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KP hierarchy for Hurwitz-type cohomological field theories*

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We generalise a result of Kazarian regarding Kadomtsev–Petviashvili integrability for single Hodge integrals to general cohomological field theories related to Hurwitz-type counting problems or hypergeometric tau-functions. The proof uses recent results on the relations between hypergeometric tau-functions and topological recursion, as well as the DOSS correspondence between topological recursion and cohomological field theories. As a particular case, we recover the result of Alexandrov of KP integrability for triple Hodge integrals with a Calabi-Yau condition.

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1. Introduction

The moduli spaces of curves are a central object in modern algebraic geometry, and have been studied intensively. In particular, their intersection theory is a subject of ongoing research. The space $\overline{\mathcal{M}}_{g,n}$ has n line bundles $\mathbb{L}_i$ whose fibres at a point are the cotangent lines at the ith point of the represented curve, and a rank-g Hodge bundle $\mathbb{E}$ whose fibres are the space of one-forms on the curve. Their Chern classes are defined to be $\psi_i := c_1(\mathbb{L}_i)$ and $\lambda_j := c_j(\mathbb{E})$, respectively. Moreover, the spaces $\overline{\mathcal{M}}_{g,n}$ for different g and n have many structure maps between them, and many classes behave well under these maps. A collection of classes on all $\overline{\mathcal{M}}_{g,n}$ satisfying certain coherence axioms with respect to the structure maps are called cohomological field theories (CohFTs), and these play an important role in enumerative geometry of curves. One well-known example is the total Hodge class $\Lambda(t) = \sum \lambda_i t^i$.

By the Witten–Kontsevich theorem [60, 43], moduli spaces of curves have many relations to areas of mathematical physics and integrable hierarchies. In particular, this theorem proves that a generating function of the intersection numbers of ψ-classes is a tau-function of the Korteweg–de Vries hierarchy.

Furthermore, the Ekedahl–Lando–Shapiro–Vainshtein formula [29] relates single Hodge integrals, i.e. intersection numbers of $\Lambda(-1)$ with ψ-classes, to simple single Hurwitz numbers, counting ramified coverings of $\mathbb{P}^1_{\mathbb{C}}$ with only simple ramifications (with profile $(2, 1, 1, 1, \ldots)$) except for one point. Hurwitz numbers themselves also give a large class of tau-functions of Toda or Kadomtsev–Petviashvili hierarchies (of which the KdV hierarchy is a reduction), as noted by Okounkov [53].

Kazarian [40] interpreted the ELSV formula as a change of variables from the generating function of single Hodge integrals to a tau-function of the Kadomtsev–Petviashvili hierarchy, using the result of Okounkov on simple single Hurwitz numbers.

All of these results have strong relations to Chekhov–Eynard–Orantin topological recursion [19, 32], a successful way of encoding many counting problems with a natural genus expansion into a spectral curve with a recursively defined collection of multidifferentials, which should be generating functions of the counts. The Witten–Kontsevich ψ-intersection numbers can be encoded this way, and this is in a sense the base case of the theory. Many types of Hurwitz numbers obey topological recursion as well, starting with [12, 10] for the first case of simple Hurwitz numbers, and culminating in the works of Bychkov–Dunin-Barkowski–Kazarian–Shadrin [18, 17], which prove topological recursion for two large families of hypergeometric KP tau-functions, encompassing nearly all previously-studied cases of Hurwitz numbers.

In another direction, there is a general correspondence between topological recursion and intersection numbers of CohFTs [31, 28], which vastly generalises the ELSV formula when combined with the results on topological recursion for Hurwitz numbers.

A particularly interesting case is the conjecture of Mariño–Vafa [51] on a further generalisation of the ELSV formula, proved independently in [47, 54]. This Mariño–Vafa formula, inspired by topological vertex amplitudes, i.e. Gromov–Witten invariants of $\mathbb{C}^3$, relates triple Hodge integrals with a Calabi–Yau condition to characters of symmetric groups. Topological recursion was conjectured for toric Calabi–Yau threefolds by Bouchard–Klemm–Mariño–Pasquetti [13]. It was first proved in [20, 63] for $\mathbb{C}^3$, as well as in [30] as an example of the general correspondence of theorem 2.13, while the general BKMP conjecture was proved in [34].

Both the space of CohFTs and the space of KP tau-functions have an action of an infinite-dimensional group, respectively the Givental group and the Heisenberg–Virasoro group. As certain elements of these spaces have been identified by Witten–Kontsevich and Kazarian, and different integrable hierarchies have been constructed for general CohFTs by Dubrovin–Zhang [24] and Buryak [14], one may ask how general the relation is with KP specifically, and the group actions are a natural tool to study this question.

Alexandrov [2] showed that in the case of a rank-one CohFT, the orbits of the Witten–Kontsevich CohFT/tau-function under these two different group actions have an intersection which is only two-dimensional, and contains exactly the triple Hodge integrals that appear in the Mariño–Vafa formula. As a consequence, Alexandrov generalises Kazarian's result to show that the generating function of Calabi–Yau triple Hodge integrals satisfies the KP hierarchy after a linear change of variables.

Results of this paper

We give a new viewpoint on the relation found by Alexandrov, by generalising Kazarian's proof in [40] to all hypergeometric KP tau-functions satisfying topological recursion, using the above results. This yields a change of variables coming from the function X for any hypergeometric tau-function preserving the KP hierarchy after removing the unstable terms of the tau-function. When topological recursion holds, this resulting tau-function can be interpreted as the generating function of the cohomological field theory.

This new viewpoint encompasses other examples of explicit CohFTs, most notably the Chiodo classes [21], which are the CohFTs associated to completed cycles orbifold Hurwitz numbers by Zvonkine's conjecture [65], proved in [25].

In general, the change of variables contains infinite linear combinations. However, we identify when the linear combinations are actually finite, and find a finite-dimensional family for each CohFT rank. In the rank one case, this recovers exactly the triple Hodge integrals, in a particular parametrisation. For higher rank, this family seems to fit within Alexandrov's deformed generalised Kontsevich model [3].

The Calabi–Yau triple Hodge integrals are an interesting special case also, because in this case the detour through topological recursion is not necessary: the Mariño–Vafa formula which generalises the ELSV formula for this case was proved independently. Therefore, we consider this case seperately in more detail, to give an explicit exposition of the general method. We do also find that the Mariño–Vafa formula fits in the more general framework, and hence give a new proof of this formula.

Interestingly, the function X may also be a Möbius transformation. In this case, there is no unstable correction term, and this can be interpreted as certain independence of the parametrisation of the spectral curve. This also resolves the meaning behind Kazarian's change of coordinates, as voiced in [40, Remark 2.6]: "The definition for the change (6) looks unmotivated. [...] The only motivation that we can provide here is that 'it works'." There is quite a freedom of choice, but the particular choice Kazarian made reduces to the finite-dimensional family indicated above.

Open questions

Single and triple Hodge integrals have been studied intensively in relation to Dubrovin–Zhang hierarchies, yielding relations to the intermediate long wave (ILW) hierarchy and the fractional Volterra hierarchy, cf. [15, 16, 49]. The relation between those results and the current work are still unclear, and will be discussed elsewhere. Between the first and second preprint versions of this paper, Liu–Wang–Zhang [48] related the ILW hierarchy to a limit of fractional Volterra hierarchy viewed as a reduction of the 2D Toda hierarchy, possibly giving a new avenue to relating to the current paper.

The family where the linear change of variables is finite seems like an interesting and natural deformation of Witten's r-spin class, keeping a single ramification point, but splitting the pole of dx. However, this family seems mostly unknown, with the exception of Alexandrov's work mentioned above. It may be interesting to investigate it more closely, in order to better understand the deformation of higher-order zeroes of dx. Moreover, the r-spin classes themselves do not fit in the scope of this paper, although they are known to satisfy the KP hierarchy (more precisely, the r-Gelfand–Dickey hierarchy, a reduction) [61, 36]. This may be amended by the use of limit arguments, but limits of topological recursion are a delicate subject, and this falls outside the scope of the current paper.

Currently, there is a gap in the literature on limits of spectral curves, which in particular limits the validity of the proof theorem 2.7, and hence the applicability of the main theorem of this paper, to dx with simple zeroes. Future work with Borot, Bouchard, Chidambaram, and Shadrin will fix this, and will investigate more generally the applicability of limit arguments for topological recursion.

For the BKP hierarchy, similar results should hold. In particular, Alexandrov and Shadrin [4] proved an adapted topological recursion for a large class of hypergeometric BKP tau-functions, analogous to theorem 2.7. The analogous ELSV–Eynard–DOSS correspondence between this kind of topological recursion and cohomological field theories has not appeared in the literature, but the special case of completed cycles spin Hurwitz numbers is treated in work of the author with Giacchetto and Lewański [37].

Outline of the paper

Section 2 contains prerequisites. In sections 2.1 and 2.2, we give a short introduction to the Kadomtsev–Petviashvili hierarchy and its space of solutions. In section 2.3, we recall the main ideas from [40], which we will generalise. In sections 2.4 and 2.5, we recall recent results on hypergeometric tau-functions and their relations to topological recursion and cohomological field theories, and state our main theorem, which is theorem 2.18. We also introduce, in section 2.6, the generating function of triple Hodge numbers, which is the main motivating example of this paper.

In section 3, we prove the main result. Firstly, in section 3.1, we find a change of variables, for any hypergeometric tau-function, that preserves the property of being a tau-function after removal of unstable terms, corollary 3.5. In section 3.2, we restrict to the case where topological recursion holds, and use this machinery to obtain tau-functions of intersection numbers, proving our main result. We also determine, in section 3.3, the exact conditions for the change of variables to be finite, in a specific sense. Finally, in section 3.4, we return to the triple Hodge integrals, and prove an explicit version of the main theorem for this case.

Notation

We work over the field of complex numbers $\mathbb{C}$. We will use the functions $\varsigma(z) := e^{\frac{z}{2}} - e^{-\frac{z}{2}}$ and $\mathcal{S}(z) = \frac{\varsigma(z)}{z}$.

Integer partitions will be denoted by μ and ν, and we will write $|\operatorname{Aut}\mu| := \prod_{i=1}^{\mu_1} m_i(\mu)!$ and $z_\mu := \prod_{i=1}^{\mu_1} i^{m_i(\mu)} m_i(\mu)!$, where $m_i(\mu)$ is the number of parts of μ of size i. We will also consistently write $n := \ell(\mu)$, also denoted $\mu \vdash n$, and $[\![n]\!] := \{1, \dots, n\}$. We write s_ν for Schur functions and χ^ν_μ for the symmetric group character of the irreducible representation ν evaluated on cycle type μ. We identify a partition ν with its Young diagram, and write $\square \in \nu$ to show $\square$ is a box in the diagram. For such a box, $c_\square$ is its content, and $h_\square$ its hook length. We denote by $\mathcal{P}$ the set of all partitions.

For a set S, we write $M \vdash S$ to indicate that M is a *set* partition of S, i.e. a collection of non-empty, disjoint subsets with union all of S. We also use $l(M)$ for the length of M, i.e. the number of subsets.

Given a set indexed by another set, e.g. $\{z_i\}_{i\in I}$, and a subset of the second set $J \subseteq I$, we write $z_J = \{z_i\}_{i\in J}$. In particular, $z_{[\![n]\!]} = \{z_1, z_2, \dots, z_n\}$.

On the origin of this paper

An earlier version of this text, only concerning triple Hodge integrals, was written in 2018, shortly after A. Alexandrov informed me of his result. That version appeared in my PhD dissertation [45, Chapter 10]. This paper is an updated and extended version of that chapter.

2. Prerequisites on the KP hierarchy and topological recursion

In this section, we review some standard notions on the KP hierarchy and its relations to the infinite Grassmannian. We give the main outline of Kazarian's proof of KP for single Hodge integrals, which we will use as a blueprint for our results. We also recall the class of hypergeometric tau-functions, which fulfills a central role in this paper, as well as its relation to topological recursion and cohomological field theories. Finally, we recall the Mariño–Vafa formula for triple Hodge integrals and show it fits in the setup.

