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On generalized stuffle relations between cell-zeta values

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We introduce a family of linear relations between cell-zeta values that have a form similar to product map relations and jointly with them imply stuffle relations between multiple zeta values.

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Introduction

Multiple zeta values are special values of multiple zeta functions on the one hand and special values of multiple polylogarithms on the other. Hence, they may be represented either as sums of number series or as integrals.

Multiple zeta values form an algebra over rational numbers. A product of two of them may be presented as a linear combination of multiple zeta values with integer coefficients by means of each representation. It gives two systems of relations which multiple zeta values obey.

The first one is called shuffle relations. They immediately follow from Fubini's theorem applied to the integral representation.

The second family is called stuffle relations and is given by rearrangement of summands in the product of number series. They are not so evident in the integral presentation. The equality of corresponding integrals is based on Fubini's theorem, relation (11), which is a form of Arnold's relation, and some coordinate transformations. These transformations are given by permutations of coordinates in cubical coordinates, but in simplicial coordinates they are birational transformations.

These relations may be extended by regularizations of some equalities with divergent series, see [IKZ06, Rac02]. This extended system of relations is called regularized double shuffle relations. The long-standing conjecture states that they imply all rational relations between multiple zeta values.

Multiple zeta values of weight n being values of integrals are periods of the pair $(\mathcal{M}^{\delta}_{0,n+3}, \mathcal{M}^{\delta}_{0,n+3} \setminus \mathcal{M}_{0,n+3})$ of a special kind ([GM04]). In [Bro09, BCS10] all periods of such pairs were studied. In [BCS10] they were called cell-zeta values.

As well as multiple zeta values, cell-zeta values obey a lot of relations over rational numbers. By the main result of [Bro09], all cell-zeta values are rational combinations of multiple zeta values. In light of this, it is natural to try to find a set of relations on cell-zeta values, which allows to express any cell-zeta value in terms of multiple zeta values and implies all known relation on multiple zeta values. We suggest a candidate for this, which consists of two families of relations. In [BCS10] another system of relations on cell-zeta values is written down. It would be very interesting to compare them.

The first family containing quadratic-linear relations was introduced in [Bro09]. It is analogous to shuffle relations and follows from Fubini's theorem. In [BCS10] these relations called product map relations. We suggest the term "generalized shuffle relations" to emphasize the similarity between our pair of families with the pair of shuffle and stuffle relations.

The second family is what we call generalized stuffle relations. This is a family of linear relations following from the relative version of Fubini's theorem. They seem to be new. These relations generalize above-mentioned manipulations with integrals in cubical coordinates. It means in particular that generalized shuffle relations and generalized stuffle relations imply usual stuffle relations. This is the main result of the paper.

The natural question is whether generalized shuffle and stuffle relations imply regularized double shuffle relations. The answer seems to be no. But if we add relations implied by Stokes' theorem to the generalized shuffle and stuffle relations, the answer becomes positive. In [KY16] a set of linear relations between multiple zeta values is introduced. It is proved there that these relations along with double shuffle relations imply regularized double shuffle relations. Thus, to confirm a positive answer to the question above one needs to show that the Kaneko–Yamamoto integral-series relations follow from Stokes' theorem and generalized stuffle relations. It is a subject of another paper [Mar].

In contrast to usual stuffle relations (see nevertheless [Sou10]), generalized stuffle relations have a clear motivic nature. A motivic version of these relations is the subject of future research.

1. Shuffle and stuffle relations

1.1. Multiple zeta values

Call a finite sequence of natural numbers $(k_1, \ldots, k_n)$ convergent if $k_1 \geq 2$. For a convergent sequence $\mathbf{k} = (k_1, \ldots, k_n)$ the multiple zeta value is defined by the integral (see e. g. [IKZ06])

$$\zeta(\mathbf{k}) = \int_{\Delta_{w(\mathbf{k})}} \omega_1(t_1) \wedge \cdots \wedge \omega_{w(\mathbf{k})}(t_{w(\mathbf{k})}), \tag{1}$$

where $\Delta_{w(\mathbf{k})} = \{1 > t_1 > \cdots > t_{w(\mathbf{k})} > 0\}$, $w(\mathbf{k}) = k_1 + \cdots + k_n$ is weight of the sequence and

$$\omega_i(t) = \begin{cases} dt/(1-t) & \text{if } i \in \{k_1, k_1+k_2, \ldots, k_1+\cdots+k_n\} \\ dt/t & \text{otherwise} \end{cases}$$

Thus multiple zeta values are iterated integrals.

A more conventional way to define multiple zeta values is by series representation ascending to Euler, but we will not need it.

