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Weyl invariant E_8 Jacobi forms and E-strings

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In 1992 Wirthmüller showed that for any irreducible root system not of type E_8 the ring of weak Jacobi forms invariant under Weyl group is a polynomial algebra. However, it has recently been proved that for E_8 the ring is not a polynomial algebra. Weyl invariant E_8 Jacobi forms have many applications in string theory and it is an open problem to describe such forms. The scaled refined free energies of E-strings with certain η-function factors are conjectured to be Weyl invariant E_8 quasi holomorphic Jacobi forms. It is further observed that the scaled refined free energies up to some powers of E_4 can be written as polynomials in nine Sakai's E_8 Jacobi forms and Eisenstein series E_2, E_4, E_6. Motivated by the physical conjectures, we prove that for any Weyl invariant E_8 Jacobi form ϕ_t of index t the function $E_4^{[t/5]}\Delta^{[5t/6]}\phi_t$ can be expressed uniquely as a polynomial in E_4, E_6 and Sakai's forms, where $[x]$ is the integer part of x. This means that a Weyl invariant E_8 Jacobi form is completely determined by a solution of some linear equations. By solving the linear systems, we determine the generators of the free module of Weyl invariant E_8 weak (resp. holomorphic) Jacobi forms of given index t when $t \leq 13$ (resp. $t \leq 11$).

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1. Introduction

In 1985 Eichler and Zagier introduced the theory of Jacobi forms in their monograph [7]. Jacobi forms are an elegant intermediate between different types of modular forms and have many applications in mathematics and physics. In 1992 Wirthmüller [25] investigated Weyl invariant Jacobi forms associated with root systems. Let R be an irreducible root system of rank r.

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A $W(R)$-invariant Jacobi form is a holomorphic function of complex variables $\tau \in \mathbb{H}$ and $\mathfrak{z} \in R \otimes \mathbb{C}$ which is modular in τ and quasi-periodic in $\mathfrak{z}$ and is invariant under the action of the Weyl group $W(R)$ on the lattice variable $\mathfrak{z}$. All $W(R)$-invariant weak Jacobi forms of integral weight and integral index form a bigraded algebra $J^{\mathrm{w},W(R)}_{*,R,*}$ over $\mathbb{C}$. When R is not of type E_8, Wirthmüller showed that $J^{\mathrm{w},W(R)}_{*,R,*}$ is a polynomial algebra generated by $r+1$ Jacobi forms over the ring $M_*(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$ of $\mathrm{SL}_2(\mathbb{Z})$-modular forms. For example, the algebra of $W(A_1)$-invariant weak Jacobi forms, that is, the ring of even-weight weak Jacobi forms introduced by Eichler and Zagier, is freely generated by two forms of index 1 and weight 0 and -2. Recently, the second named author proved in [22] that the ring of $W(E_8)$-invariant weak Jacobi forms is not a polynomial algebra. In other words, there exist some algebraic relations among generators. It is still unknown if this ring is finitely generated. In the study on the Seiberg–Witten curve of E-string theory [6], Sakai [18] constructed nine $W(E_8)$-invariant holomorphic Jacobi forms denoted A_1, A_2, A_3, A_4, A_5, B_2, B_3, B_4, B_6. The forms A_i have weight 4 and index i, and they reduce to the Eisenstein series E_4 when $\mathfrak{z} = 0$. The forms B_j have weight 6 and index j, and they reduce to E_6 when $\mathfrak{z} = 0$. In [19] Sakai conjectured that for any $W(E_8)$-invariant Jacobi form ϕ there exists a $\mathrm{SL}_2(\mathbb{Z})$-modular form $f(\tau)$ such that the product $f\phi$ can be written as a polynomial in these A_i, B_j and E_4, E_6. This conjecture was proved by the second named author in [22]. In this paper we will determine the best possible f for arbitrary index and give some further applications. Our description is inspired by some conjectures in string theory. Let us briefly introduce the physical background.