2.1. The KP hierarchy

The Kadomtsev–Petviashvili (KP) hierarchy is an infinite set of evolutionary differential equations in infinitely many variables. It is a very well-studied system, and some introductions into the subject can be found in [22, 41, 52].

Let $\underline{t} = \{t_i\}_{i\geq 1}$ be a set of independent variables and $\partial := \frac{\partial}{\partial t_1}$. Define the pseudo-differential operator (i.e. a Laurent series in ∂^{-1} with coefficients functions in $\underline{t}$ with composition defined formally)

$$L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \dots . \tag{1}$$

where the u_j are dependent variables in the t_i. For a pseudo-differential operator O, define O_+ to be its purely differential part, the part without powers of ∂^{-1}. The *Lax formulation of the KP hierarchy* is given by the system of equations

$$\frac{\partial L}{\partial t_i} = \left[(L^i)_+, L\right] . \tag{2}$$

This is a system of partial differential equations for the u_j, and they can be interpreted as the compatibility equations for the system

$$L\Psi = z\Psi\,, \qquad \frac{\partial \Psi}{\partial t_i} = (L^i)_+\Psi\,. \tag{3}$$

Equation (3) has a solution of the form

$$\Psi = \frac{\tau\big(\{t_k - \frac{z^{-k}}{k}\}\big)}{\tau(\{t_k\})} e^{\xi(\underline{t},z)}\,, \qquad \xi(\underline{t},z) = \sum_{k=1}^{\infty} t_k z^k\,, \tag{4}$$

which is called the *Baker–Akhiezer function*. Here τ is a single function, called a *tau-function*, dependent on the t_k and such that $\tau(0) = 1$, and all dependent variables can be expressed in terms of this one function. This way, the entire hierarchy can be rewritten as bilinear equations for τ called *Hirota equations*:

$$\operatorname*{Res}_{z=\infty} dz\, e^{\xi(\underline{t},z)-\xi(\underline{t}',z)} \tau\Big(\Big\{t_k - \frac{1}{kz^k}\Big\}\Big)\tau\Big(\Big\{t'_k + \frac{1}{kz^k}\Big\}\Big) = 0\,. \tag{5}$$

Writing $F = \log\tau$ (we call this a solution to the KP hierarchy), we find $u_1 = \partial^2 \log\tau$, and the first two equations are

$$0 = 3\frac{\partial^2 F}{\partial t_2^2} - 4\frac{\partial^2 F}{\partial t_3 \partial t_1} + \frac{\partial^4 F}{\partial t_1^4} + 6\Big(\frac{\partial^2 F}{\partial t_1^2}\Big)^2; \tag{6}$$

$$0 = 2\frac{\partial^2 F}{\partial t_3 \partial t_2} - 3\frac{\partial^2 F}{\partial t_4 \partial t_1} + \frac{\partial^4 F}{\partial t_2 \partial t_1^3} + 6\frac{\partial^2 F}{\partial t_2 \partial t_1}\frac{\partial^2 F}{\partial t_1^2}\,. \tag{7}$$

The first of these equations is the KP equation, after which the hierarchy is named.

2.2. Space of tau-functions and Lie action

The space of solutions of the KP hierarchy is an infinite-dimensional Grassmannian [57], which is Plücker embedded in a Fock space, i.e. a highest weight module of a certain Clifford algebra. The Hirota equations are then the Plücker relations defining the Grassmannian inside the Fock space. By the boson-fermion correspondence, this can also be expressed in terms of symmetric functions, which is the viewpoint we will adopt here.

Definition 2.1. We write $\Lambda \coloneqq \mathbb{C}[\![p_1, p_2, \dots]\!]$ for the *space of symmetric functions*, also called the *bosonic Fock space* (of type A). Here the p_k are power-sum functions $p_k = \sum_i X_i^k$ in some countably infinite variable set $\underline{X} = \{X_i\}$.

For other symmetric functions in $\underline{X}$, e.g. the Schur functions s_λ, we write $s_\lambda(\underline{p}) \coloneqq s_\lambda(\underline{X})$.

For more information on symmetric functions, see [50].

The space of symmetric functions has a projective action of the Lie algebra $\mathfrak{gl}(\infty)$, the algebra of infinite square matrices $(a_{ij})_{i,j\in\mathbb{Z}+\frac{1}{2}}$.[1] This space has a standard basis given by $E_{kl} = (\delta_{ik}\delta_{jkl})_{ij}$. Define the *vertex operator*

(8)
$$Z(z,w) = \frac{1}{z-w}\Bigg(\exp\Big(\sum_{j=1}^{\infty}(z^j - w^j)p_j\Big)\exp\Big(-\sum_{k=1}^{\infty}(z^{-k}-w^{-k})\frac{1}{k}\frac{\partial}{\partial p_k}\Big) - 1\Bigg).$$

Then expanding this vertex operator as

$$Z(z,w) = \sum_{i,j\in\mathbb{Z}+\frac{1}{2}} Z_{ij} z^{i+1/2} w^{-j-1/2}\,, \tag{9}$$

the assignment $E_{ij} \mapsto Z_{ij}$ is a projective representation of $\mathfrak{gl}(\infty)$, i.e. a representation of a central extension $\widehat{\mathfrak{gl}(\infty)}$.

The matrices $\alpha_k = \sum_{l\in\mathbb{Z}+\frac{1}{2}} E_{l-k,l}$ give rise to the following operators on Λ:

$$a_k \coloneqq \begin{cases} p_k & k>0 \\ -k\frac{\partial}{\partial p_{-k}} & k<0 \\ 0 & k=0 \end{cases}. \tag{10}$$

We also define the following operators:

$$L_m \coloneqq \frac{1}{2}\sum_{i=-\infty}^{\infty} {:}a_i a_{m-i}{:}\,, \tag{11}$$

where the : :, the *normal ordering*, means one should order the operators inside in order of decreasing index. All of these operators are in $\widehat{\mathfrak{gl}(\infty)}$.

Theorem 2.2 ([57])**.** *Under the identification $t_k = \frac{p_k}{k}$, the space of KP tau-functions is the orbit of $1 \in \Lambda$ under the action of $\widehat{\mathfrak{gl}(\infty)}$.*

[1] In order to make the Lie bracket well-defined, some decay condition is needed. A common choice is restriction to finitely many diagonals, but there are other options, see e.g. [57]. We will remain agnostic on this choice, as in this paper, the required convergence in guaranteed by our constructions.

An important class of KP tau-functions is given by the hypergeometric tau-functions [42, 55, 56], for which we will use the results and notation of [18].

Theorem 2.3 ([42, 55, 56])**.** *Given two formal power series*

$$\begin{aligned} \hat{\psi}(\hbar^2, y) &:= \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} c_{k,m} y^k \hbar^{2m} \,, \\ \hat{y}(\hbar^2, z) &:= \sum_{k=1}^{\infty} \hat{y}_k(\hbar^2) z^k := \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} s_{k,m} z^k \hbar^{2m} \,, \end{aligned} \tag{12}$$

define their associated hypergeometric KP tau-function *or* Orlov–Scherbin partition function

$$Z(\underline{p}) = e^{F(\underline{p})} = \sum_{\nu \in \mathcal{P}} \exp\Big(\sum_{\square \in \nu} \hat{\psi}(\hbar^2, -\hbar c_\square) \Big) s_\nu(\underline{p}) s_\nu\big(\{ \frac{\hat{y}_k(\hbar^2)}{\hbar} \}\big) \,. \tag{13}$$

This is a KP tau-function, as the name suggests.

For future reference, define also

$$\begin{aligned} &\psi(y) := \hat{\psi}(0, y) \,, \qquad && y(z) := \hat{y}(0, z) \,, \qquad && x(z) := \log z - \psi(y(z)) \,, \\ &X(z) := e^{x(z)} \,, && D := \frac{\partial}{\partial x} \,, && Q := z \frac{dx}{dz} \,. \end{aligned} \tag{14}$$

2.3. Single Hodge integrals

In [40], Kazarian considered the generating function for single Hodge integrals,

$$F_{\mathrm{H}}(u; T_0, T_1, T_2, \dots) := \sum_{g,n} \frac{1}{n!} \sum_{d_1, \dots, d_n \geq 0} \int_{\overline{\mathcal{M}}_{g,n}} \Lambda(-u^2) \prod_{i=1}^n \psi_i^{d_i} T_{d_i} \,, \tag{15}$$

and showed that its exponent, $Z_{\mathrm{H}} := \exp(F_{\mathrm{H}})$, is a tau-function for the KP hierarchy, after a certain change of coordinates. Explicitly, this change of coordinates is given as follows: define

$$D = (u + z)^2 z \frac{\partial}{\partial z} \,. \tag{16}$$

Then we define the T_d in terms of other coordinates q_k by the linear correspondence

$$q_k \leftrightarrow z^k\,, \qquad\qquad T_d \leftrightarrow D^d z\,. \tag{17}$$

The proof consists of three steps, and makes essential use of the ELSV formula [29] to transform this generating function into a generating function of Hurwitz numbers.

The first step, [40, Theorem 2.2], is the observation that the generating function for single simple Hurwitz numbers is a tau-function for the KP hierarchy. This is a well-known result, see [53]. In fact, the single simple Hurwitz generating function can be obtained from the trivial τ-function 1 by the action of two very explicit elements of the Lie group associated to $\widehat{\mathfrak{gl}(\infty)}$. The second step, [40, Theorem 2.3], uses the ELSV formula to rewrite the Hurwitz generating function (after subtracting the unstable geometries) as a generating function for single Hodge integrals. This introduces certain combinatorial factors, that suggest a certain change of coordinates, which is encoded by the equation $X(z) = \frac{z}{1+\beta z} e^{-\frac{\beta z}{1+\beta z}}$. After this change of coordinates, we obtain Z_{H}, viewed as a function in q's.

The third step, [40, Theorem 2.5] shows that a certain class of coordinate changes preserves solutions of the KP hierarchy, after they are modified with a quadratic function. In essence, this coordinate change is given infinitesimally by the flow along the differential part of an $A \in \widehat{\mathfrak{gl}(\infty)}$, whose polynomial part is exactly the added quadratic function. In this specific case, this quadratic function is exactly the $(0,2)$ part of the Hurwitz generating function.

In this paper, we will generalise this proof scheme to a more general setting. We will start from a general hypergeometric tau-function in the sense of theorem 2.7 below, corresponding to the first point of the proof.

We obtain a change of coordinates coming from this formalism that can always be completed to an automorphism of KP when correcting with the $H_{0,2}$ of equation (25), without any further assumption, corresponding to the third point of the proof.

If we restrict to the class of hypergeometric tau-functions satisfying topological recursion, we can use the correspondence between topological recursion and cohomological field theories of Eynard and Dunin-Barkowski–Orantin– Shadrin–Spitz [31, 28], which generalises the ELSV formula and hence gives the second step.

In the particular case of triple Hodge integrals, the role of the ELSV formula is taken by the Mariño–Vafa formula. For explanations on all the required notions and notation, see the following sections.

2.4. Topological recursion

By [18], two large families of hypergeometric KP tau-functions satisfy Eynard–Orantin topological recursion [32] (or its generalisation to non-simple ramification given by Bouchard–Eynard [11]), which we define first. We will confine ourselves to the case of rational spectral curves, as this is the appropriate setting for the Hurwitz-type problems covered.

Definition 2.4 ([32, 11])**.** A *rational spectral curve* is a quadruple $\mathcal{C} = (\Sigma = \mathbb{P}^1, dx, dy, B)$, where dx and dy are meromorphic one-forms on Σ with no common zeroes, only simple poles of dx, and without poles of dy at zeroes of dx, and $B = B(z_1, z_2) = \frac{dz_1\, dz_2}{(z_1-z_2)^2}$ is a symmetric $(1,1)$-form on $\Sigma \times \Sigma$. Write $R \subset \Sigma$ for the set of zeroes of dx, and r_a for the order of vanishing of dx at $a \in R$.

