Communications in
Number Theory and Physics
Volume 16, Number 4, 801–849, 2022

Unipotent extensions and differential equations (after Bloch-Vlasenko)

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S. Bloch and M. Vlasenko recently introduced a theory of *motivic Gamma functions*, given by periods of the Mellin transform of a geometric variation of Hodge structure. They tie properties of these functions to the monodromy and asymptotic behavior of certain unipotent extensions of the variation. In this article, we further examine their Gamma functions and the related *Apéry and Frobenius invariants* of a VHS, and establish a relationship to motivic cohomology and solutions to inhomogeneous Picard-Fuchs equations.

AMS 2000 subject classifications: Primary 14C30, 14D07, 19E15, 32G20, 32S40.
Keywords and phrases: Variations of Hodge structure, periods, Picard-Fuchs equations, motivic Gamma functions, Apéry constants, Frobenius constants, normal functions, motivic cohomology.

1. Introduction

The Frobenius method for solving linear ODEs in the neighborhood of a regular singular point (see for example [IKSY]) goes all the way back to [Fr]. The significance of the resulting basis of solutions in Hodge theory and mirror symmetry has recently been elevated by two seminal papers.

In their proof of the Gamma Conjecture for rank-one Fano threefolds [GZ], Golyshev and Zagier studied the Frobenius solutions for the regularized quantum differential equations of these Fanos, using the solutions' monodromy to define constants $\kappa_0, \kappa_1, \kappa_2, \kappa_3$ and matching those to the coefficients of the Gamma-class of each Fano; they also obtain a natural extension of the $\{\kappa_j\}$ to a (more mysterious) infinite sequence. Subsequently, Bloch and Vlasenko [BV] generalized these *Frobenius constants* to a broader class of Picard-Fuchs equations, and gave them a new interpretation, as periods of the limiting mixed Hodge structure of the underlying variation and its unipotent extensions. They also showed that the generating series $\kappa(s) := \sum_{j\geq 0} \kappa_j s^j$ is essentially a *motivic Gamma function*, that is, a period of the Mellin transform (as defined by [LS]) of the underlying $\mathcal{D}$-module.

In this paper, we study the properties of $\kappa(s)$ for a particular class of Picard-Fuchs equations, attached to polarized variations of Hodge structure over a Zariski open set $U \subset \mathbb{P}^1$ with all Hodge numbers equal to 1 (and a few other properties detailed below).

Our first main goal is simply to give a streamlined presentation of the main results of Bloch and Vlasenko in this case, making occasional technical improvements (Theorems 6.6 and 9.2), and using the polarization to make the "$\Gamma = \kappa$" result more explicit (Theorem 8.2). We also highlight how their work can be used to compute LMHSs (Example 6.8) and produce a *limiting motive* in the hypergeometric case (Remark 8.7).

Our second goal is to interpret certain features of κ in terms of motivic cohomology and admissible normal functions. For instance, if the variation has weight n (and rank $n+1$), then κ_{n+1} is the first Frobenius number not related to its LMHS; in Theorem 9.7, we obtain a motivic interpretation of the "first unipotent extension" of [BV, §5], and hence of this number, confirming a speculation in the closing pages of [loc. cit.]. In §10, we investigate the values of κ at positive integers, which we term *Apéry constants.* After characterizing them as special values of solutions to inhomogeneous equations (Theorem 10.1), we interpret them in some cases as regulators of higher cycles (Theorems 10.8 and 10.11).

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In the remainder of this Introduction, we offer a brief mathematical *dramatis personae* for the reader's reference (beginning on the next page).

To set the scene:[1] let $\Sigma = \{0, c, \ldots, \infty\} \subset \mathbb{P}^1$ be finite, with $|c| < |c'|$ for all $c' \in \Sigma \setminus \{0, c\}$. Let $\mathbb{D}$ be an open disk centered about 0 with $\mathbb{D} \cap \Sigma = \{0, c\}$; and, writing $U := \mathbb{P}^1 \setminus \Sigma$, fix $p \in \mathbb{D} \cap U$. Consider a $\bar{\mathbb{Q}}$-motivic, polarized $\mathbb{Q}$-VHS $\mathcal{M}$ on U, of weight n with Hodge numbers $h^{p,n-p} = 1$ $(0 \le p \le n)$.[2] Suppose the underlying local system has maximal unipotent monodromy at $t = 0$, and strong conifold monodromy (Remark 4.3) at $t = c$, represented by $T_0, T_c \in \mathrm{Aut}(\mathbb{M}_{\mathbb{Q},p})$ (with $N_0 := \log(T_0)$); assume in addition that $\ker(T_0 - I) \cap \ker(T_c - I) = \{0\}$. Write $\gamma_0, \gamma_c \in \pi_1(\mathbb{D} \cap U)$ for loops based at p winding once about $0, c$.

[1]For simplicity, we impose assumptions largely avoided in the text: strong conifold monodromy at c (which goes a bit beyond $\mathrm{rk}(T_c - I) = 1$, see §4), self-adjointness of L (see §§5–6), and $\mathcal{M}$ arising from a family defined over $\bar{\mathbb{Q}}$.

[2]Since $\mathcal{M}$ has a rational polarization Q, it is self-dual, so that the dual of $D^m L$ is LD^m below. We still find it useful however to formally distinguish $\mathcal{M}$ and $\mathcal{M}^\vee$ for some purposes: we use $Q(\cdot,\cdot)$ to denote the pairing on $\mathcal{M}$ (or $\mathcal{M}^\vee$), and $\langle \cdot,\cdot \rangle$ for the pairing of $\mathcal{M}$ and $\mathcal{M}^\vee$; see §4.

Betti periods Fixing $\varepsilon_0 \in (\mathbb{M}^{\vee}_{\mathbb{Q},p})^{T_0}$, there is a unique basis $\{\varepsilon_0, \ldots, \varepsilon_n\} \subset \mathbb{M}^{\vee}_{\mathbb{Q},p}$ such that $N_0 \varepsilon_j = \varepsilon_{j-1}$ and $(T_c - I)\varepsilon_j = 0$ for $j > 0$. Set $\delta := (T_c - I)\varepsilon_0 \in \mathbb{M}^{\vee}_{\mathbb{Q},p}$ and put $\mathsf{Q}_0 := Q(\varepsilon_0, \varepsilon_n)$, $\mathsf{Q}_c := Q(\varepsilon_0, \delta)$ (both in $\mathbb{Q}^{\times}$). Choose $\mu \in H^0(\mathbb{P}^1, \mathcal{F}^n_e \mathcal{M}_e)$ the (unique) section of the canonically extended Hodge line which is nowhere zero on $\mathbb{P}^1 \setminus \{\infty\}$, and normalized so that the "fundamental period"

$$\langle \varepsilon_0, \mu \rangle =: A(t) = \textstyle\sum_{k \geq 0} a_k t^k \qquad \text{(also written } \epsilon_0(t))$$

has $a_0 = 1$. Write $\psi(t) := \langle \delta, \mu \rangle$ and $\epsilon_j(t) := \langle \varepsilon_j, \mu \rangle$ ($j > 0$) for other periods, and $\epsilon_j^{\mathrm{an}}(t)$ for the analytic (at 0) part of $\epsilon_j(t)$. The left-hand column of the period matrix of the LMHS of $\mathcal{M}$ at 0 is given by $(2\pi\mathbf{i})^j \epsilon_j^{\mathrm{an}}(0)$, $0 \leq j \leq n$.

Picard-Fuchs $L := \sum_{j=0}^{d} t^j P_j(D) \in \mathbb{C}[t, D]$ is the minimal operator with $\nabla_L \mu = 0$ (hence $L\epsilon_j = 0 = L\psi$). It has order $n+1$ and degree d.

Conifold Gamma The function

$$\Gamma_c(s) := \textstyle\sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} e^{2\pi\mathbf{i}ks} \int_{\gamma_0^{-k}} \psi(t) t^s \frac{dt}{t} + (e^{2\pi\mathbf{i}s} - 1)^{n+1} \int_{\gamma_c} \epsilon_0(t) t^s \frac{dt}{t}$$

is entire, with $\sum_{j=0}^{d} P_j(-s-j)\Gamma_c(s+j) = 0$, and $\Gamma_c(-k) = (-1)^{n+1} \frac{\mathsf{Q}_c}{\mathsf{Q}_0} (2\pi\mathbf{i}) a_k$ for $k \in \mathbb{Z}_{\geq 0}$.

Frobenius periods $\Phi(s,t) = \sum_{\ell \geq 0} \phi_\ell(t) s^\ell$ is uniquely defined by $L\Phi = s^{n+1} t^s$ and $T_0 \Phi = e^{2\pi\mathbf{i}s}\Phi$. Write $\phi_\ell(t) =: \sum_{b=0}^{\ell} \frac{1}{b!} \log^b(t) \phi^{\mathrm{an}}_{\ell-b}(t)$ and $\phi^{\mathrm{an}}_\ell(t) =: \sum_{k \geq 0} a_k^{(\ell)} t^k$. Then $A_k(s) := \sum_{\ell \geq 0} a_k^{(\ell)} s^\ell$ satisfies $\Phi(s,t) = \sum_{k \geq 0} A_k(s) t^{s+k}$. (Note that $a_k^{(0)} = a_k$ and $a_0^{(\ell > 0)} = 0$, so that $A(t) = \phi_0(t)$ is the generating series of constant terms of the $\{A_k(s)\}$, and $A_0(s)$ is identically 1.) The $\phi_0(t), \ldots, \phi_n(t)$, which satisfy $L(\cdot) = 0$, are called *Frobenius periods*, as opposed to the *Betti periods* $\epsilon_0(t), \ldots, \epsilon_n(t)$.

Kappa series $(T_c - I)\Phi(s,t) =: \kappa(s)\psi(t)$, with $\kappa(s) =: \sum_{j=0}^{\infty} \kappa_j s^j$ and $\kappa(s)^{-1} =: \sum_{j=0}^{\infty} \alpha_j s^j$. We have $\kappa_0 = \alpha_0 = 1$ and $\alpha_j = (2\pi\mathbf{i})^j \epsilon_j^{\mathrm{an}}(0)$ for $0 \leq j \leq n$. Moreover, we have the asymptotic formulas $\kappa(s) = c^s \cdot \lim_{k \to \infty} \frac{A_k(s)}{a_k}$ and $\kappa_j = \sum_{j=0}^{\ell} \frac{1}{j!} \log^j(c) \cdot \lim_{k \to \infty} \frac{a_k^{(\ell-j)}}{a_k}$. The basic relation between Gamma and kappa is $\Gamma_c(s) = \frac{(1-e^{2\pi\mathbf{i}s})^{n+1}\mathsf{Q}_c}{(2\pi\mathbf{i})^n s^{n+1} \mathsf{Q}_0} \kappa(s)$ in this self-dual setting. At $s \sim -k$ we therefore have $\kappa(s) \sim \frac{(-k)^{n+1}}{(s+k)^{n+1}} a_k$.

Unipotent extension Fix $m \in \mathbb{Z}_{>0}$. There is a unique extension $0 \to \mathcal{K}_m \to \mathcal{E}_m \to \mathcal{M} \to 0$ of admissible $\mathbb{Q}$-VMHS on $\Delta_0^\times$ (a small punctured disk about 0) with underlying $\mathbb{Q}$-local system $\mathbb{E}_m$ extending to $\mathbb{D} \cap U$, underlying $\mathcal{D}$-module $\mathcal{D}/\mathcal{D}D^m L$, and with $\mathcal{K}_m$ of rank m with Hodge numbers $h^{-m,-m} = \cdots = h^{-1,-1} = 1$. The coefficients $\{\alpha_j\}_{0 \leq j \leq n+m}$ of $\kappa(s)^{-1}$ yield the left-hand column of the period matrix of the LMHS of $\mathcal{E}_m$ at 0.

Key Example 1: if $\varphi \in \bar{\mathbb{Q}}[x_1^{\pm 1}, \dots, x_{n+1}^{\pm 1}]$ is reflexive and tempered, and $f = \frac{1}{\varphi} \colon \mathcal{X} \to \mathbb{P}^1$ the resulting CY-n-fold family (with $\mathcal{M} \subseteq R^n f_* \mathbb{Q}$ as above), the **box extension** — arising from fiberwise restriction of (roughly) the symbol $\{x_1, \dots, x_{n+1}\} \in K_{n+1}^M(\bar{\mathbb{Q}}(\mathcal{X}))$ — is $\mathcal{E}_1^\vee(1)$.

Inhomogeneous equations For any $\ell \in \mathbb{Z}_{>0}$, let $V^{[\ell]}(t)$ denote the unique solution to $L(\cdot) = -t^\ell$ analytic on $\mathbb{D}$; then $\kappa(\ell) = \ell^{n+1} V^{[\ell]}(0)$. Each embedding of a Tate object $\mathbb{Q}(-a) \hookrightarrow \mathrm{IH}^1(\mathbb{P}^1 \setminus \{\infty\}, \mathbb{M})$ produces an admissible extension $0 \to \mathcal{M} \to \mathcal{V}_\mu \to \mathbb{Q}(-a) \to 0$ with higher normal function $V_\mu(t)$ of this type for $\ell \leq d$.

Key Example 2: if $d = 2$, then $\mathrm{IH}^1(\mathbb{P}^1 \setminus \{0\}, \mathbb{M}) \cong \mathbb{Q}(-a)$ for some $\frac{n+1}{2} \leq a \leq n+1$, and the resulting **higher normal function** V_μ satisfies $LV_\mu = -\mathfrak{k}t$ for some $\mathfrak{k} \in \mathbb{C}^\times$, and $\kappa(1) = \mathfrak{k}^{-1} V_\mu(0)$. Of course, $\mathcal{M}$ usually arises from a family $\mathcal{X}$ defined over $\bar{\mathbb{Q}}$, and then $\mathfrak{k} \in \bar{\mathbb{Q}}$.

Summary We record the basic properties of the kappa series, which is really a *meromorphic function* on $\mathbb{C}$ with poles at negative integers:

(1) At $s = -k \in \mathbb{Z}_{\leq 0}$, the leading term in the Laurent expansion of $\kappa(s)$ is $\kappa^*(-k) = (-k)^{n+1} a_k$. Here $\{a_k\}$ are the coefficients of the unique holomorphic period of $\mathcal{M}$ on Δ_0; in Key Example 1, $a_k = [\varphi^k]_{\underline{0}}$ are constants in the powers of the Laurent polynomial.

(2) At $s = 0$, the power series coefficients of κ (more precisely, of κ^{-1}) compute the LMHS of $\mathcal{M}$ — and, more generally, of $\mathcal{E}_m$ — at $t = 0$. These are the numbers arising in [GZ]. In Key Example 1 (with φ the Minkowski polynomial mirror to a Fano X°), by the Gamma Conjecture they should match the coefficients of powers of c_1 in the regularized $\hat{\Gamma}$-class of X° — and, more generally, of its "progenitors" (see [Go1]). (In the case of $\mathcal{E}_1$, κ_{n+1} is related to the LMHS of the box extension at $t = 0$; but this is *not* the special value of the corresponding higher normal function, which blows up at 0 in any normalization — the *extension of VMHS* cannot be specialized there.)

(3) At $s = k \in \mathbb{Z}_{>0}$, the values $\kappa(k)$ reflect the value at 0 of the unique solution to the inhomogeneous equation $L(\cdot) = -t^k$ analytic on the big disk

$\mathbb{D}$. When certain hypotheses are satisfied,[3] for small values of k these will be special values of higher normal functions arising from motivic cohomology classes on $\mathcal{X} \setminus X_\infty$. These are the numbers that arise in [Go2], and are expected to be the correct B-model interpretation of Apéry constants of homogeneous varieties tabulated in [Ga]. Moreover, they are the numbers which arise in the "spirit of Apéry" (in taking a linear combination of two exponentially increasing solutions to a recurrence that then dies exponentially).

In light of (2) and (3), it seems reasonable to call the $\{\kappa(k)\}$ Apéry numbers and the $\{\kappa_j\}$ Frobenius numbers. Evidently, these constants are global arithmetic invariants of the VHS $\mathcal{M}$.

Some mundane notational conventions: we write $\boldsymbol{\delta}_{ij}$ for the Kronecker delta, $\mathbf{i} := \sqrt{-1}$, and $D := t\frac{d}{dt}$.

2. Periods of connections

Fix a coordinate t on $\mathbb{P}^1$. We work in the setting of algebraic connections on $U := \mathbb{P}^1 \setminus \Sigma$, where Σ is a set of at least three points including 0 and ∞. That is, one has a differential operator of the form

$$L = \sum_{j=0}^{d} t^j P_j(D) = \sum_{i=0}^{r} q_{r-i}(t) D^i \in \mathbb{C}[t, D] \qquad (\gcd(\{q_\ell\}) = 1),$$

of degree d and order r, with singularities only in Σ, and accompanying $\mathcal{D} := \mathcal{D}_{\mathbb{P}^1}$-module $\mathcal{D}/\mathcal{D}L$ on $\mathbb{P}^1$ with solution sheaf $\mathrm{Hom}_{\mathcal{D}^{\mathrm{an}}}(\mathcal{D}/\mathcal{D}L, \mathcal{O}^{\mathrm{an}}_{\mathbb{P}^1})$. Its restriction to U is a *connection* $(\mathcal{M}, \nabla\colon \mathcal{M} \to \mathcal{M} \otimes \Omega^1_U)$ with underlying local system $\mathbb{M}_{\mathbb{C}} := \ker(\nabla^{\mathrm{an}}) \subset \mathcal{M}^{\mathrm{an}}$ of rank r, and solution sheaf $\mathrm{Sol}(\mathcal{M}) := \mathrm{Hom}_{\mathcal{D}_U^{\mathrm{an}}}(\mathcal{M}, \mathcal{O}_U^{\mathrm{an}}) \cong \mathbb{M}_{\mathbb{C}}^\vee$.

Write $\mu \in H^0(U, \mathcal{M})$ for the image of $1 \in \mathcal{D}/\mathcal{D}L$, so that $\nabla_L \mu = 0$. Local analytic sections ε of $\mathbb{M}_{\mathbb{C}}^\vee$ may be paired with μ to yield *periods* $\langle \varepsilon, \mu \rangle$, which are local sections of $\mathcal{O}_U^{\mathrm{an}}$ satisfying $D\langle \varepsilon, \mu\rangle = \langle \varepsilon, \nabla_D \mu\rangle$ hence $L\langle \varepsilon, \mu \rangle = \langle \varepsilon, \nabla_L \mu \rangle = 0$. On a simply connected subset $\mathcal{S} \subset U^{\mathrm{an}}$, each such period is simply the image of $1 \in \mathcal{D}/\mathcal{D}L$ under ε regarded as an element of $\mathrm{Hom}_{\mathcal{D}_{\mathcal{S}}}(\mathcal{D}/\mathcal{D}L, \mathcal{O}_{\mathcal{S}})$.

In our setup, the connection is *regular* at ∞ if $\deg(q_\ell) \leq \deg(q_0)$ $(\forall \ell)$, at $\sigma \in \Sigma^\times := \Sigma \setminus \{0, \infty\}$ if $\mathrm{ord}_\sigma(q_\ell) \geq \mathrm{ord}_\sigma(q_0) - \ell$ $(\forall \ell)$, and at 0 if $\mathrm{ord}_0(q_\ell) \geq \mathrm{ord}_0(q_0)$ $(\forall \ell)$. The latter (together with $\gcd(\{q_\ell\}) = 1$) implies that $q_0(0) \neq 0$, and we assume henceforth that $q_0(0) = 1$.

[3] namely, that $\mathrm{IH}^1(\mathbb{A}^1, \mathbb{M})$ be split Hodge-Tate (or at least have "enough" Hodge classes), as well as the Beilinson-Hodge Conjecture for the family $\mathcal{X}$ underlying $\mathcal{M}$.

Example 2.1. Let $\mathcal{X}$ be a smooth projective $(n+1)$-fold, $f\colon \mathcal{X} \to \mathbb{P}^1$ a proper morphism whose restriction $f_U\colon \mathcal{X}_U := f^{-1}(U) \to U$ is smooth, and consider the exact sequence of complexes

$$0 \to f_U^*\Omega^1_U \otimes \Omega^\bullet_{\mathcal{X}_U/U}[1] \to \Omega^\bullet_{\mathcal{X}_U} \to \Omega^\bullet_{\mathcal{X}_U/U} \to 0.$$

Applying $\mathbb{R}^k(f_U)_*$ to its terms yields a long exact sequence in which the (everywhere regular) *Gauss-Manin connection* appears as a connecting homomorphism: writing $\mathcal{M} := \mathbb{R}^n(f_U)_*\Omega^\bullet_{\mathcal{X}_U/U}$, we obtain

$$\mathcal{M} \stackrel{\nabla}{\to} \mathbb{R}^{n+1}(f_U)_*(f_U^*\Omega^1_U \otimes \Omega^\bullet_{\mathcal{X}_U/U}[1]) \cong \Omega^1_U \otimes \mathbb{R}^n(f_U)_*\Omega^\bullet_{\mathcal{X}_U/U} = \Omega^1_U \otimes \mathcal{M}.$$

Viewed in the analytic topology, ∇ annihilates $\mathbb{M}_\mathbb{K} := R^n(f_U)_*\mathbb{K}_{\mathcal{X}_U^{\mathrm{an}}}$ for any subring $\mathbb{K} \subseteq \mathbb{C}$. The solution sheaf $\mathrm{Sol}(\mathcal{M})$ identifies with the local system $\{H_n(X_t, \mathbb{C})\}_{t\in U}$.

Without loss of generality, we may assume that $\mathcal{M}$ is irreducible cyclic, so that for some $\mu \in H^0(U, \mathcal{M})$, $\mathcal{M}$ is generated as an $\mathcal{O}_U$-module by[4] $\mu, \nabla_D\mu, \nabla_D^2\mu, \ldots, \nabla_D^{r-1}\mu$. So there exists $L \in \mathcal{O}(U)[D]$, which we may normalize as above, with $\nabla_L\mu = 0$. Local analytic sections ε of $\mathbb{M}_\mathbb{K}^\vee$ may be paired with μ to yield $\mathbb{K}$-*periods* $\langle\varepsilon, \mu\rangle$, refining the ($\mathbb{C}$-)periods above.

Fix a base point $p \in U(\mathbb{C}) \cap \mathbb{R}_{>0}$ near 0, and a point $\tilde{p} \in \widetilde{U^{\mathrm{an}}}$ above p on the universal cover $\mathcal{P}\colon \widetilde{U^{\mathrm{an}}} \twoheadrightarrow U^{\mathrm{an}}$. Also fix paths γ_σ in U^{an} based at p and winding once counterclockwise about each $\sigma \in \Sigma \setminus \{\infty\}$. Write T_σ for the action of monodromy (parallel transport along γ_σ) on the stalks $\mathbb{M}_p$ and $\mathbb{M}_p^\vee$. In dual bases the matrices of these actions will be transpose-inverse to one another.

Example 2.2. Suppose only that $\mathcal{M}$ has a regular singularity at 0, and that $\mathrm{rk}(\mathbb{M}_p^{T_0}) = 1$. (Since $q_0(0) = 1$, these imply that $P_0(D) = D^r$.) Normalizing μ (and replacing L accordingly), we may assume that the unique invariant period in a neighborhood of 0 takes the form $A(t) = 1 + \sum_{k\geq 1} a_k t^k$. A first motivation for the construction of Bloch-Vlasenko Γ-functions is: can we *interpolate* the $\{a_k\}$, i.e. produce an entire function with $F(-m) = a_m$ for all $m \in \mathbb{Z}_{>0}$?