1.2. Shuffle relations

With a finite sequence of natural numbers $\mathbf{k} = (k_1, \ldots, k_n)$ associate the word $z_{\mathbf{k}} = x^{k_1-1}yx^{k_2-1}\ldots x^{k_n-1}y$ of two letters x and y. It establishes a bijection between sequences and words ending in y.

A finite multiset is an unordered finite list with possible repetitions. For a multiset M, denote by $x \cdot M$ the result of the action of operation $x\cdot$ on M elementwise.

Define shuffle product $\mathrm{sh}(\,\cdot\,,\,\cdot\,)$ of two words in letters x and y as a multiset of words given by the recursive rule

$$\mathrm{sh}(v \cdot z_1, u \cdot z_2) = v \cdot \mathrm{sh}(z_1, u \cdot z_2) \cup u \cdot \mathrm{sh}(v \cdot z_1, z_2), \tag{2}$$

where $u, v \in \{x, y\}$ and $\mathrm{sh}(1, z) = \mathrm{sh}(z, 1) = \{z\}$.

One may see that the shuffle product of two words ending in y consists of words ending in y. It defines the shuffle product of sequences of natural numbers, which we denote likewise by $\mathrm{sh}(\,\cdot\,,\,\cdot\,)$.

Proposition 1 (Shuffle relations)**.** *Let* $\mathbf{k}$ *and* $\mathbf{l}$ *be convergent sequences. Then*

$$\zeta(\mathbf{k})\,\zeta(\mathbf{l}) = \sum_{\mathbf{s} \in \mathrm{sh}(\mathbf{k},\mathbf{l})} \zeta(\mathbf{s}) \tag{3}$$

Proof. The statement is an immediate consequence of Fubini's theorem. It also follows from Proposition 3 below. □

1.3. Stuffle relations

For a finite sequence of natural numbers $\mathbf{k} = (k_1, \dots, k_{n-1})$, denote the sequence $(k_1, \dots, k_{n-1}, k_n)$ by $\mathbf{k} \cdot k_n$. Introduce the empty sequence $()$ such that $() \cdot k = (k)$.

Define stuffle product $\mathrm{st}(\,\cdot\,,\,\cdot\,)$ of two sequences as a multiset of sequences given by the recursive rule

$$\mathrm{st}(\mathbf{k} \cdot x, \mathbf{l} \cdot y) = \mathrm{st}(\mathbf{k} \cdot x, \mathbf{l}) \cdot y \cup \mathrm{st}(\mathbf{k}, \mathbf{l} \cdot y) \cdot x \cup \mathrm{st}(\mathbf{k}, \mathbf{l}) \cdot (x + y) \tag{4}$$

and by $\mathrm{st}((), \mathbf{k}) = \mathrm{st}(\mathbf{k}, ()) = \{\mathbf{k}\}$. Note that $\mathrm{st}(\mathbf{k}, \mathbf{l}) = \mathrm{st}(\mathbf{l}, \mathbf{k})$.

Stuffle relations follow easily from the series representation of multiple zeta values. The proof of them in terms of integrals may be found in [Gon02, Bro09, Sou10]. We present it in a form convenient for our purposes.

Proposition 2 (Stuffle relations). *Let* $\mathbf{k}$ *and* $\mathbf{l}$ *be convergent sequences. Then*

$$\zeta(\mathbf{k})\, \zeta(\mathbf{l}) = \sum_{\mathbf{s} \in \mathrm{st}(\mathbf{k}, \mathbf{l})} \zeta(\mathbf{s}) \tag{5}$$

Proof. Define cubical coordinates on the standard simplex $\Delta_k = \{1 > t_1 > \dots > t_k > 0\}$ by

$$x_1 = t_1 \quad x_2 = t_2/t_1 \quad \dots \quad x_k = t_k/t_{k-1}.$$

Introduce notations:

$$f_a^b = \frac{\prod_{i=a}^b x_i}{1 - \prod_{i=a}^b x_i}.$$

In cubical coordinates we can rewrite definition (1) as

$$\zeta(\mathbf{k}) = \int_\square f_1^{k_1} f_1^{k_1+k_2} \cdots f_1^{w(\mathbf{k})}\, dV, \tag{6}$$

where $dV = dx_1/x_1 \wedge dx_2/x_2 \dots$ is the standard volume form on the torus and symbol $\square$ here and below in this proof means the unit cube $\{0 < x_i < 1\}$.

Fubini's theorem gives

$$\zeta(\mathbf{k})\,\zeta(\mathbf{l}) = \int_{\square} f_1^{k_1} f_1^{k_1+k_2} \cdots f_1^{w(\mathbf{k})} \times f_{w(\mathbf{k})+1}^{w(\mathbf{k})+l_1} f_{w(\mathbf{k})+1}^{w(\mathbf{k})+l_1+l_2} \cdots f_{w(\mathbf{k})+1}^{w(\mathbf{k})+w(\mathbf{l})}\, dV \tag{7}$$

Thus we need to prove that the right hand side of (7) equals to the right hand side of (5).

