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# Elliptic threefolds with high Mordell-Weil rank

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We present the first examples of smooth elliptic Calabi-Yau threefolds with Mordell-Weil rank 10, the highest currently known value. They are given by the Schoen threefolds introduced by Namikawa; there are six isolated fibers of Kodaira Type IV. We explicitly compute the Shioda homomorphism and the induced height pairing. Compactification of F-theory on these threefolds gives an effective theory in six dimensions which contains ten abelian gauge group factors. We compute the massless matter spectrum. In particular, we show that the charged singlet matter need not reside at enhancement loci of Type $I_2$, as previously believed. We relate the multiplicities of the massless spectrum to genus-zero Gopakumar-Vafa invariants and other geometric quantities of the Calabi-Yau. We show that the gravitational and abelian anomaly cancellation conditions are satisfied. We prove a Geometric Anomaly Cancellation equation and we deduce birational equivalence for the quantities in the spectrum. We explicitly describe a Weierstrass model over $\mathbb{P}^2$ of the Calabi-Yau threefolds as a log canonical model and compare it to a construction by Elkies and classical results of Burkhardt.

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## 1. Introduction

The Mordell-Weil group of sections of an elliptically fibered Calabi-Yau variety is of considerable interest also in physics: it has a special role in establishing an upper bound on the number of massless particle species in a consistent theory of quantum gravity. In fact, the rank of the Mordell-Weil group of an elliptically fibered Calabi-Yau threefold $X$ determines the rank of the abelian (non-Cartan) gauge algebra in compactifications of F-theory (see for example [39, 7]). It is thereby directly related to aspects of quantum gravity. Consistency conditions of certain BPS strings in F-theory compactifications [18] imply various bounds on the rank of the abelian gauge group in minimally supersymmetric compactifications [24]. The results of [24] hence yield interesting implications for algebraic geometry: the bound predicted by physics implies that on an elliptic K3 surface $0 \leq \mathrm{rk}(\mathrm{MW}(\mathrm{K3})) \leq 18$ and for elliptically fibered Calabi-Yau threefolds $X \to B$, $\mathrm{rk}(\mathrm{MW}(X)) \leq 20$ if $B \neq \mathbb{P}^2$ and $\mathrm{rk}(\mathrm{MW}(X)) \leq 24$ if $B = \mathbb{P}^2$ (though it has been conjectured that both bounds can be sharpened further). For elliptic K3 surfaces the bounds are in agreement with known bounds in mathematics [6] and all such possible Mordell-Weil ranks are explicitly realized [22, 19]. Even for K3, however, it is not feasible to find explicit generators for the Mordell-Weil group for all such cases.

For elliptic Calabi-Yau threefolds, by contrast, no bound to the rank of the Mordell-Weil group is known in the mathematics literature. This highly motivates the search for elliptic fibrations for Calabi-Yau threefolds with high Mordell-Weil rank. In Section 2, we present smooth elliptic fibrations $X_i \to B$ with $\mathrm{rk}(\mathrm{MW}(X_i)) = 10$, the highest currently known value, and we investigate their properties as elliptic varieties. The discriminant of the elliptic fibration is supported on six cuspidal curves on the base $B$, the generic singular fibers are of Kodaira Type *II* and enhance to Kodaira Type *IV* over the six cusp points of the cuspidal curves (*Theorems 2.10, 2.9*).

The $X_i$ come from "the Namikawa examples" [27, 31] studied by Namikawa and Rossi for their deformation properties. They are resolutions of threefolds of the form $\bar{X} \stackrel{def}{=} B \times_{\mathbb{P}^1} B'$, with $B$ and $B'$ certain rational elliptic surfaces. These were first introduced by Schoen in [33], and are often referred to as "the Schoens". Depending on the type and relative location of the singular fibers of the two rational elliptic surfaces, $\bar{X}$ can be smooth or singular, with singularities of different types. Schoen first studied particular configurations such that $\bar{X}$ is birational to a smooth Calabi-Yau. The Schoens have interesting arithmetic properties and they have been studied also in many other contexts, from birational geometry to string theory. In the particular context of studying the Mordell-Weil rank of Calabi-Yau threefolds, the authors of [25], building on [15], present several examples of Schoen varieties with a Mordell-Weil rank of up to 9. We conjecture that the Namikawa-like examples lead to the maximal possible Mordell-Weil rank within the class of Schoen manifolds, as we point out before Section 3.1.

The geometry of a Calabi-Yau is closely related to the massless particle spectrum and the relations that the quantities in the spectrum must satisfy, the anomaly cancellation conditions. This connection brings us to four questions: 1) to establish a dictionary for the correspondence, 2) to find a geometric counterpart for the "anomaly cancellation conditions", 3) to calculate explicitly the geometric quantities in the spectrum and 4) to extract the geometric properties implied by the anomalies. In Section 4, we address

1) and 2): we review the results from physics which provide the dictionary for the correspondence, as well as for the anomaly cancellation conditions in subsection 4.1; in subsection 4.2 we define a geometric counterpart formula for the gravitational anomaly cancellation condition, the *Geometric Anomaly Equation* (4) (along the lines of [12] where we write a more general formula).

To address 3), that is to evaluate the spectrum for the Namikawa threefolds, the gravitational and gauge anomalies, and the Geometric Anomaly Equation (4), we need to explicitly determine the Poincaré pairing between $H^2(X_i, \mathbb{Z})$ and $H_2(X_i, \mathbb{Z})$ (*Propositions 3.5 and 3.6*), the Shioda map (*Corollary 3.11*), the height pairings (*Corollary 3.12*), the relative genus-zero Gopakumar-Vafa invariants of holomorphic curves (*Proposition 3.8*) and other geometric invariants of the elliptic Calabi-Yau $X_i$ (*Corollaries 3.14 and 3.13*). The computations leading to the spectrum are involved.

The results allow us to compute the spectrum (*Property 5.1*) and the U(1) charges (*Proposition 5.2*). In particular the analysis exemplifies that the charged singlet matter need not reside at enhancement loci of Type $I_2$, as previously believed. We verify that the anomaly cancellation conditions in physics are satisfied (*Proposition 5.3*) by proving the mathematical counterparts of the gravitational and U(1) anomaly equations [29], along the lines of what was stated in [12]. As a consequence we obtain birational invariants of the non $\mathbb{Q}$-factorial singularities of the Weierstrass model $\bar{X} \to B$ (*Corollary 5.5*), which answers 4).

In the last Section 6 we analyse a family of Weierstrass models $W_{\mathrm{NDE}} \to \mathbb{P}^2$, constructed by Elkies [8], with $\operatorname{rk} \mathrm{MW}(W_{\mathrm{NDE}}/\mathbb{P}^2) = 10$; one particular model shares similarities with the Namikawa threefolds. $W_{\mathrm{NDE}}$ is numerically Calabi-Yau, but Elkies does not make any statement about its minimal resolution. We compare the Weierstrass models over $\mathbb{P}^2$ of the Namikawa-Rossi threefolds with the ones constructed by Elkies by explicitly describing a Weierstrass model $W_{\mathbb{P}^2} \to \mathbb{P}^2$ of the Namikawa Calabi-Yau as a suitable log canonical model (*Corollary 6.1, Theorem 6.3 and Corollary 6.4*). Then we take the first steps in addressing the question of whether $W_{\mathrm{NDE}} \to \mathbb{P}^2$ is birationally Calabi-Yau, by building on classical results of Burkhardt, leaving the construction of a elliptic Calabi-Yau with $\mathrm{rk}(\mathrm{MW}) = 10$ in [8] conjectural.

## 2. The Namikawa-Rossi construction

Let $r : B \to \mathbb{P}^1$ be a smooth rational elliptic surface with section and 6 cuspidal fibers, that is 6 fibers of Kodaira Type *II*. $B$ is defined by the Weierstrass equation $y^2z = x^3 + bz$ in the projective bundle $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1})$, where $b \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6))$ is a general section. Let $r' : B' \to \mathbb{P}^1$ be a different copy of the same surface, with Weierstrass equation $u^2w = v^3 + bw$.

**Lemma 2.1** ([31, 27]). *The threefold* $\bar{X} \stackrel{def}{=} B \times_{\mathbb{P}^1} B'$ *is a Calabi-Yau threefold, singular at* 6 *points* $\{P_1, \cdots, P_6\}$ *of local analytic equation* $\bar{\mathrm{x}}^3 - \bar{\mathrm{v}}^3 - \bar{\mathrm{y}}^2 + \bar{\mathrm{u}}^2 = 0$.

**Lemma 2.2** ([27, 31]). *The threefold* $\bar{X} \stackrel{def}{=} B \times_{\mathbb{P}^1} B' \subset \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}) \times \mathbb{P}^1$ *is endowed with an automorphism* $\tau$ *of order* 6 *induced by the automorphism of the ambient space:*

$$\begin{aligned} \tau_{\mathcal{E}} : \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}) \times \mathbb{P}^1 &\to \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}) \times \mathbb{P}^1 \\ ([x,y,z],\ [v,u,w],\ [\lambda_0,\lambda_1]) &\mapsto ([x,y,z],\ [\epsilon v, -u, w],\ [\lambda_0,\lambda_1]), \end{aligned}$$

*with* $\epsilon$ *a primitive cube root of the unity.*

**Definition 2.3.** Let $D_i \stackrel{def}{=} \bar{X} \cap \{(\tau_{\mathcal{E}})^i([x,y,z],\ [x,y,z],\ [\lambda_0,\lambda_1])\}$, $0 \le i \le 5$.

$D_0$ is then the diagonal.