The E-string theory is a typical 6d $(1,0)$ superconformal field theory (SCFT) with E_8 flavor symmetry [26, 20, 8]. In the $S^1/\mathbb{Z}_2$ compactification of M-theory, m E-strings are realized by m M2-branes stretched between a M5-brane and a M9-brane. The bound state of m E-strings are captured by topologically twisted 4d $\mathcal{N} = 4$ $U(m)$ Yang-Mills theories on half K3, which is an elliptic surface realized as $\mathbb{P}^2$ with nine points blown up [14, 15]. From the view point of Calabi–Yau geometry, the E-string theory is equivalent to topological string theory on local half K3 Calabi–Yau threefold. By S^1 compactification, E-string theory gives marginal 5d $\mathcal{N} = 1$ $SU(2)$ gauge theory with eight fundamentals, which can flow to almost all 5d rank-one SCFTs [21]. E-string theory is also closely connected to van Diejen integrable model and elliptic Painlevé system [16, 17]. All these relations make E-string theory the mother of almost all "genus one" theories in the sense of mirror/spectral curves, or "rank one" theories in the sense of Coulomb branch dimension.

A general spacetime setting which preserves the supersymmetry for E-string theory is the so called 6d Omega background $(\mathbb{C}^2 \times T^2)_{\epsilon_1,\epsilon_2}$, where each $\mathbb{C}$ plane is rotated by $z_i \to e^{\epsilon_i} z_i$ around the cycles of the torus. For such system, the $\epsilon_{1,2}$ expansion of the total instanton free energy $F = \sum_{n,g,m\geq 0} Q^m(-\epsilon_1\epsilon_2)^{g-1}(\epsilon_1+\epsilon_2)^{2n} F_{(n,g,m)}$ defines the refined free energies of E-strings, where Q counts the number of strings. The situation when keeping all E_8 flavor fugacities $\mathfrak{z}$ is often called massive, while turning $\mathfrak{z} = 0$ massless. It is generally conjectured that the refined free energies $F_{(n,g,m)}$ of m E-strings in the massive case are $W(E_8)$-invariant quasi Jacobi forms of index m. The scaled refined free energies $(\frac{\eta^{12}}{\sqrt{q}})^m F_{(n,g,m)}$ of E-strings in the massless case were found to be quasi modular forms of weight $2(n+g)+6m-2$ on $\mathrm{SL}_2(\mathbb{Z})$ and thus can be expressed as polynomials in E_2, E_4 and E_6. It is then natural to guess that the scaled refined free energies $(\frac{\eta^{12}}{\sqrt{q}})^m F_{(n,g,m)}$ in the massive case can be written as polynomials in E_2, E_4, E_6, A_i and B_j. The process to fix all the polynomial coefficients to determine $F_{(n,g,m)}$ is called modular bootstrap and the ansatz is called modular ansatz. In [13], Huang, Klemm and Poretschkin developed the refined modular anomaly equation to efficiently compute $F_{(n,g,m)}$ of E-strings. They found that the naive modular ansatz is true for index $m \leq 4$, but usually not true for $m \geq 5$. Indeed, they found some $(\frac{\eta^{12}}{\sqrt{q}})^5 F_{(n,g,5)}$ which cannot be expressed as polynomials in Sakai's nine forms unless further multiplied by E_4. Later Del Zotto, Gu, Huang, Kashani-Poor, Klemm and Lockhart [5] discovered an exceptional $W(E_8)$-invariant holomorphic Jacobi form of weight 16 and index 5 defined by the polynomial

$$\begin{aligned} P_{16,5} = {} & 864A_1^3A_2 + 3825A_1B_2^2 - 770A_3B_2E_6 \\ & - 840A_2B_3E_6 + 60A_1B_4E_6 + 21A_5E_6^2. \end{aligned} \tag{1.1}$$

They checked numerically that $P_{16,5}$ vanishes at the zero points of E_4 for general lattice variable $\mathfrak{z}$ and then conjectured that the quotient $P_{16,5}/E_4$ is holomorphic. They did not find other similar polynomials, so they further conjectured that any Jacobi form expressed as a polynomial in A_i, B_j and E_6 which vanishes at the zero points of E_4 must be divisible by the above polynomial. In this paper we will prove their conjectures.

