

# Fibers over infinity of Landau–Ginzburg models

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We conjecture that the number of components of the fiber over infinity of Landau–Ginzburg model for a smooth Fano variety  $X$  equals the dimension of the anticanonical system of  $X$ . We verify this conjecture for log Calabi–Yau compactifications of toric Landau–Ginzburg models for smooth Fano threefolds, complete intersections in projective spaces, and some toric varieties.

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## 1. Introduction

Let  $X$  be a smooth Fano variety of dimension  $n$ . Then its *Landau–Ginzburg model* is a certain pair  $(Y, w)$  that consists of a smooth (quasi-projective) variety  $Y$  of dimension  $n$  and a regular function

$$w: Y \rightarrow \mathbb{A}^1,$$

which is called a superpotential. (More precise, Landau–Ginzburg model corresponds to a variety together with a divisor class on it, but we assume this class to be anticanonical.) Its fibers are compact and  $K_Y \sim 0$ , so that general fiber of  $w$  is a smooth Calabi–Yau variety of dimension  $n - 1$ . Homological Mirror Symmetry conjecture predicts that the derived category of singularities of the singular fibers of  $w$  is equivalent to the Fukaya category of the variety  $X$ , while the Fukaya–Seidel category of the pair  $(Y, w)$

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is equivalent to the bounded derived category of coherent sheaves on  $X$ . In short: the geometry of  $X$  should be determined by singular fibers of  $w$ .

Often, Landau–Ginzburg models of smooth Fano varieties can be constructed via their toric degenerations (see [23]). In this case, the variety  $Y$  contains a torus  $(\mathbb{C}^*)^n$ , one has  $K_Y \sim 0$ , and there exists a commutative diagram

$$(1.1) \quad \begin{array}{ccc} (\mathbb{C}^*)^n & \xrightarrow{\quad} & Y \\ \mathfrak{p} \downarrow & & \downarrow w \\ \mathbb{C} & \xlongequal{\quad} & \mathbb{C} \end{array}$$

for some Laurent polynomial  $\mathfrak{p} \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$  which is defined by an appropriate toric degeneration of the variety  $X$ . Then  $\mathfrak{p}$  is said to be a *toric Landau–Ginzburg model* of the Fano variety  $X$ , and  $(Y, w)$  is said to be its *Calabi–Yau compactification*.

If  $(Y, w)$  is a Calabi–Yau compactification of a toric Landau–Ginzburg model, then the number of reducible fibers of the morphism  $w: Y \rightarrow \mathbb{C}$  does not depend on the choice of the Calabi–Yau compactification. Likewise, the number of irreducible components of each singular fiber of  $w$  does not depend on the compactification either. Therefore, it is natural to expect that these numbers contain some information about the smooth Fano variety  $X$ . For instance, we have the following.

**Conjecture 1.2** (See [21, 26, 8]). Let  $X$  be a smooth Fano variety of dimension  $n \geq 3$ , and let  $(Y, w)$  be a Calabi–Yau compactification of its toric Landau–Ginzburg model. Then

$$h^{1,n-1}(X) = \sum_{P \in \mathbb{C}^1} (\rho_P - 1),$$

where  $\rho_P$  is the number of irreducible components of the fiber  $w^{-1}(P)$ .

Note that the toric Landau–Ginzburg models considered in Conjecture 1.2 correspond to the anticanonical divisors on Fano varieties. It may fail for other divisors. For instance, all singular fibers of Landau–Ginzburg models of smooth del Pezzo surfaces together with general divisors on them have at most ordinary double points as singularities, while ones for the anticanonical divisors are very specific. One can formulate Conjecture 1.2 replacing the Hodge number  $h^{1,n-1}(X)$  by the primitive one  $h_{pr}^{1,n-1}(X)$ , which is equal to the usual one for  $n \geq 3$  and is less by one for  $n = 2$ ; del Pezzo surfaces satisfy the corrected Conjecture 1.2.

This conjecture under some mild conditions can be derived from Homological Mirror Symmetry conjecture, cf. [15] and [10]. Recently, Conjecture 1.2 has been verified for Calabi–Yau compactifications of toric Landau–Ginzburg models of smooth Fano complete intersections and smooth Fano threefolds (see [21, 26, 3]).

In all considered cases, the commutative diagram (1.1) can be extended to a commutative diagram

$$(1.3) \quad \begin{array}{ccccc} (\mathbb{C}^*)^{n_C} & \longrightarrow & Y^C & \longrightarrow & Z \\ \mathfrak{p} \downarrow & & \downarrow \mathfrak{w} & & \downarrow \mathfrak{f} \\ \mathbb{C} & \xlongequal{\quad\quad\quad} & \mathbb{C} & \longrightarrow & \mathbb{P}^1 \end{array}$$

such that  $Z$  is a smooth proper variety that satisfies certain natural geometric conditions, e.g. the fiber  $\mathfrak{f}^{-1}(\infty)$  is reduced, it has at most normal crossing singularities, and

$$\mathfrak{f}^{-1}(\infty) \sim -K_Z.$$

Then  $(Z, \mathfrak{f})$  is called the *log Calabi-Yau compactification* of the toric Landau–Ginzburg model  $\mathfrak{p}$  (see [23, Definition 3.6]). Observe that the number of irreducible components of the fiber  $\mathfrak{f}^{-1}(\infty)$  does not depend on the choice of the log Calabi-Yau compactification. Indeed, let  $\mathfrak{f}': Z' \rightarrow \mathbb{P}^1$  be another such compactification. Then  $Z$  and  $Z'$  are smooth proper varieties such that there exists the following commutative diagram:

$$\begin{array}{ccc} Z & \overset{\psi}{\dashrightarrow} & Z' \\ \mathfrak{f} \searrow & & \swarrow \mathfrak{f}' \\ & \mathbb{P}^1 & \end{array}$$

where  $\psi$  is a birational map that is an isomorphism away from  $\mathfrak{f}^{-1}(\infty)$  and  $(\mathfrak{f}')^{-1}(\infty)$ . On the other hand, both relative canonical divisors  $K_{Z/\mathbb{P}^1}$  and  $K_{Z'/\mathbb{P}^1}$  are trivial, because

$$\begin{aligned} \mathfrak{f}^{-1}(\infty) &\sim -K_Z, \\ (\mathfrak{f}')^{-1}(\infty) &\sim -K_{Z'}. \end{aligned}$$

Then  $\psi$  is a composition of flops by [17, Theorem 1], so the number of irreducible components of  $\mathfrak{f}^{-1}(\infty)$  is independent on the choice of the log

Calabi–Yau compactification. Thus, one can expect that this number keeps some information about the Fano variety  $X$ . The following two examples confirm this.