**Lemma 2.4** ([27, 31]). *Each divisor* $D_i$ *contains the singular locus* $\{P_1, \cdots, P_6\}$ *of* $\bar{X}$.

1. *The local equation around a fixed point* $P_j \in \bar{X}$, $j = 1, \cdots, 6$, *can be written as*
$$xv[(1+\epsilon)v - \epsilon x] = yu.$$
2. *The local equations of* $D_i, D_{i+1}, D_{i+3}, D_{i+4}$, *with the indices taken mod* 6, *around* $P_j \in \bar{X}$ *can be taken respectively to be*
$$\begin{cases} x = 0 \\ y = 0 \end{cases}, \quad \begin{cases} v = 0 \\ u = 0 \end{cases}, \quad \begin{cases} x = 0 \\ u = 0 \end{cases}, \quad \begin{cases} v = 0 \\ y = 0 \end{cases}.$$

[Note a change in notation with respect to [31], in particular for $D_{i+4}$.]

*Remark* 2.5. Note in fact that $\forall\, i$, $0 \le i \le 5$, we can write the local equation around a fixed point $P_j \in \{P_1, \cdots, P_6\}$ of $\bar{X}$ as

$$(\bar{\mathrm{x}} - \bar{\mathrm{v}}) \cdot (\bar{\mathrm{x}} - \epsilon\bar{\mathrm{v}}) \cdot (\bar{\mathrm{x}} - \epsilon^2\bar{\mathrm{v}}) = (\bar{\mathrm{y}} + \bar{\mathrm{u}}) \cdot (\bar{\mathrm{y}} - \bar{\mathrm{u}})$$

with

$$\begin{cases} y = \bar{\mathrm{y}} + (-1)^i\bar{\mathrm{u}} \\ u = \bar{\mathrm{y}} - (-1)^{i+1}\bar{\mathrm{u}} \end{cases}, \quad \begin{cases} x = \bar{\mathrm{x}} - \epsilon^i\bar{\mathrm{v}} \\ v = \bar{\mathrm{x}} - \epsilon^{i+1}\bar{\mathrm{v}} \end{cases}.$$

**Theorem 2.6** ([31, 27]). *The threefold $\bar{X} \stackrel{def}{=} B \times_{\mathbb{P}^1} B'$ is a Calabi-Yau threefold, singular at 6 points $\{P_1, \cdots, P_6\}$ of local analytic equation $\bar{\mathrm{x}}^3 - \bar{\mathrm{v}}^3 - \bar{\mathrm{y}}^2 + \bar{\mathrm{u}}^2 = 0$.*

1. *$b_2(\bar{X}) = 19$ and $\rho(\bar{X}) = 19$, where $\rho(\bar{X})$ denotes the rank of the Picard group.*
2. *The singularities are terminal and not $\mathbb{Q}$-factorial.*
3. *There are 6 Weil divisors $D_i$, $0 \leq i \leq 5$, defined in Definition 2.3, which are not Cartier.*
4. *There exist 6 different small projective resolutions $\varphi_i : X_i \to \bar{X}$, $0 \leq i \leq 5$. Each $X_i$ is a smooth Calabi-Yau threefold.*
5. *$X_i$ is obtained by the consecutive blow up of the divisors $D_i$ and then of the strict transform of $D_{i+1}$. The small resolution can be described using the local equations in Lemma 2.4.*
6. *The exceptional loci of any resolution $\varphi_i : X_i \to \bar{X}$ are six disjoint pairs $\{\mathcal{P}_j^{i,A}, \mathcal{P}_j^{i,B}\}$, $1 \leq j \leq 6$, of $\mathbb{P}^1$s with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, intersecting in one point. The threefolds $X_i$ are connected to each other by flops of the exceptional curves.*
7. *$\chi_{top}(X_i) = 36$, $h^{1,1}(X_i) = \rho(X_i) = 21$, $h^{2,1}(X_i) = 3$.*

**Lemma 2.7.** *Let $\bar{\pi} : \bar{X} \to B$ be the elliptic fibration on the singular threefold induced by the projection on $B$: $\bar{X} = B \times_{\mathbb{P}^1} B' \stackrel{\bar{\pi}}{\to} B \stackrel{r}{\to} \mathbb{P}^1$. Let $\{r^{-1}(\mathfrak{p}_j)\}_{1 \leq j \leq 6} \in B$ denote the 6 cuspidal fibers of $r$, $\mathfrak{p}_j \in \mathbb{P}^1$. Let $\{p_j \in r^{-1}(\mathfrak{p}_j)\}_{1 \leq j \leq 6}$ be the cuspidal points of these fibers in $B$. Then:*

1. *$\bar{\pi}(P_j) = p_j$, i.e. the image of the singular point $P_j \in \bar{X}$ is the cuspidal point of the singular fiber $r^{-1}(\mathfrak{p}_j)$, $1 \leq j \leq 6$.*
2. *The support of the discriminant locus of the elliptic fibration $\bar{\pi}$ is the disjoint union of the 6 cuspidal curves $\{r^{-1}(\mathfrak{p}_j)\}_{1 \leq j \leq 6}$.*
3. *All the singular fibers of $\bar{\pi}$, that is the fiber over the points $q_j \in r^{-1}(\mathfrak{p}_j)$, are cuspidal curves (Kodaira type II).*
4. *The Weil divisors $D_i$ are smooth and are rational sections of the fibration $\bar{\pi}$.*

*Proof.* The statements follow from the construction and the Lemmas 2.2 and 2.4. □

**Definition 2.8.** Let $\varphi_i : X_i \to \bar{X}$ be one of the resolutions in Theorem 2.6, $0 \leq i \leq 5$.

For $0 \leq k \leq 5$, $D_k^i$ denotes the strict transform of the divisor $D_k$ by $\varphi_i$.

**Theorem 2.9.** *Let $\bar{\pi} : \bar{X} \to B$ and $X_i$ be as above and let $\bar{\pi} \circ \varphi_i \stackrel{def}{=} \pi_i : X_i \to B$ be one of the induced elliptic fibrations, $0 \le i \le 5$. The elliptic fibration $\pi_i : X_i \to B$ has $\mathrm{rk}(\mathrm{MW}(X_i/B)) = 10$.*

*Proof.* The statement follows from Theorem 2.6 and from the Tate-Shioda-Wazir Theorem [38]. The Tate-Shioda-Wazir Theorem in fact states: $\mathrm{rk}(\mathrm{MW}(X_i/B)) = \rho(X_i) - \rho(B) - 1 = 21 - 10 - 1 = 10$. □

The elliptic fibration of the smooth Calabi-Yau threefolds is described explicitly as follows:

**Theorem 2.10.** *Let $\bar{\pi} : \bar{X} \to B$ and $X_i$ be as above and let $\bar{\pi} \circ \varphi_i \stackrel{def}{=} \pi_i : X_i \to B$ be one of the induced elliptic fibrations, $0 \le i \le 5$.*

1. *$D^i_k$ is a section of the fibration $\pi_i$. $D^i_i$ and $D^i_{i+1}$ are independent elements of the free part of the Mordell-Weil group.*
2. *For all $i, j$, $\pi_i^{-1}(p_j)$, the fiber of $\pi_i$ over a singular point $p_j \in B$ of the discriminant, consists of 3 rational curves $\mathcal{P}^{i,A}_j$, $\mathcal{P}^{i,B}_j$, $\mathcal{P}^{i,0}_j$.*
3. *$\mathcal{P}^{i,A}_j$, $\mathcal{P}^{i,B}_j$, $\mathcal{P}^{i,0}_j$ intersect mutually transversely at a point (as a fiber of Kodaira type IV). $\mathcal{P}^{i,0}_j$ is the strict transform of the cuspidal curve $\bar{\pi}^{-1}(q_j)$; $\mathcal{P}^{i,A}_j$, $\mathcal{P}^{i,B}_j$ are the exceptional $\mathbb{P}^1$ for the first and the second blow up respectively.*
4. *If $q$ is a smooth point of the discriminant, $\pi_i^{-1}(q)$ is a cuspidal curve (Kodaira type II).*

*Proof.* (4) follows from Lemma 2.7. Theorem 2.6, Lemmas 2.7, 2.4 and 2.2 provide the local equations around each singular point as well as the geometric description of the singular Calabi-Yau and a resolution. We then can write the local equations of the smooth Calabi-Yau, and of $\mathcal{P}^{i,A}_j$, $\mathcal{P}^{i,B}_j$, $\mathcal{P}^{i,0}_j$.

(1) follows from the analysis of these local equations and from Theorem 2.9. A direct computation in the local equations proves (2) and (3). The linear independence of the sections $D^i_i$ and $D^i_{i+1}$ can also be checked explicitly from the intersection numbers in Proposition 3.5. □

The explicit description of the fibration in Theorem 2.10 gives directly $\chi_{top}(X_i) = 36$.

*Remark* 2.11. In the Namikawa examples studied, both elliptic rational surfaces $B$ and $B'$ in the fiber product $\bar{X} = B \times_{\mathbb{P}^1} B'$ are engineered to have six Type *II* fibers over the same points, which leads to 6 isolated singular points in $\bar{X}$. The resulting high Mordell-Weil rank of ten $\mathrm{MW}(\bar{X}/B)$ is a consequence of the fact that the 6 singular points are non $\mathbb{Q}$-factorial and that there are no other $\mathbb{Q}$-factorial singularities. The resolutions produce two additional independent curve classes in the fiber of the resolved threefold $X_i$, and no (Weil) divisor. Hence the Mordell-Weil group of the resolved threefold $X_i$ is generated by the eight generators present also on a generic Schoen manifold (with $B \neq B'$ general rational elliptic surfaces), together with two more generators associated with two independent rational sections dual to the two additional fibral curve classes from the resolution (in the Type $IV$ fibers). This is to be compared with the special threefolds studied explicitly in [33, 25] with a Mordell-Weil rank of 9: there, $B$ and $B'$ have $I_1$ fibers over the same 12 points, which leads to 12 isolated non $\mathbb{Q}$-factorial singular points in $\bar{X}$. But the resolution gives rise to one extra curve class in the fiber, leading to $8+1=9$ independent generators of the Mordell-Weil group. We believe that the collision of six Type $II$ fibers in the Namikawa threefold gives rise to the maximal possible number of independent curve classes in the fiber without inducing a singularity in codimension one, whose resolution would subtract from the Mordell-Weil group.