In a similar manner, Sakai's nine A_i, B_j forms are also used in the modular bootstrap of the elliptic genera of E-strings and $E_8 \times E_8$ Heterotic strings [12, 4, 11]. One of the main goals of the current work is to establish a rigorous foundation for modular bootstrap whenever E_8 symmetry, no matter flavor or gauge, is involved.

In [23] the second named author established the modular Jacobian criterion to give an automorphic proof of Wirthmüller's theorem. We will use this approach and the distinguished Jacobi form $P_{16,5}/E_4$ to prove the following theorem, which gives a full description of $W(E_8)$-invariant Jacobi forms in terms of Sakai's forms.

Theorem 1.1.

1. *The quotient $P_{16,5}/E_4$ is a $W(E_8)$-invariant holomorphic Jacobi form of weight 12 and index 5.*
2. *For any $W(E_8)$-invariant Jacobi form $P \in \mathbb{C}[E_6, A_1, A_2, B_2, A_3, B_3, A_4, B_4, A_5, B_6]$, if P/E_4 is holomorphic on $\mathbb{H} \times (E_8 \otimes \mathbb{C})$, then*

$$\frac{P}{P_{16,5}} \in \mathbb{C}[E_6, A_1, A_2, B_2, A_3, B_3, A_4, B_4, A_5, B_6].$$

3. *Every $W(E_8)$-invariant weak Jacobi form of index t can be expressed uniquely as*

$$\frac{\sum_{j=0}^{t_1} P_j E_4^j P_{16,5}^{t_1-j}}{\Delta^{N_t} E_4^{t_1}}, \tag{1.2}$$

where

   (i) t_1 is the integer part of $t/5$;

   (ii) $P_{t_1} \in \mathbb{C}[E_4, E_6, A_1, A_2, B_2, A_3, B_3, A_4, B_4, A_5, B_6]$;

   (iii) $P_j \in \mathbb{C}[E_6, A_1, A_2, B_2, A_3, B_3, A_4, B_4, A_5, B_6]$ for $0 \le j < t_1$;

   (iv) N_t is defined as follows

$$N_t = \begin{cases} 5t_0, & \text{if } t = 6t_0 \text{ or } 6t_0+1, \\ 5t_0+1, & \text{if } t = 6t_0+2, \\ 5t_0+2, & \text{if } t = 6t_0+3, \\ 5t_0+3, & \text{if } t = 6t_0+4 \text{ or } 6t_0+5. \end{cases}$$

The powers t_1 and N_t here are sharp. There exist $W(E_8)$-invariant Jacobi forms of arbitrary index which cannot be expressed in the form (1.2) if we replace t_1 or N_t with any smaller integer. Besides, our theorem implies that there is no $W(E_8)$-invariant Jacobi form which vanishes at the zero points of E_6 and lies in the ring $\mathbb{C}[E_4, A_1, A_2, B_2, A_3, B_3, A_4, B_4, A_5, B_6]$. This confirms the numerical search in [5].

The function (1.2) is a weak Jacobi form if and only if the numerator has Fourier expansion of the form

$$\sum_{j=0}^{t_1} P_j E_4^j P_{16,5}^{t_1-j} = O(q^{N_t}), \quad q = e^{2\pi i\tau}. \tag{1.3}$$

We can find a basis of the space of $W(E_8)$-invariant weak Jacobi forms of given weight and given index by solving the system of linear equations defined by (1.3). In [22] the second named author proved that the space $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ of $W(E_8)$-invariant weak Jacobi forms of integral weight and given index t is a free module over $M_*(\mathrm{SL}_2(\mathbb{Z}))$ whose rank is determined by a generating series. He also determined the generators of $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ for $t \leq 4$ using the differential operators on Jacobi forms. In this paper we calculate the Fourier expansions of Sakai's forms up to q^9-terms. By solving linear systems of type (1.3), we figure out the structure of $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ for $t \leq 13$. We also successfully determine the space of $W(E_8)$-invariant holomorphic Jacobi forms of arbitrary weight and index $t \leq 11$. The module $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ is quite complicated when the index t is large, such as the largest module $J^{\mathrm{w},W(E_8)}_{*,E_8,13}$ we have determined has 364 generators.