**Example 1.4.** Let  $X$  be a smooth del Pezzo surface, and let  $(Z, \mathfrak{f})$  be a log Calabi–Yau compactification of its toric Landau–Ginzburg model constructed in [2]. Then the fiber  $\mathfrak{f}^{-1}(\infty)$  consists of

$$\chi(\mathcal{O}(-K_X)) - 1 = h^0(\mathcal{O}_X(-K_X)) - 1 = K_X^2$$

irreducible rational curves.

**Example 1.5.** Let  $X$  be a smooth Fano threefold such that the divisor  $-K_X$  is very ample, and let  $(Z, \mathfrak{f})$  be a log Calabi–Yau compactification of its toric Landau–Ginzburg model constructed in [1, 22, 4]. Then  $\mathfrak{f}^{-1}(\infty)$  consists of

$$\chi(\mathcal{O}(-K_X)) - 1 = h^0(\mathcal{O}_X(-K_X)) - 1 = \frac{(-K_X)^3}{2} + 2$$

irreducible rational surfaces by [22, Corollary 35], see also [9, Theorem 2.3.14].

This example motivates the following conjecture.

**Conjecture 1.6.** Let  $X$  be a smooth Fano variety, and let  $(Z, \mathfrak{f})$  be a log Calabi–Yau compactification of its toric Landau–Ginzburg model. Then the fiber  $\mathfrak{f}^{-1}(\infty)$  consists of

$$\chi(\mathcal{O}(-K_X)) - 1 = h^0(\mathcal{O}_X(-K_X)) - 1$$

irreducible components.

In [9, Conjecture 2.3.13] this conjecture for threefolds is formulated in the equivalent form: the number of components of  $\mathfrak{f}^{-1}(\infty)$  is equal to the genus of Fano threefold  $X$  (which by definition is a genus of a generic double anticanonical section of  $X$ ) plus 1. This form suggests the generalization of the latter conjecture to higher dimensions. More precise, let  $Z$  be a generic double anticanonical section of the Fano variety  $X$  of dimension  $n$ . Then

$$h^0(\mathcal{O}_X(-K_X)) - 1 = h^0(\mathcal{O}_Z(K_Z)) + 1 = h^{0, n-2}(Z) + 1.$$

In [9, Remark 2.3.16] this observation is generalized to other Hodge numbers. That is, Mirror Symmetry expectation is that the fiber  $\mathfrak{f}^{-1}(\infty)$  is a mirror dual object to  $Z$ , and Hodge diamond for  $Z$  after the mirror  $90^\circ$ -rotation coincide to the Hodge diamond for the sheaf of vanishing cycles for  $\mathfrak{f}$  at

infinity (after appropriate shift). Thus, Conjecture 1.6 can be treated as a particular case of the conjecture alluded in [9, Remark 2.3.16], cf. [10, Theorem 3.8].

The main result of the paper is the following.

**Theorem 1.7.** Conjecture 1.6 holds for

- §2: standard rigid maximally-mutable toric Landau–Ginzburg models for smooth Fano threefolds;
- §3: Givental’s toric Landau–Ginzburg models for Fano complete intersections in projective spaces;
- §4: Givental’s toric Landau–Ginzburg models for toric varieties whose dual toric varieties admit crepant resolutions.

*Remark 1.8.* Conjecture 1.6 together with the conjectural existence of toric Landau–Ginzburg models of smooth Fano varieties [23, Conjecture 3.9] imply that

$$h^0(\mathcal{O}_X(-K_X)) \geq 2,$$

which is only known for  $\dim(X) \leq 5$ , see [12, Theorem 1.7] and [11, Theorem 1.1.1]. Let us also note that Kawamata’s [16, Conjecture 2.1] implies that  $h^0(\mathcal{O}_X(-K_X)) \geq 1$ .

Homological Mirror Symmetry conjecture suggests that the monodromy around  $f^{-1}(\infty)$  is maximally unipotent (see [15, §2.2]). Thus, if the fiber  $f^{-1}(\infty)$  in (1.3) is a divisor with simple normal crossing singularities, then its dual intersection complex is expected to be homeomorphic to a sphere of dimension  $n - 1$  (see [20, Question 7]). This follows from [20, Proposition 8] for  $n \leq 5$ . However, the following example shows that we cannot always expect  $f^{-1}(\infty)$  to be a divisor with simple normal crossing singularities.

**Example 1.9.** Let  $X$  be a smooth intersection of two general sextics in the weighted projective space  $\mathbb{P}(1, 1, 1, 2, 2, 3, 3)$ . Then  $X$  is a smooth Fano fourfold and  $-K_X = \mathcal{O}(1)$ , so that

$$h^0(\mathcal{O}_X(-K_X)) - 1 = 3 - 1 = 2.$$

A toric Landau–Ginzburg model for  $X$  is the Laurent polynomial

$$p = \frac{(x + y + 1)^6(z + t + 1)^6}{x^3yz^3t},$$

see [23, §7.2.2]. The change of variables

$$x = \frac{a^2c}{b^3d}, \quad y = \frac{ac}{b^2d} - \frac{a^2c}{b^3d} - 1, \quad z = c, \quad t = d - c - 1$$

gives us a birational map  $(\mathbb{C}^*)^4 \dashrightarrow \mathbb{C}^4$  that maps the pencil  $\mathfrak{p} = \lambda$  to the pencil of quintics in  $\mathbb{C}^4$  given by

$$d^4 = \lambda(abc - a^3c - b^3d)(d - c - 1),$$

where  $\lambda$  is a parameter in  $\mathbb{C} \cup \{\infty\}$ . Now arguing as in [3], one can construct a log Calabi–Yau compactification  $(Z, f)$  of the toric Landau–Ginzburg model  $\mathfrak{p}$ . Then  $f^{-1}(\infty)$  consists of two irreducible divisors intersecting by a singular plane cubic, and the monodromy around this fiber is maximally unipotent. All other log Calabi–Yau compactifications differ from  $(Z, f)$  by flops, so that their fibers over  $\infty$  also consist of two irreducible divisors. If one of them is a divisor with simple normal crossing singularities, then its dual intersection complex must be homeomorphic to a three-dimensional sphere by [20, Proposition 8], which is impossible for dimension reasons.

Nevertheless, all toric Landau–Ginzburg models we consider in this paper admit log Calabi–Yau compactifications such their fibers over  $\infty$  are divisors with simple normal crossing singularities. For toric Landau–Ginzburg models of smooth Fano threefolds, this follows from the construction of the log Calabi–Yau compactifications given in [22] except for the families  $\mathbb{N}^{\circ}2.1$  and  $\mathbb{N}^{\circ}10.1$ . For each of these two families, the fiber over  $\infty$  does not have simple normal crossing singularities, but one can flop the log Calabi–Yau compactification in several curves contained in this fiber such that the resulting divisor has simple normal crossing singularities.

Let us describe the structure of this paper. In Section 2, we verify Conjecture 1.6 for smooth Fano threefolds. In Section 3, we verify Conjecture 1.6 for smooth Fano complete intersections in projective spaces. In Section 4, we verify Conjecture 1.6 for some smooth toric Fano varieties.