## 3. The geometry of the spectrum

The geometry of the Calabi-Yau and its invariants are directly related to the massless particle spectrum. We review the correspondence in Section 4.

To define the dictionary between the Spectrum and the geometry, to evaluate the spectrum, the gravitational and gauge anomalies in physics, and the corresponding formula (4) in geometry, we need to determine the pairing between $H^2(X_i, \mathbb{Z})$ and $H_2(X_i, \mathbb{Z})$, the Shioda map, the height pairings and other geometric invariants of the Calabi-Yau.

### 3.1. Cohomology, homology, pairings, Gopakumar-Vafa invariants

From now on we fix a smooth resolution $X_i$ as in Theorem 2.10 and Theorem 2.9 and an index $i$.

**Definition 3.1.** (i) Let $f$ and $\mathsf{s}_k$, $0 \leq k \leq 8$, respectively denote the classes of the fiber, the zero-section and the generators of the Mordell-Weil group $\mathrm{MW}(B/\mathbb{P}^1)$; they form a basis of $H_2(B)$.

(ii) Similarly, let $\mathsf{s'}_l$, $0 \leq l \leq 8$ denote the classes of the linearly independent sections of $r' : B' \to \mathbb{P}^1$ in $H_2(B')$.

**Definition 3.2.** Let $S_l \overset{def}{=} (\bar{\pi}')^*(\mathsf{s'}_l)$, $0 \leq l \leq 8$, where $\bar{\pi}' : \bar{X} = B \times_{\mathbb{P}^1} B' \to B'$. We also denote by $S_l$ its isomorphic image in $X_i$.

We take $S_0$ to be the zero section; the sections $\{S_1, \cdots, S_8\}$ are independent generators of the Mordell-Weil group $\mathrm{MW}(\bar{X}/B)$. $S_0$ is then the zero section of the Mordell-Weil group $\mathrm{MW}(X_i/B)$ and $\{S_1, \cdots, S_8, D_i^i, D_{i+1}^i\}$ are independent sections, by Lemma 2.4.

**Definition 3.3.** Let $\mathcal{E}$ denote the class of the fiber of $\pi_i$,

$$\hat{\mathsf{s}}_k = S_0 \cdot {\pi_i}^*(\mathsf{s}_k), \quad 0 \le k \le 8,$$
$$\hat{\mathsf{f}} = S_0 \cdot {\pi_i}^*(f), \quad \text{and}$$
$$\hat{\ell}_l = S_l \cdot {\pi_i}^*(\mathsf{s}_0), \quad 1 \le l \le 8.$$

We can then conclude:

**Proposition 3.4.** *Fix any index $i$, $0 \le i \le 5$ and $j$, $1 \le j \le 6$. With the notation as in Theorem 2.10 and Definitions 3.1, 3.2 and 3.3:*

1. $\{{\pi_i}^*(f),\ {\pi_i}^*(\mathsf{s}_k),\ S_l,\ D_i^i,\ D_{i+1}^i\}$, *with $0 \le k \le 8$, $0 \le l \le 8$, is a basis of the Neron-Severi group $NS(X_i) \simeq H^2(X_i, \mathbb{Z}) \simeq c_1(Pic(X_i))$.*
2. $\{\mathcal{E},\ \hat{\mathsf{f}},\ \hat{\mathsf{s}}_k,\ \hat{\ell}_l,\ \mathcal{P}_j^{i,A},\ \mathcal{P}_j^{i,B}\}$, $0 \le k \le 8$, $1 \le l \le 8$ *is a basis of $H_2(X_i, \mathbb{Z})$.*

**Proposition 3.5.** *Fix any index $i$, $0 \le i \le 5$, and $j$, $1 \le j \le 6$. With the notation as in Theorem 2.10 and in Definitions 2.8, 3.1, 3.2 and 3.3, we find the following intersection numbers:*

| | $\mathcal{E}$ | $\hat{\mathsf{f}}$ | $\hat{\mathsf{s}}_0$ | $\hat{\mathsf{s}}_k$ | $\hat{\mathsf{s}}_{k'}$ | $\hat{\ell}_l$ | $\hat{\ell}_{l'}$ | $\mathcal{P}_j^{i,A}$ | $\mathcal{P}_j^{i,B}$ | $\mathcal{P}_j^{i,0}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\pi_i^*(f)$ | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| $\pi_i^*(\mathsf{s}_0)$ | 0 | 1 | $-1$ | 0 | 0 | $-1$ | $-1$ | 0 | 0 | 0 |
| $\pi_i^*(\mathsf{s}_k)$ | 0 | 1 | 0 | $-1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $S_0$ | 1 | 0 | $-1$ | $-1$ | $-1$ | 0 | 0 | 0 | 0 | 1 |
| $S_l$ | 1 | 0 | 0 | 0 | 0 | $-1$ | 0 | 0 | 0 | 1 |
| $D_i^i$ | 1 | 1 | $-1$ | 0 | 0 | 0 | 0 | $-1$ | 0 | 2 |
| $D_{i+1}^i$ | 1 | 1 | $-1$ | 0 | 0 | 0 | 0 | 0 | $-1$ | 2 |
| $D_{i+3}^i$ | 1 | 1 | $-1$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $D_{i+4}^i$ | 1 | 1 | $-1$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

*In the table, $k \neq k'$, $1 \le k, k' \le 8$ and $l \neq l'$, $1 \le l, l' \le 8$. Above the double line there are generators of $NS(X_i)$; we will need also the intersections below the double line.*

Note that $D_i^i \cdot (\mathcal{P}_j^{i,0} + \mathcal{P}_j^{i,A} + \mathcal{P}_j^{i,B}) = D_i^i \cdot \mathcal{E} = 1$ as it should be for a section and a fiber (similarly for $D_{i+1}^i$).

*Proof.* We need to verify the following intersections:

1. $D^i_i \cdot \mathcal{P}^{i,0}_j = D^i_{i+1} \cdot \mathcal{P}^{i,0}_j = 2$,
2. $D^i_i \cdot \mathcal{P}^{i,A}_j = D^i_{i+1} \cdot \mathcal{P}^{i,B}_j = -1$,
3. $D^i_i \cdot \mathcal{P}^{i,B}_j = D^i_{i+1} \cdot \mathcal{P}^{i,A}_j = 0$,
4. $D^i_{i+4} \cdot \mathcal{P}^{i,0}_j = D^i_{i+3} \cdot \mathcal{P}^{i,0}_j = 0$,
5. $D^i_{i+4} \cdot \mathcal{P}^{i,A}_j = D^i_{i+3} \cdot \mathcal{P}^{i,B}_j = 0$,
6. $D^i_{i+4} \cdot \mathcal{P}^{i,B}_j = D^i_{i+3} \cdot \mathcal{P}^{i,A}_j = 1$,
7. $S_k \cdot \mathcal{P}^{i,0}_j = 1$, $0 \le k \le 8$,
8. $S_k \cdot \mathcal{P}^{i,A}_j = 0$, $S_k \cdot \mathcal{P}^{i,B}_j = 0$, $0 \le k \le 8$.

(7) and (8) follow from Lemma 2.7. Theorem 2.6 and Lemma 2.4 provide the geometric description and the local equations around each singular point and of $D_i$, $D_{i+1}$, $D_{i+3}$, $D_{i+4}$. We then can write the local equations of the smooth Calabi-Yau, of $D^i_i, D^i_{i+1}, D^i_{i+3}, D^i_{i+4}$. For illustration, we exemplify (1), (2), (3) in Appendix B. (4), (5) and (6) follows from a similar analysis of these local equations. We note also that in a neighborhood of the resolutions of each singular point $D^i_i \cap D^i_{i+1} = \mathcal{P}^{i,A}_j \cup \mathcal{P}^{i,B}_j$ □

In Section 3.2, we verify the cancellation of the abelian anomalies with the Shioda-map and height pairings. To that end, we need to describe the intersections of the elements in $NS(X_i)$.

**Proposition 3.6.** *With the same hypothesis as in Proposition 3.4:*

1. $S_k \cdot S_k = -\hat{\mathsf{f}} \quad \forall k.$
2. $S_0 \cdot D^i_i = S_0 \cdot D^i_{i+1} = S_0 \cdot D^i_{i+3} = S_0 \cdot D^i_{i+4} = \hat{\mathsf{s}}_0$.
3. $S_k \cdot D^i_i = S_k \cdot D^i_{i+1} = S_k \cdot D^i_{i+3} = S_k \cdot D^i_{i+4} = \mathcal{F}_k$ *is a section of the abelian fibration* $X_i \to \mathbb{P}^1$ *such that* $\pi_{i*}(\mathcal{F}_k) = \mathsf{s}_k$.
4. $D^i_i \cdot D^i_{i+1} = \hat{\mathsf{s}}_0 + \sum_j (\mathcal{P}^{i,A}_j + \mathcal{P}^{i,B}_j)$.
5. $D^i_i \cdot D^i_{i+3} = \hat{\mathsf{s}}_0 + \hat{\mathcal{C}}$ *and* $D^i_{i+1} \cdot D^i_{i+4} = \hat{\mathsf{s}}_0 + \hat{\mathcal{C}}$.
   $\mathcal{C} = \pi_{i*}(\hat{\mathcal{C}})$ *is a smooth curve of genus* 4 *such that* $[\mathcal{C}]^2 = 9$, $\mathcal{C} \cdot \mathsf{s}_0 = 0$ *and* $\mathcal{C} \cdot f = 3$.
6. $D^i_i \cdot D^i_i = 2\pi_i{}^*(f) \cdot D^i_i + 3\hat{\mathsf{s}}_0 - \hat{\mathcal{C}}$.
7. $D^i_{i+1} \cdot D^i_{i+1} = 2\pi_i{}^*(f) \cdot D^i_{i+1} + 3\hat{\mathsf{s}}_0 - \hat{\mathcal{C}}$.

