Table 5 and Table 7 the free action of* G *on* X *lifts to a free action on a projective small resolution of* X*.*

6.3. Lifting the free action in the trivial induced action case

Up to now we considered the free action of a finite group G on a singular Schoen 3-fold where the induced action of G on the base curve $\mathbb{P}^1$ is non-trivial or equivalently the induced automorphism group $\tilde{\phi}(G)$ is $\mathbb{Z}_m$ for some $m > 1$. In this section we consider the groups G acting freely on a singular Schoen 3-fold $X = B_1 \times_{\mathbb{P}^1} B_2$ such that the induced action on $\mathbb{P}^1$ is trivial and the singularities of X are on fibers of type $I_r \times I_s$ where $r > 1$ and $s > 1$. The elements of such a group G are of the form $t_{\eta_1} \times t_{\eta_2}$ where $\eta_i \in MW(B_i)$, $i = 1, 2$ (recall that $t-\eta_i$ is the automorphism of B_i which acts as the translation by the section η_i). The rational elliptic surfaces for which the Mordell-Weil group has non-trivial torsion subgroups can be listed by scanning through the table of Mordell-Weil lattices given in [6]. For completeness we include the table of non-trivial torsion subgroups of Mordell-Weil groups of rational elliptic surfaces here as Table 9. In this table we indicate the root lattice T corresponding to the singular fibers of the elliptic surface and the generic configuration of singular fibers for surfaces with the given root lattice T.

In the below discussion when we say that $t_{\eta_1} \times t_{\eta_2}$ has intersection numbers (a, b), we mean that η_1 intersects I_r at the component θ_a and η_2 intersects the fiber I_s at the component Γ_b. In this case the action of $t_{\eta_1} \times t_{\eta_2}$ on $I_r \times I_s$ is by mapping the component $\theta_i \times \Gamma_j$ to $\theta_{i+a} \times \Gamma_{j+b}$. For a free action on $I_r \times I_s$, every element of G should have distinct intersection numbers, otherwise a non-trivial element has intersection numbers $(0, 0)$ which implies that this element has a fixed point, contradicting the action being free. As a result we obtain $|G| \leq rs$. If G contains elements with intersection numbers $(0, 1)$, $(1, 0)$, $(0, s-1)$ or $(r-1, 0)$, then the action of G does not lift to any partial resolution where the singularities on this fiber $I_r \times I_s$ are resolved by blowing up some components of $I_r \times I_s$. We explain why such an automorphism with intersection numbers $(0, 1)$ does not lift, and a similar argument works for the other three intersection numbers. Assume that $t_{\eta_1} \times t_{\eta_2}$ has intersection numbers $(0, 1)$ on an $I_r \times I_s$ fiber and assume that it lifts to an automorphism of a partial resolution $\bar{X}$ of X by small resolutions where the singularities on $I_r \times I_s$ are resolved by blowing up some components of $I_r \times I_s$. Assume that to obtain the partial resolution $\bar{X}$, the divisor which is blown up first is $\theta_i \times \Gamma_j$. Let a_1 and a_2 be the singular points of I_r on the component θ_i and let b_1 and b_2 be the singular points of I_s on the component Γ_j such that b_2 is on the component Γ_{j+1}. Since by assumption the intersection numbers is $(0, 1)$, $t_{\eta_1} \times t_{\eta_2}$ maps $\theta_i \times \Gamma_j$ to $\theta_i \times \Gamma_{j+1}$, and it maps the point $Q_1 = (a_1, b_1)$ on $\theta_i \times \Gamma_j$ to the point $Q_2 = (a_1, b_2)$ which is also on $\theta_i \times \Gamma_j$. Thus, blowing up the divisor $\theta_i \times \Gamma_j$ resolves the singularities at

Q_1 and Q_2, and the small resolutions at Q_1 and Q_2 are isomorphic to the small resolutions along $\theta_i \times \Gamma_j$. On the other hand, if $t_{\eta_1} \times t_{\eta_2}$ lifts to an automorphism of $\bar{X}$, then since it maps $\theta_i \times \Gamma_j$ to $\theta_i \times \Gamma_{j+1}$ and Q_1 to Q_2, we can conclude that the small resolution at Q_2 is also isomorphic to the small resolution at Q_2 along $\theta_i \times \Gamma_{j+1}$, which is a contradiction.