## 2. Fano threefolds

In this section, we prove Conjecture 1.6 for *standard* toric Landau–Ginzburg models of smooth Fano threefolds. More precise, by [6, Theorem 4.1], mutation-equivalence classes of *rigid maximally-mutable Laurent polynomials*

(see [6]) whose Newton polynomials are three-dimensional reflexive polytopes correspond one-to-one to the 98 deformation families of three-dimensional Fano manifolds with very ample anticanonical class. Let us call them standard. Furthermore, each of the 105 deformation families of three-dimensional Fano manifolds has a rigid maximally-mutable Laurent polynomial mirror (see [1, 22, 4]). Thus for the remaining 7 deformation families of Fano varieties with not very ample anticanonical class choose those of them that are discussed in [3] and call them standard as well. Let  $X$  be a smooth Fano threefold. Then the log Calabi–Yau compactification of its toric Landau–Ginzburg model is given by (1.3), where  $\mathbf{p}$  is standard. Let us denote by  $[f^{-1}(\infty)]$  the number of irreducible components of the fiber  $f^{-1}(\infty)$ . We have to show that

$$[f^{-1}(\infty)] = \frac{(-K_X)^3}{2} + 2.$$

The polynomial  $\mathbf{p}$  is not uniquely determined by  $X$ , but the number  $[f^{-1}(\infty)]$  does not change under mutation, and thus does depend on the choice of  $\mathbf{p}$  provided  $\mathbf{p}$  is standard. In particular, for the very ample case we may choose  $\mathbf{p}$  from [5] among any mirror partners for  $X$ . Note that conjecturally Theorem 1.7 holds for *all* rigid maximally mutable Laurent toric Landau–Ginzburg model due to [6, Conjecture 5.1].

By Example 1.5, we may assume that the anticanonical divisor  $-K_X$  is not very ample, so that  $X$  is a smooth Fano threefold  $N^{\circ}1.1$ ,  $N^{\circ}1.11$ ,  $N^{\circ}2.1$ ,  $N^{\circ}2.2$ ,  $N^{\circ}2.3$ ,  $N^{\circ}9.1$ , or  $N^{\circ}10.1$ . Here we use enumeration of deformation families of smooth Fano threefolds from [14]. Recall that the threefold  $X$  can be described as follows:

- ( $N^{\circ}1.1$ ) a smooth sextic hypersurface in  $\mathbb{P}(1, 1, 1, 1, 3)$ ;
- ( $N^{\circ}1.11$ ) a smooth sextic hypersurface in  $\mathbb{P}(1, 1, 1, 2, 3)$ ;
- ( $N^{\circ}2.1$ ) a blow up of a smooth sextic hypersurface in  $\mathbb{P}(1, 1, 1, 2, 3)$  along an elliptic curve;
- ( $N^{\circ}2.2$ ) a double cover of  $\mathbb{P}^1 \times \mathbb{P}^2$  ramified in a surface of bidegree  $(2, 4)$ ;
- ( $N^{\circ}2.3$ ) a blow up of a smooth quartic hypersurface in  $\mathbb{P}(1, 1, 1, 1, 2)$  along an elliptic curve;
- ( $N^{\circ}9.1$ )  $X \cong \mathbb{P}^1 \times \mathbf{S}_2$ , where  $\mathbf{S}_2$  is a smooth del Pezzo surface of degree 2;
- ( $N^{\circ}10.1$ )  $X \cong \mathbb{P}^1 \times \mathbf{S}_1$ , where  $\mathbf{S}_1$  is a smooth del Pezzo surface of degree 1.

Moreover, it follows from [3, §2.2], [3, §2.3], [3, §9.1], [3, §10.1] and the proof of [21, Theorem 18] that we can choose the polynomial  $\mathbf{p}$  in (1.3)

as follows:

$$p = \begin{cases} \frac{(a+b+c+1)^6}{abc} & \text{if } X \text{ is a Fano threefold } N^{\circ}1.1, \\ \frac{(a+b+1)^6}{ab^2c} + c & \text{if } X \text{ is a Fano threefold } N^{\circ}1.11, \\ \frac{(a+b+1)^6(c+1)^6}{ab^2} + \frac{1}{c} & \text{if } X \text{ is a Fano threefold } N^{\circ}2.1, \\ \frac{(a+b+c+1)^2}{a} + \frac{(a+b+c+1)^4}{bc} & \text{if } X \text{ is a Fano threefold } N^{\circ}2.2, \\ \frac{(a+b+1)^4(c+1)}{abc} + c + 1 & \text{if } X \text{ is a Fano threefold } N^{\circ}2.3, \\ \frac{(a+b+1)^4}{ab} + c + \frac{1}{c} & \text{if } X \text{ is a Fano threefold } N^{\circ}9.1, \\ \frac{(a+b+1)^6}{ab^2} + c + \frac{1}{c} & \text{if } X \text{ is a Fano threefold } N^{\circ}10.1, \end{cases}$$

where  $(a, b, c)$  are coordinates on  $(\mathbb{C}^*)^3$ .

**Proposition 2.1.** Suppose that  $X$  is a Fano threefold  $N^{\circ}1.1$ ,  $N^{\circ}1.11$ ,  $N^{\circ}2.2$ ,  $N^{\circ}2.3$ , or  $N^{\circ}9.1$ . Then  $[f^{-1}(\infty)] = \frac{(-K_X)^3}{2} + 2$ .

*Proof.* It follows from [21], [3, §2.2], [3, §2.3] and [3, §9.1] that we can choose  $p$  such that there is a pencil  $\mathcal{S}$  of quartic surfaces on  $\mathbb{P}^3$  given by

$$f_4(x, y, z, t) + \lambda g_4(x, y, z, t) = 0$$

for

$$(f_4, g_4) = \begin{cases} (x^4, yz(xt - xy - xz - t^2)) & \text{if } X \text{ is a Fano threefold } N^{\circ}1.1, \\ (x^4 + z^2(xt - xy - t^2), yz(xt - xy - t^2)) & \text{if } X \text{ is a Fano threefold } N^{\circ}1.11, \\ (xz^3 - (zt - xy - yz - t^2)z^2, xy(zt - xy - yz - t^2)) & \text{if } X \text{ is a Fano threefold } N^{\circ}2.2, \\ (x^3y + y(y+z)(xz + xt - t^2), z(y+z)(xz + xt - t^2)) & \text{if } X \text{ is a Fano threefold } N^{\circ}2.3, \\ (x^3y(y^2 + z^2)(xt - xz - t^2), yz(xt - xz - t^2)) & \text{if } X \text{ is a Fano threefold } N^{\circ}9.1 \end{cases}$$