On a rational spectral curve, define a set of symmetric multidifferentials $\{\omega_{g,n}\}_{g\geq 0, n\geq 1}$ on Σ^n via *topological recursion* as follows: first, define the unstable cases by $\omega_{0,1} \coloneqq y dx$ (this need only be defined locally near the a_i using any primitive y of dy) and $\omega_{0,2} \coloneqq B$. Then, for $2g-2+(n+1) > 0$, the stable range, define

(18)
$$\omega_{g,n+1}(z_{[\![n]\!]}, z_{n+1}) \coloneqq \sum_{a\in R} \sum_{\{0\}\subsetneq I \subset \{0,\dots,r_a-1\}} \operatorname*{Res}_{z=a} \frac{\int_a^z \omega_{0,2}(\cdot, z_{n+1})}{\prod\limits_{i\in I\setminus\{0\}} \big(\omega_{0,1}(z) - \omega_{0,1}(\sigma_a^i(z))\big)} \cdot \mathcal{W}_{g,|I|+1,n}(\sigma_a^I(z); z_{[\![n]\!]}),$$

where σ_a is a generator of the local deck transformations of a primitive of dx at a, σ_a^I is the set $\{(\sigma_a)^i \,|\, i \in I\}$, and

$$\mathcal{W}_{g,m,n}(\zeta_{[\![m]\!]}; z_{[\![n]\!]}) \coloneqq \sideset{}{'}\sum_{\substack{M \vdash [\![m]\!] \\ \bigsqcup_{k=1}^{l(M)} N_k = [\![n]\!] \\ \sum g_k = g + l(M) - n}} \prod_{k=1}^{l(M)} \omega_{g_k, |M_k|+|N_k|}(\zeta_{M_k}, z_{N_k}) \tag{19}$$

where the prime on the summation means exclusion of any term with $(g_k, |M_k| + |N_k|) = (0,1)$ for some k.

Remark 2.5. Often, the definition of spectral curves involves functions x and y, instead of their derivatives. However, these functions may not be defined globally on Σ, e.g. they may – and in this paper will – contain logarithmic terms. As most of the theory of topological recursion (with the notable exception of the global topological recursion of Bouchard–Eynard [11]) only depends on the derivatives, I have chosen to use this as a definition.

Theorem 2.6 ([6, 9])**.** *Let $\mathcal{C}$ be a rational spectral curve with simple zeroes of dx. A collection of symmetric meromorphic multidifferentials on Σ^n, $\{\omega_{g,n}\}_{g\geq 0, n\geq 1}$, with $\omega_{0,1} = ydx$ and $\omega_{0,2} = B$ satisfies topological recursion if and only if the following hold:*

- Meromorphicity: *For $2g-2+n>0$, $\omega_{g,n}$ extends to a meromorphic form on Σ^n;*
- Linear loop equation: *For any g, n, and $a \in R$,*

$$\omega_{g,n+1}(z, z_{[\![n]\!]}) + \omega_{g,n+1}(\sigma_a(z), z_{[\![n]\!]}) \tag{20}$$

  is holomorphic near $z = a$ and has a simple zero at $z = a$;
- Quadratic loop equation: *For any g, n, and $a \in R$,*

$$\omega_{g-1,n+2}(z, \sigma_a(z), z_{[\![n]\!]}) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=[\![n]\!]}} \omega_{g_1,|I|+1}(z, z_I)\omega_{g_2,|J|+1}(\sigma_a(z), z_J) \tag{21}$$

  is holomorphic near $z = a$ and has a double zero at $z = a$;
- Projection property: *For $2g-2+n>0$,*

$$\omega_{g,n}(z_{[\![n]\!]}) = \sum_{a_1,\dots,a_n\in R} \Big(\prod_{j=1}^{n} \operatorname*{Res}_{\zeta_j=a_j} \int_{w=a_j}^{\zeta_j} \omega_{0,2}(z_j, w)\Big)\omega_{g,n}(\zeta_{[\![n]\!]}) \,. \tag{22}$$

If only the meromorphicity and linear and quadratic loop equations hold, the problem is said to satisfy blobbed topological recursion, cf. [9]. In this case, the $\omega_{g,n}$ are determined by the spectral curve along with their holomorphic parts at ramification points.

One important reason to consider topological recursion is that the $\omega_{g,n}$ will often encode enumerative invariants in their Taylor series expansion around a given point of the spectral curve in a given coordinate. For us, this is also the case, as we consider the class given by the following theorem:

Theorem 2.7 ([18, 17])**.** *In the situation of theorem 2.3, write*

$$H_n(X_1, \dots, X_n) \coloneqq \sum_{k_1,\dots,k_n=1}^{\infty} \frac{\partial^n F}{\partial p_{k_1}\cdots\partial p_{k_n}}\bigg|_{p=0} X_1^{k_1}\cdots X_n^{k_n} \,. \tag{23}$$

1. *The H_n can be decomposed as*

$$H_n = \sum_{g=0}^{\infty} \hbar^{2g-2+n} H_{g,n}\,, \tag{24}$$

with $H_{g,n}$ independent of $\hbar$. Additionally, interpreting X_i as $X(z_i)$,

$$DH_{0,1}(X(z_1)) = y(z_1)\,, \qquad H_{0,2}(X(z_1),X(z_2)) = \log\Big(\frac{z_1^{-1}-z_2^{-1}}{X_1^{-1}-X_2^{-1}}\Big)\,. \tag{25}$$

2. *If moreover $\frac{d\psi(y)}{dy}\big|_{y=y(z)}$ and $\frac{dy(z)}{dz}$ have analytic continuations to meromorphic functions in z and all coefficients of positive powers of $\hbar^2$ in $\hat{\psi}(\hbar^2,y(z))$ and $\hat{y}(\hbar^2,z)$ are rational functions of z whose singular points are disjoint from the zeroes of dx, then the n-point differentials*

$$\omega_{g,n}(z_{[\![n]\!]}) \coloneqq d_1\cdots d_n H_{g,n}\big(\{X(z_i)\}_{i=1}^n\big) + \delta_{g,0}\delta_{n,2}\frac{dX(z_1)\,dX(z_2)}{(X(z_1)-X(z_2))^2}\,, \tag{26}$$

where d_i is the exterior derivative in the ith variable, can be extended analytically to $(\mathbb{P}^1)^n$ as global rational forms, and the collection of n-point differentials satisfies the linear and quadratic loop equations, i.e. blobbed topological recursion, for the curve $(\mathbb{P}^1, dx(z), dy(z), B = \frac{dz_1\,dz_2}{(z_1-z_2)^2})$.

3. *Finally, if $\hat{\psi}$ and $\hat{y}$ belong to one of the two families*

Family I $\quad \hat{\psi}(\hbar^2,y)=\mathcal{S}(\hbar\partial_y)P_1(y)+\log\Big(\frac{P_2(y)}{P_3(y)}\Big)\,; \quad \hat{y}(\hbar^2,z)=\frac{R_1(z)}{R_2(z)}\,,$

Family II $\quad \hat{\psi}(\hbar^2,y)=\alpha y\,; \quad \hat{y}(\hbar^2,z)=\frac{R_1(z)}{R_2(z)}+\mathcal{S}(\hbar z\partial_z)^{-1}\log\Big(\frac{R_3(z)}{R_4(z)}\Big)\,,$

where $\alpha\in\mathbb{C}^\times$ and the P_i and R_j are arbitrary polynomials such that $\psi(y)$ and $y(z)$ are non-zero, but vanishing at zero, and no singular points of y are mapped to branch points by x, then the n-point differentials also satisfy the projection property, and hence topological recursion, for the curve above.

Remark 2.8. It is possible to allow for constant terms in $\hat{\psi}$ in equation (12), but using quasihomogeneity of the s_ν in equation (13), this can be absorbed in a rescaling of the argument of $\hat{y}$. From the spectral curve point of view, this follows from the fact that the two curves

$$
\begin{cases} X_1(z) &= ze^{-\psi\circ y(z)+\log a} = aze^{-\psi\circ y(z)} \\ y_1(z) &= y(z) \end{cases} \quad \text{and} \quad \begin{cases} X_2(z') &= z'e^{-\psi\circ y(\frac{z'}{a})} \\ y_2(z') &= y(\frac{z'}{a}) \end{cases} \tag{27}
$$

can be identified via $z' = az$. For Family I to make sense in the generality stated in the theorem, constant terms in z of $\hat{\psi}$ should be interpreted this way.

Remark 2.9. We will consistently use the symbol x for the function which is part of the spectral curve data and X for its exponential, which is the expansion parameter for this class of Hurwitz problems.

There is one small generalisation that can be made in theorem 2.7 by using a homogeneity property, using that equation (13) is a KP tau-function identically in $\hbar$.

Lemma 2.10. *There is a $\mathbb{C}^\times$ action on pairs $(\hat{\psi}, \hat{y})$ induced by rescaling of $\hbar$ in equation (13) as follows:*

$$
\lambda \cdot \big(\hat{\psi}(\hbar^2, y), \hat{y}(\hbar^2, z)\big) = \big(\hat{\psi}(\lambda^{-2}\hbar^2, \lambda^{-1}y), \lambda\hat{y}(\lambda^{-2}\hbar^2, z)\big)\,. \tag{28}
$$

This acts on $H_{g,n}$ as $\lambda \cdot H_{g,n} = \lambda^{2-2g-n}H_{g,n}$ and on the spectral curve by

$$
\lambda \cdot (\mathbb{P}^1, dx(z), dy(z), B) = (\mathbb{P}^1, dx(z), \lambda dy(z), B)\,. \tag{29}
$$

Hence it is compatible with the homogeneity of topological recursion of e.g. [33, Section 4.1].

Corollary 2.11. *Family II of theorem 2.7 can be extended to*

$$
\hat{\psi}(\hbar^2, y) = \alpha y\,; \qquad \hat{y}(\hbar^2, z) = \frac{R_1(z)}{R_2(z)} + \lambda\mathcal{S}(\lambda^{-1}\hbar z\partial_z)^{-1}\log\Big(\frac{R_3(z)}{R_4(z)}\Big)\,, \tag{30}
$$

with the same conditions on α, the P_i, and the R_j as before, and $\lambda \in \mathbb{C}^\times$.

Proof. The constant λ has been absorbed in α and the R_i where possible. $\square$

A similar argument for Family I does not lead to an extended class, as this family is already invariant under the torus action.

2.5. Topological recursion and cohomological field theories

Topological recursion is strongly related to intersection theory of the moduli spaces of curves: there is a quite general correspondence between spectral curves and certain coherent collections of intersection classes in the moduli spaces. These coherent collections are cohomological field theories, which were originally defined by Kontsevich–Manin [44] to axiomatise Gromov–Witten theory.

Definition 2.12 ([44])**.** Let V be a vector space with a non-degenerate, symmetric bilinear form η and a distinguished vector $\mathbb{1}$. A *cohomological field theory with flat unit (CohFT)* on $(V,\eta,\mathbb{1})$ is a collection of maps

$$\Omega_{g,n}\colon V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})\,, \tag{31}$$

for all $g,n \geq 0$, such that $2g-2+n>0$, such that

- $\Omega_{g,n}$ is $\mathfrak{S}_n$-equivariant with respect to simultaneous permutation of the factors and the marked points;
- with respect to the glueing maps
  (32)
  $$\rho\colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}\,, \qquad \sigma\colon \overline{\mathcal{M}}_{g,|I|+1} \times \overline{\mathcal{M}}_{h,|J|+1} \to \overline{\mathcal{M}}_{g+h,|I\sqcup J|}\,,$$
  we get
  $$\rho^*\Omega_{g,n}\Big(\bigotimes_{k=1}^n v_k\Big) = \Omega_{g-1,n+2}\Big(\bigotimes_{k=1}^n v_k \otimes \eta^\dagger\Big)\,,$$
  $$\sigma^*\Omega_{g+h,|I|+|J|}\Big(\bigotimes_{k\in I\cup J} v_k\Big) = \Omega_{g,|I|+1}\otimes\Omega_{h,|J|+1}\Big(\bigotimes_{i\in I} v_i \otimes \eta^\dagger \otimes \bigotimes_{j\in J} v_j\Big)\,,$$
  where $\eta^\dagger \in V\otimes V$ is the bivector dual to η;
- With respect to the forgetful maps
  $$\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}\,, \tag{33}$$
  we have
  (34)
  $$\eta(v_1,v_2) = \Omega_{0,3}(v_1\otimes v_2\otimes\mathbb{1})\,, \qquad \pi^*\Omega_{g,n}\Big(\bigotimes_{k=1}^n v_k\Big) = \Omega_{g,n+1}\Big(\bigotimes_{k=1}^n v_k\otimes\mathbb{1}\Big)\,.$$

There is a large group acting on the space of CohFTs, called the Givental group [38, 58, 59]. It consists of $R(u) \in \mathrm{Id} + u\,\mathrm{End}(V)[\![u]\!]$ such that $R(u)R^\dagger(-u) = \mathrm{Id}$. It is called the unit-preserving action in case one also considers CohFTs without unit.