Using these criteria we analyze below which $I_r \times I_s$ fibers $(r > 1, s > 1)$ X can have so that a free action of G on X (with trivial action on $\mathbb{P}^1$) lifts to a free action on a projective small resolution of X.

• **Lifting the $\mathbb{Z}_3 \times \mathbb{Z}_3$ action:** In this case $I_r \times I_s = I_3 \times I_3$ and if the action is free, one element of G must have intersection numbers $(0, 1)$, hence the action does not lift by the above argument.

• **Lifting the $\mathbb{Z}_4 \times \mathbb{Z}_2$ action:** We have $|G| = 8 > 4$, hence G cannot act freely on $I_2 \times I_2$. If the action is free on $I_4 \times I_2$, then one element must have intersection numbers $(0, 1)$, hence the action does not lift. For $I_4 \times I_4$, X can have one or two $I_4 \times I_4$ fibers and we can construct the action such that without loss of generality an order 4 generator has intersection numbers $(1, 1)$ on both $I_4 \times I_4$ fibers and the second generator, which has order 2, has intersection numbers $(2, 0)$ and $(0, 2)$ on the first and the second $I_4 \times I_4$ fibers, respectively. The action lifts to the resolution obtained by blowing up the eight divisors in the orbit of $\theta_0 \times \Gamma_0$ in each $I_4 \times I_4$ fiber. The action lifts in any case, whether there are one or two $I_4 \times I_4$ fibers.

• **Lifting the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action:** If the action is free on $I_2 \times I_2$, then one element has intersection numbers $(0, 1)$, hence the action does not lift. For all cases in Table 9 for which $MW_{tors}(B)$ has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup, each I_4 fiber is intersected at the component θ_2 by two of the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$, and intersected at the component θ_0 by the other two elements including the identity (zero section). Similarly in all of these cases, each I_2 is intersected at the component θ_1 by two of the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$, and intersected at θ_0 by the other two elements. Thus, if the action on an $I_4 \times I_2$ fiber is free, then one element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ should have intersection numbers $(0, 1)$, hence the action does not lift. For $I_4 \times I_4$ fibers (X can have one or two such fibers), we can construct the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ such that the intersection numbers of the four elements are $(0, 0)$ (for the identity), $(0, 2)$, $(2, 0)$ and $(2, 2)$. The action lifts to the resolution obtained by blowing up the divisors in the orbits of $\theta_0 \times \Gamma_0$ in all $I_4 \times I_4$ fibers. For the 3-folds X where the action lifts to a resolution, the only $I_r \times I_2$ type fibers X has are one or two $I_4 \times I_4$ fibers.

• **Lifting the $\mathbb{Z}_k$ action for $k = 6, 5, 4, 3, 2$:** For the action of a cyclic group G on X if the intersection numbers of each element of G are distinct for a fiber $I_r \times I_s$, then the action is free on $I_r \times I_s$. If none of the intersection numbers for $I_r \times I_s$ is $(0,1)$, $(1,0)$, $(0,s-1)$ or $(r-1,0)$, then the action of G lifts to a partial resolution of X where all singularities on this $I_r \times I_s$ fiber are resolved as follows: Choose a divisor $\theta_i \times \Gamma_j$ and blow up all divisors in the orbit of this divisor under the G action. By Lemma 14 the action lifts to this partial resolution. If all singularities on $I_r \times I_s$ are not resolved, repeat the same process by blowing up the divisors in the orbit of the proper transform of another divisor $\theta_p \times \Gamma_q$ until all singularities are resolved. We list the allowed $I_r \times I_s$ fibers on X below so that the $G = \mathbb{Z}_k$ action lifts to a partial resolution of X where all singularities on this $I_r \times I_s$ is resolved:

For $\mathbb{Z}_6$, the allowed fibers are $I_6 \times I_6$, $I_6 \times I_3$ and $I_6 \times I_2$ (The order 6 section on B intersects each of the I_6, I_3 and I_2 at the component θ_1, hence for the G action on X, the intersection numbers of the generator of the cyclic action are $(1,1)$, and the only $I_r \times I_s$ fibers satisfying the above conditions for allowed fibers are as listed). In this case X can have one $I_6 \times I_6$, or one or two $I_6 \times I_3$, or one or two $I_6 \times I_2$, or one $I_6 \times I_3$ and one $I_6 \times I_2$.
For $G = \mathbb{Z}_5$, the only allowed $I_r \times I_s$ fiber is $I_5 \times I_5$. X can have one or two $I_5 \times I_5$ fibers.
For $G = \mathbb{Z}_4$, the allowed $I_r \times I_s$ fibers are $I_4 \times I_2$ (if the generator has intersection numbers $(1,1)$ or $(3,1)$), $I_4 \times I_4$, $I_8 \times I_2$, $I_8 \times I_4$ and $I_8 \times I_8$ (note that except for one of the I_2 fibers in the elliptic surface with singular fiber configuration $I_4^2 I_2^2$ listed under $G = \mathbb{Z}_4 \times \mathbb{Z}_2$, an order 4 section intersects the I_2 fibers at θ_1). The configurations of all allowed cases can be listed by inspecting through Table 9.
For $G = \mathbb{Z}_3$, the allowed $I_r \times I_s$ fibers are $I_9 \times I_9$, $I_9 \times I_6$, $I_9 \times I_3$, $I_9 \times I_2$, $I_6 \times I_6$, $I_6 \times I_3$, $I_6 \times I_2$ and $I_3 \times I_3$ (note that if the generator of the $\mathbb{Z}_3$ action has intersection numbers $(1,0)$, $(2,0)$, $(0,1)$ or $(0,2)$, then $I_3 \times I_3$ is not allowed. The only case where an order 3 section intersects an I_3 fiber at the component θ_0 is the elliptic surface with singular fiber configuration I_3^4 and such an order 3 section intersects only one of the four I_3 fibers at θ_0). The order 3 sections all intersect the I_2 fibers at θ_0, hence by a similar reasoning as above, we can show that $I_3 \times I_2$ and $I_2 \times I_2$ fibers are not allowed.
For $G = \mathbb{Z}_2$, the allowed fibers are $I_r \times I_s$ where $r \in \{2,4,6,8\}$ and $s \in \{2,3,4,6,8\}$ such that the intersection numbers of the generator of G are not $(0,0)$, $(1,0)$ or $(0,1)$. Note that except for some elliptic surfaces where an order 2 section intersects an I_4 fiber, an I_3 fiber, or one or two I_2 fibers at the component θ_0, in all other cases an order 2 section intersects I_8 at θ_4, I_6 at θ_3, I_4 at θ_2 and I_2 at θ_1.

We present the results of this section in the following theorem:

Theorem 17. *For singular Schoen 3-folds $X = B_1 \times_{\mathbb{P}^1} B_2$ whose singularities are on fibers of type $I_r \times I_s$ with $r > 1$ and $s > 1$, the only finite groups G which act freely on X such that the induced action on $\mathbb{P}^1$ is trivial and the action lifts to a free action on a projective small resolution of X are $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_k$ for $2 \leq k \leq 6$. For each such pair (G, X), B_1 and B_2 are surfaces which appear in Table 9 such that G is a subgroup of $MW_{tors}(B_i)$. For each G, the allowed $I_r \times I_s$ type fibers of X so that the action may lift to a free action are listed in Table 10. For the 3-folds X all of whose $I_r \times I_s$ fibers are allowed, the action of G on X is free iff at each $I_r \times I_s$ fiber the intersection numbers of the elements of G are all distinct, and the action of G lifts to a free action on a projective small resolution iff none of these intersection numbers is $(1,0)$, $(0,1)$, $(r-1,0)$ or $(0,s-1)$. In Table 11 we list all existing cases for $G = \mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_k$ for $4 \leq k \leq 6$ where the action lifts to a free action. Table 11 lists only some sample cases for $G = \mathbb{Z}_2$ and $\mathbb{Z}_3$ (due to the large number of all cases) covering all possibilities for the configuration of $I_r \times I_s$ type fibers of X for the given group G. Each line of Table 11 represents several different cases where the configuration of $I_r \times I_s$ fibers of X is a non-empty subset of the specified configuration on that line of the table (the fiber product can be formed with fewer $I_r \times I_s$ type fibers where the lifting criteria for the action of G still hold).*