(certain changes of variables can be found in [23, proof of Proposition 5.11]), that expands (1.3) to the following commutative diagram:

$$(2.2) \quad \begin{array}{ccccccc} (\mathbb{C}^*)^3 & \xrightarrow{\quad} & Y & \xrightarrow{\quad} & Z & \xleftarrow{\quad \chi \quad} & V & \xrightarrow{\quad \pi \quad} & \mathbb{P}^3 \\ \downarrow p & & \downarrow w & & \downarrow f & & \downarrow g & & \swarrow \phi \\ \mathbb{C} & \xrightarrow{\quad} & \mathbb{C} & \xrightarrow{\quad} & \mathbb{P}^1 & \xrightarrow{\quad} & \mathbb{P}^1 & & \end{array}$$

where  $\phi$  is a rational map given by  $\mathcal{S}$ , the variety  $V$  is a smooth threefold,  $\pi$  is a birational morphism described in [3], and  $\chi$  is a composition of flops. Here  $\lambda \in \mathbb{C} \cup \{\infty\}$ , where  $\lambda = \infty$  corresponds to the fiber  $f^{-1}(\infty)$ . Moreover, it follows from [3] that  $\pi$  factors through a birational morphisms  $\alpha: U \rightarrow \mathbb{P}^3$  that is uniquely determined by the following three properties:

1. the map  $\alpha^{-1}$  is regular outside of finitely many points in  $X$ ;
2. the proper transform of the pencil  $\mathcal{S}$  via  $\alpha$ , which we denote by  $\widehat{\mathcal{S}}$ , is contained in the anticanonical linear system  $| -K_U |$ ;
3. for every point  $P \in U$ , there is a surface in  $\widehat{\mathcal{S}}$  that is smooth at  $P$ .

We denote by  $\Sigma$  the (finite) subset in  $X$  consisting of all indeterminacy points of  $\alpha^{-1}$ .

Let  $S$  be the quartic surface given by  $g_4(x, y, z, t) = 0$ , let  $\widehat{S}$  be its proper transform on the threefold  $U$ , and let

$$\widehat{D} = \widehat{S} + \sum_{i=1}^k a_i E_i,$$

where  $E_1, \dots, E_k$  are  $\alpha$ -exceptional surfaces, and  $a_1, \dots, a_k$  are non-negative integers such that  $\widehat{D} \sim -K_U$ . Then  $\widehat{D} \in \widehat{\mathcal{S}}$ . Moreover, for any  $\widehat{D}' \in \widehat{\mathcal{S}}$  such that  $\widehat{D}' \neq \widehat{D}$ , we have

$$\widehat{D} \cdot \widehat{D}' = \sum_{i=1}^s m_i \widehat{C}_i,$$

where  $\widehat{C}_1, \dots, \widehat{C}_n$  are base curves of the pencil  $\widehat{\mathcal{S}}$ , and  $m_1, \dots, m_s$  are positive numbers. Without loss of generality, we may assume that the base curves of the pencil  $\mathcal{S}$  are the curves  $\alpha(\widehat{C}_1), \dots, \alpha(\widehat{C}_r)$  for some  $r \leq n$ . Then we let  $C_i = \alpha(\widehat{C}_i)$  for every  $i \leq r$ .

For every  $i \in \{1, \dots, n\}$ , let  $M_i = \text{mult}_{\widehat{C}_i}(\widehat{D})$  and

$$\delta_i = \begin{cases} 0 & \text{if } M_i = 1, \\ m_i - 1 & \text{if } M_i \geq 2. \end{cases}$$

Then it follows from [3, (1.10.8)] that

$$(2.3) \quad [f^{-1}(\infty)] = [S] + \sum_{i=1}^r \delta_i + \sum_{P \in \Sigma} D_P,$$

where  $[S]$  is the number of irreducible components of the surface  $S$ , and  $D_P$  is the *defect* of the point  $P \in \Sigma$  that is defined as

$$D_P = A_P + \sum_{\substack{i=r+1 \\ \alpha(\widehat{C}_i)=P}}^s \delta_i,$$

where  $A_P$  is the total number of indices  $i \in \{1, \dots, k\}$  such that  $a_i > 0$  and  $\alpha(\widehat{E}_i) = P$ . By [3, Lemma 1.12.1], we have  $D_P = 0$  if the rank of the quadratic form of the (local) defining equation of the surface  $S$  at the point  $P$  is at least 2.

To proceed, we need the following notation: for any subsets  $I, J$ , and  $K$  in  $\{x, y, z, t\}$ , we write  $H_I$  for the plane defined by setting the sum of coordinates in  $I$  equal to zero, we write  $L_{I,J} = H_I \cap H_J$ , and, finally, we write  $P_{I,J,K} = H_I \cap H_J \cap H_K$ .

Suppose  $X$  is a Fano threefold  $\mathbb{N}^{\circ}1.1$ . Recall that we have  $f_4 = x^4$  and  $g_4 = yz(xt - xy - xz - t^2)$ , so that the pencil  $\mathcal{S}$  is given by

$$x^4 - \lambda yz(xt - xy - xz - t^2) = 0.$$

Observe that every surface in this pencil is invariant with respect to the  $\mathbb{Z}/2\mathbb{Z}$ -action given by  $[x : y : z : t] \mapsto [x : z : y : t]$ . Moreover, the base locus of the pencil  $\mathcal{S}$  consists of the curves  $L_{\{x\},\{y\}}$ ,  $L_{\{x\},\{z\}}$ ,  $L_{\{x\},\{t\}}$ . Thus, we have  $r = 3$  and, without loss of generality, we may assume that

$$C_1 = L_{\{x\},\{y\}},$$

$$C_2 = L_{\{x\},\{z\}},$$

$$C_3 = L_{\{x\},\{t\}}.$$

Recall that  $S = \{yz(xt - xy - xz - t^2) = 0\} \subset \mathbb{P}^3$ , so that

$$S = H_{\{y\}} + H_{\{z\}} + \mathcal{Q},$$

where  $\mathcal{Q}$  is the irreducible quadric surface  $\{xt - xy - xz - t^2 = 0\} \subset \mathbb{P}^3$ , which is singular at the point  $P_{\{x\},\{t\},\{y,z\}}$ . Since  $S$  is smooth at general points

of the lines  $L_{\{x\},\{y\}}$ ,  $L_{\{x\},\{z\}}$ ,  $L_{\{x\},\{t\}}$ , we see that general surface in the pencil  $\mathcal{S}$  has isolated singularities. In particular, we have  $M_1 = 1$ ,  $M_2 = 1$ , and  $M_3 = 1$ . Moreover, if  $S'$  is another surface in the pencil  $\mathcal{S}$ , then

$$S \cdot S' = 4L_{\{x\},\{y\}} + 4L_{\{x\},\{z\}} + 8L_{\{x\},\{t\}},$$

which means that  $m_1 = 4$ ,  $m_2 = 4$  and  $m_3 = 8$ . Now, taking partial derivatives of the polynomial  $x^4 - \lambda yz(xt - xy - xz - t^2)$ , we see that all surfaces in the pencil  $\mathcal{S}$  are singular at the points  $P_{\{x\},\{y\},\{z\}}$ ,  $P_{\{x\},\{y\},\{t\}}$ ,  $P_{\{x\},\{z\},\{t\}}$ ,  $P_{\{x\},\{t\},\{y,z\}}$ , and these four points are the only singularities of a general surface in this pencil. This shows that

$$\Sigma = \left\{ P_{\{x\},\{y\},\{z\}}, P_{\{x\},\{y\},\{t\}}, P_{\{x\},\{z\},\{t\}}, P_{\{x\},\{t\},\{y,z\}} \right\}.$$