A CohFT whose image lies withing $H^0(\overline{\mathcal{M}}_{g,n})$ for all g, n – i.e. the classes are just numbers – is a topological field theory (TFT). TFTs are determined by $(V, \eta, \mathbb{1})$, and $\Omega_{0,3}$, as all other classes can be recovered from $\Omega_{0,3}$ via the glueing axiom in this case. The data of a TFT is equivalent to a commutative Frobenius algebra, with the multiplication determined by $\eta(u, v \cdot w) = \Omega_{0,3}(u, v, w)$.

Any CohFT has an underlying TFT (and hence Frobenius algebra), given by taking the degree zero part of all classes on $\overline{\mathcal{M}}_{g,n}$. By Teleman's classification [59], any CohFT with semisimple underlying Frobenius algebra can be reconstructed from its degree zero part via some element of the Givental group.

Theorem 2.13 ([31, 28, 8])**.** *Consider a rational spectral curve $(\mathbb{P}^1, dx, dy, B)$, and define V^* to be the space of residueless meromorphic one-forms on $\mathbb{P}^1$ with poles only at $a \in R$ of order at most $r_a + 1$. Choose a basis $\{d\xi^j\}_{j\in J}$ of V^* with dual basis e_j and define $d\xi^j_k = (d \circ \frac{1}{dx})^k d\xi^j$. Then*

(35)
$$\omega_{g,n}(z_1, \dots, z_n) = \sum_{j_1,\dots,j_n \in J} \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(e_{j_1} \otimes \cdots \otimes e_{j_n}) \prod_{i=1}^{n} \sum_{k_i=0}^{\infty} \psi_i^{k_i} d\xi^{j_i}_{k_i}(z_i)\,,$$

where Ω is a cohomological field theory on V, given explicitly by acting on a direct sum of Witten r_a-spin classes for all ramification points of order r_a by a Givental group element determined by the spectral curve.

Remark 2.14. The results of [31, 8] do not mention CohFTs, but rather give a relation between local spectral curves and intersection numbers. In order to obtain a CohFT, a condition is required, cf. [46, Equation (17)]. As noted in [27, Section 2.6], in case the spectral curve is compact and dx is meromorphic with simple zeroes, this condition is satisfied by [31, Appendix B]. In case of higher order zeroes, the same holds, using [8, Section 7.2.3].

In this formalism, the CohFT has a semisimple underlying Frobenius algebra if and only if x has only simple zeroes, because Witten's r-spin class is only semisimple for $r = 2$.

Remark 2.15. The space V^* is naturally related to the projection property of theorem 2.6: the $d\xi^j_k$ span the image of the projection operator. Its dimension, the rank of the CohFT, equals the degree of the divisor of zeroes of dx.

There are two common choices for the basis $d\xi^j$, depending on a local coordinate ζ_a around a ramification point a such that $x(z) = \zeta_a(z)^{r_a} + x(a)$. One is $d\xi^{a,k}(z) = \mathrm{Res}_{z'=a}\Big(\int_a^{z'} B(z,\cdot)\Big)\frac{d\zeta(z')}{\zeta(z')^k}$, with $1 \leq k \leq r_a - 1$, cf. [8, Equation (80)], while the other is $\xi^a(z) = \int^z \frac{B(\zeta_a,\cdot)}{d\zeta_a}\big|_{\zeta_a=0}$, in case $r_a = 2$, cf. [37, Equation (2.23)]. Both have merit, depending on the situation, but they are not compatible.

Furthermore, several normalisation conventions exist for the recursion operator linking $d\xi^j_k$ to $d\xi^j_{k+1}$. These different conventions can be related by rescaling Ω and the correlators, using that the integrand must be of degree $3g - 3 + n$.

So the $\omega_{g,n}$ we are concerned with can be expanded in different ways: as a formal series around $X = 0$ by theorem 2.7, and on a basis of meromorphic differentials with poles at the zeroes of dx by theorem 2.13. The change of variables we require is found by relating these different expansions.

In order to apply the Eynard–DOSS correspondence to get a good change of variables, we will want to take a different basis of V^*. It turns out to be useful to relate to powers of our preferred coordinate z, so the basis we take is $\xi^j \coloneqq \big(\frac{dx}{dz}\big)^{-1} z^j = \frac{d}{dx}\frac{z^{j+1}}{j+1}$.

By concatenating theorems 2.7 and 2.13, we obtain a class of CohFTs which are the ones we will be interested in in this paper.

Definition 2.16. The CohFTs obtained from theorem 2.13 applied to either family in theorem 2.7 are called *Hurwitz-type.*

In practise, giving an explicit form for the CohFT associated to a spectral curve via theorem 2.13 is very complicated, due to the nature of the Givental group action. But there are several cases of Hurwitz problems for which the CohFT is known. Notable examples are the Chiodo classes [21], which by Zvonkine's conjecture [65], proved in [25], are the CohFTs associated to completed cycles orbifold Hurwitz numbers. These lie in Family I, with $\psi(y) = z^r$ and $y(z) = z^q$. Two other examples (both in Family I) are weakly monotone Hurwitz numbers, $\psi(y) = -\log(1 - y)$ and $y(z) = z$, proved in [23], and strictly monotone Hurwitz numbers with even ramification, $\psi(y) = \log(1 + y)$ and $y(z) = z^2$, proved in [7]. In these two cases, the CohFTs are expressed in terms of κ and λ classes and boundary strata.

Definition 2.17. Let Ω be a cohomological field theory on a space (V, η)

with a basis $\{e_j\}_{j\in J}$. Its *generating function* G_Ω is defined as

(36)
$$G_\Omega(\{T^j_k\}_{\substack{k\geq 0\\ j\in J}}) := \sum_{\substack{g,n\\ 2g-2+n>0}} \frac{\hbar^{2g-2+n}}{n!} \sum_{j_i,\dots,j_n\in J} \int_{\overline{\mathcal{M}}_{g,n}} \Omega(e_{j_1}\otimes\cdots\otimes e_{j_n}) \prod_{i=1}^{n} \sum_{k_i=1}^{\infty} \psi_i^{k_i} T^{j_i}_{k_i}\,,$$

where we write $\{T^j\}$ for the basis of V^* dual to $\{e_j\}$ and T^j_k are associated descendent variables.

The main theorem of this paper is the following:

Theorem 2.18. *If Ω is a Hurwitz-type CohFT, then the exponential of $G_\Omega(\underline{T}(\underline{q}))$ is a KP tau-function in $\{t_d = \frac{q_d}{d}\}$, where the $T^j_k(\underline{q})$ are defined by*

$$T^j_{-1} = \frac{1}{j+1} q_{j+1}\,, \qquad T^j_{k+1} = \sum_{m=1}^{\infty}\sum_{l=0}^{\infty} m\mathcal{T}_l q_{m+l} \frac{\partial}{\partial q_m} T^j_k\,, \tag{37}$$

with $\mathcal{T}_l$ given by

$$Q(z)^{-1} = \sum_{l=0}^{\infty} \mathcal{T}_l z^l\,. \tag{38}$$

Note that we really obtain a tau-function of a single copy of KP, even in case of several ramification points. The proof of this theorem is given in proposition 3.8. The proof relies heavily on theorem 2.7: we will need different parts of that theorem for the different parts of the proof. In particular, topological recursion is needed to obtain intersection numbers.

Remark 2.19. The proof of this theorem does not use anything specific to the families mentioned, it just requires topological recursion to obtain a cohomological field theory. As soon as topological recursion is proved for another hypergeometric tau-function and the spectral curve fits in the scope of theorem 2.13, this theorem generalises.

The first two KP equations in q variables are

$$0 = \frac{\partial^2 F}{\partial q_2^2} - \frac{\partial^2 F}{\partial q_3 \partial q_1} + \frac{1}{12}\frac{\partial^4 F}{\partial q_1^4} + \frac{1}{2}\Big(\frac{\partial^2 F}{\partial q_1^2}\Big)^2\,; \tag{39}$$

$$0 = \frac{\partial^2 F}{\partial q_3 \partial q_2} - \frac{\partial^2 F}{\partial q_4 \partial q_1} + \frac{1}{6}\frac{\partial^4 F}{\partial q_2 \partial q_1^3} + \frac{\partial^2 F}{\partial q_2 \partial q_1}\frac{\partial^2 F}{\partial q_1^2}\,. \tag{40}$$

Example 2.20 (Naive single Hodge)**.** Let us consider the functions

$$\hat{\psi}(\hbar^2, y) = y\,, \qquad \hat{y}(\hbar^2, z) = z\,. \tag{41}$$

Then we find

$$x(z) = \log z - z\,, \qquad X(z) = z\,e^{-z}\,, \qquad Q = 1 - z\,. \tag{42}$$

This is the usual shape of spectral curve for simple Hurwitz numbers [12, 10], so the CohFT associated to it by theorem 2.13 is the single Hodge class $\Lambda(-1)$ via the ELSV formula [29]. This is a one-dimensional CohFT, so we drop all indices for the bases of V and V^*. In this case, writing $T_k = \sum_{m=1}^{\infty} c_{k,m} q_m$, theorem 2.18 yields

$$c_{k+1,m} = \sum_{j=0}^{m} j c_{k,j}\,. \tag{43}$$

Along with the initial condition $c_{-1,m} = \delta_{m,1}$, this shows that $c_{k,m} = \left\{ {k+m \atop m} \right\}$ for $m > -1$, the Stirling numbers of the second kind. In particular,

$$\begin{aligned} T_0 &= q_1 + q_2 + q_3 + q_4 + q_5 + \dots\,, \\ T_1 &= q_1 + 3q_2 + 6q_3 + 10q_4 + 15q_5 + \dots\,, \\ T_2 &= q_1 + 7q_2 + 25q_3 + 65q_4 + 140q_5 + \dots \end{aligned} \tag{44}$$

Note that these are infinite sums, in contrast to the ones Kazarian found in [40], cf. equations (16) and (17), even though both are related to single Hodge integrals. This phenomenon is explained by the arbitrary choice of a rational parametrisation of the spectral curve, formalised in corollary 3.6.