Introduce following transformations of the cube

$$r_a(x_i) = \begin{cases} x_{a+1-i} & \text{for } i \le a, \\ x_i & \text{for } i > a. \end{cases} \tag{8}$$

Applying $r_{w(\mathbf{k})}$ to the right hand side of (7) we get

$$\text{r.h.s. } (7) = \int_{\square} f_{1+w(\mathbf{k})-k_1}^{w(\mathbf{k})} f_{1+w(\mathbf{k})-k_1-k_2}^{w(\mathbf{k})} \cdots f_1^{w(\mathbf{k})} \times f_{w(\mathbf{k})+1}^{w(\mathbf{k})+l_1} f_{w(\mathbf{k})+1}^{w(\mathbf{k})+l_1+l_2} \cdots f_{w(\mathbf{k})+1}^{w(\mathbf{k})+w(\mathbf{l})}\, dV \tag{9}$$

To show that right hand sides of (9) and (5) are equal, we will prove a more general equality

$$\sum_{\mathbf{s}\in \mathrm{st}(\mathbf{k},\mathbf{l})} \zeta(\cdots(\mathbf{s}\cdot m_1)\cdot m_2)\cdots)\cdot m_i) = \int_{\square} f_{1+w(\mathbf{k})-k_1}^{w(\mathbf{k})} f_{1+w(\mathbf{k})-k_1-k_2}^{w(\mathbf{k})} \cdots f_1^{w(\mathbf{k})} \times f_{w(\mathbf{k})+1}^{w(\mathbf{k})+l_1} f_{w(\mathbf{k})+1}^{w(\mathbf{k})+l_1+l_2} \cdots f_{w(\mathbf{k})+1}^{w(\mathbf{k})+w(\mathbf{l})} \times f_1^{w(\mathbf{k})+w(\mathbf{l})+m_1} f_1^{w(\mathbf{k})+w(\mathbf{l})+m_1+m_2} \cdots f_1^{w(\mathbf{k})+w(\mathbf{l})+m_1+\cdots+m_i}\, dV \tag{10}$$

by induction on $w(\mathbf{k}) + w(\mathbf{l})$.

The base of induction is given by (6).

If $\mathbf{k} = ()$, there is nothing to prove. If $\mathbf{l} = ()$, the application of $r_{w(\mathbf{k})}$ to the integral proves the equality.

If both sequences are not empty, let $x = k_{w(\mathbf{k})}$ and $y = l_{w(\mathbf{l})}$ be the last terms of $\mathbf{k}$ and $\mathbf{l}$. Introduce notations $\mathbf{k} = \mathbf{k}' \cdot x$ and $\mathbf{l} = \mathbf{l}' \cdot y$.

Substituting in $\alpha = \prod_{i=1}^{a} x_i$ and $\beta = \prod_{i=a+1}^{b} x_i$ into the relation

$$\frac{1}{(1-\alpha)(1-\beta)} = \frac{\alpha}{(1-\alpha)(1-\alpha\beta)} + \frac{\beta}{(1-\beta)(1-\alpha\beta)} + \frac{1}{1-\alpha\beta} \tag{11}$$

we have for $b > a > 1$

$$f_1^a f_{a+1}^b = f_1^a f_1^b + f_{a+1}^b f_1^b + f_1^b$$

Applying this to the product of the last factors in the second and the third lines of (10) we get a sum of three terms.

The first term is

$$\int_{\square} f^{w(\mathbf{k})}_{1+w(\mathbf{k})-k_1} f^{w(\mathbf{k})}_{1+w(\mathbf{k})-k_1-k_2} \cdots f^{w(\mathbf{k})}_{1} \times \\ f^{w(\mathbf{k})+l_1}_{w(\mathbf{k})+1} f^{w(\mathbf{k})+l_1+l_2}_{w(\mathbf{k})+1} \cdots f^{w(\mathbf{k})+w(\mathbf{l}')}_{w(\mathbf{k})+1} \times \\ f_1^{w(\mathbf{k})+w(\mathbf{l}')+y} f_1^{w(\mathbf{k})+w(\mathbf{l}')+y+m_1} \cdots f_1^{w(\mathbf{k})+w(\mathbf{l}')+y+m_1+\cdots+m_i}\, dV \tag{12}$$

By the induction assumption it is equal to $\sum_{\mathbf{s}\in \mathrm{st}(\mathbf{k},\mathbf{l}')} \zeta(\cdots(\mathbf{s}\cdot y)\cdot m_1)\cdots)\cdot m_i)$.