*In homology:* $[\mathcal{C}] = [3\mathsf{s}_0 + f]$.

*Proof.* (1) follows from an argument in [9] (see (7.30) on p. 730). (2), (3) and (4) follow from the analysis of Lemma 2.2 and Lemma 2.4.

$D^i_i$ and $D^i_{i+3}$ ($D^i_{i+1}$ and $D^i_{i+4}$ respectively) intersect in the zero locus $y = v = 0$ in $\bar{X}$. The intersection locus has two components, $z = w = 0$ and a remaining curve $\bar{\mathcal{C}}$. The first component is in the resolution $X_i$ in the class $\hat{\mathsf{s}}_0$; the strict transform of $\bar{\mathcal{C}}$, $\hat{\mathcal{C}}$, is a smooth curve. Its projection to $B$ intersects the general fiber $f$ in three distinct points, and in one point at the six cusps (where $b = 0$). That is, $\mathcal{C}$ is a $3:1$ cover of $\mathbb{P}^1$ totally ramified at 6 points. It is then a curve of genus 4, by the Riemann-Hurwitz formula. The adjunction formula applied to $(\mathcal{C}, B)$, implies that $\mathcal{C}^2 = 9$. This proves (5).

To prove (6) and (7) we need the following Lemma 3.7 combined with (1)–(5): □

**Lemma 3.7.** *With the same hypothesis as in Proposition 3.4:*

1. $D^i_i = 2\pi_i^*(f) + 2\pi_i^*(\mathsf{s}_0) + 2S_0 - D^i_{i+3}$,
2. $D^i_{i+1} = 2\pi_i^*(f) + 2\pi_i^*(\mathsf{s}_0) + 2S_0 - D^i_{i+4}$.

*Proof.* We apply the pairings listed under the double lines in the Table in Proposition 3.5 and solve the systems. □

**Proposition 3.8.** *The genus-zero Gopakumar-Vafa invariants on the sublattice of curve classes generated by $\mathcal{P}^{i,A}_j$ and $\mathcal{P}^{i,B}_j$ are*

$$n_{\{0,[\mathcal{P}^{i,A}_j]\}} = 1, \quad n_{\{0,[\mathcal{P}^{i,B}_j]\}} = 1, \quad n_{\{0,[\mathcal{P}^{i,A}_j+\mathcal{P}^{i,B}_j]\}} = 1$$

*and* 0 *otherwise.*

*Proof.* This follows from [3, 2]. Note that each of these curves is super-rigid [4, page 291]. □

### 3.2. The Shioda map and height pairings

**Definition 3.9.** We denote the independent elements of the Mordell-Weil group $\mathrm{MW}(X_i/B)$ as $\mathbb{S}_a$, $a = 1, \dots, 10$ with $\mathbb{S}_l = S_l$, $1 \leq l \leq 8$, and $\mathbb{S}_9 = D^i_i$, $\mathbb{S}_{10} = D^i_{i+1}$.

**Definition 3.10.** With the notation as in Definition 3.2, the image of the set of independent sections $\mathbb{S}_a$, $1 \leq a \leq 10$, within $\mathrm{MW}(X_i/B)$ under the Shioda homomorphism

$$\sigma : \mathrm{MW}(X_i/B) \to NS(X_i) \otimes \mathbb{Q}$$

introduced in [36, 38, 29] is defined to be

$$\sigma(\mathbb{S}_a) \stackrel{def}{=} \mathbb{S}_a - S_0 - {\pi_i}^* \pi_{i*}((\mathbb{S}_a - S_0) \cdot S_0).$$

The associated height pairings take the form

$$b_{a,b} \overset{def}{=} -(\pi_i)_*(\sigma(\mathbb{S}_a)\cdot\sigma(\mathbb{S}_b))$$

and are valued in $H_2(B)$.

Proposition 3.5 enables us to prove the following Corollaries:

**Corollary 3.11.** *With the notation as in Definition 3.10, the Shioda map images take the form*

$$\begin{aligned}
\sigma(\mathbb{S}_l) &= \sigma(S_l) = S_l - S_0 - {\pi_i}^*(f), \quad 1\le l\le 8,\\
\sigma(\mathbb{S}_9) &= \sigma(D_i^i) = D_i^i - S_0 - {\pi_i}^*(\mathsf{s}_0+f),\\
\sigma(\mathbb{S}_{10}) &= \sigma(D_{i+1}^i) = D_{i+1}^i - S_0 - {\pi_i}^*(\mathsf{s}_0+f).
\end{aligned}$$

*They have the following intersections in $X_i$:*

$$\begin{aligned}
\sigma(\mathbb{S}_9)\cdot\sigma(\mathbb{S}_9) &= \hat{\mathsf{s}}_0 - \hat{\mathcal{C}} + \hat{\mathsf{f}} + \mathcal{E},\\
\sigma(\mathbb{S}_{10})\cdot\sigma(\mathbb{S}_{10}) &= \hat{\mathsf{s}}_0 - \hat{\mathcal{C}} + \hat{\mathsf{f}} + \mathcal{E},\\
\sigma(\mathbb{S}_9)\cdot\sigma(\mathbb{S}_{10}) &= \hat{\mathsf{s}}_0 - \hat{\mathsf{s}}_0 - \hat{\mathsf{s}}_0 - {\pi_i}^*(f)\cdot D_i^i - \hat{\mathsf{s}}_0 - \hat{\mathsf{f}} + \hat{\mathsf{s}}_0 + \hat{\mathsf{f}} - \hat{\mathsf{s}}_0 + \hat{\mathsf{s}}_0 - \mathcal{E}\\
&\quad + \mathcal{E} - {\pi_i}^*(f)\cdot D_{i+1}^i + \hat{\mathsf{f}} + \mathcal{E} = -\hat{\mathsf{s}}_0 + \hat{\mathsf{f}} - {\pi_i}^*(f)\cdot D_i^i\\
&\quad - {\pi_i}^*(f)\cdot D_{i+1}^i + \mathcal{E},\\
\sigma(\mathbb{S}_k)\cdot\sigma(\mathbb{S}_k) &= -3{\pi_i}^*(f)\cdot S_k + \hat{\mathsf{f}},\\
\sigma(\mathbb{S}_k)\cdot\sigma(\mathbb{S}_{k'}) &= -{\pi_i}^*(f)\cdot S_{k'} - {\pi_i}^*(f)\cdot S_k + \hat{\mathsf{f}}, \quad k\neq k',\\
\sigma(\mathbb{S}_k)\cdot\sigma(\mathbb{S}_9) &= \hat{\mathsf{s}}_k - \hat{\mathsf{s}}_0 - {\pi_i}^*(f)\cdot S_k - {\pi_i}^*(f)\cdot D_i^i + \hat{\mathsf{f}} + \mathcal{E},\\
\sigma(\mathbb{S}_k)\cdot\sigma(\mathbb{S}_{10}) &= S_k\cdot S_{10} - \hat{\mathsf{s}}_0 - {\pi_i}^*(f)\cdot S_k - {\pi_i}^*(f)\cdot D_{i+1}^i + \hat{\mathsf{f}} + \mathcal{E}.
\end{aligned}$$

**Corollary 3.12.** *The associated height-pairings are*

$$\begin{aligned}
b_{9,9} &= -\mathsf{s}_0 + \mathcal{C} - f,\\
b_{10,10} &= -\mathsf{s}_0 + \mathcal{C} - f,\\
b_{9,10} &= \mathsf{s}_0 + f,\\
b_{k',k} &= f, \quad k\neq k',\\
b_{k,k} &= 2f,\\
b_{k,9} &= \mathsf{s}_0 - \mathsf{s}_k + f,\\
b_{k,10} &= \mathsf{s}_0 - \mathsf{s}_k + f.
\end{aligned}$$

*Proof.* Note that by construction $(\pi_i)_*(\hat{\mathsf{s}}_k) = \mathsf{s}_k \in H_2(B)$. $\square$

**Corollary 3.13.** *The only non-vanishing intersection numbers of the height pairings of Corollary 3.12 are, for* $1 \leq k, l \leq 8$*:*

$$\begin{aligned} b_{9,9} \cdot b_{k,l} &= b_{10,10} \cdot b_{k,l} = 2(1 + \delta_{kl}), \\ b_{9,k} \cdot b_{9,l} &= -(1 + \delta_{kl}), \\ b_{9,9} \cdot b_{9,9} &= b_{10,10} \cdot b_{10,10} = 4, \\ b_{9,9} \cdot b_{10,10} &= 4, \\ b_{9,10} \cdot b_{9,10} &= 1, \\ b_{9,9} \cdot b_{9,10} &= b_{10,10} \cdot b_{9,10} = 2. \end{aligned}$$

**Corollary 3.14.** *The only non-vanishing intersections of the height-pairings of Corollary 3.12 with* $(-K_B)$*, the class of the anti-canonical divisor on the base* $B$*, are*

$$\begin{aligned} (-K_B) \cdot b_{9,9} &= (-K_B) \cdot b_{10,10} = 2, \\ (-K_B) \cdot b_{9,10} &= 1. \end{aligned}$$

## 4. The spectrum, charges, anomaly cancellation and geometric invariants

### 4.1. General results from F-theory

Compactification of F-theory on $X_i$ gives rise to an effective supergravity theory in six dimensions with $N = (1, 0)$ supersymmetry. Before providing the details of the effective theory, we collect general results for F-theory compactifications on elliptic threefolds that have been derived in the physics literature. For derivations and the original references we refer to the survey articles [37, 39, 7].