[4] In the sequel we make no explicit assumption about μ generating $\mathcal{M}$ in this strong sense, though in the setting imposed in §4*ff*, it will always generate $\mathcal{M}$ on a smaller Zariski open.

If $L = D + t$ then the period is $e^{-t} = \sum_{k\geq 0} \frac{(-1)^k}{k!} t^k$, and the sort of function we are after is

$$F(s) := \frac{e^{2\pi \mathbf{i} s} - 1}{2\pi \mathbf{i}} \Gamma(s), \ \text{ where } \ \Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}.$$

Since $\Gamma(s) \sim \frac{(-1)^m}{m!(s+m)}$ for $s \sim -m$, and $\frac{e^{2\pi \mathbf{i} s}-1}{2\pi \mathbf{i}} \sim s+m$, we get $F(-m) = \frac{(-1)^m}{m!}$. The Bloch-Vlasenko Γ in this case would be $(e^{2\pi \mathbf{i} s} - 1)\Gamma(s)$, see Example 3.5.

Henceforth (with the exception of Example 3.5) we shall *assume that $\mathcal{M}$ has regular singularities*. Choose a section $\mathscr{m} \in H^0(U, \mathcal{M})$, not necessarily the section μ annihilated by L. For each $\varepsilon \in \mathbb{M}^\vee_{\mathbb{K},p}$, by $\langle \varepsilon, \mathscr{m}\rangle$ we shall mean the holomorphic function on $\widetilde{U^{\mathrm{an}}}$ (or multivalued function on U^{an}) obtained by pairing $\mathscr{m}$ with the section of $\mathcal{P}^{-1}(\mathbb{M}^\vee_{\mathbb{K}})$ extending ε from $\tilde{p}$. Let $C_\bullet(\widetilde{U^{\mathrm{an}}}; \mathbb{K})$ be the complex of topological chains on the universal cover; then

$$\xi = \Big[\sum_j \gamma_j \otimes \varepsilon_j\Big] \in H_1(U^{\mathrm{an}}, \mathbb{M}^\vee_{\mathbb{K}}) := H_1(C_\bullet(\widetilde{U^{\mathrm{an}}}; \mathbb{K}) \otimes_{\mathbb{K}[\pi_1(U^{\mathrm{an}},p)]} \mathbb{M}^\vee_{\mathbb{K},p})$$

is paired with $\omega = \mathscr{m} \otimes \frac{dt}{t} \in H^1_{\mathrm{dR}}(U, \mathcal{M})$ by

$$\langle \xi, \omega\rangle := \sum_j \int_{\gamma_j^{-1}} \langle \varepsilon_j, \mathscr{m}\rangle \frac{dt}{t}.$$

This is called a *period of the connection $\mathcal{M}$*.

Remark 2.3. (i) The H_1 above also identifies with $H_1(\pi_1(U^{\mathrm{an}}, p), \mathbb{M}^\vee_{\mathbb{K},p})$ (group homology), computed by the complex $\mathcal{C}_2 \to \mathcal{C}_1 \to \mathcal{C}_0$, where $\mathcal{C}_0 := \mathbb{M}^\vee_{\mathbb{K},p}$ and (for $n = 1, 2$)

$$\mathcal{C}_n := \{\text{free abelian group on symbols } [g_1, \ldots, g_n]\} \otimes \mathbb{M}^\vee_{\mathbb{K},p}.$$

The differential is given by $\partial([\gamma_1, \gamma_2] \otimes \varepsilon) = [\gamma_2] \otimes \gamma_1^{-1}\varepsilon - [\gamma_1\gamma_2] \otimes \varepsilon + [\gamma_1] \otimes \varepsilon$ and $\partial([\gamma] \otimes \varepsilon) = \gamma^{-1}\varepsilon - \varepsilon$, which reflects the multivaluedness of the sections of $\mathbb{M}^\vee_{\mathbb{K}}$.

(ii) The pairing is well-defined: if $\xi \in \partial\mathcal{C}_2$ holds or ω is a dR-coboundary, then $\langle \xi, \omega\rangle = 0$. In the first case, this follows from

$$\int_{(\gamma_1\gamma_2)^{-1}} \langle \varepsilon, \omega\rangle \frac{dt}{t} = \int_{\gamma_2^{-1}\gamma_1^{-1}} \langle \varepsilon, \omega\rangle = \int_{\gamma_1^{-1}} \langle \varepsilon, \omega\rangle \frac{dt}{t} + \int_{\gamma_2^{-1}} \langle \gamma_1^{-1}\varepsilon, \omega\rangle \frac{dt}{t},$$

which holds because γ_1^{-1} has acted on ε before we start along γ_2^{-1}. For the second, if $\omega = \nabla\eta = \nabla_D \eta \otimes \frac{dt}{t}$ then

$$\begin{aligned}\langle \xi, \omega\rangle &= \sum_j \int_{\gamma_j^{-1}} \langle \varepsilon_j, \nabla_D \eta\rangle \frac{dt}{t} = \sum_j \int_{\gamma_j^{-1}} D\langle \varepsilon_j, \eta\rangle \frac{dt}{t} \\ &= \left\langle \sum_j (\gamma_j^{-1}\varepsilon_j - \varepsilon_j), \eta \right\rangle = \langle 0, \eta\rangle = 0\end{aligned}$$

by the Fundamental Theorem of Calculus.

3. Gamma functions and interpolation

Consider the rank-1 connection on $\mathcal{O}_U$ with $\nabla_D 1 := s$, so that the differential operator is $D - s$ and the period is t^s. By abuse of notation we write this connection as "t^s", and set $\mathcal{M}(s) := \mathcal{M} \otimes t^s$. The action of $\pi_1(U^{\mathrm{an}}, p)$ on its stalk $\mathbb{M}(s)^\vee_{\mathbb{K},p} = \mathbb{M}^\vee_{\mathbb{K},p} \otimes_{\mathbb{K}} \mathbb{K}[e^{\pm 2\pi \mathbf{i} s}]$ is the tensor product of the monodromy representation for $\mathbb{M}^\vee_{\mathbb{K}}$ with the monodromy of $t^s = e^{s \log t}$ on $\mathbb{C}^*$. (We take $1 \in \mathbb{K}[e^{\pm 2\pi \mathbf{i} s}]$ to correspond to the branch with $\log(p) \in \mathbb{R}$.) Our interest lies in certain periods of this "Mellin-transformed" connection:

Definition 3.1. Given $m \in \mathcal{M}(U)$ and

$$\xi = \left[\sum_j \gamma_j \otimes \varepsilon_j \otimes e^{2\pi \mathbf{i} n_j s}\right] \in H_1(U^{\mathrm{an}}, \mathbb{M}(s)^\vee_{\mathbb{K}}),$$

with $n_j \in \mathbb{Z}$, the associated Bloch-Vlasenko *Gamma function* is

$$\Gamma_{\xi,m}(s) := \sum_j e^{2\pi \mathbf{i} n_j s} \int_{\gamma_j^{-1}} \langle \varepsilon_j, m\rangle t^s \frac{dt}{t}.$$

It is called *motivic* if $\mathcal{M}$ arises as in Example 2.1.

Remark 3.2. (i) This function is entire: $\cup|\gamma_j|$ avoids singularities of the integrand, which is thus uniformly bounded for s in any compact set.

(ii) Given m, $\Gamma_{\xi,m}$ depends only on ξ (and not its representative) by Remark 2.3(ii) applied to $\mathcal{M}(s)$, with $\omega = m \otimes 1 \otimes \frac{dt}{t}$. Hence the set of all Gamma functions for $(\mathcal{M}, m)$ is an *image* of $H_1(U, \mathbb{M}(s)^\vee_{\mathbb{K}})$, and is finitely generated as a $\mathbb{K}[e^{\pm 2\pi \mathbf{i} s}]$-module.

Recall that μ is the section of $\mathcal{M}$ annihilated by $L = \sum_{k=0}^d t^k P_k(D)$.

Theorem 3.3. *The Gamma functions for $(\mathcal{M}, \mu)$ satisfy the difference equation*

$$\sum_{k=0}^{d} P_k(-s-k)\Gamma_{\xi,\mu}(s+k) = 0.$$

Proof. Applying the Fundamental Theorem of Calculus to

$$0 = \partial\xi = \textstyle\sum_j e^{2\pi \mathbf{i} n_j s}(\gamma_j^{-1} - 1)(\varepsilon_j \otimes 1)$$

yields

$$0 = \textstyle\sum_j e^{2\pi \mathbf{i} n_j s} \int_{\gamma_j^{-1}} D(\langle \varepsilon_j, \mu\rangle t^s)\frac{dt}{t} = \Gamma_{\xi,\nabla_D \mu}(s) + s\Gamma_{\xi,\mu}(s).$$

Moreover, $\Gamma_{\xi,t\mu}(s) = \Gamma_{\xi,\mu}(s+1)$ is evident from the definition. So

$$\sum_j t^j P_j(\nabla_D)\mu = 0$$

gives

$$\begin{aligned} 0 = \Gamma_{\xi,0}(s) &= \textstyle\sum_j \Gamma_{\xi, t^j P_j(\nabla_D)\mu}(s) \\ &= \textstyle\sum_j \Gamma_{\xi, P_j(\nabla_D)\mu}(s+j) = \sum_j P_j(-s-j)\Gamma_{\xi,\mu}(s+j). \end{aligned}$$

□

Remark 3.4 (Recurrence relations). In the setting of Example 2.2, we have

$$\begin{aligned} 0 = LA(t) &= \sum_{k=0}^{d} t^k P_k(D) \sum_{m\geq 0} a_m t^m = \sum_{k=0}^{d}\sum_{m\geq 0} P_k(m) a_m t^{m+k} \\ &= \sum_{m\geq 0}\left(\sum_{k=0}^{d} P_k(m-k) a_{m-k}\right) t^m \end{aligned}$$

hence $\sum_{k=0}^{d} P_k(m-k)a_{m-k} = 0$ for all m, which determines a_m from the $\{a_{m-k}\}_{k=1}^{\min\{m,d\}}$. Setting $s = -m$ in Theorem 3.3, we have

$$\sum_{k=0}^{d} P_k(m-k)\Gamma_{\xi,\mu}(-m+k) = 0.$$

So, if we assume $\Gamma_{\xi,\mu}(0) = 2\pi\mathbf{i}$, and $\Gamma_{\xi,\mu}(\ell) = 0$ for $\ell \in \mathbb{Z}_{>0}$, then $\Gamma_{\xi,\mu}(-m) = 2\pi\mathbf{i}a_m$. As we shall see, in the confluence of the settings of Examples 2.1

and 2.2, these formulas will turn out to be true up to a nonzero rational factor. Therefore, the Bloch-Vlasenko Γ-function *interpolates* the power-series coefficients $\{a_m\}$.

To conclude with the "simplest example", we have to break the rule about regular singularities.

Example 3.5. Let $\mathcal{M}$ be the connection on $\mathcal{O}_{\mathbb{G}_m}$ with $\nabla_D 1 = -t$. The differential operator is $D+t$, its period e^{-t} ($=\langle\varepsilon,1\rangle$ for a section ε of $\mathbb{M}^\vee$). Consider the path γ which runs from ∞ to $\epsilon>0$ along $\mathbb{R}_{>0}$, once counterclockwise around 0, then back to ∞ along $\mathbb{R}_{>0}$. Due to the subpolynomial decay of e^{-t} at ∞, $\xi=\gamma\otimes\varepsilon\otimes 1$ is a "rapid decay cycle" in $H_1^{\mathrm{RD}}(\mathbb{C}^*,\mathbb{M}(s)^\vee)$ (see [BE]), so that

$$\Gamma_{\xi,1}(s)=\int_\gamma\langle\varepsilon,1\rangle t^s\frac{dt}{t}=(e^{2\pi\mathbf{i}s}-1)\int_0^\infty e^{-t}t^{s-1}dt=(e^{2\pi\mathbf{i}s}-1)\Gamma(s)$$

as advertised. But this is "ur-Gamma" is not a *motivic* Gamma!

4. Conifold monodromy

For the remainder of this paper we work in the following setting, which is motivated and typified by the simplest $\mathcal{D}$-modules arising from Landau-Ginzburg models:

- $(\mathcal{M},\nabla)$ is *motivic*, which is to say that it underlies a sub-$\mathbb{Q}$-PVHS of an $\mathbb{R}^n(f_U)_*\Omega^\bullet_{\mathcal{X}_U/U}$ (defined as in Example 2.1). This implies:
  - $\mathcal{M}$ has regular singularities;
  - fiberwise $\mathbb{Q}$-Betti cohomology provides a $\mathbb{Q}$-local system $\mathbb{M}_{\mathbb{Q}}$ underlying $\mathbb{M}$, whose monodromies $T_\sigma=T_\sigma^{\mathrm{ss}}e^{N_\sigma}$ are thus defined over $\mathbb{Q}$;
  - fiberwise integration yields a polarization $Q(\cdot,\cdot)\colon\mathcal{M}\times\mathcal{M}\to\mathcal{O}$ sending $\mathbb{M}_{\mathbb{Q}}\times\mathbb{M}_{\mathbb{Q}}\to\mathbb{Q}$;[5] and
  - $\mathcal{M}$ has a varying Hodge flag $\mathcal{F}^\bullet$, with $\nabla\mathcal{F}^\bullet\subset\mathcal{F}^{\bullet-1}\otimes\Omega^1_U$, satisfying the Hodge-Riemann relations.

We will use $\mathcal{M}$ also to denote this PVHS in what follows.

[5]That is, Q is a morphism of VHS *of weight* $-2n$. The induced isomorphism $Q(\cdot)\colon\mathcal{M}\to\mathcal{M}^\vee$ defined by $Q(a,b)=\langle Q(a),b\rangle$ sends $\mathbb{M}_{\mathbb{Q}}\to\mathbb{M}_{\mathbb{Q}}^\vee$; and the polarization on $\mathcal{M}^\vee$ defined by $Q(a,b):=\langle a,Q^{-1}(b)\rangle$ restricts to the intersection form on $\mathbb{Q}$-Betti homology $\mathbb{M}_{\mathbb{Q}}^\vee$. (The "missing" $(2\pi\mathbf{i})^n$ twist will eventually show up.)

- $\mathcal{M}$ is *principal*: the $\mathrm{Gr}^p_{\mathcal{F}}\mathcal{M}$ are all of rank 1 for $p = 0, 1, \ldots, n$, so that $n :=$ weight of $\mathcal{M}$ and $r :=$ rank of $\mathcal{M}$ = order of L are related by $r = n + 1$.
- $\mathcal{M}$ has *maximal unipotent monodromy* at $t = 0$: $\mathrm{rk}(\mathbb{M}^{T_0}) = 1$. Accordingly, fixing $\varepsilon \in (\mathbb{M}^{\vee}_{\mathbb{Q},p})^{T_0}$ once and for all, there exists a basis $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n$ of $\mathbb{M}^{\vee}_{\mathbb{Q},p}$ with $N_0\varepsilon_i = \varepsilon_{i-1}$. Though this basis is not unique, $\mathsf{Q}_0 := Q(\varepsilon_0, \varepsilon_n) \in \mathbb{Q}^{\times}$ is independent of the choice (which in any case we will specify below).
- There is a "minimal" $c \in \Sigma^{\times}(= \Sigma \setminus \{0, \infty\})$, with $|c| < |\sigma|$ for all other $\sigma \in \Sigma^{\times}$; and $\mathcal{M}$ has *conifold monodromy* at $t = c$: $\mathrm{rk}(T_c - I) = 1$. That is, there exists $\delta \in \mathbb{M}^{\vee}_{\mathbb{Q},p}$ such that:
  - the linear span $\langle \delta \rangle = \mathrm{im}(T_c - I)\mathbb{M}^{\vee}_{\mathbb{Q},p}$;
  - for n odd, T_c is a symplectic transvection, sending $\delta \mapsto \delta$ and some $\beta \mapsto \beta + \delta$;
  - for n even, $T_c\colon \delta \mapsto -\delta$ is an orthogonal reflection; and
  - $\varepsilon \in \mathbb{M}^{\vee}_{\mathbb{Q},p}$ is invariant under T_c if and only if $Q(\varepsilon, \delta) = 0$.
- Finally, assume that $T_c\varepsilon_0 \neq \varepsilon_0$. We may then rescale δ so that $(T_c - I)\varepsilon_0 = \delta$, and set $\mathsf{Q}_c := Q(\varepsilon_0, \delta) \neq 0$.

Writing $T_\sigma = T_\sigma^{\mathrm{un}} T_\sigma^{\mathrm{ss}}$ for the Jordan decomposition and $N_\sigma := \log(T_\sigma^{\mathrm{un}})$ for the monodromy logarithms, the assumptions just made imply $T_0 = e^{N_0}$ and $N_0\varepsilon_0 = 0$, as well as:

Lemma 4.1. *δ generates $\mathbb{M}^{\vee}_p$ under N_0.*

Proof. First note that if $i + j < n$, then $n - i > j \implies N^{n-i}\varepsilon_j = 0 \implies Q(\varepsilon_i, \varepsilon_j) = Q(N^{n-i}, \varepsilon_n, \varepsilon_j) = (-1)^{n-1}Q(\varepsilon_n, N^{n-i}\varepsilon_j) = 0$. (In particular, $Q(\varepsilon_0, \varepsilon_k) = 0$ for $k < n$; and since Q is nondegenerate, we must then have $Q(\varepsilon_0, \varepsilon_n) \neq 0$ as mentioned in the third bullet above.)

Now suppose that $\delta = \sum_{i \leq k} \mathsf{c}_i \varepsilon_i$, with $k < n$. Then for any $\varepsilon \in \mathbb{M}^{\vee}_{\mathbb{Q},p}$, $Q(\varepsilon, (T_c^{-1} - I)\varepsilon_j) = Q((T_c - I)\varepsilon, \varepsilon_j) = Q(\mathsf{c}_\varepsilon \delta, \varepsilon_j) = \sum_{i \leq k} \mathsf{c}_i \mathsf{c}_\varepsilon Q(\varepsilon_i, \varepsilon_j)$ is 0 for all $j < n - k$. Hence $\varepsilon_0, \ldots, \varepsilon_{n-k-1}$ are T_c-invariant, which in the case of ε_0 contradicts the last bullet above. □

Before proceeding, we make some final calibrations to the $\mathbb{Q}$-Betti homology classes as follows:

Lemma 4.2. *Given ε_0, there exists a unique choice of $\varepsilon_1, \ldots, \varepsilon_n$ satisfying $N_0\varepsilon_j = \varepsilon_{j-1}$ and $(T_c - I)\varepsilon_j = 0$ for $j > 0$.*

Proof. Given initial choices $\varepsilon_1^\circ, \ldots, \varepsilon_n^\circ$ (and $\varepsilon_0^\circ = \varepsilon_0$) satisfying $N_0\varepsilon_j^\circ = \varepsilon_{j-1}^\circ$, write $(T_c - I)\varepsilon_k^\circ =: \mathsf{d}_k\delta$ (with $\mathsf{d}_0 = 1$), and inductively define $\varepsilon_k := \varepsilon_k^\circ - \sum_{j=1}^k \mathsf{d}_j\varepsilon_{k-j}$ for $k = 1, \ldots, n$. One easily checks the desired properties (by induction).

Suppose $\varepsilon_1', \ldots, \varepsilon_k'$ also satisfy the two properties in the statement of the Lemma. Inductively assuming that $\varepsilon_i' = \varepsilon_i$ for $i < k$, we have $N_0(\varepsilon_k' - \varepsilon_k) = \varepsilon_{k-1}' - \varepsilon_{k-1} = 0$ hence $\varepsilon_k' = \varepsilon_k + a\varepsilon_0$; whence $0 = (T_c - I)\varepsilon_k' = (T_c - I)\varepsilon_k + a(T_c - I)\varepsilon_0 = a\delta \implies a = 0$. $\square$

Remark 4.3. In the event that the geometry $\mathcal{X}_U \to U$ underlying $\mathcal{M}$ extends over c to a degeneration with smooth total space and nodal singular fiber X_c, we will say that $\mathcal{M}$ has *strong* conifold monodromy at c. In this case, there is a conifold vanishing sphere δ_0 with $Q(\delta_0, \delta_0) = (-1)^{\binom{n+1}{2}}(1+(-1)^n)$, which controls the monodromy via the Picard-Lefschetz formula $T_c\varepsilon = \varepsilon - (-1)^{\binom{n}{2}}Q(\varepsilon, \delta_0)\delta_0$. (We then have $\delta = M\delta_0$ for some $M \in \mathbb{Q}^\times$, and $\mathsf{Q}_c = -(-1)^{\binom{n}{2}}M^2$.) We shall only assume this where indicated, since there are times when one merely has a differential operator in hand.

Turning to the de Rham structure, let $\mathcal{M}_e$ be the canonical extension of $\mathcal{M}$ to $\mathbb{P}^1$, whose logarithmic connection $\nabla\colon \mathcal{M}_e \to \mathcal{M}_e \otimes \Omega^1_{\mathbb{P}^1}\langle\log\Sigma\rangle$ has residues $\mathrm{Res}_\sigma(\nabla) = -\frac{N_\sigma}{2\pi\mathbf{i}} - \mathrm{Log}(T_\sigma^{\mathrm{ss}})$ (with Log the branch of $\frac{\log}{2\pi\mathbf{i}}$ with real part in $[0,1)$). The extended Hodge sub-bundles $\mathcal{F}_e^\bullet$ satisfy $\nabla(\mathcal{F}_e) \subseteq \mathcal{F}_e^{\bullet-1} \otimes \Omega^1_{\mathbb{P}^1}\langle\log\Sigma\rangle$. In particular, the line bundle $\mathcal{F}_e^n$ is positive, and so has nonzero holomorphic sections; we *take $\mu \in H^0(\mathbb{P}^1, \mathcal{F}_e^n)$ to be the unique such section with zeroes only at ∞ and normalized so that $\langle\varepsilon_0, \mu\rangle = A(t) = \sum_{m\geq 0} a_m t^m$ has $a_0 = 1$.* The assumption that $T_c\varepsilon_0 \neq \varepsilon_0$ implies that $A(t)$ has monodromy at c, and so $\limsup_{m\to\infty} a_m^{1/m} = c^{-1}$.