Thus, using (2.3), we get

$$[\mathfrak{f}^{-1}(\infty)] = 3 + D_{P_{\{x\},\{y\},\{z\}}} + D_{P_{\{x\},\{y\},\{t\}}} + D_{P_{\{x\},\{z\},\{t\}}} + D_{P_{\{x\},\{t\},\{y,z\}}}.$$

We claim that

$$D_{P_{\{x\},\{y\},\{z\}}} = D_{P_{\{x\},\{y\},\{t\}}} = D_{P_{\{x\},\{z\},\{t\}}} = D_{P_{\{x\},\{t\},\{y,z\}}} = 0.$$

Indeed,  $P_{\{x\},\{y\},\{z\}} \notin \mathcal{Q}$  and  $P_{\{x\},\{y\},\{z\}} \in H_{\{y\}} \cap H_{\{z\}}$ , which implies that the rank of the quadratic form of the defining local equation of the surface  $S$  at the point  $P_{\{x\},\{y\},\{z\}}$  is two. Hence, we have  $D_{P_{\{x\},\{y\},\{z\}}} = 0$  by [3, Lemma 1.12.1]. Similarly, we see that the rank of the quadratic form of the defining equation of the surface  $S$  at the point  $P_{\{x\},\{t\},\{y,z\}}$  is three, because  $P_{\{x\},\{t\},\{y,z\}} \notin H_{\{y\}}$ ,  $P_{\{x\},\{t\},\{y,z\}} \notin H_{\{z\}}$ , and  $\mathcal{Q}$  has an isolated ordinary double singularity at the point  $P_{\{x\},\{t\},\{y,z\}}$ . This gives  $D_{P_{\{x\},\{t\},\{y,z\}}} = 0$ . Likewise, we have  $P_{\{x\},\{z\},\{t\}} \notin H_{\{y\}}$  and  $P_{\{x\},\{z\},\{t\}} \in H_{\{z\}} \cap \mathcal{Q}$ , but both surfaces  $H_{\{z\}}$  and  $\mathcal{Q}$  are smooth at the point  $P_{\{x\},\{z\},\{t\}}$ , and they intersect each other transversally at this point. Hence, the rank of the quadratic form of the defining equation of the surface  $S$  at the point  $P_{\{x\},\{z\},\{t\}}$  is two, which implies that  $D_{P_{\{x\},\{z\},\{t\}}} = 0$  by [3, Lemma 1.12.1]. Finally, keeping in mind the  $\mathbb{Z}/2\mathbb{Z}$ -symmetry mentioned earlier, we conclude that  $D_{P_{\{x\},\{y\},\{t\}}} = 0$ . Thus, we have

$$\begin{aligned} [\mathfrak{f}^{-1}(\infty)] &= 3 + D_{P_{\{x\},\{y\},\{z\}}} + D_{P_{\{x\},\{y\},\{t\}}} \\ &\quad + D_{P_{\{x\},\{z\},\{t\}}} + D_{P_{\{x\},\{t\},\{y,z\}}} = 3 = \frac{(-K_X)^3}{2} + 2 \end{aligned}$$

as claimed.

Now, we suppose that  $X$  is a Fano threefold  $N^{\circ}1.11$ . Recall that

$$f_4 = x^4 + z^2(xt - xy - t^2) \quad \text{and} \quad g_4 = yz(xt - xy - t^2),$$

so that  $\Sigma$  consists of the points  $P_{\{x\},\{y\},\{z\}}$ ,  $P_{\{x\},\{y\},\{t\}}$ ,  $P_{\{x\},\{z\},\{t\}}$ ; moreover,  $r = 3$ ,  $C_1 = L_{\{x\},\{z\}}$ ,  $C_2 = L_{\{x\},\{t\}}$ , and  $C_3$  is the rational quartic curve given by  $y = x^4 + txz^2 - t^2z^2 = 0$ . Then

$$M_1 = 1, \quad M_2 = 1, \quad M_3 = 1, \quad m_1 = 4, \quad m_2 = 8, \quad m_3 = 1,$$

so that

$$\begin{aligned} [f^{-1}(\infty)] &= 3 + D_{P_{\{x\},\{y\},\{z\}}} + D_{P_{\{x\},\{y\},\{t\}}} \\ &\quad + D_{P_{\{x\},\{z\},\{t\}}} + D_{P_{\{x\},\{t\},\{y,z\}}} = 3 + D_{P_{\{x\},\{y\},\{t\}}} \end{aligned}$$

by (2.3) and [3, Lemma 1.12.1]. To compute  $D_{P_{\{x\},\{y\},\{t\}}}$ , observe that (locally)  $\alpha$  is a blow up of the point  $P_{\{x\},\{y\},\{t\}}$ . Thus, we may assume that  $E_1$  is mapped to  $P_{\{x\},\{y\},\{t\}}$ . Then  $a_1 = 1$ , so that  $A_{P_{\{x\},\{y\},\{t\}}} = 1$ . Moreover, the pencil  $\widehat{\mathcal{S}}$  has a unique base curve in  $E_1$ , which is a conic in  $E_1 \cong \mathbb{P}^2$ . We may assume that this curve is  $\widehat{C}_4$ . Then  $M_4 = 2$ , which gives  $D_{P_{\{x\},\{y\},\{t\}}} = m_4$ . On the other hand, we have

$$10 = 8 + \text{mult}_{P_{\{x\},\{y\},\{t\}}}(\mathcal{C}) = \text{mult}_{P_{\{x\},\{y\},\{t\}}}(4C_1 + 8C_2 + C_3) = 4 + 2m_4,$$

which gives  $D_{P_{\{x\},\{z\},\{t\}}} = 3$ , so that  $[f^{-1}(\infty)] = 6 = \frac{(-K_X)^3}{2} + 2$ .