For simplicity of presentation and consistently with the Namikawa-Rossi example, we assume that $\pi : Y \to B$ is a smooth elliptically fibered Calabi-Yau threefold with base $B$ and zero-section $S_0$. Without loss of generality we assume that the fibration is equidimensional and that $B$ is smooth. We also assume that the Weierstrass model of $Y$, $\bar{Y} \to B$ has no singularities appearing in codimension one, that is, in physics language, the associated non-abelian gauge group associated in F-theory is trivial.

We denote by $\mathbb{S}_a$ a set of independent sections in the Mordell-Weil group $MW(Y/B)$ with Shioda map images $\sigma(\mathbb{S}_a)$ and corresponding height-pairings $b_{a,b} = -\pi_*(\sigma(\mathbb{S}_a) \cdot \sigma(\mathbb{S}_b))$, as in Definition 3.10.

**Result (Physics) 4.1** (Gauge group and spectrum). *The (abelian) gauge group $G$ of F-theory compactified on $Y$ defined above is $G = \prod_{a=1}^{r} U(1)_a$, where $r$ is the rank of the Mordell-Weil group $MW(Y/B)$. The massless physical spectrum comprises*

1. $V = h^{1,1}(Y) - h^{1,1}(B) - 1 = \mathrm{rk}(MW(Y/B))$ *vector multiplets,*
2. $T = h^{1,1}(B) - 1$ *tensor multiplets,*
3. $H = H_{unch} + H_{ch}$ *hypermultiplets, where* $H_{unch} = h^{2,1}(Y) + 1$ *is the number of uncharged multiplets and* $H_{ch}$ *the number of hypermultiplets charged under* $G$*,*
4. *one universal gravity multiplet.*

(1), (2) and (4) immediately provide a correspondence between the massless spectrum and the birational invariants of the elliptic Calabi-Yau $Y$. As for (3) we have:

**Result (Physics) 4.2** (Charged matter multiplicities). *The charged hypermultiplets $H_{ch}$ in (3) are in 1-1 correspondence with the holomorphic curves in the fiber of $Y$ with vanishing intersection with the zero-section $S_0$ (the exceptional fibers of the Weierstrass model). $H_{ch}$ is computed by either*

*a) their Gopakumar-Vafa invariants at genus zero or*
*b) the localised deformations of the singular fibration $\bar{Y} \to B$.*

*Proof.* Via duality with M-theory compactified on $Y$, massless hypermultiplets charged under $G$ in F-theory are in 1-1 correspondence with the possible wrappings of M2-branes on the exceptional fibers. The Gopakumar-Vafa index of a curve $C$ at genus zero counts the number of hypermultiplets obtained by wrapping M2-branes on $C$ [10]. See e.g. [23, 28, 20] for applications in F-theory on threefolds. The correspondence with the localised deformations of $\bar{Y}$ follows from [16]. □

**Result (Physics) 4.3** ($U(1)_a$ charges). *The $U(1)_a$ charges of the massless hypermultiplets associated with the exceptional fibers are computed as the intersections of the respective fibers with the Shioda map images $\sigma(\mathbb{S}_a)$.*

*Proof.* For a derivation via duality with M-theory see [29] as well as the reviews [39, 7]. □

**Result (Physics) 4.4** (Anomalies [13, 29]). *The gravitational, mixed gravitational $-U(1)_a - U(1)_b$ and abelian $U(1)_a - U(1)_b - U(1)_c - U(1)_d$ anomalies are cancelled by the six-dimensional Green-Schwarz mechanism if the following equations hold:*

$$H - V + 29T = 273 \qquad (1)$$

$$(2) \qquad (-K_B)\cdot b_{a,b} \;=\; \frac{1}{6}\sum_I N_I q_a^I q_b^I$$

$$(3) \qquad b_{a,b}\cdot b_{c,d} + b_{a,c}\cdot b_{d,b} + b_{a,d}\cdot b_{c,b} \;=\; \sum_I N_I q_a^I q_b^I q_c^I q_d^I.$$

*$b_{a,b}$ on the the lefthand side of (2) and (3) is defined in Section 3.2, Definition 3.10.*

In (2) and (3), the righthand side computes the anomaly coefficient for the quartic 1-loop anomalies with two and four abelian external legs, respectively, in a six-dimensional $N = (1, 0)$ supergravity with $N_I$ massless hypermultiplets of $U(1)_a$ charge $q_a^I$. The lefthand side of (2) and (3) represents the contribution to the anomaly from the Green-Schwarz counterterms.

### 4.2. The geometry of the anomaly cancellations

From a more general conjecture in [12] it follows that for the smooth elliptically fibered Calabi-Yau threefold $Y \to B$ equ. (1) translates into the relation

$$(4) \qquad \boxed{30K_B^2 + \frac{1}{2}\chi_{top}(Y) = \sum_{Q'} c_{Q'},}$$

where $\sum_{Q'} c_{Q'} = H_{ch}$ are the hypermultiplets charged only by the abelian factors $U(1)_a$.

The Geometric Anomaly Equation (4) states that the hypermultiplets charged only by the abelian factors localise at singular points $Q'$ of the discriminant with multiplicity $c_{Q'}$, giving $H_{ch} = \sum_{Q'} c_{Q'}$.

## 5. F-theory on the Namikawa-Rossi threefold

We now apply these general results to F-theory compactified on the Namikawa-Rossi threefold.

**Proposition 5.1.** *Let $X_i$ be a smooth minimal resolution of the Namikawa-Rossi threefold and consider F-theory compactified on $X_i$. The gauge group is a product of $\mathrm{rk}(\mathrm{MW}(X_i/B)) = 10$ abelian gauge group factors, $G = \prod_{a=1}^{10} U(1)_a$. Each $U(1)_a$ gauge potential is associated with the Shioda map image of one of the independent elements $\{\mathbb{S}_a\} = \{\mathbb{S}_l, \mathbb{S}_9, \mathbb{S}_{10}\}$ of $MW(X_i/B)$, as computed in Corollary (3.11). Furthermore*

*1.* $V = h^{1,1}(X_i) - h^{1,1}(B) - 1 = \mathrm{rk}(\mathrm{MW}(X_i/B)) = 10$,
*2.* $T = h^{1,1}(B) - 1 = 9$,
*3.* $H_{unch} = 4$, $H_{ch} = 18$ *and* $H = H_{unch} + H_{ch} = 22$.

*Proof.* (1) and (2) follow by constructions and from the Shioda-Wazir formula; $h^{2,1}(X_i) + 1 = 4$ by Theorem 2.6.

The holomorphic curves in the fiber of $X_i$ with vanishing intersection with the zero-section $S_0$ (the exceptional fibers) are components of the fibers of the points of the singular locus of the discriminant: $\pi_i^{-1}(p_j) = \mathcal{P}_j^{i,0} + \mathcal{P}_j^{i,A} + \mathcal{P}_j^{i,B}$, $1 \le j \le 6$, with the notation as in Theorem 2.10. Each such fiber contains 3 such holomorphic curves in class $\mathcal{P}_j^{i,A}$, $\mathcal{P}_j^{i,B}$ and $\mathcal{P}_j^{i,A} + \mathcal{P}_j^{i,B}$. By Result 4.2 (a), their genus-zero Gopakumar-Vafa invariants of Proposition 3.8 invariants compute $H_{ch}$. Each of the six singularities of the singular fibration $\bar{X}$ defined in Theorem 2.6 can be deformed to 3 nodes [31, Proposition 7].

Each node contributes +1 to $H_{ch}$, yielding $H_{ch} = 3 \times 6 = 18$ as well, by Result 4.2 (b). The deformation do not lift to global deformations of the resolution $X_i$ [27, 31]. □

**Proposition 5.2.** *Let $X_i$ be the Namikawa-Rossi threefold. Let $q_a$ denote the $U(1)_a$ charges for the hypermultiplets associated with the exceptional fibers $\mathcal{P}_j^{i,A}$, $\mathcal{P}_j^{i,B}$ and $\mathcal{P}_j^{i,A} + \mathcal{P}_j^{i,B}$. Then the non-zero $U(1)_a$ charges are computed as the respective intersections with the Shioda map images $\sigma(\mathbb{S}_a)$:*

| | $\mathcal{P}_j^A$ | $\mathcal{P}_j^B$ | $\mathcal{P}_j^A + \mathcal{P}_j^B$ |
|---|---|---|---|
| $q_9$ | $-1$ | $0$ | $-1$ |
| $q_{10}$ | $0$ | $-1$ | $-1$ |
| $q_l$ | $0$ | $0$ | $0$ |

*Proof.* We apply Proposition 3.11 and 3.5 to evaluate the charges as in Result 4.3. □

**Proposition 5.3.** *F-theory on the Namikawa-Rossi manifold satisfies the anomaly cancellation conditions as collected in Result 4.4.*

*Proof.* We evaluate the purely gravitational and the abelian and mixed gravitational-abelian anomaly conditions in turn.

*Gravitational anomalies* The condition for cancellation of the purely gravitational anomalies, equ. (1), is manifestly satisfied because $H = H_{unch} + H_{ch} = 4 + 18 = 22$, $V = 10$ and $T = 9$.

*(Mixed) Abelian anomalies* On the righthand side of (2) and (3), applied to F-theory on $X_i$, the index $I$ becomes a multi-index $I = (C, j)$, where $C \in \{A, B, A+B\}$ and $j \in \{1, \dots, 6\}$ label the curves $\mathcal{P}_j^{i,A}$, $\mathcal{P}_j^{i,B}$ and $\mathcal{P}_j^{i,A} + \mathcal{P}_j^{i,B}$ appearing in the table in Proposition 5.2. $N_I$ counts the number of massless hypermultiplets associated with each of these curves and coincides, by Result 4.2 (a), with the corresponding genus-zero Gopakumar-Vafa invariant computed in Proposition 3.8.