Henceforth $L = \sum_{j=0}^d t^j P_j(D) = \sum_{i=0}^r q_{r-i}(t)D^i$ *shall denote the* (*Picard-Fuchs*) *differential operator associated to this* μ, written so the $\{q_i\}_{i=0}^r$ have no common factor and $q_0(0) = 1$. That is, L annihilates μ and all of its periods. (From this point on we drop ∇ when convenient, writing $D\mu$, etc.) We shall be interested in the particular $\mathbb{Q}$-periods

$$\begin{aligned} \epsilon_k(t) &:= \langle\varepsilon_k, \mu\rangle \qquad\qquad (k = 0, 1, \ldots, n) \\ \psi(t) &:= \langle\delta, \mu\rangle, \end{aligned}$$

where of course $\epsilon_0(t) = A(t)$. Recalling that $g(t) \sim h(t)$ at $t = \sigma$ means $\lim_{t\to\sigma} \frac{g(t)}{h(t)} = 1$, here is what we can say about their asymptotic behavior:

Lemma 4.4. (i) *At $t = 0$, $\epsilon_k(t) \sim \frac{\log^k(t)}{k!(2\pi\mathbf{i})^k}$.*

(ii) *Write* $\mathrm{E}_n(z) := z^{\frac{n-1}{2}}$ *for* n *even and* $z^{\frac{n-1}{2}}\log(z)$ *for* n *odd. If* $\mathcal{M}$ *has strong conifold monodromy* (*Remark 4.3*), *then about* $t = c$ *we have*

$$\begin{aligned} &\epsilon_0(t) = C_0(1+\mathcal{O}(t-c))\mathrm{E}_n(t-c) + \textit{analytic function} \\ \textit{and} \quad &\psi(t) \sim C(t-c)^{\frac{n-1}{2}} \end{aligned}$$

for some constants $C_0, C \in \mathbb{C}^\times$, *and* $\mathrm{ord}_{t=c}(q_0) = 1$.

Proof. Applying repeatedly that $(2\pi\mathbf{i})D\langle\varepsilon_k,\mu\rangle = (2\pi\mathbf{i})\langle\varepsilon_k,\nabla_D\mu\rangle$ is asymptotic to $(2\pi\mathbf{i})\langle\varepsilon_k,(\mathrm{Res}_0\nabla)\mu\rangle = -\langle\varepsilon_k,N_0\mu\rangle = \langle N_0\varepsilon_k,\mu\rangle = \langle\varepsilon_{k-1},\mu\rangle$ yields (i). For (ii), the period exponent of a node $x_0^2+\cdots+x_n^2$ is $\frac{n+1}{2}$ (see [KLa, (4.6-7) and Prop. 4.1]), and by the assumptions above ε_0 maps onto the (rank one) vanishing cohomology. Since X_c is still K-trivial, and μ_c nonvanishing as a section of $\mathcal{F}^n_{e,c} \cong H^0(K_{X_c})$, the period $\epsilon_0 = \int_{\varepsilon_0}\mu$ realizes this exponent in [op. cit., (4.6)], yielding the claim about ϵ_0. For ψ, use $(T_c - I)\epsilon_0 = \psi$.

Choose a local coordinate $w \sim t-c$ so $\epsilon_0 = \mathrm{E}_n(w) + \text{analytic terms}$, and write $\partial = \frac{d}{dw}$. Then $\mathcal{M}_{e,c}$ is generated by

$$\begin{aligned} &\mu,\ \partial\mu,\ \ldots, \partial^{\frac{n-1}{2}}\mu,\ w\partial^{\frac{n+1}{2}}\mu,\ \partial w\partial^{\frac{n+1}{2}}\mu, \ldots,\ \partial^{\frac{n-1}{2}}w\partial^{\frac{n+1}{2}}\mu \qquad \text{resp.} \\ &\mu,\ \partial\mu,\ \ldots,\ \partial^{\frac{n}{2}}\mu,\ w^{\frac12}\partial w^{\frac12}\partial^{\frac{n}{2}}\mu,\ \partial w^{\frac12}\partial w^{\frac12}\partial^{\frac{n}{2}}\mu, \ldots,\ \partial^{\frac{n}{2}-1}w^{\frac12}\partial w^{\frac12}\partial^{\frac{n}{2}}\mu, \end{aligned}$$

and $\partial^{\frac{n+1}{2}}w\partial^{\frac{n+1}{2}}\mu$ resp. $\partial^{\frac{n}{2}}w^{\frac12}\partial w^{\frac12}\partial^{\frac{n}{2}}\mu$ belong to $\mathcal{M}_{e,c}$. From this one deduces that $w\partial^{n+1}\mu$ (and not $\partial^{n+1}\mu$) is a $\mathbb{C}[w]$-linear combination of $\mu, \partial\mu, \ldots, \partial^n\mu$. □

Here is a basic geometric example invoked repeatedly in §§9–10.

Example 4.5. Let $\varphi \in \mathbb{C}[x_1^{\pm1},\ldots,x_{n+1}^{\pm1}]$ be a Laurent polynomial whose Newton polytope Δ is reflexive, i.e. has integral polar polytope. (In particular, it has a unique integral interior point given by $\underline{0}$.) We shall call φ itself *reflexive* if in addition there exists a smooth blowup $\beta\colon \mathcal{X} \twoheadrightarrow \mathbb{P}_\Delta$ on which $\frac{1}{\varphi}$ extends to a proper morphism $f\colon \mathcal{X} \to \mathbb{P}^1$, $X_0 = f^{-1}(0) \subset \mathcal{X}$ is a normal-crossing divisor, and (β^* of) $\mathrm{dlog}(\underline{x}) := \frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_{n+1}}{x_{n+1}}$ extends to a nowhere-vanishing section of $\Omega^{n+1}_{\mathcal{X}}(\log X_0)$. An immediate consequence, writing Σ for the discriminant locus of f, is that (by adjunction) $X_t := f^{-1}(t)$ is a smooth CY n-fold for each $t \in \mathbb{P}^1\setminus\Sigma\,(=: U)$, given by a crepant resolution of $\overline{\{1-t\varphi(\underline{x})=0\}} \subset \mathbb{P}_\Delta$. We call $(\mathcal{X}, f)$ the *compact Landau-Ginzburg model* associated to φ [GKS, §3.1].

Put $\mathcal{X}_U := f^{-1}(U)$, so that $(\mathbb{R}^n(f_U)_*\Omega_{\mathcal{X}_U/U},\nabla)$ underlies a $\mathbb{Q}$-VHS $\mathcal{H}^n_f$, and write $\mathcal{M} \subseteq \mathcal{H}^n_f$ for the minimal sub-$\mathbb{Q}$-VHS containing the line bundle

$\mathcal{F}^n := \mathcal{F}^n\mathcal{H}^n_f$. If $\mathcal{M}$ satisfies the assumptions of the beginning of this section, we will call φ *good*. (For instance, in the $n = 2$ case where X_U is a family of $K3$ surfaces, the assumption that Hodge numbers of $\mathcal{M}$ are $(1, 1, 1)$ forces the generic Picard rank to be 19.) Taking a section μ of $\mathcal{F}^n_e$ as above with corresponding Picard-Fuchs operator L, it is enough to have L of order $n + 1$ with unique exponent 0 at $t = 0$[6] and a single integer exponent of multiplicity two or half-integer exponent of multiplicity one (for n odd resp. even) at $t = c$.

In fact, we can identify the section μ explicitly. Denoting by $\omega_f := \omega_{\mathcal{X}} \otimes f^*\omega_{\mathbb{P}^1}^{-1}$ the relative dualizing sheaf, by a result of Kollàr [Ko, Thm. 2.6] we have $\mathcal{F}^n_e \cong f_*\omega_f$. Clearly $\frac{\text{dlog}(\underline{x})}{f^*(dt/t)}$ is a section of $\omega_f \cong \omega_{\mathcal{X}}(\log X_0) \otimes f^*\omega_{\mathbb{P}^1}(\log 0)^{-1}$ vanishing to first order on X_∞ and nowhere else. Hence $\mu := \left[\frac{1}{(2\pi\mathbf{i})^n}\frac{\text{dlog}(\underline{x})}{df/f}\right] \in H^0(\mathbb{P}^1, \mathcal{F}^n_e)$ is a section with a simple zero at ∞, demonstrating that $\mathcal{F}^n_e \cong \mathcal{O}(1)$. Moreover, for each $t \in U$ we have $\frac{1}{(2\pi\mathbf{i})^n}\frac{\text{dlog}(\underline{x})}{1-t\varphi} = \frac{\mu\wedge df/f}{1-t\varphi} = \frac{\mu\wedge df}{f-t} = \mu \wedge \text{dlog}(f - t) \implies$

$$\mu_t = \frac{1}{(2\pi\mathbf{i})^n}\text{Res}_{X_t}\left(\frac{\text{dlog}(\underline{x})}{1-t\varphi(\underline{x})}\right) \in \Omega^n(X_t).$$

From this one easily shows (e.g. see [DK, (4.1)]) that a_m is the constant term in φ^m; in particular, $a_0 = 1$ as desired.

Finally, we can broaden this construction by allowing Laurent polynomials which define families with an automorphism over $t \mapsto e^{\frac{2\pi\mathbf{i}}{w}}t$ for some $w \in \mathbb{N}$, and which fail to be good only insofar as there are w conifold points of minimal modulus in Σ. Replacing $\mathcal{X}$ with its quotient by this automorphism and t by t^w, and assuming the new T_0 remains unipotent, μ still produces the desired section. In the sequel, all constructions and results stated for good reflexive polynomials φ are also valid in this setting.

5. Frobenius periods

Since $\mathbb{M}$ has maximal unipotent monodromy at $t = 0$ and $A(0) \neq 0$, it follows that L has the unique local exponent 0 there. The indicial equation $P_0(\mathtt{T}) = 0$ thus has unique root $\mathtt{T} = 0$, and so $P_0(D) = D^r$.

[6]This condition forces the underlying local system to be rational, since it implies $N_0^n \neq 0$, and a Galois-conjugate system inside $\mathbb{H}$ could not also have this property (since $h^{n,0} = 1$).

Definition 5.1. A *Frobenius deformation* for L at 0 is a formal series

$$\Phi(s,t) = \sum_{m\geq 0} \phi_m(t) s^m,$$

with each ϕ_m analytic on a neighborhood of p (and by continuation, on $\widetilde{U^{\mathrm{an}}}$), such that $L\Phi = s^r t^s$ $(= s^{n+1}t^s)$ and $T_0\Phi = e^{2\pi \mathbf{i} s}\Phi$. We shall call $\phi_0, \ldots, \phi_n$ the *Frobenius periods*, since they satisfy $L(\cdot) = 0$.

In our setting (as bulleted in §4), Φ is unique [BV]. To the author's knowledge, Frobenius deformations were first written down in the form of Definition 5.1 in [GZ, (6.4)].

Example 5.2. If L has order 3 $(n = 2)$, then

$$L\Bigg(\sum_{m\geq 0} \phi_m s^m\Bigg) = s^3 e^{s\log t} = \sum_{m\geq 3} \tfrac{\log^{m-3}(t)}{(m-3)!} s^m$$

implies $L\phi_0, L\phi_1, L\phi_2 = 0$ (morally, 3 $\mathbb{C}$-periods of a family of Picard rank 19 $K3$ surfaces) while $L\phi_3 = 1$, $L\phi_4 = \log(t)$, $L\phi_5 = \frac{\log^2 t}{2!}$, etc.

The monodromy condition $T_0\Phi = e^{2\pi\mathbf{i}s}\Phi$ forces $t^{-s}\Phi$ to be T_0-invariant (after expanding $t^{-s} = e^{-s\log(t)}$ and rearranging in powers of s). Since the ϕ_m have at worst log poles, the coefficients $\{\phi_m^{\mathrm{an}}\}$ of powers of s in $t^{-s}\Phi$ are thereby analytic in a disk about $t = 0$. Writing $\phi_m^{\mathrm{an}}(t) = \sum_{k\geq 0} a_k^{(m)} t^k$, expanding t^s gives

$$\begin{aligned}\Phi(s,t) &= \sum_{m'\geq 0} \phi_{m'}^{\mathrm{an}}(t) t^s s^{m'} = \sum_{m\geq 0} s^m \sum_{\ell=0}^{m} \tfrac{\log^\ell t}{\ell!} \phi_{m-\ell}^{\mathrm{an}}(t) \quad (m = m' + \ell) \\ &= \sum_{j,k,\ell\geq 0} s^{j+\ell} \tfrac{\log^\ell t}{\ell!} a_k^{(j)} t^k = \sum_{k\geq 0} t^k e^{s\log(t)} \sum_{j\geq 0} a_k^{(j)} s^j =: \sum_{k\geq 0} t^{k+s} A_k(s)\end{aligned}$$

in which the first line yields $\phi_m(t) = \sum_{\ell=0}^m \frac{\log^\ell t}{\ell!} \phi_{m-\ell}^{\mathrm{an}}(t)$. Furthermore, taking $t = 0$ in

$$\begin{aligned} s^r &= t^{-s} L\Phi = t^{-s}(D^r + t(\cdots))\Big(\textstyle\sum_{j\geq 0} a_0^{(j)} s^j + t(\cdots)\Big)\, t^s \\ &= \sum_{j\geq 0} a_0^{(j)} s^{j+r} + t(\cdots)\end{aligned}$$

gives $1=\sum_{j\geq 0} a_0^{(j)} s^j$, so that $a_0^{(j)}=\boldsymbol{\delta}_{0j}$. Immediate consequences are that $A_0(s)=\sum a_0^{(j)} s^j=1$, and (from $a_0^{(0)}=1$ and uniqueness of the holomorphic period) that $\phi_0(t)=\phi_0^{\mathrm{an}}(t)=\epsilon_0(t)=A(t)$, so that $a_k^{(0)}=a_k$.

Remark 5.3. *A priori* the $\{A_k(s)\}_{k>0}$ and $\Phi(s,t)$ are formal in s. However, $L\sum_{k\geq 0} t^{k+s}A_k(s)=s^r t^s$ implies the recurrence

$$A_m(s)=-(m+s)^{-r}\sum_{j=1}^{d} A_{m-j}(s)P_j(m-j+s),$$

where $\deg(P_j)\leq r$ for each j and $A_k(s):=0$ for $k<0$. This exhibits $A_m(s)$ as a rational function with poles (of order $\leq r$) in $\mathbb{Z}\cap[-m,-1]$. Moreover, the asymptotics of $A_m(s)$ as $m\to\infty$ are governed by the degree r terms of the $\{P_j\}$; these are the coefficients of $q_0(t)$, whose smallest root is nothing but c. One deduces that: for s in any compact subset of $\mathbb{C}\setminus\mathbb{Z}_{<0}$ and t in any disk about 0 of radius less than $|c|$, the series $\Phi^{\mathrm{an}}:=t^{-s}\Phi=\sum_k A_k(s)t^k$ converges uniformly to an analytic function; and Φ^{an}, Φ continue to analytic functions on $\widetilde{U^{\mathrm{an}}}\times(\mathbb{C}\setminus\mathbb{Z}_{<0})$.

We note here for reference the consequences that $\Phi(0,t)=A(t)$ and $\Phi^{\mathrm{an}}(s,0)=1$; from the latter, one has for example that $\Phi(\ell,t)$ is an analytic function vanishing at $t=0$ for each $\ell\in\mathbb{Z}_{>0}$.

Remark 5.4. In view of the equality of the 0^{th} Frobenius and $\mathbb{Q}$-Betti periods $\phi_0(t)=A(t)=\epsilon_0(t)$, it is natural to ask whether the remaining Frobenius periods are $\mathbb{Q}$-periods. It turns out that if this were the case, then the limiting mixed Hodge structure (LMHS) of $\mathcal{M}$ at $t=0$ would be $\mathbb{Q}$-split, *without* even renormalizing t! This is almost never true.

To see the relationship, recall that the LMHS is given by the limiting Hodge flag $\lim_{t\to 0} e^{\frac{\log(t)}{2\pi\mathbf{i}}N_0}\mathcal{F}_t^{\bullet}$ written with respect to the $\mathbb{Q}$-basis $\varepsilon_0^{\vee},\ldots,\varepsilon_n^{\vee}$, together with the weight monodromy filtration $W(N_0)_{2j}=\langle\varepsilon_n^{\vee},\ldots,\varepsilon_{n-j}^{\vee}\rangle$. But for computing the periods of the LMHS it is better to apply $e^{-\frac{\log(t)}{2\pi\mathbf{i}}N_0}$ to the $\mathbb{Q}$-basis and compare with $\mathcal{F}_{e,0}^{\bullet}$ in the limit. More precisely, we have the

Definition 5.5. By the *period matrix* $\Omega_{\lim}$ *of the LMHS*, we shall mean the change-of-basis matrix between[7] $\{(2\pi\mathbf{i})^{-j}e^{-\frac{\log(t)}{2\pi\mathbf{i}}N_0}\varepsilon_j^{\vee}\}_{j=0}^{n}$ (untwisted $\mathbb{Q}$-Betti) and $\mu,\nabla_D\mu,\ldots,\nabla_D^n\mu$ (de Rham) at $t=0$. Its 0^{th} column is

$$\lim_{t\to 0}\langle(2\pi\mathbf{i})^j e^{-\frac{\log(t)}{2\pi\mathbf{i}}N_0}\varepsilon_j,\mu\rangle=(2\pi\mathbf{i})^j\lim_{t\to 0}\epsilon_j^{\mathrm{an}}(t)=(2\pi\mathbf{i})^j\epsilon_j^{\mathrm{an}}(0),$$

[7]Here $(e^{-\frac{\log(t)}{2\pi\mathbf{i}}N_0}\varepsilon_j^{\vee})|_{t=0}$ belongs to $W(N_0)_{2(n-j)}\mathcal{M}_{\lim,\mathbb{Q}}$, and $\nabla_D^j\mu$ to $\mathcal{M}_{\lim}^{n-j,n-j}$; the $(2\pi\mathbf{i})^{-j}$ rescaling makes them project to the same element of $\mathrm{Gr}_{2(n-j)}^{W(N_0)}\mathcal{M}_{\lim}$.

where $\epsilon_j^{\mathrm{an}}(t)$ is the "analytic part" obtained from $\epsilon_j(t)$ by formally setting $\log(t)$ to zero. Since $N_0\varepsilon_j^\vee = -\varepsilon_{j+1}^\vee$ and $\mathrm{Res}_0(\nabla) = \frac{-N_0}{2\pi\mathbf{i}}$, each column is obtained from the previous one by shifting the entries down, yielding a lower-triangular matrix with ones on the diagonal.

If the Frobenius periods $\{\phi_j(t)\}$ were $\mathbb{Q}$-linear combinations of the Betti periods $\{\epsilon_j(t)\}$, the $\{\epsilon_j^{\mathrm{an}}(t)\}$ would be $\mathbb{Q}$-linear combinations of the $\{\phi_j^{\mathrm{an}}(t)\}$. Since $\phi_j^{\mathrm{an}}(0) = \boldsymbol{\delta}_{0j}$, all $\epsilon_j^{\mathrm{an}}(0)$ would be rational, and the $(j, j-\ell)^{\mathrm{th}}$ entries of the matrix would belong to $\mathbb{Q}(\ell)$, making the LMHS $\mathbb{Q}$-split.

6. The kappa series

We now turn to the analytic continuation of the Frobenius deformation around the conifold point. If $\mathcal{L} = \sum_{i=0}^r q_{r-i}(t)D^i$ is a differential operator underlying an algebraic connection, then its adjoint

$$\mathcal{L}^\dagger := (-1)^r \textstyle\sum_{i=0}^r (-D)^i q_{r-i}(t)$$

underlies the dual connection [BV, Lemma 34]. (In a slight abuse of notation, we shall write $\mathrm{Sol}_p(\mathcal{L})$ for the stalk $\mathrm{Sol}_p(\mathcal{D}/\mathcal{D}\mathcal{L})$ below.) Note that $(\mathcal{L}^\dagger)^\dagger = \mathcal{L}$ and $(D\mathcal{L})^\dagger = \mathcal{L}^\dagger D$.

Now remember that $\psi = \langle \delta, \mu\rangle$ denotes the period over the conifold vanishing cycle. If $\mathcal{L}$ satisfies $(T_c - I)\mathrm{Sol}_p(\mathcal{L}) = \mathbb{C}\psi$, then also $(T_c - I)\mathrm{Sol}_p(\mathcal{L}^\dagger)$ has rank one; and since $\mathrm{Sol}_p(\mathcal{L}^\dagger D) = \int \mathrm{Sol}_p(\mathcal{L}^\dagger)\frac{dt}{t}$, $(T_c - I)\mathrm{Sol}_p(\mathcal{L}^\dagger D) = \oint_{\gamma_c} \mathrm{Sol}_p(\mathcal{L}^\dagger)\frac{dt}{t}$ has rank one too. (That is, all but one function in a basis of $\mathrm{Sol}_p(\mathcal{L}^\dagger)$ is analytic at c.) Therefore $(T_c - I)\mathrm{Sol}_p(D\mathcal{L}) = \mathbb{C}\psi$. Applying this argument to get from $\mathcal{L} = D^{k-1}L$ to D^kL, we find that $(T_c - I)\mathrm{Sol}_p(D^kL) = \mathbb{C}\psi$ for all $k \in \mathbb{Z}_{\geq 0}$. But the coefficients ϕ_m of $\Phi = \sum_{m\geq 0}\phi_m s^m$ satisfy $D^\ell L\phi_m = 0$ for $m < \ell + r$, hence $(T_c - I)\phi_m = \kappa_m\psi$ for some $\kappa_m \in \mathbb{C}$. (In particular, by the normalization in §4, we have $\kappa_0 = 1$.) So the following makes sense:

Definition 6.1. The *kappa series* $\kappa(s) = \sum_{j\geq 0}\kappa_j s^j$ of L is the analytic function on $\mathbb{C}\setminus\mathbb{Z}_{<0}$ given by

$$(T_c - I)\Phi(s,t) =: \kappa(s)\psi(t).$$

The coefficients $\{\kappa_j\}$ are called the *Frobenius constants* of L.

Remark 6.2. The $\{\kappa_j\}$ were called "Apéry constants" in the original version of [BV]. In our view, this terminology is more appropriate for the values

$\kappa(\ell)$, $\ell \in \mathbb{Z} \cap [1, d-1]$; see Remark 10.3 and Example 10.4. As this paper was in the finishing stages, the final version of [op. cit.] appeared in which the language of Definition 6.1 is used.

The two Theorems that follow address (respectively) interpretation and computation of the Frobenius numbers. The intervening Lemma gives a useful asymptotic description of the power-series coefficients of periods and related functions.

Theorem 6.3. *The first $n+1$ coefficients of $\kappa(s)^{-1} =: \sum_{i\geq 0} \alpha_i s^i$ yield the LMHS periods of Remark 5.4.*

Proof. From $T_0 \sum_{j\geq 0} \phi_j s^j = e^{2\pi \mathbf{i} s} \sum_{j\geq 0} \phi_j s^j$, we have

$$N_0 \sum_{j\geq 0} \phi_j s^j = 2\pi \mathbf{i} s \sum_{j\geq 0} \phi_j s^j = 2\pi \mathbf{i} \sum_{j\geq 1} \phi_{j-1} s^j$$

and thus $N_0 \phi_j = 2\pi \mathbf{i} \phi_{j-1}$. Writing $\epsilon_n = \sum_{j=0}^{n} c_{n-j} \phi_j$ (for some constants c_i), applying N_0 repeatedly gives $(2\pi \mathbf{i})^{-k} \epsilon_{n-k} = \sum_{j=0}^{n-k} c_{(n-k)-j} \phi_j$, hence $\epsilon_{n-k}^{\mathrm{an}}(0) = (2\pi \mathbf{i})^k c_{n-k}$. Now

$$\begin{aligned} \sum_{j\geq 0} \kappa_j \psi s^j = (T_c - I) \sum_{j\geq 0} \phi_j s^j \implies \kappa_j \psi = (T_c - I)\phi_j \\ \implies \sum_{j=0}^{\ell} c_{\ell-j} \kappa_j \psi = \sum_{j=0}^{\ell} c_{\ell-j} (T_c - I) \phi_j = \tfrac{1}{(2\pi \mathbf{i})^{n-\ell}} (T_c - I) \epsilon_\ell \\ \implies (2\pi \mathbf{i})^{n-\ell} (\Sigma_{j=0}^{\ell} c_{\ell-j} \kappa_j) \delta = (T_c - I) \varepsilon_\ell = \boldsymbol{\delta}_{0\ell} \delta \\ \implies \alpha_i = (2\pi \mathbf{i})^n c_i = (2\pi \mathbf{i})^i \epsilon_i^{\mathrm{an}}(0) \ \text{ for } i = 0, \ldots, n, \end{aligned}$$

as desired. ∎

Remark 6.4. Theorem 6.3 (together with Theorem 9.2(d) below) is our version of [BV, Prop. 47]. It says that Ω_{lim} [resp. $\Omega_{\mathrm{lim}}^{-1}$] has $(i,j)^{\mathrm{th}}$ entry α_{i-j} [resp. κ_{i-j}] for $i \geq j$ and 0 for $i < j$. The proof also shows that Ω_{lim} is the change-of-basis matrix from $\{\phi_j(t)\}_{j=0}^{n}$ to $\{(2\pi \mathbf{i})^i \epsilon_i(t)\}_{i=0}^{n}$.