The paper is organized as follows. In §2 we give a brief introduce of $W(E_8)$-invariant Jacobi forms. §3 is devoted to the proof of Theorem 1.1. We calculate the generating series of $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ for index $t \leq 13$ in §4 and present their Laurent expansions in Appendix A. We discuss $W(E_8)$-invariant holomorphic Jacobi forms in §5 and construct three exceptional generators in Appendix B. In §6 we propose five conjectures on Jacobi forms and provide some evidence.

2. Preliminaries

In this section we define $W(E_8)$-invariant Jacobi forms. Let $\mathbb{N}$ be the set of non-negative integers and $(-,-)$ be the standard scalar product on $\mathbb{R}^8$. Let $W(E_8)$ denote the Weyl group of E_8. We use the model, simple roots and fundamental weights of E_8 fixed in [22, §3.1]. We say that a vector $v \in E_8$ has norm n if $(v,v) = 2n$.

Definition 2.1. Let $\varphi : \mathbb{H} \times (E_8 \otimes \mathbb{C}) \to \mathbb{C}$ be a holomorphic function and $k \in \mathbb{Z}$, $t \in \mathbb{N}$. If φ satisfies the following properties

(i) Weyl invariance:

$$\varphi(\tau, \sigma(\mathfrak{z})) = \varphi(\tau, \mathfrak{z}), \quad \sigma \in W(E_8),$$

(ii) Quasi-periodicity:

$$\varphi(\tau, \mathfrak{z} + x\tau + y) = \exp\left(-t\pi i[(x,x)\tau + 2(x,\mathfrak{z})]\right)\varphi(\tau,\mathfrak{z}), \quad x, y \in E_8,$$

(iii) Modularity:

$$\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{\mathfrak{z}}{c\tau+d}\right) = (c\tau+d)^k \exp\left(t\pi i \frac{c(\mathfrak{z},\mathfrak{z})}{c\tau+d}\right)\varphi(\tau,\mathfrak{z}),$$

with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

(iv) $\varphi(\tau,\mathfrak{z})$ has the Fourier expansion

$$\varphi(\tau,\mathfrak{z}) = \sum_{n=0}^{\infty}\sum_{\ell\in E_8} f(n,\ell)e^{2\pi i(n\tau+(\ell,\mathfrak{z}))},$$

then φ is called a $W(E_8)$-invariant weak Jacobi form of weight k and index t. If $f(n,\ell) = 0$ whenever $2nt - (\ell,\ell) < 0$, then φ is called a $W(E_8)$-invariant holomorphic Jacobi form.

Every non-zero $W(E_8)$-invariant weak Jacobi form has even weight. The $W(E_8)$-invariant weak Jacobi forms of index 0 do not depend on the lattice variable $\mathfrak{z}$ and are actually modular forms on $\mathrm{SL}_2(\mathbb{Z})$. The quasi-periodicity implies that in the above Fourier expansion $f(n_1,\ell_1) = f(n_2,\ell_2)$ if $2n_1t - (\ell_1,\ell_1) = 2n_2t - (\ell_2,\ell_2)$ and if $\ell_1 - \ell_2 \in tE_8$. We denote the vector spaces of $W(E_8)$-invariant weak and holomorphic Jacobi forms of weight k and index t respectively by

$$J^{\mathrm{w},W(E_8)}_{k,E_8,t} \supsetneq J^{W(E_8)}_{k,E_8,t}.$$

We will investigate the free $M_*(\mathrm{SL}_2(\mathbb{Z}))$-modules

$$J^{\mathrm{w},W(E_8)}_{*,E_8,t} := \bigoplus_{k\in\mathbb{Z}} J^{\mathrm{w},W(E_8)}_{k,E_8,t}, \quad J^{W(E_8)}_{*,E_8,t} := \bigoplus_{k\in\mathbb{Z}} J^{W(E_8)}_{k,E_8,t}$$

and the bigraded algebra

$$J^{\mathrm{w},W(E_8)}_{*,E_8,*} := \bigoplus_{t=0}^{\infty} J^{\mathrm{w},W(E_8)}_{*,E_8,t}.$$