The second term is

$$\int_{\square} f^{w(\mathbf{k})}_{1+w(\mathbf{k})-k_1} f^{w(\mathbf{k})}_{1+w(\mathbf{k})-k_1-k_2} \cdots f^{w(\mathbf{k})}_{1+x} \times \\ f^{w(\mathbf{k})+l_1}_{w(\mathbf{k})+1} f^{w(\mathbf{k})+l_1+l_2}_{w(\mathbf{k})+1} \cdots f^{w(\mathbf{k})+w(\mathbf{l})}_{w(\mathbf{k})+1} \times \\ f_1^{w(\mathbf{l})+w(\mathbf{k}')+x} f_1^{w(\mathbf{l})+w(\mathbf{k}')+x+m_1} \cdots f_1^{w(\mathbf{l})+w(\mathbf{k}')+x+\cdots+m_i}\, dV \tag{13}$$

Applying $r_{w(\mathbf{k})+w(\mathbf{l})}$ to it we get

$$(13) = \int_{\square} f^{w(\mathbf{l})}_{1+w(\mathbf{l})-l_1} f^{w(\mathbf{l})}_{1+w(\mathbf{l})-l_1-l_2} \cdots f^{w(\mathbf{l})}_{1} \times \\ f^{w(\mathbf{l})+k_1}_{w(\mathbf{l})+1} f^{w(\mathbf{l})+k_1+k_2}_{w(\mathbf{l})+1} \cdots f^{w(\mathbf{l})+w(\mathbf{k}')}_{w(\mathbf{l})+1} \times \\ f_1^{w(\mathbf{l})+w(\mathbf{k}')+x} f_1^{w(\mathbf{l})+w(\mathbf{k}')+x+m_1} \cdots f_1^{w(\mathbf{l})+w(\mathbf{k}')+x+\cdots+m_i}\, dV \tag{14}$$

By the induction assumption it is equal to $\sum_{\mathbf{s}\in \mathrm{st}(\mathbf{l},\mathbf{k}')} \zeta(\cdots(\mathbf{s}\cdot x)\cdot m_1)\cdots)\cdot m_i)$.

The third term is

$$(15)\quad \int_{\Box} f^{w(\mathbf{k})}_{1+w(\mathbf{k})-k_1} f^{w(\mathbf{k})}_{1+w(\mathbf{k})-k_1-k_2}\cdots f^{w(\mathbf{k})}_{1+x}\times \\ f^{w(\mathbf{k})+l_1}_{w(\mathbf{k})+1} f^{w(\mathbf{k})+l_1+l_2}_{w(\mathbf{k})+1}\cdots f^{w(\mathbf{k})+w(\mathbf{l}')}_{w(\mathbf{k})+1}\times \\ f_1^{w(\mathbf{k}')+w(\mathbf{l}')+x+y} f_1^{w(\mathbf{k}')+w(\mathbf{l}')+x+y+m_2}\cdots f_1^{w(\mathbf{k}')+w(\mathbf{l}')+x+y+\cdots+m_i}\, dV$$

Applying $r_{w(\mathbf{k})+w(\mathbf{l}')}$ to it we get

$$(16)\quad (15)=\int_{\Box} f^{w(\mathbf{l}')}_{1+w(\mathbf{l}')-l_1} f^{w(\mathbf{l}')}_{1+w(\mathbf{l}')-l_1-l_2}\cdots f^{w(\mathbf{l}')}_{1}\times \\ f^{w(\mathbf{l}')+k_1}_{w(\mathbf{l}')+1} f^{w(\mathbf{l}')+k_1+k_2}_{w(\mathbf{l}')+1}\cdots f^{w(\mathbf{l}')+w(\mathbf{k}')}_{w(\mathbf{l}')+1}\times \\ f_1^{w(\mathbf{l}')+w(\mathbf{k}')+x+y} f_1^{w(\mathbf{l}')+w(\mathbf{k}')+x+y+m_1}\cdots f_1^{w(\mathbf{l}')+w(\mathbf{k}')+x+y+\cdots+m_i}\, dV$$

By the induction assumption it is equal to $\sum_{\mathbf{s}\in \mathrm{st}(\mathbf{l}',\mathbf{k}')}\zeta(\cdots(\mathbf{s}\cdot(x+y))\cdot m_1)\cdots)\cdot m_i)$.

By (4) the sum of these three terms equals to $\sum_{\mathbf{s}\in \mathrm{st}(\mathbf{k},\mathbf{l})}\zeta(\cdots(\mathbf{s}\cdot m_1)\cdot m_2)\cdots)\cdot m_i)$, which proves (10) and thus the proposition. $\Box$

2. Generalized shuffle and stuffle relations

2.1. $\mathcal{M}_{0,S}$

For a finite set S, $|S|>2$ denote by $\mathcal{M}_{0,S}$ the moduli space of its embeddings $S\hookrightarrow \mathbb{P}^1$ to the complex projective line considered up to the action of the Möbius group. This is a smooth affine variety and it has a smooth projective compactification $\overline{\mathcal{M}}_{0,S}$, which is the moduli space of stable curves. The complement $\overline{\mathcal{M}}_{0,S}\setminus\mathcal{M}_{0,S}$ is the union of normal crossing divisors. These divisors are numerated by partitions of S in two subsets with cardinalities more than 2.