Remark. For each choice of B_1 and B_2 given in Table 11 the fiber product $B_1 \times_{\mathbb{P}^1} B_2$ can be obtained in different ways giving rise to different configurations of $I_r \times I_s$ fibers. For a configuration of $I_r \times I_s$ type fibers given in Table 11 the action of G lifts to a free action on a resolution of the 3-fold X with this configuration. Since the lifting criteria are required to hold at each $I_r \times I_s$, the criteria also hold for a fiber product whose configuration of $I_r \times I_s$ fibers is a subset of the first configuration. Each possible configuration of X where the action of G lifts is a subset of one of the configurations listed in Table 11. For some specified pairs (B_1, B_2) for $G = \mathbb{Z}_2$ in this table, we did not write down some allowed configurations if these configurations are subsets of a configuration already given in the table. Table 11 can be used to verify the χ values for $m = 1$ case in Table 13.

7. Non-simply connected Calabi-Yau 3-folds with positive Euler characteristic

We completed our analysis of the finite groups G which act freely on a singular Schoen 3-fold X whose singularities are on fibers of type $I_r \times I_s$

Table 10: Allowed $I_r \times I_s$ type fibers of X for lifting G to a free action in the trivial induced action case

C	Allowed $I_r \times I_s$ type fibers of X
$\mathbb{Z}_4 \times \mathbb{Z}_2$	$I_4 \times I_4$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$I_4 \times I_4$
$\mathbb{Z}_6$	$I_6 \times I_s$ for $s = 2, 3, 6$.
$\mathbb{Z}_5$	$I_5 \times I_5$
$\mathbb{Z}_4$	$I_r \times I_s$ for $r \in \{4, 8\}$ and $s \in \{2, 4, 8\}$.
$\mathbb{Z}_3$	$I_3 \times I_3$ and $I_r \times I_s$ for $r \in \{6, 9\}$ and $s \in \{2, 3, 6, 9\}$.
$\mathbb{Z}_2$	$I_3 \times I_t$ for $t \in \{4, 6, 8\}$ and $I_r \times I_s$ for $r, s \in \{2, 4, 6, 8\}$.

with $r > 1$ and $s > 1$. We determined when the action of G on X lifts to a free action on a projective small resolution $\hat{X}$ of X. Such a 3-fold $\hat{X}$ is a simply connected Calabi-Yau 3-fold (see [7]) and the quotient 3-fold $\hat{X}/G$ under the group action is a non-simply connected Calabi-Yau 3-fold (see [2] and [1]) with fundamental group G. The Euler characteristic of X is $e = \sum_{i=1}^{N} r_i s_i$ where the $I_r \times I_s$ type fibers of X are $I_{r_i} \times I_{s_i}$ for $i = 1, .., N$. The projective small resolution $\hat{X}$ of X has Euler characteristic $2e$ and the quotient 3-fold $\hat{X}/G$ has Euler characteristic $2e/|G|$. The non-simply connected Calabi-Yau 3-folds obtained as quotients of smooth Schoen 3-folds by Bouchard and Donagi in [2] all have Euler characteristic 0 since a smooth Schoen 3-fold has Euler characteristic 0. The non-simply connected Calabi-Yau 3-folds obtained in this paper (in the singular X case) all have positive Euler characteristic. We summarize our results in the following theorem:

Theorem 18. *Let $X = B_1 \times_{\mathbb{P}^1} B_2$ be a singular Schoen 3-fold such that the only singularities of X are on fibers of type $I_r \times I_s$ with $r > 1$ and $s > 1$. The finite groups G which act freely on X and induce a non-trivial action on the base curve $\mathbb{P}^1$ ($\tilde{\phi}(G) = \mathbb{Z}_m$ where $m > 1$) such that the action of G lifts to a free action on a projective small resolution $\hat{X}$ of X are as listed in Table 12. For a finite group G whose action on X induces a trivial action on $\mathbb{P}^1$ ($m = 1$ case), the conditions under which the action of G lifts to a free action on a projective small resolution of X are given in Theorem 17 and the results are listed in Table 11.*

For these groups G which act freely on the simply connected Calabi-Yau 3-fold $\hat{X}$, the quotient 3-fold $\hat{X}/G$ is a non-simply connected Calabi-Yau 3-fold with fundamental group G. All distinct Euler characteristic values of the non-simply connected Calabi-Yau 3-folds obtained with this construction are listed in Table 13.

Table 11: Table of Schoen 3-folds X and maximal configurations of $I_r \times I_s$ type fibers of X (cf. Remark following Theorem 17) for which the action of G lifts to a free action (trivial induced action case, $m = 1$)