Suppose that  $X$  is a Fano threefold  $N^{\circ}2.2$ . Recall that

$$f_4 = xz^3 - (zt - xy - yz - t^2)z^2 \quad \text{and} \quad g_4 = xy(zt - xy - yz - t^2),$$

so that the set  $\Sigma$  consists of the points  $P_{\{x\},\{y\},\{z\}}$ ,  $P_{\{x\},\{z\},\{t\}}$ ,  $P_{\{y\},\{z\},\{t\}}$ ; moreover,  $r = 5$ ,  $C_1 = L_{\{x\},\{z\}}$ ,  $C_2 = L_{\{y\},\{z\}}$ , and  $C_3$ ,  $C_4$ , and  $C_5$  are the conics given by

$$x = zt - yz - t^2 = 0, \quad y = xz - zt + t^2 = 0, \quad \text{and} \quad z = xy - t^2 = 0,$$

respectively. Then

$$\begin{aligned} M_1 &= 1, \quad M_2 = 1, \quad M_3 = 2, \quad M_4 = 1, \quad M_5 = 1, \\ m_1 &= 2, \quad m_2 = 2, \quad m_3 = 2, \quad m_4 = 1, \quad m_5 = 3, \end{aligned}$$

so that

$$[f^{-1}(\infty)] = [S] + 1 + D_{P_{\{x\},\{y\},\{z\}}} + D_{P_{\{x\},\{z\},\{t\}}} + D_{P_{\{y\},\{z\},\{t\}}} = 4 + D_{P_{\{y\},\{z\},\{t\}}}$$

by (2.3) and [3, Lemma 1.12.1]. To compute  $D_{P_{\{y\},\{z\},\{t\}}}$ , observe that we have  $A_{P_{\{y\},\{z\},\{t\}}} = 0$ , because  $S$  has a double point at  $P_{\{y\},\{z\},\{t\}}$ . Moreover, locally near the point  $P_{\{y\},\{z\},\{t\}}$ , the pencil  $\mathcal{S}$  is given by

$$\lambda y^2 + z^3 + z^3 t - y z^2 - \lambda y z t + \lambda y^2 z + \lambda y t^2 - y z^3 - z^2 t^2 = 0,$$

where  $P_{\{y\},\{z\},\{t\}} = (0, 0, 0)$ . Let  $\alpha_1: U_1 \rightarrow \mathbb{P}^3$  be the blow up of the point  $P_{\{y\},\{z\},\{t\}}$ , and let  $S^1$  be the proper transform on  $U_1$  of the surface  $S$ , and let  $\mathcal{S}^1$  be the proper transform on  $U_1$  of the pencil  $\mathcal{S}$ . A chart of the blow up  $\alpha_1$  is given by the coordinate change  $y_1 = \frac{y}{t}$ ,  $z_1 = \frac{z}{t}$ ,  $t_1 = t$ . In this chart, the surface  $S^1$  is given by

$$y_1(t_1 + y_1 - t_1 z_1 + t_1 y_1 z_1) = 0,$$

and the pencil  $\mathcal{S}^1$  is given by

$$\lambda y_1(t_1 + y_1) - \lambda t_1 y_1 z_1 + (\lambda t_1 y_1^2 z_1 - t_1^2 z_1^2 - t_1 y_1 z_1^2 + t_1 z_1^3) + t_1^2 z_1^3 - t_1^2 y_1 z_1^3 = 0.$$

All surfaces in this pencil are singular at the point  $(y_1, z_1, t_1) = (0, 0, 0)$ , and this is the only singular point of a general surface in the pencil  $\mathcal{S}^1$  that is contained in the  $\alpha_1$ -exceptional surface. Note also that the  $\alpha_1$ -exceptional surface contains unique base curve of the pencil  $\mathcal{S}^1$ . Without loss of generality, we may assume that its proper transform on  $U$  is the curve  $\widehat{C}_6$ . Then  $M_6 = 2$ . Furthermore, since the rank of the quadratic form of the (local) defining equation of the surface  $S^1$  at the point  $(y_1, z_1, t_1) = (0, 0, 0)$  is two, we can apply arguments of the proof of [3, Lemma 1.12.1] to the pencil  $\mathcal{S}^1$  to deduce the equality  $D_{P_{\{x\},\{z\},\{t\}}} = \delta_6 = m_6 - 1$ . On the other hand, we have

$$4 + m_6 = \text{mult}_{P_{\{y\},\{z\},\{t\}}} \left( 3C_5 + 2C_1 + 2C_2 + 2C_3 + C_4 \right) = 6,$$

so that  $m_6 = 2$ . This gives  $D_{P_{\{y\},\{z\},\{t\}}} = 1$ . Hence, we have

$$[f^{-1}(\infty)] = 5 = \frac{(-K_X)^3}{2} + 2.$$

Suppose that  $X$  is a Fano threefold  $N^{\circ}2.3$ . Recall that

$$f_4 = x^3 y + y(y + z)(xz + xt - t^2) \quad \text{and} \quad g_4 = z(y + z)(xz + xt - t^2).$$

In this case, the set  $\Sigma$  consists of the points

$$P_{\{x\},\{z\},\{t\}}, P_{\{x\},\{y\},\{z\}}, P_{\{x\},\{t\},\{y,z\}};$$

moreover,  $r = 5$  and

$$C_1 = L_{\{x\},\{t\}}, C_2 = L_{\{y\},\{z\}}, C_3 = L_{\{x\},\{y,z\}},$$

the curve  $C_4$  is given by  $z = x^3 + xyt - yt^2 = 0$ , and the curve  $C_5$  is the conic  $y = xz + xt - t^2 = 0$ . Then

$$\begin{aligned} M_1 &= 1, & M_2 &= 2, & M_3 &= 1, & M_4 &= 1, & M_5 &= 1, \\ m_1 &= 6, & m_2 &= 2, & m_3 &= 3, & m_4 &= 1, & m_5 &= 1, \end{aligned}$$

so that

$$[f^{-1}(\infty)] = [S] + 1 + D_{P_{\{x\},\{z\},\{t\}}} + D_{P_{\{x\},\{y\},\{z\}}} + D_{P_{\{x\},\{t\},\{y,z\}}} = 4 + D_{P_{\{x\},\{z\},\{t\}}}$$

by (2.3) and [3, Lemma 1.12.1]. Arguing as in the case N<sup>o</sup>1.11, we can get  $D_{P_{\{x\},\{z\},\{t\}}} = 2$ , so that  $[f^{-1}(\infty)] = 6 = \frac{(-K_X)^3}{2} + 2$ .