Present S as a union of the three-element set $\{\underline{0},\underline{\infty},\underline{1}\}$ and a set with n elements, where and below $|S|=n+3$. For a point of $\mathcal{M}_{0,S}$ introduce on $\mathbb{P}^1$ the coordinate such that coordinates of points labeled by $\underline{0},\underline{\infty},\underline{1}$ are 0, ∞ and 1 correspondingly. Coordinates of points labeled by the finite set with n elements are called simplicial coordinates on $\mathcal{M}_{0,S}$. Thus a point of $\mathcal{M}_{0,S}$ with simplicial coordinates $(t_1,\ldots,t_n)$ is $(0,\infty,1,t_1,\ldots t_n)$.

The algebra of regular differential forms on $\mathcal{M}_{0,S}$ with logarithmic singularities at infinity is generated by 1-forms

$$(17)\qquad \omega_{ij}=d\,log(t_i-t_j)\quad 0\le i<j\le n+1\quad ij\neq 0n+1,$$

where t_i are simplicial coordinates, $t_0 = 1$ and $t_{n+1} = 0$. The only relations between them are Arnold's relations:

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ik} + \omega_{ik} \wedge \omega_{ij} = 0$$

Rewrite these relations in new coordinates $\alpha = t_j/t_i$, $\beta = t_k/t_j$ and $t = t_i$:

$$\begin{aligned}(18)\quad &\left(-\frac{d\alpha}{1-\alpha} + \frac{dt}{t}\right) \wedge \left(-\frac{d\beta}{1-\beta} + \frac{d\alpha}{\alpha} + \frac{dt}{t}\right) + \\ &\left(-\frac{d\beta}{1-\beta} + \frac{d\alpha}{\alpha} + \frac{dt}{t}\right) \wedge \left(-\frac{d(\alpha\beta)}{1-\alpha\beta} + \frac{dt}{t}\right) + \\ &\left(-\frac{d(\alpha\beta)}{1-\alpha\beta} + \frac{dt}{t}\right) \wedge \left(-\frac{d\alpha}{1-\alpha} + \frac{dt}{t}\right) = 0\end{aligned}$$

Collecting terms with $d\alpha \wedge d\beta$ we get relation (11), which is crucial in the proof of Proposition 2.

2.2. $\mathcal{M}^{\delta}_{0,S}$

Let S be a cyclically ordered set. In [Bro09] the space $\mathcal{M}^{\delta}_{0,S}$ is introduced and described in great detail. It may be thought as a partial compactification of $\mathcal{M}_{0,S}$

$$\mathcal{M}_{0,S} \subset \mathcal{M}^{\delta}_{0,S} \subset \overline{\mathcal{M}}_{0,S},$$

which contains only those compactification divisors for which corresponding partitions of S respect the cyclic order. It is proved in [Bro09] that $\mathcal{M}^{\delta}_{0,S}$ is a smooth affine variety.

For any subset $T \subset S$ the forgetful map

$$\mathcal{M}^{\delta}_{0,S} \to \mathcal{M}^{\delta}_{0,T}$$

is defined, where the cyclic order on T is the restriction of the one from S. Being restricted on $\mathcal{M}_{0,S}$ this map forgets points labeled by elements of $S \setminus T$.

As above present S as a union of the three-element set $\{\underline{0}, \underline{\infty}, \underline{1}\}$ and a set with n elements, and introduce a cyclic order on it as follows: $[\underline{0}, \underline{\infty}, \underline{1}, 1, \ldots, n]$. Consider the standard simplex $\Delta_n = \{1 > t_1 > \cdots > t_n > 0\}$ in $\mathcal{M}_{0,S}$, where t_i are simplicial coordinates. This set depends on the order of the labeling set, but one may see that it depends only on the cyclic order. Then, the closure of the standard simplex in $\mathcal{M}^{\delta}_{0,S}$ is compact; this is the Stasheff polytope. Denote this subset of $\mathcal{M}^{\delta}_{0,S}$ by $\Delta(S)$ to emphasize its dependence on the cyclic order.