G	Fib. of B_1	Fib. of B_2	Fibers of X (are a subset of)
$\mathbb{Z}_4 \times \mathbb{Z}_2$	$I_4^2 I_2^2$	$I_4^2 I_2^2$	$2(I_4 \times I_4)$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$I_4^2 I_2^2$	$I_4^2 I_2^2$	$2(I_4 \times I_4)$
	$I_4^2 I_2^2$, $I_4 I_2^4$	$I_4 I_2^4$	$I_4 \times I_4$
$\mathbb{Z}_6$	$I_6 I_3 I_2 I_1$	$I_6 I_3 I_2 I_1$	$I_6 \times I_6$ or $2(I_6 \times I_3)$ or $2(I_6 \times I_2)$ or $I_6 \times I_3 + I_6 \times I_2$
$\mathbb{Z}_5$	$I_5^2 I_1^2$	$I_5^2 I_1^2$	$2(I_5 \times I_5)$
$\mathbb{Z}_4$	$I_8 I_2 I_1^2$	$I_8 I_2 I_1^2$	$I_8 \times I_8$ or $2(I_8 \times I_2)$
	$I_8 I_2 I_1^2$	$I_4^2 I_2^2$, $I_4^2 I_2 I_1^2$	$I_8 \times I_4 + I_4 \times I_2$ or $I_8 \times I_2 + I_4 \times I_2$
	$I_8 I_2 I_1^2$	$I_1^* I_4 I_1$	$I_8 \times I_4$ or $I_4 \times I_2$
	$I_4^2 I_2^2$, $I_4^2 I_2 I_1^2$	$I_4^2 I_2^2$, $I_4^2 I_2 I_1^2$	$2(I_4 \times I_4)$ or $I_4 \times I_4 + 2(I_4 \times I_2)$
	$I_4^2 I_2^2$, $I_4^2 I_2 I_1^2$	$I_1^* I_4 I_1$	$I_4 \times I_4$ or $I_4 \times I_2$
	$I_1^* I_4 I_1$	$I_1^* I_4 I_1$	$I_4 \times I_4$
$\mathbb{Z}_3$	$I_9 I_1^3$	$I_9 I_1^3$	$I_9 \times I_9$
	$I_9 I_1^3$	$I_6 I_3 I_2 I_1$	$I_9 \times I_6$ or $I_9 \times I_3$ or $I_9 \times I_2$
	$I_6 I_3 I_2 I_1$	$I_6 I_3 I_2 I_1$	$I_6 \times I_6 + I_3 \times I_3$ or $I_6 \times I_3 + I_6 \times I_2$ or $2(I_6 \times I_2) + I_3 \times I_3$ or $2(I_6 \times I_3)$
	$I_6 I_3 I_2 I_1$	$I_3^3 I_1^3$	$I_6 \times I_3 + I_3 \times I_3$
	$I_3^3 I_1^3$	$I_3^3 I_1^3$	$3(I_3 \times I_3)$
$\mathbb{Z}_2$	$I_8 I_2 I_1^2$	$I_8 I_2 I_1^2$	$I_8 \times I_8$ or $2(I_8 \times I_2)$
	$I_8 I_2 I_1^2$	$I_6 I_3 I_2 I_1$	$I_8 \times I_3 + I_6 \times I_2$ or $I_8 \times I_2 + I_6 \times I_2$ or $I_8 \times I_6$
	$I_8 I_2 I_1^2$	$I_4^2 I_2 I_1^2$	$I_8 \times I_4 + I_4 \times I_2$ or $I_8 \times I_2 + I_4 \times I_2$
	$I_8 I_2 I_1^2$	$I_4 I_3 I_2^2 I_1$	$I_8 \times I_3 + I_4 \times I_2$
	$I_6 I_3 I_2 I_1$	$I_6 I_3 I_2 I_1$	$2(I_6 \times I_3) + I_2 \times I_2$ or $I_6 \times I_6 + I_2 \times I_2$
	$I_6 I_3 I_2 I_1$	$I_6 I_2^2 I_1^2$	$I_6 \times I_3 + I_6 \times I_2 + I_2 \times I_2$
	$I_6 I_2^2 I_1^2$	$I_6 I_2^2 I_1^2$	$2(I_6 \times I_2) + I_2 \times I_2$
	$I_6 I_3 I_2 I_1$	$I_4^2 I_2^2$	$I_6 \times I_4 + I_4 \times I_3 + I_2 \times I_2$ or $I_6 \times I_2 + I_4 \times I_3 + I_4 \times I_2$
	$I_6 I_2^2 I_1^2$	$I_4^2 I_2^2$	$I_6 \times I_4 + I_4 \times I_2 + I_2 \times I_2$ or $I_6 \times I_2 + 2(I_4 \times I_2)$
	$I_6 I_3 I_2 I_1$	$I_4 I_3 I_2^2 I_1$	$I_6 \times I_3 + I_4 \times I_3 + I_2 \times I_2$ or $I_6 \times I_2 + I_4 \times I_3 + I_2 \times I_2$
	$I_6 I_2^2 I_1^2$	$I_4 I_3 I_2^2 I_1$	$I_6 \times I_3 + I_4 \times I_2 + I_2 \times I_2$ or $I_6 \times I_2 + I_4 \times I_2 + I_2 \times I_2$
	$I_4^2 I_2^2$	$I_4^2 I_2^2$	$4(I_4 \times I_2)$ or $I_4 \times I_4 + 2(I_4 \times I_2)$ or $2(I_4 \times I_4) + 2(I_2 \times I_2)$
	$I_4 I_2^4$	$I_4 I_2^4$	$2(I_4 \times I_2) + 2(I_2 \times I_2)$ or $4(I_2 \times I_2)$
	$I_4^2 I_2^2$	$I_4 I_2^4$	$I_4 \times I_4 + I_4 \times I_2 + 2(I_2 \times I_2)$
	$I_4^2 I_2^2$	$I_4 I_3 I_2^2 I_1$	$I_4 \times I_4 + I_4 \times I_3 + 2(I_2 \times I_2)$ or $I_4 \times I_3 + 2(I_4 \times I_2)$
	$I_4 I_3 I_2^2 I_1$	$I_4 I_3 I_2^2 I_1$	$2(I_4 \times I_3) + 2(I_2 \times I_2)$ or $I_4 \times I_3 + I_4 \times I_2 + I_2 \times I_2$

Table 12: Finite groups G which act freely on a singular $X = B_1 \times_{\mathbb{P}^1} B_2$ with non-trivial induced action on $\mathbb{P}^1$ where the action lifts to a free action on a projective small resolution of X