Lemma 6.5. *Suppose a power-series $B(t) = \sum_{m\geq 0} B_m t^m$ with radius of convergence $|c|$ extends to an analytic function on $\widetilde{U^{an}}$, that the restriction of its modulus $|B(t)|$ (or $|\int_0 B(t)dt|$) to the cut disk*

$$\mathbb{D}_\epsilon := \{t \mid |t| < |c| + \epsilon,\ \tfrac{t}{c} \notin [1, 1 + \tfrac{\epsilon}{|c|})\}$$

is bounded (for some $\epsilon > 0$) by $\beta \in \mathbb{R}_{>0}$, and that its monodromy satisfies

$$\Lambda := (T_c - I)B \sim \lambda(t-c)^{w-1} \quad \textit{near} \quad t = c$$

for some $\lambda \in \mathbb{C}^\times$ and $w \in \frac{1}{2}\mathbb{Z}_{\geq 2}$. Then

$$B_m \sim \frac{\lambda c^{w-1}\Gamma(w)}{2\pi \mathbf{i}} \times \frac{1}{c^m m^w} \quad \textit{as} \quad m \to \infty.$$

Proof. Write $\mathfrak{e}_m := |c|(w+1)\frac{\log(m)}{m}$, and take $m \in \mathbb{N}$ sufficiently large that $\mathfrak{e}_m < \epsilon$. By Cauchy, we have

$$2\pi \mathbf{i} B_m = \int_{\partial \mathbb{D}_{\mathfrak{e}_m}} \frac{B(t)}{t^{m+1}} dt = \oint_{|t|=|c|+\mathfrak{e}_m} \frac{B(t)}{t^{m+1}} dt + \int_c^{c(1+\frac{\mathfrak{e}_m}{|c|})} \frac{\Lambda(t)}{t^{m+1}} dt.$$

The first term's modulus is bounded by

$$\tfrac{2\pi\beta}{(|c|+\mathfrak{e}_m)^m} = \tfrac{2\pi\beta}{|c|^m\left(1+(w+1)\frac{\log(m)}{m}\right)^m} \sim \tfrac{2\pi\beta}{|c|^m m^{w+1}} =: B'_m.$$

The second term is asymptotic to

$$\begin{aligned} &\lambda \int_c^{c(1+\frac{\mathfrak{e}_m}{|c|})} \frac{(t-c)^{w-1}}{t^{m+1}} dt = \tfrac{\lambda c^{w-1}}{c^m} \sum_{j\geq 0} \tfrac{(-1)^j \binom{w-1}{j}}{m+j+1-w} \underbrace{\{1 - (1+\tfrac{\mathfrak{e}_m}{|c|})^{w-(m+j+1)}\}}_{\sim 1} \\ &\sim \tfrac{\lambda c^{w-1}}{c^m} \int_0^1 X^{m-w}(1-X)^{w-1} dX = \tfrac{\lambda c^{w-1}}{c^m} \mathrm{B}(m-w+1, w) \\ &\sim \tfrac{\lambda c^{w-1}}{c^m} \tfrac{\Gamma(w)}{m^w} =: B''_m, \end{aligned}$$

where the last line used Stirling's approximation for the beta function. Since $\frac{B'_m}{|B''_m|} \to 0$, we conclude that $2\pi \mathbf{i} B_m \sim B''_m$.

If $B(t)$ is not bounded on $\mathbb{D}_\epsilon$, but $\int_0 B(t)dt = \sum_{m\geq 1} \frac{B_{m-1}}{m} t^m$ is (e.g. when $w = 1$ and $B(t) \sim \frac{\lambda}{2\pi\mathbf{i}} \log(t-c)$ as $t \to c$), then the argument gives $2\pi\mathbf{i}\frac{B_{m-1}}{m} \sim \frac{\lambda c^w}{w c^m} \frac{\Gamma(w+1)}{m^{w+1}}$, which again gives $2\pi\mathbf{i}B_m \sim B''_m$. □

Theorem 6.6. *If $\mathcal{M}$ has strong conifold monodromy,*[8] *then*

(i) $\kappa(s) = c^s \lim_{k\to\infty} \frac{A_k(s)}{a_k}$, *and thus*

[8] All we need is the consequence of Lemma 4.4(ii). The final revision of [BV] includes a result (their Lemma 24) of the same form as our Theorem 6.6, but with much more restrictive conditions which Lemma 6.5 allows us to avoid.

(ii) $\kappa_m = \sum_{j=0}^m \frac{\log^j c}{j!} \lim_{k\to\infty} \frac{a_k^{(m-j)}}{a_k}$.

Proof. Observe that $\tilde{\Phi} := \Phi - \kappa\phi_0$ has no monodromy about $t = c$ for any fixed $s = s_0$, so that

$$\hat{\Phi}_{s_0}(t) := \Phi(s_0, t) - \tfrac{t^{s_0}}{c^{s_0}}\kappa(s_0)\phi_0(t) = \tilde{\Phi}(s_0, t) + (1 - \tfrac{t^{s_0}}{c^{s_0}})\kappa(s_0)\phi_0(t)$$

has $(T_c - I)\hat{\Phi}_{s_0} = (1 - \frac{t^{s_0}}{c^{s_0}})\kappa(s_0)\psi$. The function

$$\begin{aligned}\mathcal{B}(t) := t^{-s_0}\hat{\Phi}_{s_0} = \Phi^{\mathrm{an}}(s_0, t) - \tfrac{\kappa(s_0)}{c^{s_0}}\phi_0(t) &= \sum_{k\geq 0} A_k(s_0)t^k - \tfrac{\kappa(s_0)}{c^{s_0}}\sum_{k\geq 0} a_k t^k \\ &= \sum_{k\geq 0}(A_k(s_0) - \tfrac{\kappa(s_0)}{c^{s_0}}a_k)t^k,\end{aligned}$$

which is clearly invariant about $t = 0$, then has

$$(T_c - I)\mathcal{B}(t) = (t^{-s_0} - c^{-s_0})\kappa(s_0)\psi(t) \sim -\tfrac{s_0\kappa(s_0)}{c^{s_0+1}}(t - c)\psi(t)$$

for t near c, while $(T_c - I)\phi_0(t) \sim \psi(t)$.

By Lemma 4.4(ii) we have $\psi(t) \sim C(t-c)^{\frac{n-1}{2}}$, as well as the boundedness of $\phi_0(t) = \sum_{m\geq 0} a_m t^m$ (or its integral) and $\mathcal{B}(t) =: \sum_{m\geq 0} b_m^{\mathcal{B}} t^m$ required for the application of Lemma 6.5. This yields

$$a_m \sim \frac{\mathsf{C}'}{c^m m^{\frac{n+1}{2}}} \quad \text{and} \quad b_m^{\mathcal{B}} \sim \frac{\mathsf{C}''}{c^m m^{\frac{n+3}{2}}},$$

and so $\lim_{m\to\infty} \frac{b_m^{\mathcal{B}}}{a_m} = \frac{\mathsf{C}''}{\mathsf{C}'} \lim_{m\to\infty} \frac{1}{m} = 0$. That is,

$$0 = \lim_{m\to\infty} \frac{A_m(s_0) - \frac{\kappa(s_0)}{c^{s_0}}a_m}{a_m} = \lim_{m\to\infty}\left(\frac{A_m(s_0)}{a_m} - \frac{\kappa(s_0)}{c^{s_0}}\right)$$

which gives (i). In fact, since $\frac{\mathsf{C}''}{\mathsf{C}'} = -\frac{n+1}{2}\frac{s_0\kappa(s_0)}{c^{s_0}}$, this limit is uniform in s in a neighborhood of $s = 0$; we may thus expand c^s and equate power-series coefficients, whence (ii). □

Remark 6.7. The flavor here is that, while a_m and $A_m(s_0)$ have similar growth rate, the particular linear combination $A_m(s_0) - \frac{\kappa(s_0)}{c^{s_0}}a_m$ has somewhat slower growth. This characterization of $\frac{\kappa(s_0)}{c^{s_0}}$ is vaguely reminiscent to that of $\zeta(3)$ in Apéry's proof, though what happens at positive integer values of s_0 is much closer to the Apéry phenomenon; see Remark 10.3 and Example 10.4.

Example 6.8. When L is a hypergeometric operator (cf. [BV, §3]), the results of this section suffice to *compute* the matrix Ω_{lim} from Definition 5.5. Suppose that L arises as in §4, with strong conifold monodromy (cf. Remark 4.3) at $c=1$, and takes the form $L = D^r + tP_1(D)$, with $P_1(D) = -\prod_{j=1}^{r}(D+\mathfrak{a}_j)$. Then $q_0 = 1-t$ implies $\Sigma^{\times} = \{1\}$ and (via Prop. 7.1(vi) below) $L^{\dagger} = L$, whence $\{\mathfrak{a}_j\} = \{1-\mathfrak{a}_j\}$ as sets and $\sum \mathfrak{a}_j = \frac{r}{2}$. By the recurrence in Remark 5.3, we have

$$A_k(s) = \prod_{j=1}^{r} \frac{\Gamma(k+s+\mathfrak{a}_j)\Gamma(s+1)}{\Gamma(s+\mathfrak{a}_j)\Gamma(k+s+1)},$$

and so Theorem 6.6(i) together with Stirling's formula yields

$$\kappa(s)^{-1} = \lim_{k\to\infty} \frac{a_k}{A_k(s)} = \lim_{k\to\infty} \frac{A_k(0)}{A_k(s)} = \prod_{j=1}^{r} \frac{\Gamma(s+\mathfrak{a}_j)}{\Gamma(s+1)\Gamma(\mathfrak{a}_j)}.$$

This is enough to recover, for instance, the LMHSs for the complete intersection CY families in [DM], previously computed (using Iritani's mirror theorem [Ir1]) in [dSKP, §4].

To illustrate, consider the mirror quintic family ($\mathbb{P}^4[5]$ in [op. cit.]),[9] with $r=4$ and $\underline{\mathfrak{a}} = (\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5})$. Taking the power-series expansion of $\prod_{j=1}^{4} \frac{\Gamma(s+\frac{j}{5})}{\Gamma(s+1)\Gamma(\frac{j}{5})}$, we obtain ${}^t(\alpha_0,\alpha_1,\alpha_2,\alpha_3) =$

$${}^t\left(1, -5\log 5, 10\zeta(2) + \tfrac{25}{2}\log^2 5, -40\zeta(3) - 50(\log 5)\zeta(2) - \tfrac{125}{6}\log^3 5\right)$$

for the 0^{th} column of Ω_{lim}. One arrives at the more standard form of this data by renormalizing the LMHS with respect to the local coordinate $\frac{t}{5^5}$, which means multiplying the column vector by $e^{(5\log 5)[N_0]_{\underline{\varepsilon}}}$; this yields

$${}^t(1, 0, 10\zeta(2), -40\zeta(3)).$$

Moreover, the correct integral basis of the dual local system is not $\underline{\varepsilon} = (\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3)$ but rather $(\varepsilon_0,\varepsilon_1,5\varepsilon_2,5\varepsilon_3)$; this leads us to multiply the last two entries of the vector by 5. The resulting invariants $50\zeta(2)$ and $-200\zeta(3)$ correspond exactly to $a=50$ and $b=-200$ in the table in [op. cit.].

[9] Take $\varphi = \sum_{i=1}^{4} x_i + \prod_{i=1}^{4} x_i^{-1}$ and replace t by t^5 as at the end of Ex. 4.5.

7. Conifold Gamma

The main theorem of [BV], a variant of which is given in the next section, is a precise relationship between $\kappa(s)$ and a specific Gamma function $\Gamma_c(s)$. The latter involves particular choices of section $\mathfrak{m}_c \in H^0(U, \mathcal{F}^n)$ and homology class $\xi_c \in H_1(U, \mathbb{M}(s)^\vee_{\mathbb{Q}})$. We first explain where the section comes from.

Let $\{e_j\}_{j=0}^n \subset \mathcal{M}^\vee(U)$ be the dual basis of $\{D^i\mu\} \subset \mathcal{M}(U)$. Since the latter are meromorphic as sections of $\mathcal{M}_e$ on $\mathbb{P}^1$, the former are meromorphic sections of $\mathcal{M}_e^\vee$. Using $De_j + e_{j-1} = \frac{q_{r-j}}{q_0}e_n$, one checks as in [BV, §4] that $L^\dagger(\frac{e_n}{q_0}) = 0$. Moreover, by definition e_n pairs to zero with generators of $\mathcal{F}^1\mathcal{M}$, and so it belongs to[10] $\mathcal{F}^0\mathcal{M}^\vee = Q(\mathcal{F}^n\mathcal{M})$, whence $e_n = \frac{Q(\mu)}{Y}$ for some $Y \in \mathbb{C}(t)^\times$. As $\langle e_n, D^n\mu\rangle = 1$,

$$Y = Y\langle e_n, D^n\mu\rangle = \langle Q(\mu), D^n\mu\rangle = Q(\mu, D^n\mu)$$

is the *Yukawa coupling*. Besides being a rational function, it has the following properties:

Proposition 7.1. *In the setting of §4, we have:*

(i) $Y(0) = \frac{1}{(2\pi\mathbf{i})^n \mathsf{Q}_0}$.

(ii) $DY = -\frac{2}{r}\frac{q_1}{q_0}Y$ *(recall $r = n+1$).*

(iii) $q := \frac{q_0 Y}{Y(0)}$ *is a polynomial with $q(0) = 1$.*

(iv) *The adjoint operator is given by $L^\dagger = \frac{1}{q}Lq$.*

(v) *If $\mathcal{M}$ has strong conifold monodromy (Remark 4.3) at c, then $q(c) \neq 0$.*

(vi) *The conditions $q \equiv 1$, $Y = \frac{Y(0)}{q_0}$, $L^\dagger = L$, and $q_1 = \frac{r}{2}Dq_0$ are equivalent. They hold in particular when $|\Sigma^\times| = d$ and $\mathcal{M}$ has strong conifold monodromy at each point of $\Sigma^\times$.*

Sketch. (i) Applying D^n to Lemma 4.4(i)[11] gives $\langle \varepsilon_k, D^n\mu\rangle \sim (2\pi\mathbf{i})^{-n}\boldsymbol{\delta}_{kn}$ as $t \to 0$ hence $D^n\mu \sim (2\pi\mathbf{i})^{-n}\varepsilon_n^\vee$. So we have $(2\pi\mathbf{i})^n Y \sim Q(\mu, \varepsilon_n^\vee) \sim (-1)^n\langle Q(\varepsilon_n^\vee), \mu\rangle = \langle \frac{\varepsilon_0}{\mathsf{Q}_0}, \mu\rangle \sim \frac{1}{\mathsf{Q}_0}$.

[10] We remind the reader that the polarization $Q(\cdot,\cdot)$ induces an isomorphism $Q(\cdot)\colon \mathcal{M} \to \mathcal{M}^\vee$ by $Q(a,b) =: \langle Q(a), b\rangle$. This is used throughout the remaining sections. Here the point is simply that $\mathcal{F}^1\mathcal{M}$ is the orthogonal complement of $\mathcal{F}^n\mathcal{M}$ under $Q(\cdot,\cdot)$ and $\mathcal{F}^0\mathcal{M}^\vee$ under $\langle\cdot,\cdot\rangle$.

[11] Since $\varepsilon_k(t)$ is a period and the monodromy at 0 is unipotent, the Lemma says that $\langle \varepsilon_k, \mu\rangle - \frac{\log^k(t)}{k!(2\pi\mathbf{i})^k}$ is a sum of smaller (than k) powers of $\log(t)$ times functions analytic at 0.

(ii) Take $m = \lfloor \frac{n}{2} \rfloor$. Applying D to $Q(D^{i-1}\mu, D^{n-i}\mu) = 0$ for $1 \leq i \leq k$ yields $Q(D^k\mu, D^{n-k}\mu) = (-1)^k Y$; whence

$$\begin{aligned} DY &= Q(D\mu, D^n\mu) + Q(\mu, -\Sigma_{i=0}^n \tfrac{q_{r-i}}{q_0} D^i\mu) \\ &= -mDY + (-1)^m Q(D^{m+1}\mu, D^{n-m}\mu) - \tfrac{q_1}{q_0} Y, \end{aligned}$$

in which the middle term is 0 for n odd and $\frac{1}{2}DY$ for n even.

(iii) At $\sigma \in \Sigma^\times$, $\mathrm{ord}_\sigma q_0 \geq \mathrm{rk}(T_\sigma - I) = \mathrm{rk}(\mathrm{Res}_\sigma(\nabla)) \geq -\mathrm{ord}_\sigma Y$.

(iv) Writing $L^\dagger$ and $\frac{1}{q}Lq$ in the form $\sum_i p_{r-i}(t)D^i$, they have the same p_0. But then they are equal because both kill $\frac{e_n}{q_0} = Q(\frac{\mu}{q})$: we have $Lq(\frac{e_n}{q_0}) = Q(Lq\frac{\mu}{q}) = Q(L\mu) = 0$.

(v) Using $Y = \pm Q(D^m\mu, D^{n-m}\mu)$ from (ii) above with Lemma 4.4(ii) shows that Y has a simple pole at $t = c$; this cancels the zero of q_0.

(vi) The equivalence is clear. By (ii), q_0 has a zero at each zero or pole of Y, and d strong conifolds exhausts the zeroes of q_0 (as $\deg(q_0) \leq d$). So on $\mathbb{P}^1 \setminus \{\infty\}$, Y has d simple poles at these points, and no other zeroes or poles. □

Accordingly, we shall set

$$\mathcal{m}_c := \tfrac{1}{q}\mu \in H^0(U, \mathcal{F}^n\mathcal{M}) \tag{7.1}$$

and $\tilde{A}(t) = \sum_{m\geq 0} \tilde{a}_m t^m := \langle \varepsilon_0, \mathcal{m}_c \rangle = \frac{A(t)}{q(t)}$. Notice that $\mathcal{m}_c$ and thus $\tilde{A}$ are annihilated by $L^\dagger$. However, we also point out that the situation in (vi) is both easy to check and quite common for LG-models; and in that case, $\mathcal{m}_c = \mu$ and $\tilde{A} = A$.

Remark 7.2. In view of Prop. 7.1(iv), we say that L is *essentially self-adjoint* (cf. [vS, §2.4]); this reflects the self-duality $\mathcal{M}^\vee \cong \mathcal{M}(n)$. But the operator $\hat{L} := \frac{1}{\sqrt{q}} L \sqrt{q}$ satisfies $\hat{L}^\dagger = \hat{L}$, i.e. it is *self-adjoint* on the nose. Why don't we replace L by this? First, q may not be a square, even for something as simple as a family of elliptic curves with an I_0^* fiber; in this case, $\hat{L}$ corresponds to a quadratic twist of $\mathcal{M}$ (not $\mathcal{M}$ itself). Second, even if q is a square, $\hat{L}$ corresponds to $\frac{\mu}{\sqrt{q}}$ (in place of μ), which is a *strictly meromorphic* section of $\mathcal{M}_e$ (unless of course $q \equiv 1$). We prefer to work with the true *Picard-Fuchs equation* of $\mathcal{M}$, i.e. the one corresponding to μ as we normalized it in §4.

However, we feel obliged to point out that in the LG-model setting of Example 4.5, L itself turns out to be self-adjoint (i.e. $q \equiv 1$) with striking frequency. Though one can certainly cook up counterexamples (e.g. see

Remark 10.7(ii)), consider the fact that this holds for all 23 of the PF operators of order 3 arising in the table of "3D Minkowski period sequences" in [Fano]. So the reader mainly interested in this case might consider ignoring the daggers from here on out.

Turning to the homology class, we write

$$P(x) := (x-1)^r = \sum_m \lambda_m x^m$$

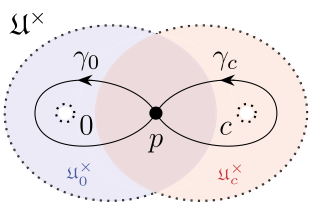
and set[12]

$$\xi_c := \left[\sum \lambda_m \gamma_0^m \otimes \delta \otimes e^{2\pi\mathbf{i}ms} + \gamma_c^{-1} \otimes \varepsilon_0 \otimes P(e^{2\pi\mathbf{i}s})\right] \in H_1(U, \mathbb{M}(s)^\vee_{\mathbb{Q}}).$$

This is well-defined since applying ∂ to the bracketed expression yields

$$\begin{aligned}&\sum_m \lambda_m \gamma_0^{-m}(\delta \otimes e^{2\pi\mathbf{i}ms}) - \sum_m \lambda_m \delta \otimes e^{2\pi\mathbf{i}ms} + (\gamma_c - 1)\varepsilon_0 \otimes P(e^{2\pi\mathbf{i}s}) \\ &\quad = \sum_m \lambda_m \gamma_0^{-m} \delta \otimes 1 - \delta \otimes P(e^{2\pi\mathbf{i}s}) + \delta \otimes P(e^{2\pi\mathbf{i}s}) \\ &\quad = (\gamma_0^{-1} - 1)^r \delta \otimes 1 = 0.\end{aligned}$$

Definition 7.3. The *conifold Gamma* is $\Gamma_c(s) := \Gamma_{\xi_c, \mathscr{m}_c}(s)$.



Let $\mathfrak{U}_0$ and $\mathfrak{U}_c$ be neighborhoods of 0 and c containing γ_0 and γ_c respectively (and no other roots of q_0); then $\mathfrak{U}_0 \cap U = \mathfrak{U}_0^\times := \mathfrak{U}_0 \setminus \{0\}$ and $\mathfrak{U}_c \cap U = \mathfrak{U}_c^\times := \mathfrak{U}_c \setminus \{c\}$, and $p \in \mathfrak{U}_0 \cap \mathfrak{U}_c$. Write

$$\mathfrak{U} := \mathfrak{U}_0^\times \cup \mathfrak{U}_c \quad \text{and} \quad \mathfrak{U}^\times := \mathfrak{U}_0^\times \cup \mathfrak{U}_c^\times.$$

Notice that ξ_c is supported on $\mathfrak{U}^\times$.