Using the intersection numbers

$$\int_{\bar{\mathcal{M}}_{0,3}} 1 = \int_{\bar{\mathcal{M}}_{0,4}} \psi_i = 1\,, \tag{45}$$

$$\int_{\bar{\mathcal{M}}_{1,1}} \lambda_1 = \int_{\bar{\mathcal{M}}_{1,1}} \psi_1 = \int_{\bar{\mathcal{M}}_{1,2}} \psi_i^2 = \int_{\bar{\mathcal{M}}_{1,2}} \psi_1\psi_2 = \int_{\bar{\mathcal{M}}_{1,2}} \lambda_1\psi_1 = \frac{1}{24}\,, \tag{46}$$

we see that

$$\begin{aligned} G_{\Lambda(-1)}(\underline{T}) = \hbar\Big(\frac{1}{6}T_0^3 + \frac{1}{24}T_1 - \frac{1}{24}T_0\Big) \\ + \hbar^2\Big(\frac{1}{6}T_0^3 T_1 + \frac{1}{48}T_1^2 + \frac{1}{24}T_0T_2 - \frac{1}{24}T_0T_1\Big) + \mathcal{O}(\hbar^3)\,. \end{aligned} \tag{47}$$

From this, we obtain

$$\begin{aligned}
\frac{\partial^2 G_{\Lambda(-1)}(\underline{T}(\underline{q}))}{\partial q_2^2} &= \hbar \sum_{k>0} q_k \\
&\quad + \hbar^2 \Big(\sum_{k,l>0} \Big(\frac{l(l+1)}{2} + 3 \Big) q_k q_l + \frac{2 \cdot 3^2}{48} + \frac{2 \cdot 7}{24} - \frac{2 \cdot 3}{24} \Big) \\
&\quad + \mathcal{O}(\hbar^3)\,, \\
\frac{\partial^2 G_{\Lambda(-1)}(\underline{T}(\underline{q}))}{\partial q_1 q_3} &= \hbar \sum_{k>0} q_k \\
&\quad + \hbar^2 \Big(\sum_{k,l>0} \Big(\frac{l(l+1)}{2} + \frac{7}{2} \Big) q_k q_l + \frac{2 \cdot 6}{48} + \frac{25+1}{24} - \frac{6+1}{24} \Big) \\
&\quad + \mathcal{O}(\hbar^3)\,, \\
\frac{\partial^4 G_{\Lambda(-1)}(\underline{T}(\underline{q}))}{\partial q_1^4} &= \hbar^2 \frac{24}{6} + \mathcal{O}(\hbar^3)\,, \\
\frac{\partial^2 G_{\Lambda(-1)}(\underline{T}(\underline{q}))}{\partial q_1^2} &= \hbar \sum_{k>0} q_k + \mathcal{O}(\hbar^2)\,,
\end{aligned}$$

which does show that $G_{\Lambda(-1)}(\underline{T}(\underline{q}))$ solves equation (39) up to second order in $\hbar$.

I would like to thank P. Norbury for using this example to check the results of this paper.

2.6. The Mariño–Vafa formula and KP for topological vertex amplitudes

A particularly interesting family of hypergeometric tau-functions is given by the theory of the topological vertex, or triple Hodge integrals. For the triple Hodge integrals, the ELSV-type formula required is the Mariño–Vafa formula [51]. Because this formula was proved without using topological recursion [54, 47] (in fact before topological recursion was first formulated), we can use it directly without going through the detour of topological recursion.

On the other hand, this theory is the particular case for $\mathbb{C}^3$ of the Gromov–Witten theory of toric Calabi–Yau threefolds, which was conjectured by Bouchard–Klemm–Mariño–Pasquetti [13] to satisfy topological recursion. The case we are interested in was proved in [20, 63], as well as in [30] as an example of the general correspondence of theorem 2.13, while the general BKMP conjecture was proved in [34]. Combining this with theorem 2.7, we actually obtain a new proof of the Mariño–Vafa formula.

In this section, we use the triple Hodge integrals as an example of our general theory, using methods slightly adapted to this special case. We will see in section 3.3 why this case is particularly nice.

Definition 2.21. The *triple Hodge cohomological field theory with Calabi–Yau condition* is the one-dimensional CohFT $\mathrm{TH}_{g,n}(a,b,c) = \Lambda(a)\Lambda(b)\Lambda(c)$, where the parameters a, b, c satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$.

We write

$$G_{\mathrm{TH}}(a,b,c;\underline{T}) \coloneqq G_{\mathrm{TH}(a,b,c)}(\underline{T})\,. \tag{48}$$

An adapted application of theorem 2.18 is given in the following theorem. This theorem has already been proved by Alexandrov [2], here we give a new proof.

Theorem 2.22 ([2, Theorem 2])**.** *Define* $T_0(\underline{q}) \coloneqq q_1$, $T_{k+1}(\underline{q}) \coloneqq \sum_{m=1}^\infty m(u^2 q_m + u\frac{w+2}{\sqrt{w+1}} q_{m+1} + q_{m+2})\frac{\partial}{\partial q_m} T_k$. *Then*

$$G_{\mathrm{TH}}\Big(-u^2, -u^2 w, \frac{u^2 w}{w+1}; \{T_k(\underline{q})\}\Big) \tag{49}$$

is a solution of the KP hierarchy with respect to the variables $\{t_d = \frac{q_d}{d}\}$*, identically in* u *and* w*.*

In this particular case, we will make slightly different choices to end up with the formulation above.

Remark 2.23. Note that the triple $a = -u^2$, $b = -u^2 w$, $c = \frac{u^2 w}{w+1}$ does indeed satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$, and moreover any triple satisfying this condition can be written this way.

Remark 2.24. In the limit $w \to 0$, this theorem reduces to the main theorem, 2.1, of [40]. In the limit $u \to 0$, it reduces to the Witten–Kontsevich theorem [60, 43]: in that limit $T_d \to (2d-1)!!q_{2d+1}$ and independence of even parameters reduces the KP hierarchy to the KdV hierarchy.

Before giving the Mariño–Vafa formula, note that in genus zero

$$\int_{\overline{\mathcal{M}}_{0,n}} \frac{\Lambda(a)\Lambda(b)\Lambda(c)}{\prod_{i=1}^n 1 - \mu_i \psi_i^{d_i}} = |\mu|^{n-3} \tag{50}$$

for $n \geq 3$, and this serves as a definition for $n = 1, 2$. These terms are not included in G_{TH}.

Theorem 2.25 (Mariño–Vafa formula, [51, 47, 54]). *There is a relation between triple Hodge integrals and characters of symmetric groups, as follows:*

$$\begin{aligned} \exp\Bigg(\sum_{\substack{\mu\in\mathcal{P}\\ g\geq 0}} \frac{(w+1)^{g+n-1}}{|\operatorname{Aut}\mu|}\prod_{i=1}^{n}\frac{\prod_{j=1}^{\mu_i-1}(\mu_i+jw)}{(\mu_i-1)!} \\ \cdot\int_{\overline{\mathcal{M}}_{g,n}}\frac{\Lambda(-1)\Lambda(-w)\Lambda\big(\frac{w}{w+1}\big)}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}\,\beta^{2g-2+n+|\mu|}\,p_\mu\Bigg) \\ = \sum_{m=0}^{\infty}\sum_{\mu,\nu\vdash m}\frac{\chi^\nu_\mu}{z_\mu}e^{(1+\frac{w}{2})\beta f_2(\nu)}\prod_{\square\in\nu}\frac{\beta w}{\varsigma(\beta w h_\square)}p_\mu\,. \end{aligned} \tag{51}$$

On the right-hand side the sum is over all partitions μ, ν of equal size m, and $f_2(\nu) = \frac{1}{2}\sum_j(\nu_j - j + \frac{1}{2})^2 - (-j + \frac{1}{2})^2$ is the shifted symmetric sum of squares.

Remark 2.26. Even though it seems the triple Hodge class in this formula only depends on one parameter, w, the parameter β can be interpreted in this way as well, entering as a cohomological grading parameter. Hence, the formula does govern the entire generating function of triple Hodge integrals.

In the limit $w \to 0$, the Mariño–Vafa formula reduces to the ELSV formula, as the product over boxes simplifies to the hook length formula for the dimension of the $\mathfrak{S}_{|\mu|}$-representation associated to ν.

Remark 2.27. This formula is perfectly well-behaved for $w = -1$, but theorem 2.22 does not make sense in this case. From the general theorem 2.18, we will see that in this case X is a Möbius transformation, and hence conforms to corollary 3.6.

By symmetry in the arguments of the Λ-classes, the point $w = -1$ is equivalent to the limit $w \to \infty$, which in the conventional formulation of the Mariño–Vafa formula is the initial condition for the cut-and-join equation used to prove the formula, see [62, Theorem 3.3]. In this case, the integral reduces to $\int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \psi^{2g-2}$ by Mumford's relation. These integrals were calculated by Faber–Pandharipande [35].

The right-hand side of the Mariño–Vafa formula is a hypergeometric KP tau-function, which can be seen explicitly by the following lemma. In essence this lemma was used by both [47, 54] to prove the Mariño–Vafa formula.

Lemma 2.28. *Relabel parameters in equation* (51) *by* $\beta = \frac{\hbar}{w}$ *and* $p_k = \left(\frac{\gamma w}{\hbar}\right)^k r_k$ *to obtain*

$$\exp\Bigg(\sum_\mu \sum_{g=0}^\infty \frac{(w+1)^{g+n-1}}{|\operatorname{Aut}\mu|} \prod_{i=1}^n \frac{\prod_{j=1}^{\mu_i-1}(\mu_i + jw)}{(\mu_i-1)!} \cdot \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda(-1)\Lambda(-w)\Lambda\left(\frac{w}{w+1}\right)}{\prod_{i=1}^n(1-\mu_i\psi_i)} \hbar^{2g-2+n}\gamma^{|\mu|}w^{|\mu|+2-2g-n} r_\mu\Bigg) = \sum_{m=0}^\infty \sum_{\mu,\nu\vdash m} \frac{\chi_\mu^\nu}{z_\mu} e^{(\frac{1}{w}+\frac{1}{2})\hbar f_2(\nu)} \prod_{\square\in\nu} \frac{\gamma w}{\varsigma(\hbar h_\square)} r_\mu . \tag{52}$$

This right-hand side (and hence also the left-hand side) is equal to a hypergeometric KP tau-function $Z(\underline{r})$ *of equation* (13), *with*

$$\hat\psi(\hbar^2,y) = -\frac{y}{w}, \qquad \hat y(\hbar^2,z) = \sum_{k=1}^\infty \frac{1}{k\mathcal{S}(\hbar k)}(\gamma w z)^k, \qquad X(z) = z(1-\gamma wz)^{1/w}. \tag{53}$$

This pair $(\hat\psi,\hat y)$ *is an element of Family II, with* $\alpha = -\frac{1}{w}$, $R_1(z) = R_2(z) = R_4(z) = 1$, $R_3(z) = \gamma wz$, *and hence, by theorem* 2.7, *topological recursion holds for this case.*

Proof. By basic theory of symmetric functions, $\sum_{\mu\vdash m}\frac{\chi_\mu^\nu}{z_\mu}p_\mu = s_\nu(\underline{p})$. Also, by [54, Equations (0.6), (0.7)],

$$\frac{1}{\prod_{\square\in\nu} q^{h_\square/2} - q^{-h_\square/2}} = q^{-|\nu|/2 - f_2(\nu)/2} s_\nu(1, q^{-1}, q^{-2}, \dots)$$

where here the q^{-k} are the 'usual' variables, i.e. the ones in which s_ν is symmetric, not the power sum variables.