Since $\Delta(S)$ is compact, a regular differential form of top degree may be integrated by it. For a regular differential form with logarithmic singularities at infinity ω on $\mathcal{M}^{\delta}_{0,S}$ the integral $\int_{\Delta(S)} \omega$ is a period of the pair $(\mathcal{M}^{\delta}_{0,S}, \mathcal{M}^{\delta}_{0,S} \setminus \mathcal{M}_{0,S})$, see [GM04]. In [BCS10] these numbers are called cell-zeta values. By the very definition (1), multiple zeta values are examples of such numbers, convergence of sequence $\mathbf{k}$ implies that the form has no poles on divisors from $\mathcal{M}^{\delta}_{0,S} \setminus \mathcal{M}_{0,S}$.

2.3. Generalized shuffle relations

Let $\underline{\mathbf{3}}$ be a cyclically ordered set with three elements. Define a 3-pointed cyclically ordered set $\mathcal{T}$ as a pair of a cyclically ordered set T and a monotonic embedding $\imath\colon \underline{\mathbf{3}} \hookrightarrow T$.

Let $\mathcal{T}_{1,2}$ be a pair of 3-pointed cyclically ordered sets and $\imath_{1,2}\colon \underline{\mathbf{3}} \hookrightarrow T_{1,2}$ are corresponding embeddings. Let $T_1 \coprod_{\underline{\mathbf{3}}} T_2$ be the colimit of the diagram in the category of sets given by these embeddings. Denote by $\mathfrak{sh}(\mathcal{T}_1, \mathcal{T}_2)$ the set of cyclically ordered sets given by all cyclic orders on $T_1 \coprod_{\underline{\mathbf{3}}} T_2$ for which projections on T_1 and T_2 are monotonic.

For any $C \in \mathfrak{sh}(\mathcal{T}_1, \mathcal{T}_2)$ consider the map

$$\beta_C\colon \mathcal{M}^{\delta}_{0,C} \to \mathcal{M}^{\delta}_{0,T_1} \times \mathcal{M}^{\delta}_{0,T_2}, \tag{19}$$

which is the forgetful map on each factor. In [Bro09, 2.7] this map is called the product map.

The following proposition is taken from [Bro09, BCS10], where it is called product map relations.

Proposition 3 (Generalized shuffle relations)**.** *Using notations as above let ϕ and ψ be regular top-degree differential forms on $\mathcal{M}^{\delta}_{0,T_1}$ and $\mathcal{M}^{\delta}_{0,T_2}$ correspondingly. Then*

$$\left(\int_{\Delta(T_1)} \phi\right) \cdot \left(\int_{\Delta(T_2)} \psi\right) = \sum_{C \in \mathfrak{sh}(\mathcal{T}_1, \mathcal{T}_2)} \int_{\Delta(C)} \beta^*_C(\phi \boxtimes \psi) \tag{20}$$

Proof. By Fubini's theorem and because β is an embedding containing the domain of integration, the left hand side of (20) equals to the integral of $\beta^*_C(\phi \boxtimes \psi)$ by $\beta^{-1}(\Delta(T_1) \times \Delta(T_2))$. The decomposition of the latter set in simplices corresponding to elements of $\mathfrak{sh}(\mathcal{T}_1, \mathcal{T}_2)$ proves the statement. For more details see [Bro09, Corollary 7.10]. □

2.4. Generalized stuffle relations

Let $\underline{\mathbf{4}}$ be a cyclically ordered set with four elements. Define a 4-pointed cyclically ordered set $\mathcal{T}$ as a pair of a cyclically ordered set T and a monotonic embedding $\imath\colon \underline{\mathbf{4}} \hookrightarrow T$.

The Klein four-group V acts on $\underline{\mathbf{4}}$. Half of this group respects the cyclic order and the other half reverses it. For $\nu \in V$ and a 4-pointed cyclically ordered set $\mathcal{T} = (T, \imath)$ denote by $\mathcal{T}^\nu$ the 4-pointed cyclically ordered set with the embedding equal to $\imath$ composed with ν and with the cyclic ordered set equal to T or to T^{op} depending on whether ν respects cyclic order on $\underline{\mathbf{4}}$ or not, where $\cdot^{op}$ means the same set with the opposite order. Denote the latter cyclically ordered set by T^ν.

Let $\mathcal{T}_{1,2}$ be a pair of 4-pointed cyclically ordered sets and $\imath_{1,2}\colon \underline{\mathbf{4}} \hookrightarrow T_{1,2}$ are corresponding embeddings. Let $T_1 \coprod_{\underline{\mathbf{4}}} T_2$ be the colimit of the diagram in the category of sets given by these embeddings. Denote by $\mathfrak{st}(\mathcal{T}_1, \mathcal{T}_2)$ the set of cyclically ordered sets given by all cyclic orders on $T_1 \coprod_{\underline{\mathbf{4}}} T_2$ for which projections on T_1 and T_2 are monotonic.