Proposition 7.4. *Suppose $\mathcal{M}$ has strong conifold monodromy at c. Then the $\mathbb{Q}[e^{2\pi\mathbf{i}s}]$-module of Gamma functions (for $\mathscr{m}_c$) arising from $H_1(\mathfrak{U}^\times, \mathbb{M}(s)^\vee_{\mathbb{Q}})$ has rank one and is spanned by ξ_c.*

Proof. By Proposition 7.1(v), $\mathscr{m}_c$ is a holomorphic section of $\mathcal{F}_e^n$ on $\mathfrak{U}$ (actually on $\mathfrak{U}_0 \cup \mathfrak{U}_c$). Let $\mathcal{X}_{\mathfrak{U}} \to \mathfrak{U}$ be the extension of $f^{-1}(\mathfrak{U}^\times) \to \mathfrak{U}^\times$ guaranteed by strong conifold monodromy, with nodal fiber over c; then $\mathscr{m}_c \otimes \frac{dt}{t}$ belongs to $\Omega^{n+1}(\mathcal{X}_{\mathfrak{U}})$, so its pairing with $H_1(\mathfrak{U}^\times, \mathbb{M}(s)^\vee_{\mathbb{Q}})$ factors through

[12] This is a rational multiple of the cycle written ξ_0 in [BV, Cor. 33], but for the *dual* local system.

$\mathrm{IH}_1(\mathfrak{U}, \mathbb{M}(s)^\vee_{\mathbb{Q}})$. Since $H^0(\mathfrak{U}, \mathbb{M}(s)_{\mathbb{Q}}) = \{0\}$, Euler-Poincaré says the rank of $\mathrm{IH}^1(\mathfrak{U}, \mathbb{M}(s)_{\mathbb{Q}})$ (and its dual) is $(r - \mathrm{rk}(\mathbb{M}(s)^{T_c})) - r\chi(\mathfrak{U}) = 1 - 0 = 1$. Similarly, we have that $\mathrm{IH}_1(\mathfrak{U}_c, \mathbb{M}(s)^\vee_{\mathbb{Q}}) = \{0\}$ and so $\mathrm{IH}_1(\mathfrak{U}, \mathbb{M}(s)^\vee_{\mathbb{Q}}) \hookrightarrow \mathrm{IH}_1(\mathfrak{U}, \mathfrak{U}_c; \mathbb{M}(s)^\vee_{\mathbb{Q}}) \cong H_1(\mathfrak{U}_0^\times, \{p\}; \mathbb{M}(s)^\vee_{\mathbb{Q}}) \underset{\cong}{\overset{T_0 - I}{\to}} H_0(\{p\}, \mathbb{M}(s)^\vee_{\mathbb{Q}})$ (where $T_0 - I$ is an isomorphism thanks to the action on t^s). The image of ξ_c under the whole composition is just $\partial(\sum \lambda_m \gamma_0^m \otimes \delta \otimes e^{2\pi \mathbf{i} m s}) = -\delta \otimes P(e^{2\pi \mathbf{i} s})$, which is certainly nonzero. □

Remark 7.5. Under the same hypothesis, for $\Re(s) > 0$ we have that $\Gamma_c(s) = -P(e^{2\pi \mathbf{i} s}) \int_0^c \frac{\psi(t)}{q(t)} t^{s-1} dt$ [BV, Prop. 15]. However, this is not particularly useful for computing the derivatives of Γ_c at $s = 0$ (which interest us below), since the corresponding integrals do not converge. See Example 8.3 below for a small but amusing exception.

8. Gamma = kappa

Our main objective in this section is to present Theorem 30 of [BV] in a more precise form that accounts for the self-duality of $\mathcal{M}$, relating the conifold Gamma for $\mathcal{M}$ to the kappa series for L. The proof is similar to that in [op. cit.], but with sufficiently many changes that we summarize it here.

Let $\{\rho_i\}_{i=0}^n \subset \mathcal{M}(U)$ be the dual basis of $\{D^j(\frac{e_n}{q_0})\}_{j=0}^n$. Arguing as in §7 (for e_n), ρ_n belongs to $\mathcal{F}^n$ hence equals $F\mu$ for some $F \in \mathbb{C}(t)^\times$. To find it, write

$$\begin{aligned} 1 = \langle D^n(\tfrac{e_n}{q_0}), \rho_n \rangle &= Q(D^n \tfrac{\mu}{q_0 Y}, \rho_n) = \tfrac{(-1)^n}{q_0 Y} Q(\rho_n, D^n \mu) \\ &= \tfrac{(-1)^n F}{q_0 Y} Q(\mu, D^n \mu) = \tfrac{(-1)^n F}{q_0} \quad \Longrightarrow \quad \rho_n = (-1)^n q_0 \mu. \end{aligned}$$

Next, write $L^\dagger = \sum_{j=0}^r p_{r-j}(t) D^j$ (where $p_0 = q_0$), and define

$$\begin{aligned} &\eta \colon \mathcal{O}^{\mathrm{an}} \to \mathcal{M}^{\vee, \mathrm{an}} \quad \text{by} \quad \eta(\phi) := \textstyle\sum_{i=0}^n (D^i \phi) e_i \qquad \text{and} \\ &\chi \colon \mathcal{O}^{\mathrm{an}} \to \mathcal{M}^{\mathrm{an}} \quad \text{by} \quad \chi(\theta) := \tfrac{(-1)^n}{Y(0)} \textstyle\sum_{i=0}^n (D^i \tfrac{\theta}{q}) \rho_i. \end{aligned}$$

Using $De_i + e_{i-1} = \frac{q_{r-i}}{q_0} e_n$ and (dually) $D\rho_i + \rho_{i-1} = \frac{p_{r-i}}{q_0} \rho_n = (-1)^n p_{r-i} \mu$, one easily computes that

$$D(\eta(\phi)) = (L\phi) \tfrac{e_n}{q_0} \qquad \text{and} \qquad D(\chi(\theta)) = (L^\dagger \tfrac{\theta}{q}) \tfrac{\mu}{Y(0)} = (L\theta) \tfrac{\mu}{Y(0) q}.$$

Defining the bracket

$$[\ ,\ ] \colon \mathcal{O}^{\mathrm{an}} \times \mathcal{O}^{\mathrm{an}} \to \mathcal{O}^{\mathrm{an}} \qquad \text{by} \qquad [\phi, \theta] := \langle \eta(\phi), \chi(\theta) \rangle ,$$

we have the crucial

Lemma 8.1. (i) $D[\phi,\theta]=\frac{1}{Y(0)p}\{\phi L\theta+(-1)^n\theta L\phi\}$.

(ii) *If α,β are local sections of $\mathbb{M}_{\mathbb{C}}^{\vee}$, with periods $\pi_\alpha=\langle\alpha,\mu\rangle$ and $\pi_\beta=\langle\beta,\mu\rangle$, then $\eta(\pi_\alpha)=\alpha$, $Q(\chi(\pi_\beta))=\beta$, and $[\pi_\alpha,\pi_\beta]=Q(\alpha,\beta)$.*

Proof. (i) follows immediately from writing $\langle D\eta(\phi),\chi(\theta)\rangle+\langle\eta(\phi),D\chi(\theta)\rangle=\frac{(-1)^nL\phi}{Y(0)}\sum_{i=0}^n(D^i\frac{\theta}{q})\langle\frac{e_n}{q_0},\rho_i\rangle+\frac{L\theta}{Y(0)q}\sum_{i=0}^n(D^i\phi)\langle e_i,\mu\rangle$, since $\langle\frac{e_n}{q_0},\rho_i\rangle=\boldsymbol{\delta}_{i0}=\langle e_i,\mu\rangle$. For (ii), notice that $L\pi_\alpha=0=L\pi_\beta\implies D(\eta(\pi_\alpha))=0=D(\chi(\pi_\beta))\implies\eta(\pi_\alpha)$ and $Q(\chi(\pi_\beta))$ are sections of $\mathbb{M}_{\mathbb{C}}^{\vee}$. To see *which* sections, we pair them with μ: $\langle\eta(\pi_\alpha),\mu\rangle=\sum_{i=0}^n(D^i\pi_\alpha)\langle e_i,\mu\rangle=\pi_\alpha$; and $\langle Q(\chi(\pi_\beta)),\mu\rangle=(-1)^n\langle Q(\mu),\chi(\pi_\beta)\rangle=(-1)^nYq_0\langle\frac{e_n}{q_0},\chi(\pi_\beta)\rangle=\pi_\beta$. Hence $\langle\eta(\pi_\alpha),\chi(\pi_\beta)\rangle=\langle\alpha,Q^{-1}(\beta)\rangle=Q(\alpha,\beta)$. □

Theorem 8.2. *In the setting of §4,*

$$\kappa(s)=\frac{\mathsf{Q}_0}{\mathsf{Q}_c}\frac{(2\pi\mathbf{i})^ns^r}{(1-e^{2\pi\mathbf{i}s})^r}\Gamma_c(s).$$

Proof. Rewriting our representative of ξ_c in the form $\sum_j\gamma_j\otimes\varepsilon_j\otimes e^{2\pi\mathbf{i}n_js}$, we compute

$$\mathscr{G}(s):=\sum_j e^{2\pi\mathbf{i}n_js}\int_{\gamma_j^{-1}}D[\epsilon_j,\Phi]\frac{dt}{t}$$

in two different ways. First, since $L\Phi=s^rt^s$ and $L\epsilon_j=0$,

$$D[\epsilon_j,\Phi]=\frac{\epsilon_js^rt^s}{Y(0)q}=(2\pi\mathbf{i})^n\mathsf{Q}_0s^r\langle\varepsilon_j,\mathfrak{m}_c\rangle t^s$$

by Lemma 8.1(i) and $\mathscr{G}(s)=(2\pi\mathbf{i})^n\mathsf{Q}_0s^r\Gamma_c(s)$. Second, by the Fundamental Theorem of Calculus

$$\begin{aligned}\mathscr{G}(s)&=\textstyle\sum_je^{2\pi\mathbf{i}n_js}(\gamma_j^{-1}-1)[\epsilon_j,\Phi]\\&=\textstyle\sum_m\lambda_me^{2\pi\mathbf{i}ms}(\gamma_0^{-m}-1)[\psi,\Phi]+P(e^{2\pi\mathbf{i}s})(\gamma_c-1)[\epsilon_0,\Phi]\\&=\textstyle\sum_m\lambda_me^{2\pi\mathbf{i}ms}([\gamma_0^{-m}\psi,e^{-2\pi\mathbf{i}ms}\Phi]-[\psi,\Phi])\\&\qquad\qquad+P(e^{2\pi\mathbf{i}s})([\epsilon_0+\psi,\Phi+\kappa\psi]-[\epsilon_0,\Phi])\\&=[P(\gamma_0^{-1})\psi,\Phi]-P(e^{2\pi\mathbf{i}s})[\psi,\Phi]\\&\qquad\qquad+P(e^{2\pi\mathbf{i}s})[\psi,\Phi]+P(e^{2\pi\mathbf{i}s})\kappa[\epsilon_0+\psi,\psi]\\&=P(e^{2\pi\mathbf{i}s})\kappa(s)[\epsilon_0+\psi,\psi]\end{aligned}$$

since $P(\gamma_0^{-1}) = 0$ on $\mathbb{M}^\vee_{\mathbb{Q},p}$. By Lemma 8.1(ii), we have $[\epsilon_0 + \psi, \psi] = Q(\varepsilon_0 + \delta, \delta) = Q(T_c\varepsilon_0, (-1)^{n+1}T_c\delta) = (-1)^{n+1}Q(\varepsilon_0, \delta) = (-1)^r \mathsf{Q}_c$. $\square$

Example 8.3. Here is the simplest real example: let $\mathcal{X} \to \mathbb{P}^1$ be the family of "CY 0-folds" arising as in Example 4.5 from $\varphi = -x + 2 - x^{-1}$, and $\mathcal{M}$ its reduced fiberwise H^0. We have $L = D - 4t(D + \frac{1}{2}) = L^\dagger$, $\mathsf{Q}_0 = 2$, $\mathsf{Q}_c = -4$, $c = \frac{1}{4}$, $A(t) = (1-4t)^{-\frac{1}{2}} = -\frac{1}{2}\psi(t)$, and (from Remark 7.5) $\Gamma_c(s) = 2(e^{2\pi \mathbf{i} s} - 1)\int_0^c A(t)t^{s-1}dt$. Applying Theorem 8.2 gives $\kappa(s) = s\int_0^{\frac{1}{4}} \frac{t^{s-1}dt}{\sqrt{1-4t}} = 4^{-s}s\mathrm{B}(s, \frac{1}{2}) = \frac{\Gamma(1+s)^2}{\Gamma(1+2s)} = \exp\left(2\sum_{k\geq 2} \frac{(-1)^{k-1}}{k}(2^{k-1}-1)\zeta(k)s^k\right)$.

Corollary 8.4. *Writing $L^\dagger = \sum_{i=0}^d t^i Q_i(D)$, the difference equation*

$$\sum_{k=0}^d \frac{Q_k(-s-k)}{(s+k)^r}\kappa(s+k) = 0$$

holds.

Proof. Divide Theorem 8.2 by s^r and apply Theorem 3.3. $\square$

Corollary 8.5. *We have $\Gamma_c(0) = (-1)^r\frac{\mathsf{Q}_c}{\mathsf{Q}_0}2\pi\mathbf{i}$; and for $m \in \mathbb{Z}_{>0}$, $\Gamma_c(m) = 0$, $\Gamma_c(-m) = \Gamma_c(0)\tilde{a}_m$,*[13] *and $\kappa(s) \sim \frac{(-m)^r}{(s+m)^r}\tilde{a}_m$ at $s = -m$.*

Proof. In addition to Theorem 8.2, use Theorem 3.3 and Remark 3.4 (applied to $(L^\dagger, m\!\!\!m_c)$). $\square$

The remarks that follow address the implications of Theorem 8.2 for the LMHS of $\mathcal{M}$ at 0, whose periods turn out to be given by derivatives of (a variant of) the conifold Gamma at $s = 0$.

Remark 8.6. Replacing L by $L^\dagger$, μ by $m\!\!\!m_c$, and ψ by $\psi^\dagger := \langle \delta, m\!\!\!m_c\rangle$, we may define $\Phi^\dagger$ and $\kappa^\dagger$ as in Definitions 5.1 and 6.1. (Note that we are *not* replacing $\mathcal{M}$ by $\mathcal{M}^\vee$.) Then Theorem 6.3 remains true; and since $m\!\!\!m_c(0) = \mu(0)$ in $\mathcal{F}^n_{e,0}$, we find that $\kappa_j^\dagger = \kappa_j$ for $j = 0, \ldots, n$ (but not $j \geq r$). Moreover, Theorem 8.2 and Corollary 8.5 hold replacing κ by $\kappa^\dagger$, $\tilde{a}_m$ by a_m, and Γ_c by $\Gamma_{\mathcal{M}} := \Gamma_{\xi_c,\mu}$. (To see this, replace $[\epsilon_j, \Phi]$ by $[\epsilon_j, p\Phi^\dagger]$ in the proof.) It follows that ${}^t(\kappa_0, \ldots, \kappa_n)$ is the product of a rational lower-triangular matrix by ${}^t\left(\frac{\Gamma_{\mathcal{M}}(0)}{2\pi\mathbf{i}}, \frac{\Gamma'_{\mathcal{M}}(0)}{(2\pi\mathbf{i})^2}, \ldots, \frac{\Gamma^{(n)}_{\mathcal{M}}(0)}{(2\pi\mathbf{i})^{n+1}}\right)$.

Now by Theorem 6.3, ${}^t(\kappa_0, \ldots, \kappa_n)$ is the leading column of a period matrix for the dual of the LMHS of $\mathcal{M}$ at 0. As the LMHS of a polarized VHS is (up to twist) self-dual, we conclude that there exists a $\mathbb{Q}$-basis $\{e_j \in$

[13] See (7.1)ff for $\tilde{a}_m$ (which equals a_m if $L^\dagger = L$).

$W(N_0)_j\}_{j=0}^n$ of $\mathbb{M}_{\mathbb{Q},p}$ such that $\mu(0)=\sum_{j=0}^n(2\pi\mathbf{i})^{-j-1}\Gamma_{\mathcal{M}}^{(j)}(0)\tilde{e}_j(0)$ in $\mathcal{M}_{e,0}$, where $\tilde{e}_j(t):=e^{-\frac{\log(t)}{2\pi\mathbf{i}}N_0}e_j$.

Since $\frac{d^j}{dt^j}P(e^{2\pi\mathbf{i}s})|_{s=0}=0$ for $j<r$, one finds that

$$\frac{\Gamma_{\mathcal{M}}^{(j)}(0)}{(2\pi\mathbf{i})^{j+1}}=\sum_{m=0}^{r}\lambda_m\int_{\gamma_0^{-m}}\psi(t)\left(\frac{\log(t)}{2\pi\mathbf{i}}+m\right)^j\frac{dt}{2\pi\mathbf{i}t}\,,$$

taking $\log(p)\in\mathbb{R}$ at the start of each path. As a formula for actually computing the LMHS this seems closely related to the "Cauchy integral method" in [dSKP, §5], though more unwieldy. Rather, its importance is theoretical, as the next Remark demonstrates.

Remark 8.7 (Limiting motive). The family $L_t:=(\mathbb{G}_m,\{1,t\})$ of relative motives underlies the rank-2 connection $\mathcal{D}/\mathcal{D}D^2$ in $\mathbb{G}_m$, with periods 1 and $\frac{\log(t)}{2\pi\mathbf{i}}$ over the cycles S^1 and $[1,t]$ in $H_1(L_t)$. Write $\mathcal{M}[n]$ for the VMHS $\mathcal{M}\otimes\mathrm{Sym}^nH^1(L_t)$ on U, and $\Xi_j\in H_1(U^m,\mathbb{M}[n]_{\mathbb{Q}}^{\vee})$ for the class of the cycle $\sum_m\lambda_m\gamma_0^m\otimes\delta\otimes([1,t]+mS^1)^j(S^1)^{n-j}$ (closed for $j<r$). Putting $\varpi:=\mu\otimes(\frac{dz_1}{2\pi\mathbf{i}z_1}\wedge\cdots\wedge\frac{dz_n}{2\pi\mathbf{i}z_n})\otimes\frac{dt}{2\pi\mathbf{i}t}\in H^1_{\mathrm{dR}}(U,\mathcal{M}[n])$, for $0\leq j\leq n$ we recover $(2\pi\mathbf{i})^{-j-1}\Gamma_{\mathcal{M}}^{(j)}(0)$ as periods $\langle\Xi_j,\varpi\rangle$ of the connection.

These are also periods of a relative variety. Inside our smooth total space $\mathcal{X}\xrightarrow{f}\mathbb{P}^1$, consider $\mathcal{X}_{\mathbb{G}_m}:=f^{-1}(\mathbb{G}_m)$. Let $\mathbb{D}[n]\subset\mathbb{G}_m^n\times\mathbb{G}_m$ be the divisor defined by $\prod_{i=1}^n(z_i-1)(z_i-t)$, and write $\mathrm{X}[n]:=\mathcal{X}_{\mathbb{G}_m}\times\mathbb{G}_m^n$, $\mathrm{D}[n]:=\mathcal{X}_{\mathbb{G}_m}\times_{\mathbb{G}_m}\mathbb{D}[n]$, and $\mathrm{X}[n]_{\mathrm{rel}}:=(\mathrm{X}[n],\mathrm{D}[n])$. Then recalling that μ is a *holomorphic* section of $\mathcal{F}_e^n$, we may regard Ξ_j and ϖ as classes in $H_{2n+1}(\mathrm{X}[n]_{\mathrm{rel}},\mathbb{Q})$ and $F^{2n+1}H^{2n+1}(\mathrm{X}[n]_{\mathrm{rel}},\mathbb{C})$ respectively. A further refinement is obtained by observing that $\mathrm{IH}^1(\mathbb{G}_m,\mathbb{M}[n])$ yields a sub-MHS/motive of $H^{2n+1}(\mathrm{X}[n]_{\mathrm{rel}})$, of which the $\langle\Xi_j,\varpi\rangle$ remain periods.

Now in general these are only some of the periods, not all of the periods, of this MHS. (Alas, the part of $\mathrm{X}[n]_{\mathrm{rel}}$ over $\mathfrak{U}$ is not a motive.) But there *is* a case in which the $\{\Xi_j\}$ *span* $\mathrm{IH}_1(\mathbb{G}_m,\mathbb{M}[n])$, and that is when $|\Sigma^{\times}|=1$: indeed, by Euler-Poincaré we find that $\mathrm{rk}(\mathrm{IH}^1(\mathbb{G}_m,\mathbb{M}[n]))=n+1$. So in this "hypergeometric" case, we obtain a *motive with Hodge realization equal to the LMHS of* $\mathcal{M}$ *at* $t=0$.

Naturally, we have left aside the messiness of constructing a log-resolution of $\left(\overline{\mathrm{X}[n]},\overline{\mathrm{D}[n]}\cup(\overline{\mathrm{X}[n]}\setminus\mathrm{X}[n])\right)$ and the required projectors, but it is clear that this can be done. Moreover, despite various "limiting motive" constructions, this is the first of which we are aware *with the desired Hodge realization* outside of the weight-one setting [Ha], further illustrating the power of the approach of Bloch and Vlasenko.

9. The unipotent extensions

Closely related to the Frobenius deformation in §5 is an inverse limit of VMHSs whose periods are annihilated by $D^m L(\cdot)$ for some m [BV, §5]. Our initial intention in this section was to investigate these VMHSs, but (given our choice of μ and thus L) it turns out to be more natural to consider $D^m L^\dagger$, essentially because the periods of *its* adjoint LD^m integrate the periods of μ. The warning here is that while L and $L^\dagger$ define isomorphic $\mathcal{D}$-modules, $D^m L$ and $D^m L^\dagger$ do *not* – unless, of course, $L^\dagger = L$. While we won't need to make this "self-adjointness" assumption here, we remind the reader that the assumptions made at the beginning of §4 – e.g., that 0 is a point of maximal unipotent monodromy – do remain in force.

Fix $m \in \mathbb{Z}_{>0}$, and consider the exact sequence of connections

$$0 \to \mathcal{K} \to \mathcal{E} \overset{\pi}{\to} \mathcal{M} \to 0$$

on U given by $\mathcal{D}/\mathcal{D}D^m \underset{L^\dagger(\cdot)}{\hookrightarrow} \mathcal{D}/\mathcal{D}D^m L^\dagger \twoheadrightarrow \mathcal{D}/\mathcal{D}L^\dagger$. The dual sequence

$$0 \to \mathcal{M}^\vee \to \mathcal{E}^\vee \to \mathcal{K}^\vee \to 0$$

is given by $\mathcal{D}/\mathcal{D}L \underset{D^m(\cdot)}{\hookrightarrow} \mathcal{D}/\mathcal{D}LD^m \twoheadrightarrow \mathcal{D}/\mathcal{D}D^m$, and the solution sheaves by

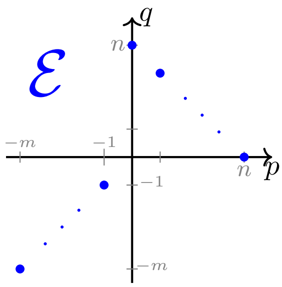
$$0 \to \mathbb{M}^\vee_{\mathbb{C}} \overset{\imath}{\to} \mathbb{E}^\vee_{\mathbb{C}} \to \mathbb{K}^\vee_{\mathbb{C}} \to 0.$$

Via $\imath$, the basis $\varepsilon_0, \ldots, \varepsilon_n$ of $\mathbb{M}^\vee_{\mathbb{Q},p}$ may be regarded as elements of $\mathbb{E}^\vee_{\mathbb{C},p}$. Let $\Omega \in \mathcal{E}(U)$ denote the image of $1 \in \mathcal{D}/\mathcal{D}D^m L^\dagger$, so that $\pi(\Omega) = \mathfrak{m}_c$.