Writing $q = e^\hbar$ and using that $f_2(\nu) = \sum_{\square\in\nu} c_\square$ gives

$$\sum_{m=0}^\infty \sum_{\mu,\nu\vdash m} \frac{\chi_\mu^\nu}{z_\mu} e^{(\frac{1}{w}+\frac{1}{2})\hbar f_2(\nu)} \prod_{\square\in\nu} \frac{\gamma w}{\varsigma(\hbar h_\square)} r_\mu = \sum_{m=0}^\infty \sum_{\nu\vdash m} s_\nu(\underline{r}) e^{\sum_{\square\in\nu}\frac{\hbar}{w}c_\square} s_\nu\left(\left\{\frac{\gamma w}{e^{\hbar(l+\frac{1}{2})}}\right\}_{l=0}^\infty\right).$$

Comparing with theorem 2.7, we immediately find $\hat\psi(\hbar^2,y) = -\frac{y}{w}$. To find $\hat y$, we must revert to power-sum variables in the Schur functions. We use that

$$
\begin{aligned}
p_k\Big(\Big\{\frac{\gamma w}{e^{\hbar(l+\frac{1}{2})}}\Big\}_{l=0}^{\infty}\Big) &= \sum_{l=0}^{\infty}\Big(\frac{\gamma w}{e^{\hbar(l+\frac{1}{2})}}\Big)^k \\
&= (\gamma w)^k e^{-\frac{\hbar k}{2}} \sum_{l=0}^{\infty} e^{-\hbar k l} \\
&= \frac{(\gamma w)^k}{e^{\frac{\hbar k}{2}}}\frac{1}{1-e^{-\hbar k}} \\
&= \frac{(\gamma w)^k}{\varsigma(\hbar k)}\,.
\end{aligned}
$$

By equation (13), $\hat{y}_k = \hbar p_k$, as the p_k are the power-sum arguments of the Schur functions, so

$$
\hat{y}(\hbar^2, z) = \sum_{k=1}^{\infty} \hat{y}_k z^k = \sum_{k=1}^{\infty} \frac{(\gamma w)^k}{\varsigma(\hbar k)}\hbar z^k = \sum_{k=1}^{\infty} \frac{1}{k\mathcal{S}(\hbar k)}(\gamma w z)^k\,,
$$

which yields the value of $\hat{y}$. Finally,

$$
\begin{aligned}
X(z) &= e^{x(z)} \\
&= z e^{-\hat{\psi}(0,\hat{y}(0,z))} \\
&= z \exp\Big(\frac{1}{w}\sum_{k=1}^{\infty}\frac{1}{k}(\gamma w z)^k\Big) \\
&= z \exp\Big(\frac{\log(1-\gamma w z)}{w}\Big) \\
&= z(1-\gamma w z)^{\frac{1}{w}}\,.
\end{aligned}
$$

□

If one were to relabel instead by $\beta = \tilde{\gamma}\hbar$ and $p_k = \hbar^{-k}\tilde{r_k}$, i.e. just to naively introduce a parameter $\hbar$, one would not end up in Family II, but in the extension of corollary 2.11. These two choices are related by lemma 2.10, for $\lambda = \gamma w$.

Corollary 2.29. *Topological recursion yields a new proof of the Mariño–Vafa formula.*

Proof. Topological recursion for these triple Hodge integrals, i.e. the left-hand side of equation (52), was already proved in [20, 63, 30]. Lemma 2.28 shows the right-hand side satisfies topological recursion for the same spectral curve, equation (53). Hence, both sides must be equal. □

Zhou [64] also explored this relation between triple Hodge integrals and integrable hierarchies, extending it to the 2-Toda hierarchy and to certain relative Gromov–Witten theories.

3. KP hierarchy for intersection numbers

In this section, we will prove the main theorem 2.18, as proposition 3.8, generalising Kazarian's method to the generating functions of intersection numbers coming from hypergeometric tau-functions.

3.1. The change of variables

We will interpret any $X(z)$ defined by equations (12) and (14) as giving a change of coordinates. For this, define a linear correspondence Θ between power series in X or z on the one hand and linear series in p or q on the other by

$$p_k \leftrightarrow X^k\,, \qquad\qquad q_m \leftrightarrow z^m\,. \tag{54}$$

In other words, Θ is a pair of linear isomorphisms

$$\Theta^z \colon \mathbb{C}[\![z]\!] \xrightarrow{\sim} \prod_{m=1}^{\infty} \mathbb{C}q_m \colon z^m \mapsto q_m\,, \tag{55}$$

$$\Theta^X \colon \mathbb{C}[\![X]\!] \xrightarrow{\sim} \prod_{k=1}^{\infty} \mathbb{C}p_k \colon X^k \mapsto p_k\,. \tag{56}$$

This map is meant to express the correspondence of equation (23), so a homogeneous polynomial of degree n in the p_k will correspond to a symmetric polynomial in $X_1, \dots, X_n$.

This defines a change of coordinates as follows:

Definition 3.1. We define a linear morphism between power series in $\{p_m\}_{m\geq 1}$ and $\{q_d\}_{d\geq 1}$ by

$$p_k(\underline{q}) = \sum_{m=k}^{\infty} c_k^m q_m \qquad \text{with } c_k^m \text{ given by} \qquad X^k = \sum_{m=k}^{\infty} c_k^m z^m\,. \tag{57}$$

This is the map $\Theta^X \circ X \circ (\Theta^z)^{-1}$.

In order to make this change of coordinates and remain within the realm of solutions of the KP hierarchy, we should flow along the action of the infinite general linear algebra. Hence, we should find the infinitesimal flow associated to this change. For this, we introduce a flow parameter β by

$$X_\beta(z) \coloneqq \frac{1}{\beta} X(\beta z) = z e^{-\psi(y(\beta z))}\,, \tag{58}$$

such that $X_0(z) = z$ and $X_1(z) = X(z)$.

Lemma 3.2. *For $X_\beta(z) \coloneqq \frac{1}{\beta}X(\beta z)$, where $X(z) = z + \mathcal{O}(z^2)$, and with $Q(z) \coloneqq \frac{z}{X(z)}\frac{dX}{dz}(z)$, the flow along β of the function X_β is given by*

$$\frac{\partial X_\beta}{\partial \beta} = \Big(1 - \frac{1}{Q(\beta z)}\Big)\frac{z}{\beta}\frac{\partial X_\beta}{\partial z} = \frac{1}{\beta}\big(Q(\beta z) - 1\big)X_\beta\,. \tag{59}$$

Proof. By definition of Q, $X = \frac{z}{Q(z)}\frac{dX}{dz}$. Therefore,

$$\begin{aligned}\frac{\partial X_\beta}{\partial \beta} &= \frac{\partial}{\partial \beta}\Big(\frac{1}{\beta}X(\beta z)\Big) = \frac{z}{\beta}\frac{dX}{dz}\Big|_{z\to\beta z} - \frac{1}{\beta^2}X(\beta z)\\ &= \frac{z}{\beta}\frac{dX}{dz}\Big|_{z\to\beta z} - \frac{1}{\beta^2}\Big(\frac{z}{Q(z)}\frac{dX}{dz}\Big)\Big|_{z\to\beta z} = \Big(1 - \frac{1}{Q(\beta z)}\Big)\frac{z}{\beta}\frac{\partial X_\beta}{\partial z}\,. \qquad \square\end{aligned}$$

We will use this with [40, Theorem 2.5], which uses the $\widehat{\mathfrak{gl}(\infty)}$ action on τ-functions:

Theorem 3.3 ([40])**.** *In the situation of a correspondence like equation (57), there is a quadratic function $Q(\underline{p})$ such that the transformation sending an arbitrary series $\Phi(\underline{p})$ to the series $\Psi(\underline{q}) = (\Phi + Q)\big|_{p\to p(\underline{q})}$ is an automorphism of the KP hierarchy: it sends solutions to solutions.*

The function $Q(\underline{p})$ is not unique.

Proposition 3.4. *In the general situation of theorem 2.7, without analytic assumptions, the quadratic function for the change of variables of definition 3.1 can be taken to be $-\frac{1}{2}\Theta(H_{0,2})$.*

Proof. We will use the β-flow defined above to flow from Φ at $\beta = 0$ to Ψ at $\beta = 1$. Consider the more general linear correspondence Θ_β between power series in X_β or z on the one hand and linear series in p or q on the other by

$$p_k \leftrightarrow X_\beta(z)^k\,, \qquad\qquad q_m \leftrightarrow z^m\,.$$

This gives a linear morphism between power series in $\{p_m\}_{m\geq 1}$ and $\{q_d\}_{d\geq 1}$ by

$$p_k(\beta;\underline{q}) = \sum_{m=k}^{\infty} c_k^m q_m \qquad \text{with } c_k^m \text{ given by} \qquad X_\beta^k = \sum_{m=k}^{\infty} c_k^m z^m\,, \tag{60}$$

such that $p_k(0;\underline{q}) = q_k$.

Under Θ_β, the operator $z^{m+1}\frac{\partial}{\partial z}$ transforms into $\sum_{k=1}^{\infty} k q_{m+k}\frac{\partial}{\partial q_k}$, which is the differential part of $L_m(\underline{q})$. The polynomial part of this operator is

$$\frac{1}{2}\sum_{k=1}^{m-1} q_k q_{m-k}\,,$$

which under the correspondence transforms into

$$\frac{1}{2}\sum_{k=1}^{m-1} z_1^k z_2^{m-k} = \frac{1}{2} z_1 z_2 \frac{z_1^{m-1} - z_2^{m-1}}{z_1 - z_2} = -\frac{1}{2}\frac{z_1^{m-1} - z_2^{m-1}}{z_1^{-1} - z_2^{-1}}\,.$$

Therefore, the correction to be made to lemma 3.2 to obtain a KP-preserving flow is found by the substitution $f(z)z\frac{\partial}{\partial z} \to -\frac{1}{2}\frac{1}{z_1^{-1}-z_2^{-1}}\big(z_1^{-1}f(z_1) - z_2^{-1}f(z_2)\big)$ for a series $f(z) \in z\mathbb{C}[\![z]\!]$. Note that $\frac{1}{\beta}\Big(1 - \frac{1}{Q(\beta z)}\Big) \in z\mathbb{C}[\![z]\!]$, and we find that the differential operator of lemma 3.2 needs to be completed by

$$\begin{aligned} &-\frac{1}{2\beta(z_1^{-1} - z_2^{-1})}\Big(z_1^{-1}\big(1 - \frac{1}{Q(\beta z_1)}\big) - z_2^{-1}\big(1 - \frac{1}{Q(\beta z_2)}\big)\Big) \\ &\quad = \frac{1}{2\beta(z_1^{-1} - z_2^{-1})}\Big(\frac{1}{z_1 Q(\beta z_1)} - \frac{1}{z_2 Q(\beta z_2)}\Big) - \frac{1}{2\beta}\,. \end{aligned}$$

By a similar calculation as for lemma 3.2, if we take X to be β-independent, and see equation (60) as giving the β-dependence of z,

$$\begin{aligned} \frac{\partial z(X_\beta)}{\partial \beta}\Big|_{X \text{ const.}} &= \frac{\partial}{\partial \beta}\Big(\frac{1}{\beta} z(\beta X)\Big)\Big|_{X\text{const.}} \\ &= \Big(-\frac{1}{\beta^2} z(\beta X) + \frac{1}{\beta}\frac{dz(X)}{dX}\Big|_{X\to\beta X} X\Big) \\ &= \Big(-\frac{1}{\beta^2} z(\beta X) + \frac{1}{\beta^2}\Big(\frac{dz(X)}{dX} X\Big)\Big|_{X\to\beta X}\Big) \\ &= \Big(-\frac{1}{\beta} z(X_\beta) + \frac{1}{\beta^2}\Big(\frac{1}{Q(z(X))} z(X)\Big)\Big|_{X\to\beta X}\Big) \end{aligned}$$

$$
\begin{aligned}
&= \Big(-\frac{1}{\beta} z(X_\beta) + \frac{1}{\beta}\frac{1}{Q(\beta z(X_\beta))} z(X_\beta)\Big) \\
&= \frac{1}{\beta}\Big(\frac{1}{Q(\beta z(X_\beta))} - 1\Big) z(X_\beta)\,.
\end{aligned}
$$

It follows that, using equation (25),

$$
\begin{aligned}
-\frac{\partial H_{0,2}}{\partial \beta}\Big|_{X_i \text{ const.}} &= -\frac{\partial}{\partial \beta}\log\Big(\frac{z_1^{-1}-z_2^{-1}}{X_1^{-1}-X_2^{-1}}\Big)\Big|_{X_i \text{ const.}} \\
&= \frac{1}{z_1^{-1}-z_2^{-1}}\Big(z_1^{-2}\frac{\partial z_1}{\partial \beta}\Big|_{X_i \text{ const.}} - z_2^{-2}\frac{\partial z_2}{\partial \beta}\Big|_{X_i \text{ const.}}\Big) \\
&= \frac{1}{\beta(z_1^{-1}-z_2^{-1})}\Big(\frac{1}{z_1 Q(\beta z_1)} - \frac{1}{z_2 Q(\beta z_2)}\Big) - \frac{1}{\beta}\,,
\end{aligned}
$$

which, up to a factor 2, is exactly the polynomial correction needed.