For any $\nu \in V$, $C \in \mathfrak{st}(\mathcal{T}_1, \mathcal{T}_2)$ and $C_\nu \in \mathfrak{st}(\mathcal{T}_1, \mathcal{T}_2^\nu)$ consider maps

$$\begin{array}{ccc} \gamma_C\colon \mathcal{M}^\delta_{0,C} & \searrow & \\ & & \mathcal{M}^\delta_{0,T_1} \times \mathcal{M}^\delta_{0,T_2} \\ \gamma_{C_\nu}\colon \mathcal{M}^\delta_{0,C_\nu} & \nearrow & \end{array} \tag{21}$$

which are forgetful map on each factor.

Proposition 4 (Generalized stuffle relations)**.** *Using notations as above let ν be a non-trivial element of the Klein four-group and ϕ and ψ be regular differential forms on $\mathcal{M}^\delta_{0,T_1}$ and $\mathcal{M}^\delta_{0,T_2}$ correspondingly such that*

$$\deg\phi + \deg\psi = |T_1| + |T_2| - 7$$

Then

$$\sum_{C \in \mathfrak{st}(\mathcal{T}_1,\mathcal{T}_2)} \int_{\Delta(C)} \gamma_C^*(\phi \boxtimes \psi) = \epsilon \cdot \sum_{C_\nu \in \mathfrak{st}(\mathcal{T}_1,\mathcal{T}_2^\nu)} \int_{\Delta(C_\nu)} \gamma_{C_\nu}^*(\phi \boxtimes \psi), \tag{22}$$

where $\epsilon = (-1)^{\frac{(|T_2|-3)(|T_2|-2)}{2}}$ if ν reverses the cyclic order on $\underline{\mathbf{4}}$ and $\epsilon = 1$ if not.

Proof. Note that there are only two possibilities: ϕ is a top-degree form and degree of ψ is one less and vice versa.

Consider the forgetful projection $\mathcal{M}^{\delta}_{0,C} \to \mathcal{M}^{\delta}_{0,\underline{\mathbf{4}}}$. By Fubini's theorem,

$$\int_{\Delta(C)} \gamma_C^*(\phi \boxtimes \psi) = \int_{\Delta(\underline{\mathbf{4}})} \left(\int_{\Delta(C)/\Delta(\underline{\mathbf{4}})} \gamma_C^*(\phi \boxtimes \psi) \right), \tag{23}$$

where $\int_{\Delta(C)/\Delta(\underline{\mathbf{4}})}$ is the fiber-wise integral of the projection. By the relative analog of Proposition 3, we have

$$\sum_{C \in \mathfrak{st}(\mathcal{T}_1,\mathcal{T}_2)} \int_{\Delta(C)/\Delta(\underline{\mathbf{4}})} \gamma_C^*(\phi \boxtimes \psi) = \left(\int_{\Delta(T_1)/\Delta(\underline{\mathbf{4}})} \phi \right) \cdot \left(\int_{\Delta(T_2)/\Delta(\underline{\mathbf{4}})} \psi \right)$$

Because the cross-ratio of four points is invariant under the Klein four-group, action of the Klein four-group on four points of projective line may be lifted to the whole projective line. It follows the equality of forms on $\mathcal{M}^{\delta}_{0,\underline{\mathbf{4}}}$

$$\int_{\Delta(T_2)/\Delta(\underline{\mathbf{4}})} \psi = \epsilon \cdot \int_{\Delta(T_2^{\nu})/\Delta(\underline{\mathbf{4}})} \psi$$

Thus we get

$$\sum_{C \in \mathfrak{st}(\mathcal{T}_1,\mathcal{T}_2)} \int_{\Delta(C)/\Delta(\underline{\mathbf{4}})} \gamma_C^*(\phi \boxtimes \psi) = \left(\int_{\Delta(T_1)/\Delta(\underline{\mathbf{4}})} \phi \right) \cdot \left(\int_{\Delta(T_2)/\Delta(\underline{\mathbf{4}})} \psi \right) =$$
$$\epsilon \cdot \left(\int_{\Delta(T_1)/\Delta(\underline{\mathbf{4}})} \phi \right) \cdot \left(\int_{\Delta(T_2^{\nu})/\Delta(\underline{\mathbf{4}})} \psi \right) = \epsilon \cdot \sum_{C_\nu \in \mathfrak{st}(\mathcal{T}_1,\mathcal{T}_2^{\nu})} \int_{\Delta(C_\nu)/\Delta(\underline{\mathbf{4}})} \gamma_{C_\nu}^*(\phi \boxtimes \psi)$$

Integrating both sides by $\Delta(\underline{\mathbf{4}})$ and using (23) we get a proof of the proposition. □

Theorem 1. *Generalized shuffle relations (20) and generalized stuffle relations (22) jointly imply shuffle relations (3) and stuffle relations (5).*