With this and the charges as in the table in Proposition 5.2, and $1 \le l \le 8$, $1 \le a \le 10$, equ. (2) becomes the requirement that

$$\begin{aligned} U(1)_9^2 - \text{grav:} &\quad (-K)_B \cdot b_{9,9} = 2 \\ U(1)_9 - U(1)_{10} - \text{grav:} &\quad (-K)_B \cdot b_{9,10} = 1 \\ U(1)_{10} - U(1)_{10} - \text{grav:} &\quad (-K)_B \cdot b_{10,10} = 2 \\ U(1)_l - U(1)_a - \text{grav:} &\quad (-K)_B \cdot b_{l,a} = 0, \end{aligned}$$

and equ. (3) becomes

$$\begin{aligned} U(1)_9^4\text{:} &\quad b_{9,9} \cdot b_{9,9} = 4 \\ U(1)_9^3 - U(1)_{10}\text{:} &\quad b_{9,9} \cdot b_{9,10} = 2 \\ U(1)_9^2 - U(1)_{10}^2\text{:} &\quad b_{9,9} \cdot b_{10,10} + 2 b_{9,10} \cdot b_{9,10} = 6 \\ U(1)_9 - U(1)_{10}^3\text{:} &\quad b_{9,10} \cdot b_{10,10} = 2 \\ U(1)_{10}^4\text{:} &\quad b_{10,10} \cdot b_{10,10} = 4 \\ U(1)_l - U(1)_a - U(1)_b - U(1)_c\text{:} &\quad b_{l,a} \cdot b_{b,c} + b_{l,b} \cdot b_{c,a} + b_{l,c} \cdot b_{a,b} = 0. \end{aligned}$$

These equations are manifestly satisfied with the help of Corollaries 3.14 and 3.13. □

**Proposition 5.4.** *The Namikawa-Rossi manifolds satisfy the Geometric Anomaly Cancellation equation (4).*

*Proof.* Indeed, $K_B^2 = 0$, $\chi_{top}(X_i) = 36$ by Theorems 2.6, 2.9 and $\sum_{Q'} c_{Q'} = 6 \times 3 = 18$, by Proposition 3.8. □

**Corollary 5.5.** *$\sum_{Q'} c_{Q'}$ is a birational invariant of the minimal model of the elliptic fibration.*

*Proof.* In fact, the left hand side of the equation (4) is a birational invariant of the minimal model [12]. □

$\sum_{Q'} c_{Q'}$ is a birational invariant of the non $\mathbb{Q}$-factorial terminal singularities of the Weierstrass model $\bar{X}$, in the sense that it is a birational invariant of the $\mathbb{Q}$-factorialization.

## 6. The Weierstrass model over $\mathbb{P}^2$, Elkies' birational example

In this Section we take the first steps in addressing the question of whether the model $W_{\text{NDE}} \to \mathbb{P}^2$ constructed in [8] is birationally Calabi-Yau. We prove that the Weierstrass models over $\mathbb{P}^2$ of the Namikawa-Rossi threefolds are not the ones constructed by Elkies.

### 6.1. Summary of [8]

In the 2018 seminar talk [8] Elkies gave a construction of a family of elliptically fibered threefolds in Weierstrass form, $W_{\text{NDE}} \to \mathbb{P}^2$, with $K_{W_{\text{NDE}}} \equiv 0$ and $\text{rk}(\text{MW}(W_{\text{NDE}}/\mathbb{P}^2)) = 10$. [8] does not address the question of whether the minimal resolutions are Calabi-Yau threefolds.

The starting point of the construction is what Elkies calls an "excellent family", that is elliptic fibrations which depend on the parameter $\zeta$:

$$y^2 = x^3 + (p_4\zeta^4 + p_{10}\zeta)x + \zeta^9 + p_6\zeta^6 + p_{12}\zeta^3 + p_{18}. \tag{5}$$

In Elkies' construction the variables $(x, y, \zeta)$ have weights $(6, 9, 2)$ and the coefficients $p_j$ are the invariant forms of degree $j$ in $\mathbb{P}^4$ for the Shephard-Todd unitary reflection group $\text{ST}_{33}$ in $\mathbb{C}^5$ [35], which we discuss below. Then Elkies obtains elliptic threefolds $W_{\text{NDE}}$ by restricting the coefficients $p_j$ to a general $\mathbb{P}^2$ and by taking $\zeta$ to be a quadratic form in that $\mathbb{P}^2$.

The discriminant locus of each fibration $W_{\text{NDE}} \to \mathbb{P}^2$ is then a curve of degree 36, and $K_{W_{\text{NDE}}} \equiv 0$; $h^1(\mathcal{O}_{W_{\text{NDE}}}) = 0, h^2(\mathcal{O}_{W_{\text{NDE}}}) = 0$ by construction. The threefolds $W_{\text{NDE}}$ are potentially birational Calabi-Yau. However, it is easy to construct Calabi-Yau Weierstrass models with the same numerical properties with log canonical singularities which are not birationally equivalent to a Calabi-Yau with terminal singularities. The example of [8] might a priori fall into this class.

Elkies' excellent family extends Shioda's excellent families for rational elliptic surfaces. Here "excellent" refers to the explicit generators of the Mordell-Weil group of sections [34]. The particular structure of the excellent family implies that $\text{rk}\,\text{MW}(W_{\text{NDE}}/\mathbb{P}^2) = 10$.

The coefficients $p_j$ are of geometric interest in their own right; in fact $\text{ST}_{33} \simeq \mathbb{Z}/2\mathbb{Z} \times \text{PSp}(4, \mathbb{F}_3)$, where $\text{PSp}(4, \mathbb{F}_3) \simeq G_{25920}$ is the Burkhardt group [14]. Shephard and Todd [35] prove that the invariants $p_j$ of $\text{ST}_{33}$ are the same invariants as for the Burkhardt group $G_{25920}$. The latter were originally computed by Burkhardt [5]. In particular Burkhardt shows that possible coefficients $p_{18}$ are either the product of lower degree invariants or an irreducible polynomial of degree 18, or a linear combination thereof.

As will become clear in the following section, Elkies' special choice $\zeta = 0$ in the family (5) could be a candidate for the Weierstrass model of the Namikawa-Rossi manifolds. We will now give an explicit construction of the Weierstrass models and then compare to Elkies' model for $\zeta = 0$.

### 6.2. Weierstrass models over $\mathbb{P}^2$ of the Namikawa threefolds

**Proposition 6.1.** *Let $X_i$ be one fixed smooth resolution of the Namikawa threefolds as in Theorem 2.6 with elliptic fibration $\pi_i : X_i \to B$. Let $\bar{\pi}_i$ be the morphism induced by the contraction $B \to \mathbb{P}^2$ of the rational curves $\{\mathsf{s}_0, \cdots, \mathsf{s}_8\}$:*

$$\begin{array}{ccc} X_i & & \\ {\scriptstyle \pi_i}\downarrow & \overset{\bar{\pi}_i}{\searrow} & \\ B & \longrightarrow & \mathbb{P}^2 \end{array}$$

*Then there exists a diagram*

$$\begin{array}{ccccc} X_i & \overset{\psi_1}{\dashleftarrow\dashrightarrow} & X'_i & \overset{\psi_2}{\longrightarrow} & Z_i \\ {\scriptstyle \pi_i}\downarrow & \overset{\bar{\pi}_i}{\searrow} & \downarrow & \overset{\pi_{Z_i}}{\swarrow} & \\ B & \longrightarrow & \mathbb{P}^2 & & \end{array}$$

*such that the following holds:*

1. *$\psi_1 : X_i \dashrightarrow X'_i$, with $X'_i$ smooth, is a birational map constructed as the composition of the 81 flops of the rational curves ${\pi_i}^*(\mathsf{s}_k) \cdot S_\ell$, $0 \leq \ell \leq 8$, $0 \leq k \leq 8$. The discriminant locus of $\bar{\pi}_i$ consists of 6 irreducible cuspidal curves which intersect pairwise transversely in* 9 *distinct smooth points $\{z_0, \cdots, z_8\}$. The fiber over each $z_j$ is the surface $\psi_{1*}({\pi_i}^*(\mathsf{s}_k)) \simeq \mathbb{P}^2$, $0 \leq k \leq 8$.*
2. *The elliptic fibration $X'_i \to \mathbb{P}^2$ has* 11 *linearly independent sections (i.e. the rank of* $\mathrm{MW}(X'_i/\mathbb{P}^2)$ *is* 10*), the strict transforms of the sections of $\pi_i$:*
   $S'_{l,\mathbb{P}^2} \overset{def}{=} \psi_{1*}(S_\ell)$, $0 \leq \ell \leq 8$, ${D'}^i_i \overset{def}{=} \psi_{1*}(D^i_i)$ *and* ${D'}^i_{i_1} \overset{def}{=} \psi_{1*}(D^i_{i+1})$.
3. *$\psi_2 : X'_i \to Z_i$ is a composition of* 9 *birational contractions with exceptional loci $\{\psi_{1*}({\pi_i}^*(\mathsf{s}_k)) \simeq \mathbb{P}^2, 0 \leq k \leq 8\}$. The Calabi-Yau $Z_i$ has* 9 *canonical (but not terminal) isolated singularities.*
4. *$Z_i$ is rigid.*
5. *The elliptic fibration $Z_i \to \mathbb{P}^2$ has* 11 *linearly independent sections (i.e. the rank of* $\mathrm{MW}(Z_i/\mathbb{P}^2)$ *is* 10*).*

6. *The elliptic fibration $\pi_{Z_i} : Z_i \to \mathbb{P}^2$ is equidimensional.*
7. *For every $k$, the singular fiber $\pi_{Z_i}^{-1}(z_k)$ consists of 9 rational curves meeting at the point of canonical singularity of $Z_i$.*

*Proof.* The statements (1) and (3) follow from the contraction theorems and the existence of log flips for threefolds, stated in Appendix A for convenience: To obtain the flops in (1) in Theorem A.2 we take $Y = \mathbb{P}^2$, $\mathcal{D} = \epsilon{\pi_i}^*(\mathsf{s}_k)$ for a fixed $k$, $0 \le k \le 8$, $\epsilon \ll 1$ and $R = {\pi_i}^*(s_k) \cdot S_\ell$, $0 \le \ell \le 8$. Each of these log-flips is a flop. For each of the contraction morphism in (3), we take in Theorem A.1 $\mathcal{D} = \psi_{1*}(\pi_i^*(\mathsf{s}_k)) \simeq \mathbb{P}^2$ and $R$ any line in $\mathbb{P}^2$. (4) follows from [32, 1] and the survey [30]. (2), (5), (6) and (7) follow from the construction. □

We now give an intrinsic description of the Weierstrass model over $\mathbb{P}^2$ of the Namikawa-Rossi manifolds.