From these calculations, we find that

$$
A \coloneqq \Big(1 - \frac{1}{Q(\beta z)}\Big)\frac{z}{\beta}\frac{\partial}{\partial z} - \frac{1}{2}\frac{\partial H_{0,2}}{\partial \beta}\Big|_{X_i \text{ const.}}
$$

corresponds to a linear combination of L_m under Θ_β, and hence preserves KP. Now consider a KP τ-function $Z = \exp\Phi(\underline{p})$ and define the function $Z(\beta,\underline{q}) = \exp\big(\Phi(\underline{p}) - \frac{1}{2}\Theta(H_{0,2})\big)\big|_{\underline{p}=\underline{p}(\beta,\underline{q})}$. We see that $Z(0,\underline{q}) = Z$, as $\underline{p}(0,\underline{q}) = \underline{q}$, and therefore $\Theta(H_{0,2}) = 0$. Also, $Z(1,\underline{q}) = \exp\big(\Psi(\underline{q})\big)$. Moreover,

$$
\begin{aligned}
&\frac{\partial}{\partial \beta} Z = \Big(\sum_{k=1}^{\infty}\frac{\partial p_k(\beta,\underline{q})}{\partial \beta}\frac{\partial}{\partial p_k} - \frac{1}{2}\Theta\Big(\frac{\partial H_{0,2}}{\partial \beta}\Big|_{X_i \text{ const.}}\Big)\Big) Z && (61)\\
&= \Theta\Big(\Big(1-\frac{1}{Q(\beta z)}\Big)\frac{z}{\beta}\frac{\partial}{\partial z} - \frac{1}{2}\frac{\partial H_{0,2}}{\partial \beta}\Big|_{X_i \text{ const.}}\Big) Z(\beta) && (62)\\
&= \Theta(A)\, Z(\beta) && (63)
\end{aligned}
$$

As $\Theta(A)$ preserves τ-functions of KP, this automorphism does indeed preserve solutions, so Φ is a solution if and only if Ψ is. □

Corollary 3.5. *For $Z(\underline{p})$ defined by equation (13), $Z(\underline{p})\exp\big(-\Theta(\hbar^{-1}H_{0,1} + \frac{1}{2}H_{0,2})\big)\big|_{p\to p(\underline{q})}$ is also a KP tau-function, whose logarithm does not contain unstable terms. Here $H_{0,1}$ and $H_{0,2}$ are defined by equation (25).*

Proof. As all equations in the KP hierarchy only contain at least second derivatives of F, addition of a linear term $-\Theta(\hbar^{-1}H_{0,1})$ preserves solutions. By proposition 3.4, subtracting the $(0,2)$ term and changing $p \mapsto p(\underline{q})$ is an automorphism as well. $\square$

Corollary 3.6. *In case $X(z)$ is a Möbius transformation with the shape of equation (14), i.e. $X(z) = \frac{az}{1+bz}$ (taking into account remark 2.8), this quadratic function can be taken to be* 0.

Proof. By direct calculation,

$$H_{0,2}(z_1,z_2) = \log\Big(\frac{z_1^{-1}-z_2^{-1}}{X(z_1)^{-1}-X(z_2)^{-1}}\Big) = \log a\,.$$

Comparing this with the proof of proposition 3.4, the quadratic correction is needed to complete the operator A, which only depends on $\frac{\partial H_{0,2}}{\partial \beta}$. As this vanishes in the present case, we may as well omit the entire correction. $\square$

Remark 3.7. The usual B-function of topological recursion,

$$B(z_1,z_2) = \frac{dz_1\,dz_2}{(z_1-z_2)^2} = d_1 d_2 \log(z_1^{-1}-z_2^{-1})\,, \tag{64}$$

is invariant under all Möbius transformations, so $d_1 d_2 H_{0,2}$ vanishes if X is any Möbius transformation. However, this is not the case for $H_{0,2}$ itself: it is invariant under a one-dimensional subgroup, changes by a constant under the two-dimensional subgroup above, but under other Möbius transformations also changes by addition of terms like $\log z_i$.

Viewed another way, these more general Möbius transformations would take us out of the realm of formal power series in z. However, in a space of functions, a shift $z \mapsto z+c$ does preserve the KP hierarchy, so if the formal power series converges to a function on a large enough domain, this shift does preserve KP. This argument is essentially taken from [40, Section 8]. In particular, under the 'natural analytic assumptions' of [17, section 1.3], i.e. the assumptions in the second part of theorem 2.7, the $H_{g,n}$ do extend to rational functions on all of $\mathbb{P}^1$, so this shift is well-defined.

In case the Möbius transformation is a dilation, $X(z) = az$, the invariance of KP is just quasi-homogeneity: in this case equation (57) reduces to $p_k = a^k q_k$.

3.2. KP for intersection numbers

Now we will restrict to the cases where topological recursion, and hence theorem 2.13, can be used, in order to relate to intersection numbers. In this case, the following holds from equation (35).

(65)

$$F(\underline{p}) = \Big(\sum_{2g-2+n>0} \frac{\hbar^{2g-2+n}}{n!} \sum_{j_1,\dots,j_n \in J} \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(e_{j_1} \otimes \cdots \otimes e_{j_n}) \prod_{i=1}^{n} \sum_{k_i=0}^{\infty} \psi_i^{k_i} \xi_{k_i}^{j_i}(z_i) + \hbar^{-1} H_{0,1} + \frac{1}{2} H_{0,2} \Big) \Big|_{X_i^{k_i} \to p_{k_i}},$$

if we define $\xi_k^j(z) \coloneqq \int_{z'=r_j}^{z} d\xi_k^j(z')$, noting that due to the shape of the $H_{g,n}$ in equations (23) and (24) and $X(z)$ in equation (14), the $H_{g,n}$ have no constant terms in z_i.

Under the correspondence $p_k \leftrightarrow X^k$, $q_m \leftrightarrow z^m$ of definition 3.1, we define T_k^j by

$$T_k^j(\underline{q}) \leftrightarrow \frac{1}{dx} d\xi_k^j(z) = D^{k+1} \frac{z^{j+1}}{j+1}, \tag{66}$$

with D as in equation (14). Explicitly, we define

$$T_{-1}^j = \frac{1}{j+1} q_{j+1}, \qquad T_{k+1}^j = \sum_{m=1}^{\infty} \sum_{l=0}^{\infty} m \mathcal{T}_l q_{m+l} \frac{\partial}{\partial q_m} T_k^j, \tag{67}$$

with $\mathcal{T}_l$ given by

$$Q(z)^{-1} = \sum_{l=0}^{\infty} \mathcal{T}_l z^l. \tag{68}$$

Note that, even though the recursion operator for the T_k^j may have infinitely many terms, its alternate description via equation (66) ensures they are well-defined in Λ.

Therefore, by definition,

(69)

$$\Theta\big(\hbar^{-1} H_{0,1} + \frac{1}{2} H_{0,2}\big) + G_\Omega(T(\underline{p}))$$

$$= \Theta\big(\hbar^{-1}H_{0,1} + \frac{1}{2}H_{0,2}\big)$$
$$+ \sum_{2g-2+n>0} \frac{\hbar^{2g-2+n}}{n!} \sum_{j_1,\dots,j_n\in J} \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(e_{j_1}\otimes\cdots\otimes e_{j_n}) \prod_{i=1}^{n}\sum_{k_i=0}^{\infty} \psi_i^{k_i} T_{k_i}^{j_i}(\underline{q})$$

is the logarithm of a tau-function, where $\{e_j\}$ is the dual basis to the basis $\{d\xi_0^j\}$ of V^*.

Proposition 3.8. *Suppose the pair of functions $(\hat{\psi},\hat{y})$ lies in family I or II of theorem 2.7, and let Ω be the cohomological field theory associated to the related topological recursion via theorem 2.13. Then*

$$Z_\Omega(\underline{q}) = e^{G_\Omega(T(\underline{q}))} \tag{70}$$

is a KP tau-function.

Proof. Apply corollary 3.5 to the exponent of equation (69). □

3.3. Finiteness of the transformation

The operator A in the proof of proposition 3.4 corresponds to a finite sum of L_m if and only if $Q(z)^{-1}$ is a polynomial in z. As this case seems particularly nice, we will investigate it here. Note that this condition is dependent on the parameter z on the spectral curve, cf. the difference between example 2.20 and section 2.3.

Write $P(z) = Q(z)^{-1}$ for this polynomial, and write $r+1$ for its degree. From equation (14), it follows that $P(0)=1$, so we may write

$$P(z) = \prod_{j=1}^{r+1}(1-c_jz)\,. \tag{71}$$

We immediately see that $dx(z) = \frac{dz}{zP(z)}$, and hence the spectral curve has a unique ramification point, ∞, of ramification index r. This is also the rank of the associated Frobenius algebra. But we can do better. By calculating the residues in v of

$$\frac{v^{r+1}dv}{(1-vz)\prod_{k=1}^{r+1}(v-c_k)} \tag{72}$$

and using that they sum to zero, one can check that (if all c_j are distinct)[2]

$$\frac{dx}{dz} = \frac{1}{z} + \sum_{j=1}^{r+1} \frac{c_j^{r+1}}{\prod_{k\neq j}(c_j - c_k)} \frac{1}{1 - c_j z}\,, \tag{73}$$

from which we see that

$$x(z) = \log z - \sum_{j=1}^{r+1} \prod_{k\neq j} \big(1 - \frac{c_k}{c_j}\big)^{-1} \log(1 - c_j z)\,. \tag{74}$$

If $r = 0$, dx has two (simple) poles, and hence no zeroes. In fact, in this case, X is a Möbius transformation.

If $r = 1$, this recovers the triple Hodge curve, studied in section 3.4 below, after identifying $c_1 = -\beta$, $c_2 = -(w+1)\beta$.

If $r > 1$, the related Frobenius algebra is not semi-simple, cf. remark 2.14: it is a deformation of the algebra corresponding to Witten's $r+1$-spin cohomological field theory, which is given by $x = y^{r+1}$, cf. [61, 26, 5]. This class fits in Alexandrov's theory of the deformed generalised Kontsevich model [3]: it seems like it is a complementary subspace of the polynomial deformations of the Witten $r+1$-spin theory. Note that the Witten spin classes themselves do not fit in the scope of this paper, as the curve is not of the shape of equation (14).

Interestingly, except for special choices of c_j, these cases seem not to be covered in the two families in theorem 2.7 for which topological recursion is proved (for any choice of y). Even the $r = 1$ case does not fall in that scope, unless $\frac{c_1}{c_2} \in \mathbb{Q}$.

3.4. The case of triple Hodge integrals

Let us now consider the special case of triple Hodge integrals. The approach taken in this section overlaps with the previous results, but is also slightly different in details, adapted to this specific problem. For example, we do not use the formal parameter $\hbar$, but make another convenient choice. In this case, the ELSV-type formula is completely explicit, and there is no need to take the detour via topological recursion.

[2]If some c_j coincide, the residue argument still holds, but the result changes.