Finally, we consider the case when  $X$  is a smooth Fano threefold N<sup>o</sup>9.1. In this case, we have

$$f_4 = x^3y(y^2 + z^2)(xt - xz - t^2) \quad \text{and} \quad g_4 = yz(xt - xz - t^2).$$

Then  $\Sigma$  consists of the points  $P_{\{x\},\{z\},\{t\}}$  and  $P_{\{x\},\{y\},\{z\}}$ ; moreover,  $r = 4$ , and  $C_1 = L_{\{x\},\{t\}}$ ,  $C_2 = L_{\{y\},\{z\}}$ , the curve  $C_3$  is given by  $y = xt - xz - t^2 = 0$ , and  $C_4$  is given by  $z = x^3 + yt(x + t) = 0$ . Observe that

$$\begin{aligned} M_1 &= 1, & M_2 &= 2, & M_3 &= 2, & M_4 &= 1, \\ m_1 &= 6, & m_2 &= 3, & m_3 &= 2, & m_4 &= 1. \end{aligned}$$

Thus, using (2.3) and [3, Lemma 1.12.1], we get

$$[f^{-1}(\infty)] = 6 + D_{P_{\{x\},\{z\},\{t\}}} + D_{P_{\{x\},\{y\},\{z\}}} = 6 + D_{P_{\{x\},\{z\},\{t\}}}.$$

Arguing as in the case N<sup>o</sup>1.11, we get  $D_{P_{\{x\},\{z\},\{t\}}} = 2$ , so that

$$[f^{-1}(\infty)] = \frac{(-K_X)^3}{2} + 2$$

as claimed. □

**Proposition 2.4.** Suppose that  $X$  is a Fano threefold  $N^{\circ}2.1$  or a Fano threefold  $N^{\circ}10.1$ . Then  $[f^{-1}(\infty)] = \frac{(-K_X)^3}{2} + 2$ .

*Proof.* It follows from [3, §2.1] that the following commutative diagram exists:

$$(2.5) \quad \begin{array}{ccccccc} V & \xrightarrow{\pi} & \mathbb{P}^2 \times \mathbb{P}^1 & \longleftarrow & \mathbb{C}^3 & \overset{\gamma}{\dashrightarrow} & \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \longrightarrow Y \longrightarrow Z \\ & \searrow g & \downarrow \phi & & \downarrow \mathfrak{q} & & \downarrow \mathfrak{p} & \downarrow w & \downarrow f \\ & & \mathbb{P}^1 & \longleftarrow & \mathbb{C}^1 & \xlongequal{\quad} & \mathbb{C}^1 & \xlongequal{\quad} & \mathbb{C}^1 \longrightarrow \mathbb{P}^1 \end{array}$$

where  $\mathfrak{q}$  is a surjective morphism,  $\gamma$  is a birational map that is described in [3, §2.1],  $\pi$  is a birational morphism,  $V$  is a smooth threefold, the map  $g$  is a surjective morphism such that  $-K_V \sim g^{-1}(\infty)$ , and  $\phi$  is a rational map that is given by the pencil  $\mathcal{S}$  given by

$$f_{2,3}(x, y, a, b, c) + \lambda g_{2,3}(x, y, a, b, c) = 0,$$

where  $([x : y], [a : b : c])$  is a point in  $\mathbb{P}^1 \times \mathbb{P}^2$ , both  $f_{2,3}$  and  $g_{2,3}$  are bi-homogeneous polynomials of bi-degree  $(2, 3)$ , and  $\lambda \in \mathbb{C} \cup \{\infty\}$ . The diagram (2.5) is similar to (2.2), so that we will follow the proof of Proposition 2.1 and use its notation. The only difference is that  $\mathbb{P}^3$  is now replaced by  $\mathbb{P}^1 \times \mathbb{P}^2$ , and  $S$  is the surface given by  $g_{2,3}(x, y, a, b, c) = 0$ .

Suppose that  $X$  is a Fano threefold  $N^{\circ}2.1$ . Then

$$\begin{aligned} f_{2,3} &= x(x + y)c^3 - y^2(abc - b^2c - a^3), \\ g_{2,3} &= y(x + y)(abc - b^2c - a^3). \end{aligned}$$

Then  $\Sigma$  consists of the point  $P_{\{y\},\{a\},\{c\}}$ , and the base locus of the pencil  $\mathcal{S}$  consists of the curve  $C_1$  given by  $x + y = abc - b^2c - a^3 = 0$ , the curve  $C_2$  given by  $x = abc - b^2c - a^3 = 0$ , the curve  $C_3$  given by  $y = c = 0$ , and the curve  $C_4$  given by  $a = c = 0$ . Hence, we have

$$[f^{-1}(\infty)] = 4 + D_{P_{\{y\},\{a\},\{c\}}} = 4 = \frac{(-K_X)^3}{2} + 2$$

by (2.3) and [3, Lemma 1.12.1], since  $M_1 = 2, M_2 = 1, M_3 = 1, M_4 = 1$ , and  $m_1 = 2$ .

Suppose that  $X$  is a Fano threefold  $N^{\circ}10.1$ . Then

$$\begin{aligned} f_{2,3} &= xyc^3 + (x^2 + y^2)(abc - b^2c - a^3), \\ g_{2,3} &= xy(abc - b^2c - a^3). \end{aligned}$$

In this case, we have  $\Sigma = \emptyset$ , and the base locus of the pencil  $\mathcal{S}$  consists of the curve  $C_1$  given by  $x = abc - b^2c - a^3 = 0$ , the curve  $C_2$  given by  $y = abc - b^2c - a^3 = 0$ , and the curve  $C_3$  given by  $a = c = 0$ . Moreover, one has  $M_1 = 2, M_2 = 2, M_3 = 1, m_1 = 2$  and  $m_2 = 2$ . Hence, using (2.3), we get

$$[f^{-1}(\infty)] = 5 = \frac{(-K_X)^3}{2} + 2$$

as claimed. □

### 3. Fano complete intersections in projective spaces

Let  $X$  be a Fano complete intersection in  $\mathbb{P}^N$  of hypersurfaces of degrees  $d_1, \dots, d_k$ , let  $i_X$  be its Fano index, and let  $\mathfrak{p}$  be the Laurent polynomial

$$\frac{\prod_{i=1}^k (x_{i,1} + \dots + x_{i,d_i-1} + 1)^{d_i}}{\prod_{i=1}^k \prod_{j=1}^{d_i-1} x_{i,j} \prod_{j=1}^{i_X-1} y_j} + y_1 + \dots + y_{i_X-1} \in \mathbb{C}[x_{i,j}^{\pm 1}, y_s^{\pm 1}],$$

which we consider as a regular function on  $(\mathbb{C}^*)^n$ , where  $n = \dim(X)$ . Let  $\Delta$  be the Newton polytope of  $\mathfrak{p}$  in  $\mathcal{N} = \mathbb{Z}^n$ , let  $T_\Delta$  be the toric Fano variety whose fan polytope (convex hull of generators of rays of the fan of  $T_\Delta$ ) is  $\Delta$ . In other words, cones of the fan that defines  $T_\Delta$  are cones of faces of  $\Delta$ . Let

$$\nabla = \left\{ x \mid \langle x, y \rangle \geq -1 \text{ for all } y \in \Delta \right\} \subset \mathcal{M}_{\mathbb{R}} = \mathcal{N}^\vee \otimes \mathbb{R}$$

be the dual to  $\Delta$  polytope. Then  $\nabla$  and  $\Delta$  are *reflexive* (see [24]). Let  $M$  be the matrix