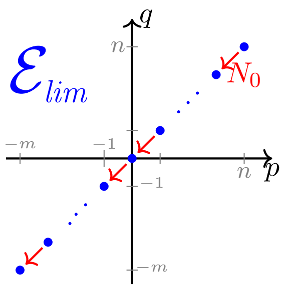
Definition 9.1. The connection $\mathcal{E}$ (or its restriction to a subset of U) *underlies a $\mathbb{Q}$-VMHS* if there is a $\mathbb{Q}$-local system $\mathbb{E}_{\mathbb{Q}} \subset \mathbb{E}_{\mathbb{C}} = \ker(\nabla)$ with $\mathbb{E}_{\mathbb{Q}} \otimes \mathbb{C} \cong \mathbb{E}_{\mathbb{C}}$, a flag $\mathcal{F}^\bullet \subset \mathcal{E}$ of holomorphic sub-bundles with $D\mathcal{F}^\bullet \subset \mathcal{F}^{\bullet-1}$, and a weight filtration $\mathcal{W}_\bullet$ on $\mathbb{E}_{\mathbb{Q}}$, such that the pointwise restrictions of $(\mathbb{E}_{\mathbb{Q}}, \mathcal{W}_\bullet, \mathcal{F}^\bullet\mathcal{E})$ define $\mathbb{Q}$-MHSs.

Here we shall mainly be concerned with the restriction of $\mathcal{E}$ to the punctured neighborhood $\mathfrak{U}_0^\times$ and (provided this underlies a VMHS) its LMHS at 0, in which $\mathcal{W}_\bullet$ is replaced by the relative monodromy weight filtration $W(N_0, \mathcal{W})_\bullet$ (whose existence is not an issue here).

Theorem 9.2. *$\mathcal{E}|_{\mathfrak{U}_0^\times}$ underlies a $\mathbb{Q}$-VMHS which is the* ***unique*** *one on $\mathfrak{U}_0^\times$ with underlying $\mathcal{D}$-module $\mathcal{D}/\mathcal{D}D^m L^\dagger$ and having the properties:*



- (i) *Ω belongs to $\mathcal{F}^n$;*
- (ii) *$\imath(\varepsilon_0)$ belongs to $\mathbb{E}^\vee_{\mathbb{Q},p}$;*
- (iii) *$\mathbb{E}^{(\vee)}_{\mathbb{Q}}$ extends to $\mathfrak{U}$ (i.e. is closed under T_c); and*
- (iv) *$\mathrm{rk}(\mathcal{E}^{k,n-k}) = 1$ for $0 \leq k \leq n$, $\mathrm{rk}(\mathcal{E}^{-k,-k}) = 1$ for $1 \leq k \leq m$, and all other $\mathcal{E}^{p,q}$ are zero.*



This VMHS satisfies, in addition, the following:

- (a) *$\pi|_{\mathfrak{U}_0^\times}$ is a morphism of $\mathbb{Q}$-VMHS;*
- (b) *$(T_c - I)\mathbb{E}^\vee_{\mathbb{Q}} \subset \mathbb{Q}\imath(\delta)$;*
- (c) *the LMHS $\mathcal{E}_{\lim}$ of $\mathcal{E}$ at 0 is Hodge-Tate, with $N_0^{n+m} \neq 0$; and*
- (d) *the first $n+m+1$ power-series coefficients of $\kappa^\dagger(s)^{-1}$ yield the LMHS periods at 0 (extending Theorem 6.3/Remark 8.6).*

Proof. The Hodge filtration $\mathcal{F}^{n-k}\mathcal{E} = \mathcal{O}\langle \Omega, D\Omega, \ldots, D^k\Omega\rangle$, as well as the weight filtration $\mathcal{W}_n\mathcal{E} = \mathcal{E}$, $\mathcal{W}_{n-1}\mathcal{E} = \mathcal{W}_{-2}\mathcal{E} = \mathcal{K}$,

$$W_{-2n+2k}\mathcal{E} = \mathcal{O}\langle D^{m-k-1}L^\dagger\Omega, \ldots, D^{m-1}L^\dagger\Omega\rangle \quad (k = 1, \ldots, m),$$

are forced upon us by Griffiths transversality, $D^m L^\dagger \mathcal{E} = \{0\}$, and (iv). Sending $\Omega \mapsto m_c$ projects $(\mathcal{E}, \mathcal{F}^\bullet) \twoheadrightarrow (\mathcal{M}, \mathcal{F}^\bullet)$. We need to construct the $\mathbb{Q}$-local system and show that $\mathcal{W}_\bullet$ is compatible with the resulting $\mathbb{Q}$-structure; this will be carried out on the dual.

Writing $\Phi^\dagger = \sum_k \phi_k^\dagger s^k$ (cf. Remark 8.6), we find exactly as in §6 that $(T_c - I)\phi_k^\dagger = \kappa_k^\dagger \psi^\dagger$, $N_0\phi_k^\dagger = 2\pi\mathbf{i}\phi_{k-1}^\dagger$, and $L^\dagger\phi_{n+j}^\dagger = \frac{\log^{j-1} t}{(j-1)!} \implies D^j L^\dagger \phi_{n+j} = 0$. In particular, this yields identifications

$$\mathbb{M}^\vee_{\mathbb{C},p} \underset{\langle\cdot, m_c\rangle}{\overset{\cong}{\to}} \mathrm{Sol}_p(L^\dagger) = \mathbb{C}\langle \phi_0^\dagger, \ldots, \phi_n^\dagger\rangle$$

and

$$\mathbb{E}^\vee_{\mathbb{C},p} \underset{\langle\cdot, \Omega\rangle}{\overset{\cong}{\to}} \mathrm{Sol}_p(D^m L^\dagger) = \mathbb{C}\langle \phi_0^\dagger, \ldots, \phi_{n+m}^\dagger\rangle$$

for the $\mathbb{C}$-local systems. Omitting "$\imath(\cdot)$" for simplicity, we must extend the $\mathbb{Q}$-basis $\{\varepsilon_0, \ldots, \varepsilon_n\}$ of $\mathbb{M}^\vee_{\mathbb{Q},p}$ by some $\varepsilon_{n+1}, \ldots, \varepsilon_{n+m} \in \mathbb{E}^\vee_{\mathbb{Q},p}$. Recalling from the proof of Theorem 6.3 (with daggers inserted) that $\epsilon_k^\dagger = (2\pi\mathbf{i})^{-k}\sum_{j=0}^k \alpha_{k-j}^\dagger \phi_j^\dagger$ for $k = 0,\dots,n$, we can simply use this formula to *define* $\epsilon_k^\dagger$ and $\varepsilon_k := \langle\cdot,\Omega\rangle^{-1}(\epsilon_k^\dagger)$ for $k = n{+}1,\dots,n{+}m$. Then we automatically get $N_0\varepsilon_k = \varepsilon_{k-1}$, and

$$(T_c - I)\varepsilon_k = \left(\tfrac{1}{(2\pi\mathbf{i})^k}\textstyle\sum_{j=0}^k \alpha_{k-j}^\dagger \kappa_j^\dagger\right)\delta = \begin{cases}\delta, & k=0\\ 0, & k>0.\end{cases}$$

The LMHS periods are just the

$$(2\pi\mathbf{i})^k\epsilon_k^{\dagger,\mathrm{an}}(0) = \textstyle\sum_{j=0}^k \alpha_{k-j}^\dagger \phi_j^{\dagger,\mathrm{an}}(0) = \sum_{j=0}^k \alpha_{k-j}^\dagger \boldsymbol{\delta}_{0j} = \alpha_k^\dagger.$$

The weight filtration dual to $\mathcal{W}_\bullet$ may be described as $\mathcal{W}_{-n}^\vee = \mathcal{W}_1^\vee = \mathbb{M}_{\mathbb{Q}}^\vee$ and $\mathcal{W}_{2k}^\vee = \mathcal{W}_{2k+1}^\vee = \mathbb{M}_{\mathbb{Q}}^\vee + \mathbb{Q}\langle\varepsilon_{n+1},\dots,\varepsilon_{n+k}\rangle$ (the point being that this $\mathcal{W}_{2k}^\vee = \mathrm{im}(N_0^{m-k})$ hence kills $\mathcal{W}_{-2k-2}\mathcal{E} = \mathcal{O}\langle D^kL^\dagger\Omega,\dots,D^{m-1}L^\dagger\Omega\rangle$ because $\langle N_0^{m-k}(\cdot), D^{\geq k}L^\dagger\Omega\rangle = \langle(\cdot), D^{\geq m}L^\dagger\Omega\rangle = 0$). This completes the proof of existence of the $\mathbb{Q}$-VMHS and properties (a)–(d).

For uniqueness, suppose another $\hat{\mathcal{E}}$ satisfies (i)–(iv). Again $\mathcal{F}^\bullet$ and $\mathcal{W}_\bullet$ are forced upon us, so that $\hat{\mathcal{E}}$ and $\mathcal{E}$ are the same as bifiltered $\mathcal{D}$-modules. To show $\hat{\mathbb{E}}_{\mathbb{Q},p}^\vee = \mathbb{E}_{\mathbb{Q},p}^\vee$ inside $\mathbb{E}_{\mathbb{C},p}^\vee$, write $E_k := \ker(N_0^k) \subset \mathbb{E}_{\mathbb{C},p}^\vee$ and assume inductively $E_{k-1}\cap\mathbb{E}_{\mathbb{Q},p}^\vee = E_{k-1}\cap\hat{\mathbb{E}}_{\mathbb{Q},p}^\vee$ (with (ii) providing the "base case" $k=1$). We have an isomorphism[14]

$$(N_0, T_c - I)\colon E_k \overset{\cong}{\to} E_{k-1}\oplus\mathbb{C}\imath(\delta),$$

under which any choice of $\mathbb{Q}$-structure on the left-hand side consistent with (iii) must go to $(E_{k-1}\cap\mathbb{E}_{\mathbb{Q},p}^\vee)\oplus\mathbb{Q}\imath(\delta)$ on the right. So $E_k\cap\mathbb{E}_{\mathbb{Q},p}^\vee = E_k\cap\hat{\mathbb{E}}_{\mathbb{Q},p}^\vee$. □

Corollary 9.3. *Given a $\mathbb{Q}$-VMHS $\mathcal{E}'$ over U of type* (iv), *with a surjective morphism to the $\mathbb{Q}$-VHS $\mathcal{M}$ sending $\omega\in H^0(U,\mathcal{F}^n\mathcal{E}')$ to $\mathfrak{m}_c$, and $D^mL^\dagger\omega = 0$. Then $\mathcal{E}'|_{\mathfrak{U}_0^\times}\cong\mathcal{E}_{\mathrm{Thm.\ 9.2}}$ as a $\mathbb{Q}$-VMHS, and in particular* (b) *resp.* (c)–(d) *hold for $\mathbb{E}'^\vee_{\mathbb{Q}}$ resp. $\mathcal{E}_{\lim}$.*

Proof. Clearly (i)–(iii) are immediate from the hypotheses. □

[14] We are not *using* (a)–(c) here (as we must not!), only (ii)–(iii) and the differential equation $D^mL^\dagger(\cdot) = 0$. Since the latter is essentially D^{m+n+1} at 0, and $N_0 = -2\pi\mathbf{i}\mathrm{Res}_0(\nabla_D)$, we get $N_0^{n+m}\neq 0$ directly. We saw at the beginning of §6 that $(T_c - I)\mathbb{E}_{\mathbb{C},p}^\vee\subset\mathbb{C}\delta$ (from the differential equation only). The map is an isomorphism because we have $E_0 = \ker(N_0)\underset{T_c-I}{\overset{\cong}{\to}}\mathbb{C}\imath(\delta)$ by our earlier assumptions on $\mathbb{M}_{\mathbb{C}}^\vee$ in §4.

There is a plentiful source of such $\mathbb{Q}$-VMHS in the case $m=1$. Let φ be a reflexive Laurent polynomial. With notation as in Example 4.5, and writing $\mathcal{X}^{\times} := \beta^{-1}(\mathbb{G}_m^n)$, we can take the cup-product of the $\beta^* x_i \in \mathcal{O}^{\times}(\mathcal{X}^{\times}) \cong H^1_{\mathrm{M}}(\mathcal{X}^{\times}, \mathbb{Q}(1))$ $(i=1,\ldots,n+1)$ to get a motivic cohomology class $\{\underline{x}\} \in H^{n+1}_{\mathrm{M}}(\mathcal{X}^{\times}, \mathbb{Q}(n+1))$ called the *coordinate symbol*.[15]

Definition 9.4. We say that φ is *tempered* if $\{\underline{x}\}$ extends to a class $\zeta \in H^{n+1}_{\mathrm{M}}(\mathcal{X} \setminus X_0, \mathbb{Q}(n+1))$. (One may assume without loss of generality that $\varphi \in \bar{\mathbb{Q}}[x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}]$, since — up to scale — this is a necessary condition for temperedness [DK, Prop. 4.16]. Minkowski polynomials are expected to be tempered in general; this is known for $n \leq 2$ [dS]. See [DK, §3] for further discussion.)

Recall that a (graded-polarizable) $\mathbb{Q}$-VMHS $\mathcal{V}$ on U is called *admissible* (with respect to $\mathbb{P}^1$) if it is the restriction of a polarizable mixed Hodge module from $\mathbb{P}^1$. Admissibility always holds for geometric variations, and guarantees that a LMHS exists at each $\sigma \in \Sigma$; henceforth these are written $\psi_\sigma \mathcal{V}$.[16]

Definition 9.5. An admissible VMHS of the form

$$0 \to \mathcal{H} \to \mathcal{V} \to \mathbb{Q}_U(0) \to 0,$$

where $\mathcal{H}$ is a $\mathbb{Q}$-PVHS on U, is called an *admissible normal function*; we write $\mathcal{V} \in \mathrm{ANF}(\mathcal{H})$. (These are only interesting, i.e. can be non-split, for $\mathcal{H}$ of weight ≤ -1. If the weight is < -1, they are called *higher normal functions* since Bloch's higher Chow groups, or equivalently motivic cohomology, are the standard source.) Using $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), \mathbb{H}_t) \cong \mathbb{H}_{t,\mathbb{C}} / \left(F^0 \mathbb{H}_{t,\mathbb{C}} + \mathbb{H}_{t,\mathbb{Q}}\right)$, pointwise restriction yields a holomorphic section $\mathcal{V}_t$ of the generalized Jacobian bundle $J(\mathcal{H}) := \mathcal{H} / \left(F^0 \mathcal{H} + \mathbb{H}_{\mathbb{Q}}\right)$; it is in this sense that $\mathcal{V}$ is a "function".

If φ is tempered and good (cf. Example 4.5), we may construct a higher normal function by applying (to $\zeta|_{\mathcal{X}_U}$) the composition

$$H^{n+1}_{\mathrm{M}}(\mathcal{X}_U, \mathbb{Q}(n+1)) \overset{c_{\mathscr{H}}}{\to} H^{n+1}_{\mathscr{H}}(\mathcal{X}_U, \mathbb{Q}(n+1))$$

[15] For the reader unfamiliar with higher cycles/motivic cohomology and Abel-Jacobi maps on them, the accounts in [KLi], [DK, §1], and (regarding limits) [7K, §§5–6] may be useful.

[16] The LMHS is only well-defined with a choice of local parameter vanishing to first order at σ, and this parameter would usually be written as the subscript; for us, the parameter is always $t-\sigma$ (σ finite) or t^{-1} ($\sigma = \infty$).

$$\cong \mathrm{Ext}^{n+1}_{\mathrm{MHM}(\mathcal{X}_U)^{\mathrm{ps}}_{\mathcal{X}}}(\mathbb{Q}_{\mathcal{X}_U}(0), \mathbb{Q}_{\mathcal{X}_U}(n+1))$$
$$\overset{\mathrm{Gr}^1_{\mathcal{L}}}{\twoheadrightarrow} \mathrm{Ext}^1_{\mathrm{AVMHS}(U)}(\mathbb{Q}_U(0), \mathcal{H}^n_f(n+1)) \twoheadrightarrow \mathrm{Ext}^1_{\mathrm{AVMHS}(U)}(\mathbb{Q}_U(0), \mathcal{M}(n+1))$$
$$\cong \mathrm{ANF}(\mathcal{M}(n+1))$$

of the absolute-Hodge cycle-class map [KLe, §3], the projection to the bottom nonzero Leray-graded piece, and the projection from $\mathcal{H}^n_f$ to its direct summand $\mathcal{M}$. In more concrete terms, the corresponding section[17] of $J(\mathcal{M}(n+1)) \cong \mathcal{M}/\mathbb{M}(n+1)$ is evaluated at $t \in U$ by applying the (fiberwise) Abel-Jacobi map

$$\mathrm{AJ}\colon H^{n+1}_{\mathrm{M}}(X_t, \mathbb{Q}(n+1)) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^n_t(n+1))$$
$$\cong H^n(X_t, \mathbb{C}/\mathbb{Q}(n+1))$$

to $\zeta_t := \imath^*_{X_t}\zeta$ and projecting to $\mathbb{M}_{t,\mathbb{Q}} \otimes \mathbb{C}/\mathbb{Q}(n+1)$.

Definition 9.6. This higher normal function, written

$$\mathcal{V}_\varphi \in \mathrm{ANF}(\mathcal{M}(n+1)),$$

is called the *box extension* associated to φ.

Theorem 9.7. *If $\mathcal{M}$ arises from a reflexive, good, tempered Laurent polynomial φ, then the dual box extension provides a geometric realization of the unipotent extension with $m = 1$:*

$$\mathcal{V}^\vee_\varphi(1)|_{\mathfrak{U}_0^\times} \cong \mathcal{E}_{\text{Thm. 9.2}}.$$

Consequently, the periods of $\psi_0\mathcal{V}^\vee_\varphi$ and $\psi_0\mathcal{V}_\varphi$ are given by $\{\alpha^\dagger_0, \ldots, \alpha^\dagger_{n+1}\}$ and $\{\kappa^\dagger_0, \ldots, \kappa^\dagger_{n+1}\}$ respectively.[18]

Proof. Since $\mathcal{V}_\varphi$ is an extension of $\mathbb{Q}_U(0)$ by $\mathcal{M}(n+1) \cong \mathcal{M}^\vee(1)$, $\mathcal{V}^\vee_\varphi(1)$ is an extension of $\mathcal{M}$ by $\mathbb{Q}(1)$, and is of the form (iv), with dual maps $\pi\colon \mathcal{V}^\vee_\varphi(1) \to \mathcal{M}$ and $\imath\colon \mathbb{M}^\vee \to \mathbb{V}_\varphi(-1)$. Let $\omega \in H^0(U, \mathcal{F}^n\mathcal{V}^\vee_\varphi(1))$ be the unique section mapping to $\mathscr{m}_c$. We must show that $DL^\dagger$ annihilates all periods of ω. Clearly $\langle \imath(\varepsilon_j), \omega\rangle = \langle \varepsilon_j, \pi(\omega)\rangle = \langle \varepsilon_j, \mathscr{m}_c\rangle$ is killed by $L^\dagger$ for $j = 0, \ldots, n$; so it remains to check that the remaining independent period (which will not be killed by $L^\dagger$) is killed by $DL^\dagger$.

[17] i.e., produced by taking pointwise restrictions as in Defn. 9.5

[18] Recall that $\alpha^\dagger_i$ and $\kappa^\dagger_i$ are, by definition, the power series coefficients of $\kappa^\dagger(s)^{-1}$ resp. $\kappa^\dagger(s)$ (and we can drop daggers if $L = L^\dagger$); see Thm. 9.2 (and Rem. 8.6).

Let $\tilde{\mathcal{R}}_F \in H^0(U, \mathcal{F}^0\mathcal{V}_\varphi)$ and $\tilde{\mathcal{R}}_\mathbb{Q} \in H^0(\widetilde{U^{\mathrm{an}}}, \mathbb{V}_{\varphi,\mathbb{Q}})$ be sections mapping to $1 \in \mathbb{Q}(0)$; their difference $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}_\mathbb{Q} - \tilde{\mathcal{R}}_F$ is a multivalued section of $\mathcal{M}$ whose image in $J(\mathcal{M}(n+1))$ "is" $\mathcal{V}_\varphi$ (as a normal function). By [DK, Cor. 4.1] we have $D\tilde{\mathcal{R}} = (2\pi\mathbf{i})^n\mu$.[19] This implies that $D^k\tilde{\mathcal{R}} \in \mathcal{F}^{n+1-k}$ for $k > 0$ so that $Q(D^k\tilde{\mathcal{R}}, \mathfrak{m}_c) = 0$ for $0 < k < n+1$.

Now consider the (holomorphic, multivalued) truncated higher normal function

$$V_\varphi(t) := Q(\tilde{\mathcal{R}}, \mathfrak{m}_c),$$

and calculate

$$\begin{aligned} L^\dagger V_\varphi &= q_0 Q(D^{n+1}\tilde{\mathcal{R}}, \mathfrak{m}_c) + Q(\tilde{\mathcal{R}}, \cancelto{0}{L^\dagger \mathfrak{m}_c}) \\ &= (2\pi\mathbf{i})^n q_0 Q(D^n\mu, \tfrac{\mu}{q}) = \tfrac{(2\pi\mathbf{i})^n q_0}{q}(-1)^n Q(\mu, D^n\mu) \\ &= \tfrac{(-2\pi\mathbf{i})^n q_0 Y}{Y(0)^{-1}q_0 Y} = \tfrac{(-2\pi\mathbf{i})^n}{(2\pi\mathbf{i})^n \mathsf{Q}_0} = \tfrac{(-1)^n}{\mathsf{Q}_0}. \end{aligned}$$

On the other hand, the duality pairing $\mathcal{V}_\varphi \times \mathcal{V}_\varphi^\vee(1) \to \mathcal{O}(1)$ sends $\mathcal{F}^0 \times \mathcal{F}^n$ to zero, so that $\langle \tilde{\mathcal{R}}_F, \omega\rangle = 0$ and

$$V_\varphi = \langle \tilde{\mathcal{R}}, \pi(\omega)\rangle = \langle \tilde{\mathcal{R}}, \omega\rangle = \langle \tilde{\mathcal{R}}_\mathbb{Q}, \omega\rangle$$

is a period, independent from $\{\langle \varepsilon_j, \omega\rangle\}_{j=0}^n$, and killed by $DL^\dagger$. □

Example 9.8. The (reflexive, good, tempered) Laurent polynomial $\varphi = (1 - x_1 - x_2 + x_1x_2 - x_1x_2x_3)\prod_{i=1}^3(1 - x_i^{-1})$ appears in the algebro-geometrization of Apéry's irrationality proof for $\zeta(3)$ [Ke1, §5.3]. Its Picard-Fuchs operator $L = D^3 - t(34D^3 + 51D^2 + 27D + 5) + t^2(D+1)^3$ is self-adjoint, and we have $\kappa_1 = 0$, $\kappa_2 = -2\zeta(2)$, $\kappa_3 = \frac{17}{6}\zeta(3)$ [GZ]. At the end of [BV], Bloch and Vlasenko "speculate" that the dual box extension $\mathcal{V}_\varphi^\vee(1)|_{\mathfrak{U}_0^\times}$ coincides with their unipotent extension $\mathcal{E}$ (with $m = 1$) in this case. So Theorem 9.7 confirms this speculation.