Sakai's forms A_1, A_2, B_2, A_3, B_3, A_4, B_4, A_5, B_6 were first constructed in [18, Appendix A.1]. It was proved in [22, Theorem 4.1] that the nine

Jacobi forms A_i and B_j are algebraically independent over $\mathbb{C}[E_4, E_6]$. Due to its importance, we briefly explain how Sakai constructed these forms. One starts with the Jacobi theta function of E_8

$$\vartheta_{E_8}(\tau, \mathfrak{z}) = \sum_{\ell \in E_8} e^{\pi i(\ell,\ell)\tau + 2\pi i(\ell,\mathfrak{z})}$$

which is the unique $W(E_8)$-invariant holomorphic Jacobi form of weight 4 and index 1. Acting the index raising Hecke operators $T_-(t)$ on ϑ_{E_8} (see e.g. [22, Lemma 3.6]), one obtains $W(E_8)$-invariant holomorphic Jacobi forms of weight 4 and arbitrary index t

$$X_t(\tau, \mathfrak{z}) = 1 + O(q), \quad q = e^{2\pi i\tau}. \tag{2.1}$$

Sakai's forms A_j are constructed as

$$A_j(\tau, \mathfrak{z}) = X_j(\tau, \mathfrak{z}),\ j = 1, 2, 3, 5, \quad A_4(\tau, \mathfrak{z}) = \vartheta_{E_8}(\tau, 2\mathfrak{z}). \tag{2.2}$$

To construct B_t, one first takes an appropriate modular form g_t of weight 2 on the congruence subgroup $\Gamma_0(t)$ of $\mathrm{SL}_2(\mathbb{Z})$. Then the trace sum of $g_t(\tau)\vartheta_{E_8}(t\tau, t\mathfrak{z})$ with respect to the cosets of $\Gamma_0(t)\backslash \mathrm{SL}_2(\mathbb{Z})$ defines a $W(E_8)$-invariant holomorphic Jacobi form of weight 6 and index t. That is the desired B_t.

The Fourier expansion characterizes Jacobi forms. Due to the Weyl invariance, the Fourier expansion of any $W(E_8)$-invariant Jacobi form can be expressed in terms of Weyl orbits of vectors in E_8. We review some useful facts following [22]. For any $v \in E_8$ we define the Weyl orbit of v as

$$\mathrm{orb}(v) = \sum_{\sigma \in W(E_8)/W(E_8)_v} e^{2\pi i(\sigma(v),\mathfrak{z})}, \tag{2.3}$$

where $W(E_8)_v$ is the stabilizer subgroup of $W(E_8)$ with respect to v. For any non-negative integer n the q^n-term of φ

$$[\varphi]_{q^n} = \sum_{\ell \in E_8} f(n, \ell) e^{2\pi i(\ell,\mathfrak{z})}$$

can be written as a $\mathbb{C}$-linear combination of Weyl orbits. Let α_i and w_i, $1 \le i \le 8$, be the simple roots and fundamental weights of E_8 respectively (see [22, §3.1] for their coordinates). The eight fundamental Weyl orbits $\mathrm{orb}(w_i)$ are algebraically independent over $\mathbb{C}$. Moreover, every Weyl orbit

$\mathrm{orb}(v)$ can be expressed as a polynomial in these fundamental $\mathrm{orb}(w_i)$. By [2], every Weyl orbit meets the set Λ_+ in exactly one point, where

$$\begin{aligned}\Lambda_+ &= \{m \in E_8 : (\alpha_i, m) \geq 0, 1 \leq i \leq 8\} \\ &= \left\{ m = \sum_{i=1}^{8} m_i w_i : m_i \in \mathbb{N}, 1 \leq i \leq 8 \right\}\end{aligned}$$

is the closure of a Weyl chamber. Thus we only need to consider the Weyl orbits of vectors in Λ_+. With respect to the partial order on E_8, we have the decomposition

$$\mathrm{orb}(m) = \prod_{i=1}^{8} \mathrm{orb}(w_i)^{m_i} + \sum_{\substack{l \in \Lambda_+ \\ l < m}} c_{l,m}\, \mathrm{orb}(l),$$

where $c_{l,m}$ are some integers. Let us define

$$T(m) := (m, w_8) = 2m_1 + 3m_2 + 4m_3 + 6m_4 + 5m_5 + 4m_6 + 3m_7 + 2m_8. \tag{2.4}$$