Proof. From (17) we see that for a sequence $\mathbf{k}$ the differential form under the integral sign in (1) is a top-degree form on $\mathcal{M}_{0,w(\mathbf{k})+3}$. If sequence $\mathbf{k}$ is convergent, then the form comes from $\mathcal{M}^{\delta}_{0,w(\mathbf{k})+3}$ for the standard cyclic order $[0, \infty, 1, t_1, \ldots, t_{w(\mathbf{k})}]$ on the labeling set. Given a pair of such forms corresponding to sequences $\mathbf{k}$ and $\mathbf{l}$, applying Proposition 3 to them with

$$T_1 = [0, \infty, 1, t_1, \ldots, t_{w(\mathbf{k})}] \quad \text{and} \quad T_2 = [0, \infty, 1, s_1, \ldots, s_{w(\mathbf{l})}]$$

and the common subset $[0, \infty, 1]$ we get (2). Thus generalized shuffle relations imply shuffle relations.

Now show that generalized shuffle and generalized stuffle relations jointly imply stuffle relations. To do it we need to verify that all steps of the proof of Proposition 2 follow from generalized shuffle and generalized stuffle relations.

As it is mentioned in [Bro09, Sou10], the first relation (7) of the proof follows from the Proposition 3 for differential forms corresponding to sequences $\mathbf{k}$ and $\mathbf{l}$ and for

$$T_1 = [0, \infty, 1, t_1, \ldots, t_{w(\mathbf{k})}] \quad \text{and} \quad T_2 = [0, \infty, t_{w(\mathbf{k})}, s_1, \ldots, s_{w(\mathbf{l})}]$$

with the common subset $[0, \infty, t_{w(\mathbf{k})}]$.

The rest of the proof of Proposition 2 depends on two statements: invariance of integrals under transformations (8) and relation (11). The second one is a form of Arnold's relations by (18). Invariance of an integral under transformation r_a follows from Proposition 4 for

$$T_1 = [0, \infty, 1, t_a, t_{a+1}, \ldots, t_n] \quad \text{and} \quad T_2 = [0, \infty, 1, t_1, t_2, \ldots, t_a]$$

with the common subset $[0, \infty, 1, t_a]$ and the involution ν, which interchanges 0 and ∞. There are three places where invariance under these transformations is used in the proof of Proposition 2: (9), (14) and (16). Corresponding differential forms ϕ and ψ from the statement of Proposition 4 may be found from these formulae, for (9) and (14) the first form is being of top degree and for (16) the second form is. □

2.5. Formal algebra of periods of $(\mathcal{M}^\delta_{0,*}, \mathcal{M}^\delta_{0,*} \setminus \mathcal{M}_{0,*})$

If differential forms in Propositions 3 and 4 are regular with logarithmic singularities at infinity, then forms under integral signs in (20) and (22) are also regular with logarithmic singularities at infinity. Thus Propositions 3 and 4 impose quadratic-linear and linear conditions on periods of $(\mathcal{M}^\delta_{0,*}, \mathcal{M}^\delta_{0,*} \setminus \mathcal{M}_{0,*})$ or cell-zeta values, which are integrals $\int_{\Delta(S)} \omega$ of a regular differential form with logarithmic singularities at infinity ω on $\mathcal{M}^\delta_{0,S}$ by the standard simplex. By Proposition 3 they form an algebra.

One may consider the formal algebra of periods of $(\mathcal{M}^\delta_{0,*}, \mathcal{M}^\delta_{0,*} \setminus \mathcal{M}_{0,*})$, which is the one generated by symbols representing integrals as above on which all natural relations such as Stokes' theorem, and generalized shuffle and generalized stuffle relations are imposed. Formal multiple zeta values are elements of this algebra corresponding to iterated integrals (1).

The main theorem of [Bro09] states that all cell-zeta values are rational combinations of multiple zeta values. The long-standing conjecture ([IKZ06, Conjecture 1]) states that all rational relations between multiple zeta values are given by regularized double shuffle relations. This leads us to the following conjecture.

Conjecture. *The formal algebra of periods of* $(\mathcal{M}^{\delta}_{0,*}, \mathcal{M}^{\delta}_{0,*} \setminus \mathcal{M}_{0,*})$ *is generated by formal multiple zeta values and the system of relations to which they obey is equivalent to regularized double shuffle relations.*

An analogous conjecture was formulated in [BCS10]. The formal algebra of cell-zeta values defined there has the same generators, but relations differ. Its system of relations contains product map relations, which are the same as generalized shuffle relations, dihedral relations and shuffles with respect to one element. It would be interesting to compare these algebras. Note that dihedral relations follow from generalized stuffle relations for $|T_1| = 4$ and $\phi = 1$.

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