The coordinate change we want to perform is inspired by the Mariño–Vafa formula.

$$\begin{aligned} F\big(w,\beta;\underline{p}\big) &= \log\Bigg(\sum_{m=0}^{\infty} \sum_{\mu,\nu\vdash m} \frac{\chi^\nu_\mu}{z_\mu} e^{(1+\frac{w}{2})\beta f_2(\nu)} \prod_{\square\in\nu} \frac{\beta w}{\varsigma(\beta w h_\square)} p_\mu \Bigg) \\ &= \sum_\mu \sum_{g=0}^{\infty} \frac{(w+1)^{g+n-1}}{|\operatorname{Aut}\mu|} \prod_{i=1}^{n} \frac{\prod_{j=1}^{\mu_i-1}(\mu_i+jw)}{(\mu_i-1)!} \\ &\qquad \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda(-1)\Lambda(-w)\Lambda\big(\frac{w}{w+1}\big)}{\prod_{i=1}^n (1-\mu_i\psi_i)} \beta^{2g-2+n+|\mu|} p_\mu\,. \end{aligned} \tag{75}$$

As $2g-2+n+|\mu| = \frac{2}{3}\dim\overline{\mathcal{M}}_{g,n} + \sum_{i=1}^n(\mu_i+\frac{1}{3})$ and $g+n-1 = \frac{1}{3}\dim\overline{\mathcal{M}}_{g,n} + \sum_{i=1}^n \frac{2}{3}$, we get after rewriting $u \coloneqq \beta^{\frac{1}{3}}(w+1)^{\frac{1}{6}}$

(76)

$$\begin{aligned} F\big(w,\beta;\underline{p}\big) &= \sum_\mu \frac{1}{|\operatorname{Aut}\mu|} \sum_{g=0}^{\infty} \prod_{i=1}^{n} u^4 \frac{\prod_{j=1}^{\mu_i-1}(\mu_i+jw)\beta}{(\mu_i-1)!} \\ &\qquad \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda(-u^2)\Lambda(-u^2w)\Lambda\big(\frac{u^2w}{w+1}\big)}{\prod_{i=1}^n (1-\mu_i u^2\psi_i)} p_\mu \\ &= \sum_{g=0}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Lambda(-u^2)\Lambda(-u^2w)\Lambda\Big(\frac{u^2w}{w+1}\Big) \prod_{i=1}^{n}\sum_{d=0}^{\infty} \tilde{T}_d(\underline{p})\psi_i^d \\ &= G_{\mathrm{TH}}\Big(-u^2, -u^2w, \frac{u^2w}{w+1}; \tilde{T}(\underline{p})\Big) + H_{0,1} + \frac{1}{2}H_{0,2}\,, \end{aligned}$$

where

$$\tilde{T}_d(\underline{p}) \coloneqq \sum_{m=1}^{\infty} \frac{\prod_{j=1}^{m-1}(m+jw)}{(m-1)!} m^d u^{2d+4}\beta^{m-1} p_m\,. \tag{77}$$

Hence, our goal is to show that this change of variables and addition of the unstable terms preserves solutions of the KP hierarchy.

Lemma 3.9. *The following two expressions are inverse to each other:*

$$\begin{aligned} X_\beta(z) &= \frac{z}{1+(w+1)\beta z}\Big(\frac{1+\beta z}{1+(w+1)\beta z}\Big)^{\frac{1}{w}}\,; \\ z(X_\beta) &= \sum_{m=1}^{\infty} \frac{\prod_{j=1}^{m-1}(m+jw)}{(m-1)!}\beta^{m-1}X_\beta^m\,. \end{aligned} \tag{78}$$

Proof. This can be proved by a residue calculation. Start from the formula for $X(z) = X_1(z)$ and write $z(X) = \sum_{m=1}^{\infty} C_m X^m$. Then $C_m = \mathrm{Res}_{X=0}\, z\, X^{-m} \frac{dX}{X}$, and

$$\begin{aligned} \frac{dX}{X} &= \frac{dz}{z} + \frac{d(1+z)^{\frac{1}{w}}}{(1+z)^{\frac{1}{w}}} + \frac{d(1+(w+1)z)^{-\frac{w+1}{w}})}{(1+(w+1)z)^{-\frac{w+1}{w}}} \\ &= \frac{dz}{z} + \frac{1}{w}\frac{dz}{1+z} - \frac{(w+1)^2}{w}\frac{dz}{1+(w+1)z} \\ &= \frac{dz}{z(1+z)(1+(w+1)z)}\,. \end{aligned}$$

Therefore,

$$\begin{aligned} C_m &= \operatorname*{Res}_{X=0} z\, X^{-m} \frac{dz}{z(1+z)(1+(w+1)z)} \\ &= \operatorname*{Res}_{z=0} z^{-m}(1+z)^{-\frac{m}{w}-1}(1+(w+1)z)^{m\frac{w+1}{w}-1} dz \\ &= \operatorname*{Res}_{z=0} z^{-m} \sum_{k=0}^{\infty} \frac{\prod_{i=0}^{k-1}(-\frac{m}{w}-1-i)}{k!} z^k \sum_{l=0}^{\infty} \frac{\prod_{j=0}^{l-1}(m\frac{w+1}{w}-1-j)}{l!}(w+1)^l z^l dz \\ &= \operatorname*{Res}_{z=0} z^{-m} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{k}(\frac{m}{w}+i)}{k!}(-1)^k z^k \sum_{l=0}^{\infty} \frac{\prod_{j=m-l}^{m-1}(\frac{m}{w}+j)}{l!}(w+1)^l z^l dz \\ &= \sum_{k=0}^{m-1} \frac{\prod_{i=1}^{k}(\frac{m}{w}+i)}{k!} \frac{\prod_{j=k+1}^{m-1}(\frac{m}{w}+j)}{(m-k-1)!}(-1)^k (w+1)^{m-k-1} \\ &= \frac{\prod_{j=1}^{m-1}(\frac{m}{w}+j)}{(m-1)!} \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^k (w+1)^{m-k-1} \\ &= \frac{\prod_{j=1}^{m-1}(\frac{m}{w}+j)}{(m-1)!} w^{m-1} = \frac{\prod_{j=1}^{m-1}(m+jw)}{(m-1)!}\,. \end{aligned}$$

Finally, β can be introduced in this formula by scaling $z \to \beta z$, $X \to \beta X_\beta$. ☐

Corollary 3.10. *The expressions for $X(z)$ in lemma 2.28 and $X_\beta(z)$ in lemma 3.9 are related by Möbius transformations:*

$$X\Big(\frac{\beta z}{\gamma(1+(w+1)\beta z)}\Big) = \frac{\beta}{\gamma} X_\beta(z) \tag{79}$$

Hence, by corollary 3.6, they require the same correction term for their induced linear change of variables.

We see that in this particular case we may obtain the function X in two different ways: from the general theory of theorem 2.7, or from the specific shape of the Mariño–Vafa formula, theorem 2.25. In fact, the second choice is nothing but choosing the spectral curve coordinate z to equal ξ (which is unique in this case), or in other words $T_0 = q_1$. The scaling factor $\frac{\beta}{\gamma}$ is exactly the scaling factor between r_k and p_k in lemma 2.28.

Remark 3.11. Under the correspondence of theorem 2.13, the rank of the cohomological field theory corresponds to the number of zeroes of dx, counted with multiplicities. So for rank one, dx can only have one zero, and hence must have three poles. By Möbius transformation, we may place the zero at infinity, and two of the poles at 0 and -1, from which we find that dx must correspond to the $\frac{dX}{X}$ found in the proof of lemma 3.9. This may explain in part why Alexandrov [2] finds only the triple Hodge CohFT in the intersection of the orbits of the Givental and Heisenberg–Virasoro groups. However, dx is not the only datum of a spectral curve, and while $\mathbb{P}^1$ is rigid and has a unique B, it is not clear why there is no freedom in the choice of dy.

Lemma 3.12. *The series $X_\beta(z)$ from lemma 3.9 satisfies the differential equation*

$$\frac{\partial X_\beta}{\partial \beta}(z) = -\big((w+2)z + (w+1)\beta z^2\big) z \frac{\partial X_\beta}{\partial z}(z)\,. \tag{80}$$

Proof. For $X(z) = \frac{z}{1+(w+1)z}\Big(\frac{1+z}{1+(w+1)z}\Big)^{\frac{1}{w}}$, we get $Q(z)^{-1} = (1+z)(1+(w+1)z)$, which using lemma 3.2 immediately yields the result. ☐

We use this lemma in combination with the linear correspondence of definition 3.1, slightly adapted as follows: define a linear correspondence Θ between power series in X or z on the one hand and linear series in p or $\tilde{q}$ on the other by

$$p_k \leftrightarrow X^k\,, \qquad\qquad \tilde{q}_m \leftrightarrow z^m\,. \tag{81}$$

Definition 3.13. We define a linear morphism between power series in $\{p_m\}_{m\geq 1}$ and $\{\tilde{q}_d\}_{d\geq 1}$ by

$$p_k(\underline{\tilde{q}}) = \sum_{m=k}^{\infty} c_k^m \tilde{q}_m \qquad \text{with } c_k^m \text{ given by} \qquad X^k = \sum_{m=k}^{\infty} c_k^m z^m\,. \tag{82}$$

Under the correspondence $p_k \leftrightarrow X^k$, $\tilde{q}_m \leftrightarrow z^m$, we have

$$\text{(83)}\quad \tilde{T}_d(\underline{p}) \leftrightarrow (u^2 D)^d u^4 z\,; \quad D := X\frac{\partial}{\partial X} = \big(1+\beta z\big)\big(1+(w+1)\beta z\big) z\frac{\partial}{\partial z}\,.$$

In terms of $\tilde{q}$-variables, this gives

(84)

$$\tilde{T}_d = u^2 \sum_{m=1}^{\infty} m\big(\tilde{q}_m + (w+2)\beta\tilde{q}_{m+1} + (w+1)\beta^2\tilde{q}_{m+2}\big)\frac{\partial}{\partial\tilde{q}_m}\tilde{T}_{d-1}\,; \quad \tilde{T}_0 = u^4\tilde{q}_1\,.$$

If we write $\tilde{q}_m(\underline{q}) := u^{-4m}q_m$, and using $\beta = \frac{u^3}{\sqrt{w+1}}$, we may express $T_d(\underline{q}) := \tilde{T}_d\big(p(\tilde{q}(\underline{q}))\big)$ as

(85)

$$T_d = \sum_{m=1}^{\infty} m\Big(u^{4m+2}\tilde{q}_m + \frac{u^3(w+2)}{\sqrt{w+1}}u^{4m+2}\tilde{q}_{m+1} + u^6 u^{4m+2}\tilde{q}_{m+2}\Big)\frac{1}{u^{4m}}\frac{\partial}{\partial\tilde{q}_m}T_{d-1}$$

(86)

$$= \sum_{m=1}^{\infty} m\Big(u^2 q_m + \frac{u(w+2)}{\sqrt{w+1}}q_{m+1} + q_{m+2}\Big)\frac{\partial}{\partial q_m}T_{d-1}\,;$$

(87)

$$T_0 = q_1\,.$$

This is exactly the definition given in theorem 2.22.

Corollary 3.14. *For* $X_\beta(z) = \frac{z}{1+(w+1)\beta z}\big(\frac{1+\beta z}{1+(w+1)\beta z}\big)^{\frac{1}{w}}$*, the quadratic correction of theorem 3.3 is* $Q = -\frac{1}{2}\Theta(H_{0,2})$*.*

Proof. The function $X_\beta(z)$ satisfies the conditions of theorem 2.7, so we may apply proposition 3.4. □

Now we are ready to prove the main result on KP integrability of triple Hodge integrals.

Proof of theorem 2.22. By lemma 2.28, $\exp F$ is a tau function of the KP hierarchy in the variables $s_k := \big(\frac{\hbar}{\gamma w}\big)^k\frac{r_k}{k} = \frac{p_k}{k}$. By quasi-homogeneity of the KP hierarchy, it is also a tau function in the variables $t_k = \frac{p_k}{k}$. By equation (76), in combination with corollary 3.14 and theorem 3.3,

$$\text{(88)}\qquad G_{\mathrm{TH}}\Big(-u^2, -wu^2, \frac{wu^2}{w+1}; \{T_d(\underline{q})\}\Big)$$

is a solution of the KP hierarchy in the variables $\frac{\tilde{q}_m}{m}$. Again using quasi-homogeneity of the KP hierarchy, rescaling $\tilde{q}_m \to q_m$ preserves solutions. This completes the proof. $\square$

Remark 3.15. The result in this subsection do hold for $w = -1$ (ignoring powers of u), but in this specific case $X(z)$ is a Möbius transformation, so it reduces to the setting of corollary 3.6. From another point of view, in this case the change of coordinates equation (57) is an isomorphism, whereas it gives a half-dimensional subspace in all other cases. Equations for this half-dimensional space, in the linear Hodge case, were found in [1], cf. also [39] for a reformulation. These can be viewed as a deformation of the reduction from KP to KdV. Similar equations should exist for triple Hodge integrals as well, but clearly none of this works for $w = -1$.

In light of section 3.3, one may expect a deformation of the reduction from KP to r-KdV or r-Gelfand-Dickey for the families found there.

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