Since w_8 is the highest root of E_8, the condition $l \leq m$ implies that $T(l) \leq T(m)$. We then derive the further decomposition

$$\mathrm{orb}(m) = \sum_{\substack{l \in \Lambda_+ \\ T(l) \leq T(m)}} c_l \prod_{i=1}^{8} \mathrm{orb}(w_i)^{l_i}, \quad c_m = 1. \tag{2.5}$$

In this paper we first compute the Fourier expansions of Sakai's forms in terms of Weyl orbits. This can be calculated efficiently by their definitions. We then use (2.5) to express their Fourier coefficients as polynomials in the eight fundamental Weyl orbits. This expression is very convenient for calculating the Fourier expansions of monomials in Sakai's generators. The q^0-term plays a crucial role in the study of weak Jacobi forms. We recall a useful result. [22, Lemma 4.2] states that the q^0-term of any $W(E_8)$-invariant weak Jacobi form of index t can be expressed as

$$\sum_{\substack{m \in \Lambda_+ \\ T(m) \leq t}} c_m \prod_{i=1}^{8} \mathrm{orb}(w_i)^{m_i}. \tag{2.6}$$

The number of monomials including the constant term in the above sum is equal to the rank of the free module $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ (see [22, Theorem 4.1] and

its proof). This number also equals the number of orbits of the quotient group E_8/tE_8 under the action of $W(E_8)$, and $\{m \in \Lambda_+ : T(m) \leq t\}$ gives a representative set of these orbits.

Finally, we describe the number of elements in a Weyl orbit $\mathrm{orb}(m)$, which equals the index of the stabilizer $W(E_8)_m$ in $W(E_8)$. We remind that Weyl orbits of the same norm do not necessarily have distinct number of elements (see Appendix B for some examples).

Lemma 2.2. *Let $m = \sum_{i=1}^{8} m_i w_i \in \Lambda_+$. Then the stabilizer of m in $W(E_8)$ is given by the intersection of all stabilizers of fundamental weights w_i with $m_i \neq 0$.*

Proof. Let $g \in W(E_8)$. Then $g(m) = m$ if and only if $\sum_{i=1}^{8} m_i g(w_i) = \sum_{i=1}^{8} m_i w_i$. With respect to the partial order on E_8, we have that $g(w_i) \leq w_i$ for $1 \leq i \leq 8$ (see [2]). Therefore, $g(m) = m$ if and only if $g(w_i) = w_i$ for all i such that $m_i \neq 0$. □

3. The proof of Theorem 1.1

In this section we prove Theorem 1.1. We divide its proof into several lemmas.

As explained in [22, §3.2, pp. 529–530], we take $g(\tau) = (5E_2(5\tau) - E_2(\tau))/4$ which is a modular form of weight 2 on the congruence subgroup $\Gamma_0(5)$, and define $\widehat{B}_5$ analogous to Sakai's B_j as

$$\widehat{B}_5(\tau,\mathfrak{z}) = \frac{5^4}{5^4-1}\left[g(\tau)\vartheta_{E_8}(5\tau,5\mathfrak{z}) - \frac{1}{5^5}\sum_{k=0}^{4} g\Big(\frac{\tau+k}{5}\Big)\vartheta_{E_8}\Big(\frac{\tau+k}{5},\mathfrak{z}\Big)\right]. \tag{3.1}$$

We note that $\widehat{B}_5$ is a $W(E_8)$-invariant holomorphic Jacobi form of weight 6 and index 5 whose reduction is $\widehat{B}_5(\tau,0) = E_6$.