Remark 9.9. If we view the $\{\varepsilon_j\}_{j=0}^n$ as rational classes in $\mathbb{M}(n+1)_\mathbb{Q} \cong \mathbb{M}_{\mathbb{Q}(n+1)}$ via $(2\pi\mathbf{i})^{n+1}Q^{-1}(\cdot)\colon \mathbb{M}_\mathbb{Q}^\vee \to \mathbb{M}_{\mathbb{Q}(n+1)}$, then in the proof of Theorem 9.7 one may choose $\tilde{\mathcal{R}}_\mathbb{Q} = \mathsf{Q}_0^{-1}(2\pi\mathbf{i})^{n+1}\varepsilon_{n+1}$ and extend $e_0, \ldots, e_n$

[19] The proof there is long and uses regulator currents; here is a sketch of a more hands-off proof: we can go from $H_{\mathscr{H}}(\mathcal{X}_U, \mathbb{Q}(n+1))$ to $H^1_{\mathrm{dR}}(U, \mathcal{M})$ by (a) mapping to $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^{n+1}(\mathcal{X}_U, \mathbb{Q}(n+1)))$ and taking the first Leray graded piece, or (b) taking fiberwise restrictions to get a section of $J(\mathcal{M}(n+1))$ and applying ∇. It is a standard exercise to show that these two compositions are equal; and (a) is given by $\mathrm{dlog}(\underline{x}) = (2\pi\mathbf{i})^n\mu \otimes \frac{dt}{t}$, while (b) is exactly $\nabla\tilde{\mathcal{R}} = D\tilde{\mathcal{R}} \otimes \frac{dt}{t}$.

by $e_{n+1} = \mathsf{Q}_0\tilde{\mathcal{R}}_F$. In precise terms, the Theorem is saying that $\omega(0) = \sum_{j=0}^{n+1}(2\pi\mathbf{i})^{-j}\alpha_j^\dagger\tilde{\varepsilon}_j^\vee(0)$ and $e_{n+1}(0) = \sum_{j=0}^{n+1}(2\pi\mathbf{i})^j\kappa_{n+1-j}^\dagger\tilde{\varepsilon}_j(0)$, where the tilde means to apply $e^{-\frac{\log(t)}{2\pi\mathbf{i}}N_0}$. More usefully, these can be recast as formulas for

$$V_\varphi \equiv \tfrac{1}{\mathsf{Q}_0}\sum_{k=0}^{n+1}\alpha_{n+1-k}^\dagger\tfrac{\log^k(t)}{k!} \qquad \text{and}$$
$$Q(\tilde{\mathcal{R}}) \equiv \tfrac{1}{\mathsf{Q}_0}\Big((2\pi\mathbf{i})^{n+1}\varepsilon_{n+1} - \textstyle\sum_{j=0}^{n+1}(2\pi\mathbf{i})^j\kappa_{n+1-j}^\dagger\tilde{\varepsilon}_j\Big)$$

modulo $\mathcal{O}(t\log^{n+1}t)$,[20] i.e. terms which limit to zero with t. In particular, we have that $V_\varphi^{\mathrm{an}}(0) = \frac{\alpha_{n+1}^\dagger}{\mathsf{Q}_0}$.

It was pointed out in [Ke2] that in Example 9.8, one can use a variant of [DK, (9.29)] to check that $\mathsf{Q}_0V_\varphi^{\mathrm{an}}(0) = -\frac{17}{6}\zeta(3)$ (where $\mathsf{Q}_0 = -\frac{1}{12}$). Clearly this laborious partial confirmation of the "speculation" of [BV] is superseded by Theorem 9.7.

Example 9.10. Writing $\varphi_r(\underline{x}) := (1+\sum_{i=1}^r x_i)(1+\sum_{i=1}^r x_i^{-1})$, the Feynman integral $I_r(t) := \int_{\mathbb{R}_{\geq 0}^{\times r}}\frac{\mathrm{dlog}(\underline{x})}{1-t\varphi(\underline{x})}$ arising from the r-banana graph with equal masses can (up to a $\mathbb{Q}$-period of the graph-hypersurface pencil) be interpreted as V_φ (with $L = L^\dagger$) by the methods of [BKV]. So $I_r^{\mathrm{an}}(0)$ is (up to products of lower-weight terms) a rational multiple of the relevant α_r, which in turn should be the top-degree coefficient of the (inverted, regularized) $\hat{\Gamma}$-class [GZ, §5] of the degree-$(1,1,\ldots,1)$ Fano hypersurface in $(\mathbb{P}^1)^{\times(r+1)}$. See also [Ir2] and [BFKNS].

10. Inhomogeneous equations and normal functions

Recall from the proof of Theorem 6.6 that $\tilde{\Phi}(s,t) = \Phi(s,t) - \kappa(s)\phi_0(t)$ has no monodromy about $t = c$ for any fixed s. Taking $s = \ell \in \mathbb{Z}_{>0}$,

$$L\tilde{\Phi}(\ell,t) \;=\; L\Phi(\ell,t) - \kappa(\ell)\cancelto{0}{L\phi_0(t)} = \; \ell^r t^\ell.$$

Moreover, $\tilde{\Phi}(\ell,t) = \sum_{k\geq 0}A_k(\ell)t^{k+\ell} - \kappa(\ell)\phi_0(t)$ is analytic at 0. The set of solutions to $L(\cdot) = \ell^r t^\ell$ which are analytic at 0 is clearly then $\{\tilde{\Phi}(\ell,t) + z\phi_0(t)\}_{z\in\mathbb{C}}$, and if $z \neq 0$ these solutions have monodromy about c. Since ℓ is a positive integer and $\Phi(\ell,t) = \sum_{k\geq 0}A_k(\ell)t^{k+\ell}$, we have $\Phi(\ell,0) = 0$; and recalling in addition that $\phi_0(0) = 1$ gives $\tilde{\Phi}(\ell,0) = -\kappa(\ell)$. This proves the

[20]Here $\varepsilon_{n+1} - \tilde{\varepsilon}_{n+1} = \sum_{k=1}^{n+1}\frac{(-1)^k}{k!}\big(\frac{\log(t)}{2\pi\mathbf{i}}\big)^k\varepsilon_{n+1-k}$ belongs to $\mathcal{M}^\vee$; so the formula for $\tilde{\mathcal{R}}$ makes sense.

Theorem 10.1. *Let $V^{[\ell]}(t)$ be the unique solution to the inhomogeneous equation $L(\cdot) = -t^{\ell}$ analytic at 0 with no monodromy about c. Then $\kappa(\ell) = \ell^r V^{[\ell]}(0)$.*

Definition 10.2. The values $\{\kappa(\ell)\}_{\ell\in\mathbb{N}}$ are called the *Apéry constants* of L.

Remark 10.3. If we take $\ell \in [1, d-1] \cap \mathbb{Z}$,[21] then

$$b_k^{[\ell]} := \begin{cases} 0, & k < \ell \\ \frac{1}{\ell^r} A_{k-\ell}(\ell), & k \geq \ell \end{cases}$$

is evidently a solution to the recurrence attached to L. (If $L \in \mathbb{Q}[t, D]$, then the $b_k^{[\ell]}$ are also rational.) Its generating series $\frac{1}{\ell^r}\Phi(\ell, t)$ and the holomorphic period $\phi_0(t) = \sum_{k\geq 0} a_k t^k$ both have monodromy about c, but $V^{[\ell]}(t) = \sum_{k\geq 0}(\frac{\kappa(\ell)}{\ell^r} a_k - b_k^{[\ell]})t^k =: \sum_{k\geq 0} v_k t^k$ does not. So if the a_k are nonzero for sufficiently large k, we have

$$0 = \lim_{k\to\infty} \frac{v_k}{a_k} \qquad \Longrightarrow \qquad \frac{\kappa(\ell)}{\ell^r} = \lim_{k\to\infty} \frac{b_k^{[\ell]}}{a_k}.$$

Note that in the strong conifold monodromy case, the nonvanishing of $a_{k\gg 0}$ is guaranteed by the asymptotics in the proof of Theorem 6.6; moreover, the description of $\kappa(\ell)$ just given is consistent with Theorem 6.6(i) since $\lim_{k\to\infty} \frac{a_{k-\ell}}{a_k} = c^{\ell}$ by those same asymptotics.

Example 10.4. Revisiting Example 9.8 ($d = n = 2$) and taking $\ell = 1$, Remark 10.3 reproduces the pair of sequences $\{a_k\} = 1, 5, 73, 1445, \ldots$ and[22] $\{b_k\} := \{b_k^{[1]}\} = 0, 1, 2106, \frac{125062}{3}, \ldots$ in Apéry's irrationality proof for $\zeta(3)$, with limit $\kappa(1) = \lim_{k\to\infty} \frac{b_k}{a_k} = \frac{\zeta(3)}{6}$. Though the difference between this and $\kappa_3 = \frac{17}{6}\zeta(3)$ may seem trivial, this is an artifact of the VHS $\mathcal{M}$ underlying Apéry possessing an "involution" under $t \mapsto \frac{1}{t}$ (cf. [Ke1, §5.3] and [GKS, §5.2]). In general, the Apéry constants $\{\kappa(\ell)\}$ and Frobenius constants $\{\kappa_j\}$ describe completely different things. The $\{\kappa(\ell)\}$ are closely related, as we shall see, to *special values at* 0 *of normal functions* nonsingular at 0, as well as to Galkin's Apéry constants of Fano varieties [Ga]. The $\{\kappa_j\}$ are extension-class invariants of the LMHS at 0 of the unipotent extensions of §9 (but *cannot* be evaluated as the *limit of an extension* at 0), and are closely tied to the Gamma constants of Fano varieties [GGI, GZ].

[21] Remember that d is the degree of L (in t).

[22] In most of the literature, the second sequence is multiplied by 6. Note that any solution to the recurrence is determined by its first two terms.

We shall conclude this article by saying something about these special values of normal functions. Given a polarized $\mathbb{Q}$-VHS $\mathcal{H}$ on U (of negative weight), there are *singularity invariants*

$$\operatorname{sing}_\sigma \colon \mathrm{ANF}(\mathcal{H}) \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), (\psi_\sigma \mathcal{H})_{T_\sigma}(-1)) \tag{10.1}$$

attached to each $\sigma \in \Sigma$ [KP, §2.12]. (This is essentially the restriction map $H^1(U, \mathbb{H}) \to H^1(\Delta_\sigma^\times, \mathbb{H})$ applied to Hodge-$(0,0)$ classes, where $\Delta_\sigma^\times$ is a punctured disk about σ.) One says that $\mathcal{V} \in \mathrm{ANF}(\mathcal{H})$ is *singular* at σ if $\operatorname{sing}_\sigma(\mathcal{V}) \neq 0$.

Example 10.5. Given a cycle $\Xi \in \mathrm{CH}^a(\mathcal{X}_U, b) \cong H_{\mathrm{M}}^{2a-b}(\mathcal{X}_U, \mathbb{Q}(a))$ with $2a - b - 1 = n$, taking fiberwise Abel-Jacobi maps produces a normal function $\mathcal{V} \in \mathrm{ANF}(\mathcal{M}(a))$. (Recall that $\mathcal{M}$ is a sub-VHS of the n^{th} = middle cohomology of the fibers; so $\mathcal{M}(a)$ has weight $-b-1$.) The composition

$$\mathrm{CH}^a(\mathcal{X}_U, b) \overset{\mathrm{Res}_\sigma}{\to} \mathrm{CH}^{a-1}(X_\sigma, b-1) \overset{\mathrm{cl}}{\to} \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H_n(X_\sigma)(b-a))$$

of residue and cycle-class maps factors through (10.1) (with $\mathcal{H} = \mathcal{M}(a)$). If Ξ is the restriction of a cycle in $\mathrm{CH}^a(\mathcal{X}_U \cup X_\sigma, b)$,[23] then $\mathrm{Res}_\sigma(\Xi) = 0$ and $\operatorname{sing}_\sigma(\mathcal{V}) = 0$. See [7K, Thm. 5.2] for more details.

Writing $h := \deg(\mathcal{F}_e^n)$ for the degree of the Hodge line bundle, we have the

Lemma 10.6 ([GKS]). *Given $a \in \mathbb{Z}_{>0}$ and $\boldsymbol{v} \in \mathrm{ANF}(\mathcal{M}(a)) \setminus \{0\}$ nonsingular away from ∞, let $\tilde{\boldsymbol{v}}$ be a (multivalued) lift to $\mathcal{M}$ of the associated section of the generalized Jacobian bundle $J(\mathcal{M}(a))$, and $\mathsf{v}(t) := Q(\tilde{\boldsymbol{v}}, \mu)$ the resulting (multivalued, holomorphic) truncated HNF on U. Then $L\mathsf{v}$ is a nonzero polynomial in t vanishing at $t = 0$, of degree $\leq d - h$.*

The following is a technical remark related to how Lemma 10.6 will be applied in the proofs and examples below; it makes use of the polynomial q from Proposition 7.1, and may be skipped on a first reading.

Remark 10.7. (i) If $\boldsymbol{v}$ is nonsingular away from 0 instead, the result in [GKS] (which works in greater generality than our setting) says that $\deg(L\mathsf{v}) \leq d - h - 1$ if T_∞ is unipotent and $\leq d - h$ otherwise. (However, $L\mathsf{v}$ need not vanish at 0.) The box extensions $\mathcal{V}_\varphi$ from §9 are of this type, with $h = 1$,

[23] Here $\mathcal{X}_U \cup X_\sigma$ is the union inside $\mathcal{X}$ (gluing that singular fiber back in). Note that "classical" algebraic cycles (case $b = 0$) never have residues — they always extend by taking Zariski closure. The corresponding fact for normal functions is that when $\mathcal{H}$ has weight -1, the invariants (10.1) are always zero.

and the proof of Theorem 9.7 shows — writing $\mathsf{v}_\varphi := Q(\tilde{\mathcal{R}}, \mu) = qV_\varphi$ — that $L\mathsf{v}_\varphi = qLV_\varphi = \frac{(-1)^n}{\mathsf{Q}_0} q$. So in the setting of Definition 9.4, we get that $\deg(q) \leq d-2$ resp. $d-1$ (depending on T_∞).

(ii) Continuing with this setting, there are immediate consequences for the lowest degrees. Clearly if $d = 1$ then $\deg(q) = 0$, T_∞ is non-unipotent, and $L^\dagger = L$. In fact, if $d = 2$ we also have $L^\dagger = L$. To see this, write c' for the second root of q_0. If $c' \in U$, then $\mu, \partial\mu, \ldots, \partial^n\mu$ do not span $\mathcal{M}_{c'}$ ($\partial^{n+1}\mu$ is not an $\mathcal{O}_{c'}$-linear combination of them); so there is a gap in the Kodaira-Spencer maps and $Y(c') = 0$. A similar argument shows $Y(c') = 0$ if $c' = c$ ($\mathrm{ord}_c(q_0) = 2$). Either way, q has (at least) a double zero at c', contradicting $\deg(q) \leq 1$. So $\Sigma^\times = \{c, c'\}$ and $1 = \mathrm{ord}_{c'}(q_0) \geq \mathrm{rk}(T_{c'} - I)$ forces conifold monodromy at c'; moreover, no Kodaira-Spencer maps vanish anywhere[24] on $\mathbb{C}^\times$. So $q_0 Y$ is constant $\implies q \equiv 1 \implies L^\dagger = L$. (In contrast, if $d = 3$ there are examples like the family generated by $\varphi = \frac{(1+x_1+x_2^2)^2}{x_1 x_2} - 8$, with $\Sigma^\times = \{-\frac{1}{16}, -\frac{1}{8}\}$, $q_0 = (1+16t)(1+8t)^2$, and $q = 1+8t$. The trouble is the I_0^* fiber at $t = -\frac{1}{8}$.)

(iii) If $\boldsymbol{v}$ is nonsingular everywhere (and nontrivial), then $\mathcal{M}(a)$ must have weight -1 ($\iff n$ odd and $a = \frac{n+1}{2}$), which corresponds to "classical" normal functions. In this case, $\deg(L\mathsf{v}) \leq d-h-1$ resp. $d-h$ *and* $L\mathsf{v}$ vanishes at 0. See Example 10.12(b) below.

We shall use the Lemma to prove a result which, together with Theorem 10.1, produces an interpretation of (some) $\kappa(\ell)$'s as special values (at $t = 0$) of normal functions.

Theorem 10.8. *Suppose there exists an embedding*

$$\vartheta \colon \mathbb{Q}(-a) \hookrightarrow \mathrm{IH}^1(\mathbb{A}^1, \mathbb{M}),$$

where $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$. Then there is a normal function[25]

$$\boldsymbol{v}_\vartheta \in \mathrm{ANF}(\mathcal{M}(a)) \setminus \{0\},$$

nonsingular away from ∞, with $\mathsf{v}_\vartheta := Q(\tilde{\boldsymbol{v}}_\vartheta, \mu)$ satisfying $L\mathsf{v}_\vartheta(t) = tP_\vartheta(t)$, where $P_\vartheta \in \mathbb{C}[t] \setminus \{0\}$ has $\deg(P_\vartheta) \leq d-h-1$. The lift $\tilde{\boldsymbol{v}}_\vartheta$ can be chosen so that v_ϑ is analytic on a disk of radius $> |c|$ about the origin.

[24]Any vanishing of a K-S map *at* a conifold monodromy point is away from the center, hence duplicated by the self-duality; so Y has odd order. Any vanishing of a K-S map (equivalently, of Y) on U makes q_0 vanish.

[25]See the proof for the precise correspondence with ϑ.

Proof. Recall that $\mathcal{M}$ is a summand of the n^{th} cohomology of some $f_U\colon \mathcal{X}_U \to U$, or more precisely of its quotient H^n_{var} by $H^n_{\text{fix}} = H^0(U, R^n(f_U)_*\mathbb{Q}_{\mathcal{X}_U})$ (the so-called "fixed part"). Let $\mathcal{X} \supset \mathcal{X}_U$ be our smooth compactification, and consider the extension in AVMHS(U) with fibers

$$0 \to H^n_{\text{var}}(X_t) \to H^{n+1}(\mathcal{X} \setminus X_\infty, X_t) \to \ker\{H^{n+1}(\mathcal{X} \setminus X_\infty) \to H^{n+1}_{\text{fix}}\} \to 0.$$

Pushing forward by $H^n_{\text{var}} \twoheadrightarrow \mathcal{M}$ on the left and pulling back by the composition of ϑ with the inclusion of $\mathrm{IH}^1(\mathbb{A}^1, \mathbb{M})$ on the right, we get an element $\mathcal{V}_\vartheta \in \mathrm{Ext}^1_{\mathrm{AVMHS}(U)}(\mathbb{Q}(-a), \mathcal{M}) \cong \mathrm{Ext}^1_{\mathrm{AVMHS}(U)}(\mathbb{Q}(0), \mathcal{M}(a))$. Its topological invariant $[\mathcal{V}_\vartheta] \in \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^1(U, \mathbb{M})(a))$ is tautologically the (nonzero) image of 1 under $\mathbb{Q}(0) \overset{\vartheta}{\hookrightarrow} \mathrm{IH}^1(\mathbb{A}^1, \mathbb{M}(a)) \hookrightarrow H^1(U, \mathbb{M}(a))$. In particular, it has no singularities on $\mathbb{A}^1$. Apply Lemma 10.6 to this $\mathcal{V}_\vartheta$.

It remains to check existence of a lift $\tilde{\mathcal{V}}_\vartheta$ with no monodromy on $\mathfrak{U}^\times$. This boils down to whether $[\mathcal{V}_\vartheta]$ restricts to zero in $H^1(\mathfrak{U}^\times, \mathbb{M})$. Writing $\bar{\mathfrak{U}} = \mathfrak{U}^\times \cup \{0, c\}$, $[\mathcal{V}_\vartheta]|_{\mathfrak{U}^\times}$ clearly lies in the image of $\mathrm{IH}^1(\bar{\mathfrak{U}}, \mathbb{M})$. But in the Mayer-Vietoris sequence

$$\mathbb{M}_p^{T_0} \oplus \mathbb{M}_p^{T_c} \to \mathbb{M}_p \to \mathrm{IH}^1(\bar{\mathfrak{U}}, \mathbb{M}) \to \mathrm{IH}^1(\mathfrak{U}_0, \mathbb{M}) \oplus \mathrm{IH}^1(\mathfrak{U}_c, \mathbb{M})$$

the first arrow is surjective (replace $\mathbb{M}$ by $\mathbb{M}^\vee$ and argue that $(\mathbb{M}_p^\vee)^{T_0}$ contains ε_0 and $(\mathbb{M}_p^\vee)^{T_c}$ contains $\varepsilon_1, \ldots, \varepsilon_n$) and the final term is zero; so we are done. □

Remark 10.9. The existence of the single-valued lift on $\mathfrak{U}^\times$ is made out to be a harder result in more special cases in [DK, BKV, Ke1]; but this is because for the applications in those works, an exact identification of the current representing the lift was required.

Definition 10.10. The extension of $\mathbb{Q}$-VMHS

$$0 \to \mathcal{M} \to \mathcal{V}_\vartheta \to \mathbb{Q}(-a) \to 0$$

corresponding to the normal function in Theorem 10.8 is called an *Apéry extension.*

Since $\mathcal{V}_\vartheta$ is nonsingular at 0, the associated section of $J(\mathcal{M}(a))$ has a well-defined "limit" (or value at 0) in the sense of [7K, Thm. 5.2(a)]. If $\mathcal{V}_\vartheta$ comes from an algebraic cycle $\mathfrak{Z}_\vartheta \in \mathrm{CH}^a(\mathcal{X} \setminus X_\infty, 2a-n-1)_{\mathbb{Q}}$, as predicted by the Beilinson-Hodge conjecture, then this leads to an explicit prescription (up to $\mathbb{Q}(a)$) for the special value $\mathsf{v}_\vartheta(0)$ [7K, Cor. 5.3]. We shall now spell out what this means.