	Cases from Table 5	
G	B_1	B_2
$\mathbb{Z}_3 \times \mathbb{Z}_3$	1,[9].	1,[9].
$\mathbb{Z}_4 \times \mathbb{Z}_2$	2,[10].	2,[10].
	[11]	[11]
$\mathbb{Z}_6$	[14]	[14]
	[15]	[15]
$\mathbb{Z}_4$	2,[10].	2,[10].
	[11],[12],[19],[20].	[11],[12],[19],[20].
$\mathbb{Z}_3$	1,16.	1,16.
	[9],[14],[31].	[9],[14],[31].
	1,16,19.	[9],[31].
$\mathbb{Z}_2 \times \mathbb{Z}_2$	2,21,24,25.	2,21,24,25.
	[10],[11],[21]–[26],[28].	[10],[11],[21]–[26],[28].
	2,21–26,28,29.	[10],[21],[22],[23].
$\mathbb{Z}_2$	2,21,24,25,32,38,40.	2,21,24,25,32,38,40.
	2,21–26,28,29,32,36,38,40,42,46,48,52.	[10],[21],[22],[23],[35],[37]
	[10],[11],[15],[21]–[26], [28],[35],[37],[38],[41],[43].	[10],[11],[15],[21]–[26], [28],[35],[37],[38],[41],[43].
	Cases from Table 6	
$\mathbb{Z}_4 \times \mathbb{Z}_2$	3	3,[12].
$\mathbb{Z}_4$	3,11,13.	3,11,13.
	3,11,12,13.	[11],[12],[19].
$\mathbb{Z}_2 \times \mathbb{Z}_2$	2,24,28.	28,[25],[26].
	2,24.	22
	2,21,24,25,27.	27
	2,21–29.	[27]
$\mathbb{Z}_2$	2,4,6,8–10,21–53.	[25]–[27],[39],[42].
	2,4,6,8–10,24,28,33,37,39,40,43,45, 47–49,51,53.	[11],[24],[36],[38],[41].
	2,4,8,10,21–25,27–33,35–40,44–49,53.	28,45.
	2,4,8,10,21,22,24,25,27,28,30,32,33, 35,38–40,44,45,49.	22,27,35,44.
	2,4,8,10,24,28,33,39,40,45,49.	23,29,31,36,46.
	2,4,8,10,24,28,33,37,39,40,45,47–49,53.	37,47,48,53.
	Cases from Table 7	
$\mathbb{Z}_4$	3,11.	3,11,[11],[12],[19].
$\mathbb{Z}_2 \times \mathbb{Z}_2$	2,24.	22
	2,24,28.	28,[25],[26].
$\mathbb{Z}_2$	2,21–25,28,29,32,36,38,40,46,48.	28
	2,21,22,24,25,28,32,38,40.	22
	2,24,28,40	23,29,36,46.
	2,24,28,40,48.	48,[11],[24],[38],[41].
	2,21–26,28,29,32,36,38,40,42,46,48,52.	[25],[26].

Table 13: Fundamental groups and Euler characteristics of the Calabi-Yau 3-folds obtained as quotients of projective small resolutions of singular Schoen 3-folds ($m > 1$ case refers to non-trivial action on $\mathbb{P}^1$ and $m = 1$ case refers to trivial action on $\mathbb{P}^1$)

π_1	χ values in the $m > 1$ case	χ values in the $m = 1$ case
$\mathbb{Z}_3 \times \mathbb{Z}_3$	6	–
$\mathbb{Z}_4 \times \mathbb{Z}_2$	4,8.	4,8.
$\mathbb{Z}_6$	4,6.	4,6,8,10,12.
$\mathbb{Z}_5$	–	10,20.
$\mathbb{Z}_4$	4,8,12,16,20.	4,8,12,16,20,32.
$\mathbb{Z}_3$	8,12,18.	6,8,12,14,16,18,20,22,24,30,36,54.
$\mathbb{Z}_2 \times \mathbb{Z}_2$	4,8,12,16,20.	8,16.
$\mathbb{Z}_2$	8,12,16,18,20,24,28,32,36, 40,48,64.	4,8,12,16,18,20,22,24,26,28, 30,32,34,36,40,48,64.

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