Lemma 3.1. *The Jacobi form $\widehat{B}_5$ satisfies the identity*

$$179712\Delta E_4 \widehat{B}_5 = E_6 P_{16,5} + E_4 Q_{18,5}, \tag{3.2}$$

where

$$\begin{aligned} Q_{18,5} = &-2880A_1^3B_2 + 1350A_1A_2B_2E_4 + 1920A_1^2B_3E_4 - 70A_3B_2E_4^2 \\ &- 600A_2B_3E_4^2 - 60A_1B_4E_4^2 - 567A_1A_2^2E_6 + 672A_1^2A_3E_6 \\ &- 2400B_2B_3E_6 - 504A_2A_3E_4E_6 - 21A_5E_4^2E_6. \end{aligned}$$

Proof. Both sides of the above identity are $W(E_8)$-invariant Jacobi forms of weight 22 and index 5. We check by computer that their Fourier coefficients are the same up to q^4-terms. Thus their difference divided by Δ^4 defines a $W(E_8)$-invariant weak Jacobi form of weight -26 and index 5, which has to be zero because the weight of a non-zero $W(E_8)$-invariant weak Jacobi form of index 5 is at least -18 (see [22, Proposition 5.17]). This proves the desired identity. □

We remark that $P_{16,5}/E_4^2$ is non-holomorphic because $P_{16,5}$ reduces to $864E_4^4 + 2296E_4E_6^2$ when $\mathfrak{z} = 0$.

In [23] the second named author defined the Jacobian of Jacobi forms and used this tool to give a simple proof of Wirthmüller's theorem. We refer to [23, Proposition 2.2, Proposition 2.3] for details. We will apply this method to the case of E_8. Let us first calculate the Jacobian of Sakai's generators defined as the determinant

$$\frac{1}{(2\pi i)^9}\begin{vmatrix} A_1 & 2A_2 & 2B_2 & 3A_3 & 3B_3 & 4A_4 & 4B_4 & 5A_5 & 6B_6 \\ \frac{\partial A_1}{\partial z_1} & \frac{\partial A_2}{\partial z_1} & \frac{\partial B_2}{\partial z_1} & \frac{\partial A_3}{\partial z_1} & \frac{\partial B_3}{\partial z_1} & \frac{\partial A_4}{\partial z_1} & \frac{\partial B_4}{\partial z_1} & \frac{\partial A_5}{\partial z_1} & \frac{\partial B_6}{\partial z_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial A_1}{\partial z_8} & \frac{\partial A_2}{\partial z_8} & \frac{\partial B_2}{\partial z_8} & \frac{\partial A_3}{\partial z_8} & \frac{\partial B_3}{\partial z_8} & \frac{\partial A_4}{\partial z_8} & \frac{\partial B_4}{\partial z_8} & \frac{\partial A_5}{\partial z_8} & \frac{\partial B_6}{\partial z_8} \end{vmatrix}, \tag{3.3}$$

where z_i, $i = 1, \dots, 8$, are the standard basis of $\mathbb{R}^8$.

Lemma 3.2. *The modular Jacobian of Sakai's forms satisfies the identity*

$$J := J(A_1, A_2, B_2, A_3, B_3, A_4, B_4, A_5, B_6) = c\Delta^{14}E_4 \cdot \Phi_{E_8}, \qquad c = -\frac{3^3}{5^4 7^2}, \tag{3.4}$$

where Φ_{E_8} is the theta block associated to E_8 (see [10] for the general theory of theta blocks developed by Gritsenko, Skoruppa and Zagier)

$$\Phi_{E_8}(\tau, \mathfrak{z}) = \prod_r \frac{\vartheta(\tau, (r, \mathfrak{z}))}{\eta^3(\tau)}.$$

The above product takes over all positive roots of E_8 and ϑ is the odd Jacobi theta function

$$\vartheta(\tau, z) = q^{\frac{1}{8}}(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}) \prod_{n=1}^{\infty} (1-q^n\zeta)(1-q^n\zeta^{-1})(1-q^n), \quad q = e^{2\pi i\tau},\ \zeta = e^{2\pi iz}.$$