First, $\mathfrak{Z}_\vartheta(0) := \imath^*_{X_0}\mathfrak{Z}_\vartheta \in H^{n+1}_{\mathrm{M}}(X_0, \mathbb{Q}(a))$ has

$$\mathrm{AJ}_{X_0}(\mathfrak{Z}_\vartheta(0)) \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^n(X_0, \mathbb{Q}(a))).$$

Next, the composition $\mathbb{Q}(0) \cong (\psi_0\mathcal{M}^\vee)_{T_0} \hookrightarrow H^n_{\lim}(X_t)_{T_0}(n) \stackrel{\mathrm{sp}}{\to} H_n(X_0)$ of MHS-morphisms sends $1 \mapsto Q(\mu_0) \mapsto \mathrm{Res}_{X_0}(\frac{\mathrm{dlog}(\underline{x})}{(2\pi\mathbf{i})^n}) =: \mu_{X_0}$; and so pairing with μ_{X_0} induces a projection $H^n(X_0, \mathbb{Q}) \twoheadrightarrow \mathbb{Q}(0)$. By [loc. cit.] we therefore have

$$\begin{aligned} \mathsf{v}_\vartheta(0) &\equiv \lim_{t\to 0} Q(\mathrm{AJ}_{X_t}(\mathfrak{Z}_\vartheta(t)), \mu_t) \\ &\equiv \langle \mathrm{AJ}_{X_0}(\mathfrak{Z}_\vartheta(0)), \mu_{X_0}\rangle \in \mathbb{C}/\mathbb{Q}(a) \cong \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), \mathbb{Q}(a)). \end{aligned}$$

In this scenario, the second line typically factors through the "Borel" regulator $H^1_{\mathrm{M}}(\mathrm{Spec}(K), \mathbb{Q}(a)) \to \mathbb{C}/\mathbb{Q}(a)$, with K the field of definition of $\mathfrak{Z}_\vartheta$. When $K = \mathbb{Q}$, one then has $\mathsf{v}_\vartheta(0) \in \mathbb{Q}\zeta(a)$. Note that for families of $K3$ surfaces ($n = 2$), there are two possibilities: $a = 3$ and $a = 2$. Both do occur ([GKS]; and see Example 10.13 below). Similarly, for elliptic curves ($n = 1$), Example 10.12 below shows that both $a = 2$ and $a = 1$ happen.

Putting everything together, *provided one can find enough embeddings* ϑ, and either assuming the BHC or constructing the cycles, one would obtain that:

- $\kappa(1), \ldots, \kappa(d-h)$ are of the form $\mathsf{v}_\vartheta(0) \underset{\mathbb{Q}(0)}{\equiv} \langle \mathrm{AJ}_{X_0}(\mathfrak{Z}_\vartheta(0)), \mu_{X_0}\rangle$, hence are periods; and
- with an assumption on the field of definition, they are actually rational multiples of Riemann zeta values.

However, we caution the reader that there are several obstacles to the existence of such embeddings (especially multiple, independent ones), the first of which is that $\mathrm{IH}^1(\mathbb{A}^1, \mathbb{M})$ may not be Hodge-Tate. Even if it is, it can possess nontrivial extension classes which "obstruct" such embeddings (which are after all Hodge classes), meaning that one must consider biextensions of VMHS on U; though in that case it is likely that the resulting $\kappa(j)$'s can still be analyzed in terms of (higher) cycles on Zariski-open subsets of the fibers. Our assumption that $\mathcal{M}$ be of type $(1, 1, \ldots, 1)$ also imposes severe limitations: if n is even, then there can be at most one[26] Hodge class in $\mathrm{IH}^1(\mathbb{A}^1, \mathbb{M})$; but this just means that a more general study is in order.

[26] This follows from the proof of Theorem 10.11 below.

We finish with one (still fairly broad) case where we only want *one* embedding, and that embedding fortunately *must* exist. This involves certain families of CY n-folds with 2 conifold points:

Theorem 10.11. *Assume $\mathcal{M}$ arises from a good, reflexive, tempered Laurent polynomial φ (Example 4.5), and that $d = 2$. Then we have an isomorphism $\mathrm{IH}^1(\mathbb{A}^1, \mathbb{M}) \underset{\vartheta}{\overset{\cong}{\leftarrow}} \mathbb{Q}(-a)$ for some $a \in [\frac{n+1}{2}, n+1] \cap \mathbb{Z}$. The resulting admissible normal function satisfies $L\mathsf{v}_\vartheta = -\mathfrak{k}t$ for some $\mathfrak{k} \in \bar{\mathbb{Q}}^\times$, and $\kappa(1) = \frac{1}{\mathfrak{k}}\mathsf{v}_\vartheta(0)$.*

Proof. Note that by Remark 10.7(ii), $\Sigma^\times$ comprises two conifold points (and also $L^\dagger = L$). By Euler-Poincaré, the rank of $\mathrm{IH}^1(\mathbb{A}^1, \mathbb{M})$ is given by

$$\textstyle\sum_{\sigma \in \Sigma \setminus \{\infty\}} \mathrm{rk}(T_\sigma - I) - r\chi(\mathbb{A}^1) = n + 1 + 1 - (n + 1) = 1.$$

So one of the end terms in the exact sequence of MHS

$$0 \to \mathrm{IH}^1(\mathbb{P}^1, \mathbb{M}) \to \mathrm{IH}^1(\mathbb{A}^1, \mathbb{M}) \to (\psi_\infty \mathcal{M})_{T_\infty}(-1) \to 0$$

is zero, and the other has rank one. (Applying E-P to the first term, either $\mathrm{rk}(T_\infty - I) = n$ and the first term vanishes, or it $= n + 1$ and the last term vanishes.) A rank-one MHS is of the form $\mathbb{Q}(-a)$; and the first term can only have weight $n + 1$, while the last term can have weights between $n + 2$ and $2n + 2$. □

Example 10.12. For $n = 1$ (and $d = 2$), we demonstrate the two possibilities ($a = 1$ or 2) in Theorem 10.11:

(a) $\underline{\mathrm{IH}^1(\mathbb{P}^1, \mathbb{M}) = \{0\}}$: $\varphi = (1 - x_1^{-1})(1 - x_2^{-1})(1 - x_1 - x_2)$ yields the "little Apéry" family of elliptic curves associated with irrationality of $\mathsf{v}_\vartheta(0) = \zeta(2)$, where $\mathfrak{Z}_\vartheta \in \mathrm{CH}^2(\mathcal{X} \setminus X_\infty, 2)$ is obtained by pulling the box cycle ζ back along the involution $(x_1, x_2, t) \mapsto (\frac{x_1}{x_1 - 1}, \frac{1 - x_2}{1 - x_1 - x_2}, -\frac{1}{t})$ [Ke1, §5.2]. (So we have $a = 2$.) In direct analogy to [GKS, §5.2], one can show that

$$\mathsf{v}_\vartheta(t) = \int_{\mathbb{R}^{\times 2}_{\leq 0}} \frac{\mathrm{dlog}(\underline{x})}{t + \varphi(\underline{x})} = \sum_{j \geq 0} (-t)^j \int_{\mathbb{R}^{\times 2}_{\leq 0}} \frac{\mathrm{dlog}(\underline{x})}{\varphi(\underline{x})^{j+1}} = \zeta(2) + (3\zeta(2) - 5)t + \cdots .$$

Applying $L = D^2 - t(11D^2 + 11D + 3) - t^2(D + 1)^2$ and invoking Theorem 10.11, we find $\mathfrak{k} = 5$ hence $\kappa(1) = \frac{\zeta(2)}{5}$.

(b) $\underline{\mathrm{IH}^1(\mathbb{P}^1, \mathbb{M}) \neq \{0\}}$ ($\Longrightarrow \quad a = 1$): $\varphi = x_1^{-1}x_2^{-1}(1 + x_1 + x_2 + x_2^2)^2$ yields a family of elliptic curves with singular fibers of types $\mathrm{I}_4, \mathrm{I}_1, \mathrm{I}_1, \mathrm{I}_0^*$

at $0, \frac{1}{12}, -\frac{1}{4}, \infty$ respectively. It has a nontorsion[27] section given by $\mathfrak{Z}_\vartheta = [(0,\zeta_3)] - [(0,\zeta_3^2)] \in \mathrm{CH}^1(\mathcal{X})$, where $\zeta_3 := e^{\frac{2\pi\mathbf{i}}{3}}$. The Abel-Jacobi map yields

$$\mathsf{v}_\vartheta(t) = \int_{(0,\zeta_3^2)}^{(0,\zeta_3)} \mu = \frac{1}{2\pi\mathbf{i}} \int_{\zeta_3^2}^{\zeta_3} \oint_{|x_1|=\epsilon} \frac{dx_1/x_1}{1-t\varphi(\underline{x})} \frac{dx_2}{x_2} = \sum_{j\geq 0} t^j \int_{-\mathbf{i}}^{\mathbf{i}} [\varphi^k]_{x_1} \frac{dx_2}{x_2}$$
$$= -\tfrac{2}{3}\pi\mathbf{i} + (4\sqrt{3}\mathbf{i} - \tfrac{4}{3}\pi\mathbf{i})t + (18\sqrt{3}\mathbf{i} - 12\pi\mathbf{i})t^2 + \cdots,$$

where $[\varphi^k]_{x_1}$ means terms constant in x_1. Applying $L = (1-8t-48t^2)D^2 - (8t+96t^2)D - (2t+36t^2)$ and invoking Theorem 10.11 once more, we find $\mathfrak{k} = -4\sqrt{3}\mathbf{i}$ and $\kappa(1) = \frac{\pi}{6\sqrt{3}}$.

(c) The simplest example of what we mean by an "obstruction" occurs for $n=1$ and $d=3$, for the polynomial $\varphi = x_1^{-1}x_2^{-1}(1+x_1+x_2^2)^2 - 8$ from the end of Remark 10.7(ii) (with an I_1 at ∞). As in (b), there is a nontorsion section $\mathfrak{Z} = [(0,\mathbf{i})] - [(0,-\mathbf{i})] \in \mathrm{CH}^1(\mathcal{X})$, which limits in particular to $\frac{\mathbf{i}(\sqrt{2}-1)}{\mathbf{i}(\sqrt{2}+1)} = 3-2\sqrt{2} \in \mathbb{C}^\times$ in the group law on X_∞^{sm}.[28] The difference is that in this case $\mathrm{IH}^1(\mathbb{A}^1,\mathbb{M})$ has rank 2, and is a (nonsplit) extension of $(\psi_\infty\mathcal{M})_{T_\infty}(-1) \cong \mathbb{Q}(-2)$ by $\mathrm{IH}^1(\mathbb{P}^1,\mathbb{M}) \cong \mathbb{Q}(-1)$ with class $\log(3-2\sqrt{2}) \in \mathbb{C}/\mathbb{Q}(1) \cong \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(-2),\mathbb{Q}(-1))$. So there is no morphism $\mathbb{Q}(-2) \hookrightarrow \mathrm{IH}^1(\mathbb{A}^1,\mathbb{M})$, and one must deal with biextensions. This still may be treated via a higher cycle, but this cycle lives in $\mathrm{CH}^2(\mathcal{X}\setminus\{X_\infty \cup |\mathfrak{Z}|\},2)$ and does not lift to $\mathrm{CH}^2(\mathcal{X}\setminus X_\infty,2)$.

Example 10.13. The regularized differential operators in Golyshev's article [Go2] underlie variations $\mathcal{M}$ of the type described in Theorem 10.11, with $n=2$ and $d=2$. Geometrically, these correspond to families of (generic) Picard-rank 19 $K3$ surfaces with 2 conifold points. They exhibit both of the possibilities in the Theorem, namely $a=2$ or 3. (In both cases, $\mathrm{IH}^1(\mathbb{P}^1,\mathbb{M}) = \{0\}$.) The corresponding (higher) normal functions are constructed explicitly in [GKS]. We briefly summarize two of these constructions here, and direct the interested reader to §§5.3–5.5 of [op. cit.] for more details.

(a) $\underline{a=3}$: Taking φ as in Examples 9.8 and 10.4, $\mathcal{X} \to \mathbb{P}^1$ is the "big Apéry" family of $K3$ surfaces, with singular fibers at $(t=)\ 0, \infty$, and $(\sqrt{2}\pm 1)^4$. The

[27]Observe that $x_1 = 3(u-1)^{-4}(u-\mathbf{i}\sqrt{3})^2(u-\frac{\mathbf{i}}{\sqrt{3}})^2$, $x_2 = (u-1)^{-2}(u+1)^2$ yields a normalization $\mathbb{P}^1 \to X_{\frac{1}{12}}$ sending $u = 0,\infty$ to the node $(3,1)$. The preimage of the cycle is $[\frac{\mathbf{i}}{\sqrt{3}}] - [\mathbf{i}\sqrt{3}]$, and $\frac{\mathbf{i}/\sqrt{3}}{\mathbf{i}\sqrt{3}} = \frac{1}{3} \in \mathbb{C}^\times$ has infinite order.

[28]Normalize X_∞ by $x_1 = 2(u-1)^{-4}(u-\mathbf{i}(\sqrt{2}+1))^2(u-\mathbf{i}(\sqrt{2}-1))^2$, $x_2 = (u-1)^{-2}(u+1)^2$; the preimage of $\mathfrak{Z}$ is $[(\mathbf{i}(\sqrt{2}-1)] - [(\mathbf{i}(\sqrt{2}+1)]$.

relevant cycle $\mathfrak{Z}_\vartheta \in \mathrm{CH}^3(\mathcal{X} \setminus X_\infty, 3)$ is constructed by pulling the box cycle ζ back along the birational involution

$$(x_1, x_2, x_3, t) \mapsto \left(\tfrac{x_3}{1-x_3}, \tfrac{-(1-x_1)(1-x_2)}{1-x_1-x_2+x_1x_2-x_1x_2x_3}, \tfrac{x_1}{1-x_1}, \tfrac{1}{t}\right)$$

of $\mathcal{X}$, and it is shown in [GKS, §5.4] that

$$\mathsf{v}_\vartheta(t) = \int_{\mathbb{R}^3_{\le 0}} \frac{\mathrm{dlog}(\underline{x})}{t - \varphi(\underline{x})} = 2\zeta(3) + (-12 + 10\zeta(3))t + \cdots .$$

Applying L (from Example 9.8) and invoking Theorem 10.11, we get $\mathfrak{k} = 12$ and $\kappa(1) = \frac{1}{6}\zeta(3)$.

(b) $\underline{a = 2}$: $\varphi = x_1^{-1}x_2^{-1}(1 - x_3^{-1})(1 - x_1 - x_3)(1 - x_2 - x_3)(1 - x_1 - x_2 - x_3)$ defines a family of $K3$ surfaces with singular fibers at $0, \infty$, and $\frac{-11\pm 5\sqrt{5}}{4}$. Each smooth fiber X_t intersects $x_3 = 0$ in a cycle $C_1 \cup \cdots \cup C_5$ of $\mathbb{P}^1$'s. Writing z_i for coordinates on them, with $z_i = 0$ and ∞ at intersections (so that the sum of divisors in X_t is zero), $\sum_i (C_i, z_i)$ yields an element of $\mathrm{CH}^2(X_t, 1)$. These are fiberwise restrictions of a "global" higher Chow cycle $\mathfrak{Z}_\vartheta \in \mathrm{CH}^2(\mathcal{X} \setminus X_\infty, 1)$. In [GKS, §5.3], it is shown that $\mathsf{v}_\vartheta(t) = \zeta(2) + (-10 + 6\zeta(2))t + \cdots$, whence $\mathfrak{k} = 10$ and $\kappa(1) = \frac{1}{10}\zeta(2)$.

(c) The Landau-Ginzburg models for Fano 3-folds V_{16} and V_{18} give two more examples similar to (a). But in the V_{18} case, as noticed by [dS], there is a crucial difference: we have $\mathfrak{k} \notin \mathbb{Q}$; in fact $\mathfrak{k} = \sqrt{-3}$, and $\mathsf{v}_\vartheta(0) \in (2\pi\mathbf{i})^3\mathbb{Q}$. Though the family is defined over $\mathbb{Q}$, the normalization of X_∞ (and consequently $\mathfrak{Z}_\vartheta$) is only defined over $\mathbb{Q}(\sqrt{-3})$. See [GKS, §5.5].

Remark 10.14. We have argued above that $\kappa(1), \ldots, \kappa(d-1)$ are interesting invariants of $\mathcal{M}$ related to algebraic cycles; the natural reaction is to wonder if $\kappa(d)$, $\kappa(d+1)$, etc. are similarly interesting. In fact, to expand on [BV, Rem. 32] a bit, they are not: taking $L \in K[t, D]$, they are always contained in $K[\kappa(1), \ldots, \kappa(d-1)]$ in view of Corollary 8.4. For example, if $L^\dagger = L$ ($\implies Q_j = P_j$) then

$$\kappa(d) = \frac{-d^r}{P_d(-d)} \sum_{j=0}^{d-1} j^{-r} P_j(-j)\kappa(j),$$

where "$0^{-r}P_0(0)$" is to be read as $\lim_{s\to 0} s^{-r}P_0(-s) = \lim_{s\to 0} \frac{(-s)^r}{s^r} = (-1)^r$. So in the Apéry $\zeta(3)$ case (Examples 9.8, 10.4, and 10.13(a)), where $\kappa(1) = \frac{\zeta(3)}{6}$ (and $\kappa(0) = 1$), we find $\kappa(2) = -8 + \frac{5}{6}\zeta(3)$; one can also show (in the notation of Theorem 10.1) that the solutions of the inhomogeneous equations satisfy $V^{[2]}(t) = -1 - \frac{35}{48}\zeta(3)A(t) + 5V^{[1]}(t)$.

Appendix A. On "The Frobenius method"

Given a linear ODE with a regular singular point (say, $t = 0$), the classical Frobenius theorem tells us the form of a local basis of solutions at 0 in terms of the roots of the indicial polynomial. The "Frobenius basis" $\{\phi_m(t)\}_{m=0}^n$ of the present article is, in the case where 0 is a point of maximal unipotent monodromy (with indicial equation $T^{n+1} = 0$), the simplest basis consistent with this theorem. It is uniquely specified by the properties $\phi_m(t) \sim \frac{1}{m!}\log^m(t)$ (so that $\phi_0 = 1+O(t)$ is the unique solution holomorphic at 0) and $\phi_m(t) - \frac{\log^m(t)}{m!} \to 0$ as $t \to 0$.

At least in the hypergeometric setting, where power series coefficients a_k of a holomorphic solution are given in terms of finite products of fractional values $\Gamma(\mathfrak{a}+k)$ ($\mathfrak{a} \in \mathbb{Q}$) of the Gamma function, the "Frobenius theorem" is closely connected to the "Frobenius method". That is, one perturbs $k \mapsto k+s$ everywhere in the solution, differentiates one or more times in s, and then sets $s = 0$, thereby obtaining additional solutions. When 0 is a MUM point, this gives a *basis* of solutions, and it is natural to ask how this relates to the bases of this article. We shall now show that it actually yields the $\mathbb{Q}$-*Betti basis* $\{\epsilon_k(t)\}$.

To begin in some greater generality, suppose we have a VHS $\mathcal{M}$ over $\mathbb{P}^1 \setminus \Sigma$ as in §3 — of weight n, with Hodge numbers $(1, 1, \dots, 1)$, MUM at 0, etc. — and *assume in addition that* $L = L^\dagger$. The Frobenius deformation $\Phi(s, t)$ is defined by $L\Phi = s^{n+1}t^s$ and $T_0\Phi = e^{2\pi \mathbf{i} s}\Phi$. Define the kappa series by $(T_c - I)\Phi = \kappa(s)\psi(t)$, and the Frobenius periods by $\Phi(s, t) =: \sum_{m\geq 0} \phi_m(t)s^m$.

The Betti periods $\epsilon_k(t)$ are defined in §4 for $m = 0, \dots, n$ and for $m > n$ in §9, as periods of a uniquely determined $\mathbb{Q}$-VMHS that satisfy $D^\infty L(\cdot) = 0$. The holomorphic period is

$$\epsilon_0(t) = \phi_0(t) = \sum_{k\geq 0} a_k t^k. \tag{A.1}$$

By results in §6, §9, the "Betti-period generating series" satisfies

$$\mathcal{E}(s, t) := \sum_{m\geq 0} (2\pi\mathbf{i})^m \epsilon_m(t)s^m = \frac{\Phi(s, t)}{\kappa(s)}. \tag{A.2}$$

(While this is not stated there, it can be read off from the statement that $\sum_{i=0}^{\ell} \epsilon^{\mathrm{an}}_{\ell-i}(0)\kappa_i = \boldsymbol{\delta}_{0\ell}$ in the proof of Theorem 9.2; again we remind the reader that $\kappa(0)=1$.)

Next, recall that the Frobenius generating series can be rewritten as

$$\Phi(s,t) = \sum_{k\geq 0} A_k(s)t^{k+s}, \tag{A.3}$$

where $A_0(s)\equiv 1$, $A_k(0)=a_k$, and the $\{A_k(s)\}$ satisfy the recurrence relation from Remark 5.3. From (A.2)–(A.3) we obviously have that

$$\mathcal{E}(s,t) = \sum_{k\geq 0} \frac{A_k(s)}{\kappa(s)} t^{k+s}. \tag{A.4}$$

Now to state the obvious, if we take s-derivatives of $\mathcal{E}$ [resp. Φ] then set $s=0$, we get Betti [resp. Frobenius] periods. *So in (A.1), if we replace a_k by $a^{\mathit{Frob}}_{k+s} := A_k(s)$ and t^k by t^{k+s}, we obtain Φ (hence Frobenius periods); while if we replace a_k by $a^{\mathit{Betti}}_{k+s} := \frac{A_k(s)}{\kappa(s)}$ and t^k by t^{k+s}, we get $\mathcal{E}$ (hence Betti periods).*

Finally, we specialize to the (hypergeometric) setting of Example 6.8, with $\{\mathfrak{a}_j\}_{j=0}^n \subset \mathbb{Q}$ centered about $\frac{1}{2}$ and $L = D^{n+1} - t\prod_{j=0}^n (D+\mathfrak{a}_j)$. Here we can compute everything in closed form: namely,

$$\begin{aligned} A_k(s) &= \prod_{j=0}^n \frac{\Gamma(k+s+\mathfrak{a}_j)\Gamma(s+1)}{\Gamma(s+\mathfrak{a}_j)\Gamma(k+s+1)}, \qquad a_k = \prod_{j=0}^n \frac{\Gamma(k+\mathfrak{a}_j)}{\Gamma(\mathfrak{a}_j)\Gamma(k+1)} \\ \kappa(s) &= \prod_{j=0}^n \frac{\Gamma(s+1)\Gamma(\mathfrak{a}_j)}{\Gamma(s+\mathfrak{a}_j)}, \quad \text{and} \quad \frac{A_k(s)}{\kappa(s)} = \prod_{j=0}^n \frac{\Gamma(k+s+\mathfrak{a}_j)}{\Gamma(\mathfrak{a}_j)\Gamma(k+s+1)}. \end{aligned} \tag{A.5}$$

Substituting $k+s$ for k in the formula for a_k clearly gives $\frac{A_k(s)}{\kappa(s)} = a^{\mathrm{Betti}}_{k+s}$, justifying our earlier assertion that the Frobenius method yields $\mathbb{Q}$-Betti solutions.

As a simple example, consider the PF operator $L = D^2 - t(D+\frac{1}{2})^2$ for the Legendre family of elliptic curves, with

$$a_k = \frac{1}{16^k}\binom{2k}{k}^2 = \frac{1}{16^k}\frac{\Gamma(2k+1)^2}{\Gamma(k+1)^4} = \left(\frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(k+1)}\right)^2.$$

Replacing k by $k+s$ in the right-hand expression gives exactly

$$\frac{A_k(s)}{\kappa(s)} = \left(\frac{\Gamma(k+s+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(k+s+1)}\right)^2,$$

hence the $\mathbb{Q}$-Betti deformation (A.4). Not only does this produce the other $\mathbb{Q}$-period $\epsilon_1(t)$ of the Legendre family, but all the mixed periods $\epsilon_{m>1}(s)$ as well.

Acknowledgments

We thank S. Bloch, V. Golyshev, and A. Klemm for many helpful discussions, and the two referees for numerous suggestions which have helped to clarify the exposition. This work was partially supported by Simons Collaboration Grant 634268 and NSF Standard Grant DMS-2101482.

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Received January 14, 2021
Accepted September 21, 2022