**Lemma 6.2.** *Let $S'_{0,\mathbb{P}^2} = \psi_{1*}(S_0)$ be a fixed section for $\bar\pi'_i : X'_i \to \mathbb{P}^2$. There exists a crepant birational morphism $\psi_3$ such that the following diagram commutes:*

$$\begin{array}{ccc} X'_i & \xrightarrow{\psi_3} & W_{\mathbb{P}^2} \\ \downarrow{\scriptstyle \bar\pi'_i} & \swarrow{\scriptstyle \pi_{W_{\mathbb{P}^2}}} & \\ \mathbb{P}^2 & & \end{array}$$

*$\pi_{W_{\mathbb{P}^2}} : W_{\mathbb{P}^2} \to \mathbb{P}^2$ is the Weierstrass model of $X'_i$ with marked section $S_{\mathbb{P}^2} \stackrel{def}{=} {\psi_3}_* S'_{0,\mathbb{P}^2}$.*

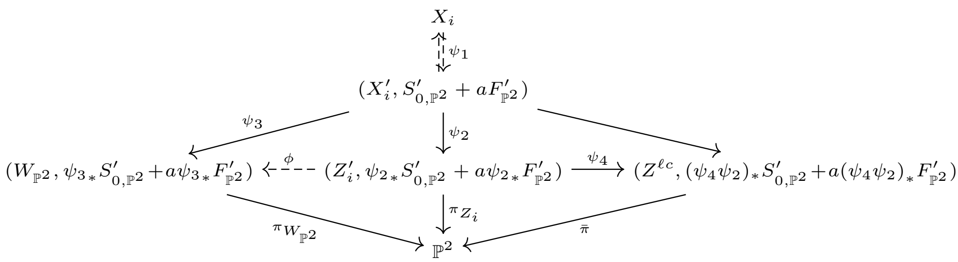
*In addition, $K_{X'_i} + S'_{0,\mathbb{P}^2} = {\psi_3}^*(K_{W_{\mathbb{P}^2}} + S_{\mathbb{P}^2})$.*

*Proof.* The existence of the Weierstrass model and of the commutative diagram such that $S'_{0,\mathbb{P}^2} = {\psi_3}^*(S_{\mathbb{P}^2})$ is proved in [26]. The morphism $\psi_3$ is crepant because, with $\Lambda_{\mathbb{P}^2}$ the support of the discriminant, $\mathcal{O}_{X'_i} \simeq K_{X'_i} \simeq (\bar\pi'_i)^*(K_{\mathbb{P}^2} + \Lambda_{\mathbb{P}^2})$ and $\mathcal{O}_{W_{\mathbb{P}^2}} \simeq (\bar\pi_{W_{\mathbb{P}^2}})^*(K_{\mathbb{P}^2} + \Lambda_{\mathbb{P}^2}) \simeq K_{W_{\mathbb{P}^2}}$ since $\bar\pi'_i$ and $\pi_{W_{\mathbb{P}^2}}$ have the same discriminant. □

The construction of the Weierstrass model in Lemma 6.2 is not explicit, so we use the construction of the relative log canonical model instead.

**Theorem 6.3.** *Let $h$ be a general line in $\mathbb{P}^2$, $F'_{\mathbb{P}^2} \stackrel{def}{=} (\bar\pi'_i)^*(h)$ and $0 < a \le 1$. The Weierstrass model $W_{\mathbb{P}^2} \to \mathbb{P}^2$ of the Namikawa-Rossi threefold is the relative log canonical model of $(X'_i, S'_{0,\mathbb{P}^2} + aF'_{\mathbb{P}^2})$ described in Proposition 6.1. $W_{\mathbb{P}^2}$ is obtained from $X_i$ by the composition of $\psi_1$, $\psi_2$ and the birational contractions of the flops of the rational curves ${\pi_i}^*(s_k) \cdot S_\ell$, $0 \le \ell \le 8$,*

$0 \leq k \leq 8$, *the* 6 *pairs of curves* $\{\mathcal{P}_j^{i,A}, \mathcal{P}_j^{i,B}\}$. $W_{\mathbb{P}^2}$ *is also the relative log canonical model of* $(Z'_i, \psi_{2*}S'_{0,\mathbb{P}^2} + a\psi_{2*}F'_{\mathbb{P}^2})$.




*Proof.* For $0 \leq a \leq 1$, $(X'_i, S'_{0,\mathbb{P}^2} + aF'_{\mathbb{P}^2})$ is a log canonical pair. General results from the minimal model program together with the existence of abundance in dimension 3 [17] ensure the existence of the log canonical model $(Z^{\ell c}, S^{\ell c} + aF^{\ell c}))$ for the pair $(X'_i, S'_{0,\mathbb{P}^2} + aF'_{\mathbb{P}^2})$, $0 < a \leq 1$, relative to the fibration $\bar{\pi}'_i$. $K_{Z'_i} + \psi_{2*}S'_{0,\mathbb{P}^2} + a\psi_{2*}F'_{\mathbb{P}^2}$ is $\pi_{Z_i}$-nef. Abundance [17] gives the birational morphism $\psi_4$ to the log canonical model $(Z^{\ell c}, \psi_{4*}S'_{0,\mathbb{P}^2} + a\psi_{4*}F'_{\mathbb{P}^2})$ (Definition A.4). $\psi_4$ contracts the flops of the rational curves ${\pi_i}^*(s_k) \cdot S_\ell$, $0 \leq \ell \leq 8$, $0 \leq k \leq 8$ and the 6 pairs of curves $\{\mathcal{P}_j^{i,A}, \mathcal{P}_j^{i,B}\}$. $K_{W_{\mathbb{P}^2}} + \psi_{3*}S'_{0,\mathbb{P}^2} + a\psi_{3*}F'_{\mathbb{P}^2}$ is $\pi_{W_{\mathbb{P}^2}}$-ample. $(X'_i, S'_{0,\mathbb{P}^2} + aF'_{\mathbb{P}^2})$ is a common log resolution of the three log canonical pairs $(W_{\mathbb{P}^2},\ \psi_{3*}S'_{0,\mathbb{P}^2} + a\psi_{3*}F'_{\mathbb{P}^2})$, $(Z'_i,\ \psi_{2*}S'_{0,\mathbb{P}^2} + a\psi_{2*}F'_{\mathbb{P}^2})$ and $(Z^{\ell c},\ \psi_{4*}S'_{0,\mathbb{P}^2} + a\psi_{4*}F'_{\mathbb{P}^2})$. The morphisms $\psi_2$, $\psi_3$ and $\psi_4$ are isomorphisms onto their images when restricted to $S'_{0,\mathbb{P}^2} + aF'_{\mathbb{P}^2}$. Then $K_{X'_i} + S'_{0,\mathbb{P}^2} + aF'_{\mathbb{P}^2} \simeq (\psi_2)^*(K_{Z'_i} + \psi_{2*}S'_{0,\mathbb{P}^2} + a\psi_{2*}F'_{\mathbb{P}^2})$ and $K_{X'_i} + S'_{0,\mathbb{P}^2} + aF'_{\mathbb{P}^2} \simeq (\psi_4 \cdot \psi_2)^*(K_{Z^{\ell c}} + (\psi_4\psi_2)_*S'_{0,\mathbb{P}^2} + a(\psi_4\psi_2)_*F'_{\mathbb{P}^2})$ by construction while $K_{X'_i} + S'_{0,\mathbb{P}^2} + aF'_{\mathbb{P}^2} \simeq {\psi_3}^*(K_{W_{\mathbb{P}^2}} + \psi_{3*}S'_{0,\mathbb{P}^2} + a\psi_{3*}F'_{\mathbb{P}^2})$ by Theorem 6.2. In particular $(W_{\mathbb{P}^2}, \psi_{3*}S'_{0,\mathbb{P}^2} + a\psi_{3*}F'_{\mathbb{P}^2})$ satisfies the conditions to be a log canonical model, Definition A.4. We conclude as in Section I.4.1. in [40] by recalling that the log canonical model is unique [21, Theorem 3.52]. □

Summarizing:

**Corollary 6.4.** $\pi_{W_{\mathbb{P}^2}} : W_{\mathbb{P}^2} \to \mathbb{P}^2$ *has affine equation* $y^2 = x^3 + \beta(s,t)$, *where* $\beta(s,t)$ *is the equation of the* 6 *general cuspidal curves in the pencil of* $\mathbb{P}^2$ *which give rise to the smooth general rational elliptic surface with* 6 *type II fibers* $r : B \to \mathbb{P}^1$. *The* 6 *cuspidal curves intersect in the points* $\{z_0, \cdots, z_8\}$. *The Weierstrass model is non-minimal of type* $(*, 6, 12)$ *at each of the points* $\{z_0, \cdots, z_8\} \subset \mathbb{P}^2$. $W_{\mathbb{P}^2}$ *has* $\mathbb{Q}$*-factorial canonical, but not terminal singularities in the fibers over* $\{z_0, \cdots, z_8\} \subset \mathbb{P}^2$. $W_{\mathbb{P}^2}$ *has non* $\mathbb{Q}$*-factorial terminal* *singularities in the fibers over the* 6 *cuspidal points. The singular locus of the reduced discriminant consists of* 15 *points.*