$$\left( \begin{array}{cccc|cccc|ccc} i_X & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & -1 & \dots & -1 \\ 0 & i_X & \dots & 0 & \dots & 0 & 0 & \dots & 0 & -1 & \dots & -1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & i_X & \dots & 0 & 0 & \dots & 0 & -1 & \dots & -1 \\ -i_X & -i_X & \dots & -i_X & \dots & 0 & 0 & \dots & 0 & -1 & \dots & -1 \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline 0 & 0 & \dots & 0 & \dots & i_X & 0 & \dots & 0 & -1 & \dots & -1 \\ 0 & 0 & \dots & 0 & \dots & 0 & i_X & \dots & 0 & -1 & \dots & -1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & i_X & -1 & \dots & -1 \\ 0 & 0 & \dots & 0 & \dots & -i_X & -i_X & \dots & -i_X & -1 & \dots & -1 \\ \hline 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & i_X & -1 & \dots & -1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 & -1 & \dots & i_X & -1 \end{array} \right),$$

which is formed from  $k$  blocks of sizes  $(d_i - 1) \times d_i$  and one last block of size  $(i_X - 1) \times (i_X - 1)$ . Then it follows from [24] that the vertices of  $\nabla$  are the rows of the matrix  $M$ . Note that there is a mistake in the size of the last block in [24].

It has been shown in [13, 24] that  $\mathfrak{p}$  is a toric Landau–Ginzburg model of the variety  $X$  that admits a log Calabi–Yau compactification  $(Z, \mathfrak{f})$ .

**Theorem 3.1** (cf. [24, Problem 11]). The number of irreducible components of the fiber  $\mathfrak{f}^{-1}(\infty)$  equals  $h^0(\mathcal{O}_X(-K_X)) - 1$ .

*Proof.* By [13, Theorem 2.2], the toric variety  $T_\Delta$  is a flat degeneration of  $X$ . Since this degeneration is flat, one has

$$\chi((\mathcal{O}_X(-K_X)) = \chi((\mathcal{O}_{T_\Delta}(-K_{T_\Delta})).$$

On the other hand,  $T_\Delta$  is Fano by construction. Moreover, the singularities of  $T_\Delta$  are Kawamata log terminal by [18, Proposition 3.7]. Thus, by Kodaira vanishing (see e.g. [19, Theorem 2.70]), one has  $h^i(\mathcal{O}_{T_\Delta}(-K_{T_\Delta})) = 0$  for  $i > 0$ . Similarly, applying Kodaira vanishing on a smooth Fano variety  $X$ , we see that  $h^i(\mathcal{O}_X(-K_X)) = 0$  for  $i > 0$ . Therefore, we obtain

$$h^0(\mathcal{O}_X(-K_X)) = h^0(\mathcal{O}_{T_\Delta}(-K_{T_\Delta})).$$

It is well known (see, for instance, [7, §6.3]) the anticanonical linear system of  $T_\Delta$  can be described as the linear system of Laurent polynomials supported on its dual polytope  $\nabla$ . Since  $\nabla$  is reflexive, the dimension  $h^0(-K_{T_\Delta}) - 1$  of this linear system equals to the number of integral points on the boundary of  $\nabla$ . By [24, Theorem 1], the log Calabi–Yau compactification  $(Z, \mathfrak{f})$  is constructed via a crepant toric resolution of  $T_\Delta$  and a sequence of blow ups in smooth centers such that exceptional divisors of these blow ups do not lie over  $\infty$ . In particular, the number of irreducible components of the fiber  $\mathfrak{f}^{-1}(\infty)$  is equal to the number of boundary divisors of the crepant resolution of  $T_\Delta$ , which is equal to the number of integral points in the boundary of  $\nabla$ , since  $\Delta$  is reflexive. This gives the assertion of the theorem. □

*Remark 3.2.* Theorem 3.1 seems to hold in a much more general case of smooth Fano weighted complete intersections. The problem is that the Newton polytope  $\Delta$  in this case is usually is not reflexive, so that  $\nabla$  is not integral. This means that the lattice points count in  $\nabla$  is not enough for the claim, because the log Calabi–Yau compactification procedure (construction of the diagram (1.3)) from [22] does not work. However, at least in some

cases, this procedure can be modified: one can construct the compactification in the face fan of the (non-integral) polytope  $\nabla$  and blow down some of the components of the fiber over infinity. It turns out that the blown down components correspond exactly to the non-integral vertices of  $\nabla$ , so that the arguments of Theorem 3.1 work in these cases. For details, see [25].

### 4. Toric Fano varieties

Let  $X$  be a smooth toric Fano variety of dimension  $n$ , let  $\Delta$  be its fan polytope, and let  $\nabla$  be the dual (integral) polytope, and let  $X^\vee$  be the dual toric variety, i.e. the Fano variety whose fan polytope is  $\nabla$ . Note that  $X^\vee$  can be singular. Suppose that  $X^\vee$  admits a crepant (toric) resolution  $\tilde{X}^\vee \rightarrow X^\vee$ . Let  $\mathbf{p}$  be the Laurent polynomial given by the sum of monomials corresponding to vertices of  $\Delta$ . Then it follows from [22] that  $\mathbf{p}$  defines a toric Landau–Ginzburg model of the Fano variety  $X$  that admits a log Calabi–Yau compactification  $(Z, f)$  such that the exists the following commutative diagram:

$$\begin{array}{ccccc}
 \tilde{X}^\vee & \longleftarrow & (\mathbb{C}^*)^n & \longrightarrow & Z \\
 \downarrow \phi & & \downarrow \mathbf{p} & & \downarrow f \\
 \mathbb{P}^1 & \longleftarrow & \mathbb{C} & \longrightarrow & \mathbb{P}^1
 \end{array}$$

where  $\phi$  is a rational map given by an anticanonical pencil  $\mathcal{S}$  on the (weak Fano) variety  $\tilde{X}^\vee$ . Note that the toric boundary divisor  $\tilde{X}^\vee \setminus (\mathbb{C}^*)^n$  is contained in  $\mathcal{S}$ .

**Proposition 4.1.** The fiber  $f^{-1}(\infty)$  consists of  $h^0(\mathcal{O}_X(-K_X)) - 1$  irreducible components.

*Proof.* Since  $X$  is smooth, every irreducible toric boundary divisor of  $\tilde{X}^\vee$  is isomorphic to a projective space, and the restriction of base locus of the pencil  $\mathcal{S}$  on this divisor is a hyperplane that does not contain torus invariant points. Thus, to obtain  $Z$ , we can blow up (consecutively) irreducible components of the base locus of the pencil  $\mathcal{S}$ , which implies that  $f^{-1}(\infty)$  is the proper transform of the the toric divisor  $\tilde{X}^\vee \setminus (\mathbb{C}^*)^n$ . In particular, the number of irreducible components of the fiber  $f^{-1}(\infty)$  equals the number of integral points of  $\nabla$  minus one. This number is exactly  $h^0(\mathcal{O}_X(-K_X)) - 1$ , which can be described as a linear system of Laurent polynomials supported by  $\nabla$ . □

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