*Proof.* The affine equation is $y^2 = x^3 + \beta(s,t)$ because $j(W_{\mathbb{P}^2}) = 0$. The zero locus of $\beta(s,t)$ is the reduced discriminant, which by (1) in Proposition 6.1 and Theorem 6.3 consists of the 6 type *II* fibers in pencil in $\mathbb{P}^2$ which give rise to the smooth general rational elliptic surface $r : B \to \mathbb{P}^1$. The type of the Weierstrass model then follows, in fact if $y^2 = x^3 + \alpha x + \beta$ is a local Weierstrass equation and $\delta$ is the equation for the discriminant then the triplet $(\nu(\alpha(P)), \nu(\beta(P)), \nu(\delta(P)))$ is given by the vanishing orders at $P$ of $\alpha$, $\beta$ and $\delta$. It is non-minimal by definition. The contraction $\psi_2$ gives rise to canonical but non-terminal singularities, by part (3) in Proposition 6.1, while the contraction $\psi_4$ results in non $\mathbb{Q}$-factorial terminal singularities (see the proof of Theorem 6.3). ☐

### 6.3. Comparison with Elkies' construction

We now compare the Weierstrass model $W_{\mathbb{P}^2}$ of the Namikawa-Rossi threefolds, which we described explicitly in Theorem 6.3 and Corollary 6.4, to Elkies' Weierstrass model (5) for $\zeta = 0$, $W_{\mathrm{NDE},0} : y^2 = x^3 + {p'}_{18}$. It is clear that if ${p'}_{18}$ is taken to be irreducible, the two Weierstrass models are different. For more general invariants ${p'}_{18}$ one must answer the question whether the defining equation $\beta(s,t)$ appearing in $W_{\mathbb{P}^2}$ in Corollary 6.4 is the restriction of an invariant of the Burkhardt group to $\mathbb{P}^2$. We pursue this investigation in an upcoming paper [11].

## Appendix A. Review of background material

We review some foundational results in birational geometry which can be found for example in [21]. Applications to relative log canonical models of elliptic fibrations can be found in Chapter I of [40].

**Theorem A.1** (Contraction morphism)**.** *Let $\pi : Z \to Y$ be a morphism, $Z$ a threefold, $\mathcal{D}$ an effective $\mathbb{Q}$-divisor. If $(Z, \mathcal{D})$ has $\mathbb{Q}$-factorial klt singularities and $K_Z + \mathcal{D}$ is not $\pi$-nef, that is $(K_Z + \mathcal{D}) \cdot R < 0$, for some extremal ray $R \in NE(Z/B)$, then there exists a morphism $\bar{\phi} : Z \to \bar{Z}$, contracting all the curves in the numerical equivalence (homology) class of $[R]$ such that the following diagram is commutative:*

$$\begin{array}{ccc} (Z,\mathcal{D}) & \xrightarrow{\bar{\phi}} & (\bar{Z},\bar{\mathcal{D}}) \\ {\scriptstyle\pi}\downarrow & \swarrow{\scriptstyle\bar{\pi}} & \\ Y & & \end{array}$$

*$\bar{Z}$ is a normal variety and* $\dim \mathrm{NE}(Z/B) > \dim \mathrm{NE}(\bar{Z}/B)$.

**Theorem A.2** (The flops)**.** *Let $(Z, \mathcal{D})$ a variety with $\mathbb{Q}$-factorial klt singularities. Let $\bar{\phi}$ be a $(K_Z + \mathcal{D})$ contraction of an extremal ray $R$ as in Theorem A.1. Assume that $\bar{\phi}$ is small. Then there exists a log flip $\psi$ : $(Z, \mathcal{D}) \dashrightarrow (Z', \bar{\mathcal{D}}')$ of $R$. That is, $K_{Z'} + \mathcal{D}'$ is $\pi$-nef (i.e. $(K_{Z'} + \mathcal{D}') \cdot R' > 0$, $\forall\ R' \in \mathrm{NE}(Z'/\bar{Z})$) and the following diagram is commutative*

$$\begin{array}{ccc} (Z,\mathcal{D}) & \overset{\psi}{\longleftarrow\!\dashrightarrow} & (Z',\bar{\mathcal{D}}') \\ \downarrow \bar{\phi} & \swarrow \bar{\phi}' & \\ (\bar{Z},\bar{\mathcal{D}}) & & \end{array}$$

*$(Z', \mathcal{D}')$ has $\mathbb{Q}$-factorial klt singularities.*

There is also a relative version.

**Definition A.3.** Let $Z, Y$ be normal varieties, $f : X \to Z$ a birational morphism and $(Z, \mathcal{D})$ a pair such that $K_Z + \mathcal{D}$ is $\mathbb{Q}$-Cartier. Let $\{E_j\}$ be the collection of the exceptional divisors; then the formula

$$K_Y + (f^{-1})_*(\mathcal{D}) \equiv f^*(K_X + \mathcal{D}) + \sum_j a(E_j, Z, \mathcal{D}) E_j$$

defines $a(E_j, Z, \mathcal{D})$.

$(Z, \mathcal{D})$ is a log canonical pair if and only if $\inf_j a(E_j, Z, \mathcal{D}) \geq -1$.

**Definition A.4.** Let $(Z, \mathcal{D})$ be a log canonical pair and $\pi : Z \to Y$ a proper morphism. $(Z^{\ell c}, \mathcal{D}^{\ell c})$ is the log canonical model over $Y$ if in the following diagram:

$$\begin{array}{ccc} (Z,\mathcal{D}) & \overset{\phi}{\dashrightarrow} & (Z^{\ell c},\mathcal{D}^{\ell c}) \\ \pi \downarrow & \swarrow \bar{\pi} & \\ Y & & \end{array}$$

1. $\bar{\pi}$ is proper
2. $\phi^{-1}$ has no exceptional divisor
3. $\phi_*(\mathcal{D}) = \mathcal{D}^{\ell c}$
4. $K_{Z^{\ell c}} + \mathcal{D}^{\ell c}$ is $\bar{\pi}$-ample
5. for every $\phi$-exceptional divisor $E \subset Z$, $a(E, Z, \mathcal{D}) \leq a(E, Z^{\ell c}, \mathcal{D}^{\ell c})$.

## Appendix B. Derivation of intersection numbers

In this appendix, we exemplify the derivation of the intersection numbers presented in Proposition 3.5. These intersections are in a neighborhood of the resolution of each singular point. We derive (1), (2) and (3) in the proof using geometry, the local equations around a singular point of the threefold and its resolution around the exceptional loci.

With the notation from Theorem 2.10 and Definition 2.8 we recall that the section $D^i_{i+1}$ on $X_i$ is by construction the strict transform of $D_{i+1}$ on $\bar{X}$ by the resolution $\varphi_i$. In a neighborhood of the exceptional loci, $D^i_i \cap D^i_{i+1} = \cup_j \mathcal{P}^{i,A}_j \cup \mathcal{P}^{i,B}_j$. We note also that $D_{i+1}$ inherits from $r : B \to \mathbb{P}$ the structure of a rational elliptic surface with six fibers of type *II*. $\varphi_i|_{D^i_{i+1}}$ induces two blow ups of the type *II* fibers at the cuspidal points. $D^i_{i+1}$ is then a non minimal rational elliptic surface $E_{0,D^i_{i+1}}$ be the strict transform of the cuspidal fiber in $D^i_{i+1}$, and $E_{D^i_{i+1}}$ be the general fiber (note that $\pi_i(E_{D^i_{i+1}}) = f \in B$). Then $E_{D^i_{i+1}} \equiv (E_{0,D^i_{i+1}} + 2\mathcal{P}^{i,A}_j + 3\mathcal{P}^{i,B}_j)|_{D^i_{i+1}}$ with $(\mathcal{P}^{i,A}_j \cdot \mathcal{P}^{i,A}_j)_{|D^i_{i+1}} = -2$, $(\mathcal{P}^{i,B}_j \cdot \mathcal{P}^{i,B}_j)_{|D^i_{i+1}} = -1$, for any $1 \leq j \leq 6$. The three component curves $E_{0,D^i_{i+1}}$, $\mathcal{P}^{i,A}_j$ and $\mathcal{P}^{i,B}_j$ intersect in one point.

Hence we obtain the following intersection numbers:

$$\begin{aligned}
D^i_i \cdot \mathcal{P}^{i,A}_j &= (D^i_i \cdot \mathcal{P}^{i,A}_j)_{|D^i_{i+1}} \\
&= \frac{1}{2}(\mathcal{P}^{i,A}_j + \mathcal{P}^{i,B}_j) \cdot (E_{D^i_{i+1}} - E_{0,D^i_{i+1}} - 3\mathcal{P}^{i,B}_j)|_{D^i_{i+1}} \\
&= \frac{1}{2}(0 - 2 - 3 + 3) = -1, \\
D^i_i \cdot \mathcal{P}^{i,B}_j &= \frac{1}{3} D^i_i \cdot (E_{D^i_{i+1}} - E_{0,D^i_{i+1}} - 2\mathcal{P}^{i,A}_j)|_{D^i_{i+1}} \\
&= \frac{1}{3}(\mathcal{P}^{i,A}_j + \mathcal{P}^{i,B}_j) \cdot (E_{D^i_{i+1}} - E_{0,D^i_{i+1}} - 2\mathcal{P}^{i,A}_j)|_{D^i_{i+1}} \\
&= \frac{1}{3}(0 - 2 - 2(-2 + 1)) = 0.
\end{aligned}$$

Either from the local equations of the resolved Calabi-Yau, or from the above intersections together with $D^i_i \cdot \mathcal{E} = 1$ we find also

$$D^i_i \cdot \mathcal{P}^{i,0}_j = 2.$$

The intersection numbers with $D^i_{i+1}$ follow similarly, noting however that the strict transform of $D_i$ after the first blow up acquires $A_1$ singularities, which are then resolved in the second blow up.

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