Proof. By [23, §4], Φ_{E_8} is a weak E_8 Jacobi form of weight -120 and index 30 which is anti-invariant under the action of $W(E_8)$ (i.e. invariant up to the determinant character). Moreover, it vanishes precisely on

$$\{(\tau, \mathfrak{z}) \in \mathbb{H} \times (E_8 \otimes \mathbb{C}) : (r, \mathfrak{z}) \in \mathbb{Z} + \mathbb{Z}\tau \text{ for some positive root } r\}$$

with multiplicity one. By [23, Proposition 2.2], J also vanishes on the above set. Thus J/Φ_{E_8} is holomorphic and defines a $W(E_8)$-invariant weak Jacobi form of weight 172 and index 0, which yields that J/Φ_{E_8} is a $\mathrm{SL}_2(\mathbb{Z})$-modular form of weight 172. We notice that every partial derivative of A_i or B_j cancels the q^0-term. By calculating the Fourier expansions of A_i and B_j up to q^7-terms, it is sufficient to calculate the q^{14}-term of J. We find that $J/\Phi_{E_8} = cq^{14} + O(q^{15})$, which implies that $J/\Phi_{E_8} = c\Delta^{14}E_4$. This completes the proof. □

Remark 3.3. By [23, Proposition 2.2 (2)], the above Jacobian J is not identically zero if and only if the forms A_i and B_j are algebraically independent over $\mathbb{C}[E_4, E_6]$. Thus Lemma 3.2 yields a new proof of the algebraic independence of Sakai's forms.

Lemma 3.4. *For any $W(E_8)$-invariant weak Jacobi form ϕ, there exists an integer N such that*

$$\Delta^N \phi \in \mathbb{C}[E_4, E_6, A_1, A_2, B_2, A_3, B_3, A_4, B_4, \widehat{A}_5, B_6],$$

where

$$\widehat{A}_5 := \frac{P_{16,5}}{E_4} - 21E_4^2 A_5.$$

Proof. From the definition of the modular Jacobian, we see that

$$\begin{aligned} &J(A_1, A_2, B_2, A_3, B_3, A_4, B_4, \widehat{A}_5, B_6) \\ &= J(A_1, A_2, B_2, A_3, B_3, A_4, B_4, P_{16,5} - 21E_4^3 A_5, B_6)/E_4 \\ &= -21 \cdot 1728\Delta J(A_1, A_2, B_2, A_3, B_3, A_4, B_4, A_5, B_6)/E_4. \end{aligned}$$

By Lemma 3.2, we have

$$J(A_1, A_2, B_2, A_3, B_3, A_4, B_4, \widehat{A}_5, B_6)/\Phi_{E_8} = -36288c\Delta^{15}.$$

We then prove the claim by applying the criterion established in [23, Proposition 2.3 (2)]. □

We also need the following eight specific $W(E_8)$-invariant weak Jacobi forms of weight 4 whose q^0-terms are a single fundamental Weyl orbit.

Lemma 3.5. *The following $W(E_8)$-invariant weak Jacobi forms $\varphi_{-,t}$ of weight 4 and index t with indicated q^0-term exist.*

$$\begin{aligned}
\varphi_{4a,2} &= P_1(E_4,E_6,A_i,B_j)/\Delta = \mathrm{orb}(w_1)+O(q),\\
\varphi_{4b,2} &= P_2(E_4,E_6,A_i,B_j)/\Delta = \mathrm{orb}(w_8)+O(q),\\
\varphi_{4a,3} &= P_3(E_4,E_6,A_i,B_j)/\Delta = \mathrm{orb}(w_7)+O(q),\\
\varphi_{4b,3} &= P_4(E_4,E_6,A_i,B_j)/\Delta^2 = \mathrm{orb}(w_2)+O(q),\\
\varphi_{4a,4} &= P_5(E_4,E_6,A_i,B_j)/\Delta^3 = \mathrm{orb}(w_3)+O(q),\\
\varphi_{4b,4} &= P_6(E_4,E_6,A_i,B_j)/\Delta^3 = \mathrm{orb}(w_6)+O(q),\\
\varphi_{4,5} &= P_7(E_4,E_6,A_i,B_j)/\Delta^3 = \mathrm{orb}(w_5)+O(q),\\
\varphi_{4,6} &= P_8(E_4,E_6,A_i,B_j)/\Delta^5 = \mathrm{orb}(w_4)+O(q),
\end{aligned}$$

where

$$\begin{aligned}
P_1 &= \frac{1}{4}\left(-12A_1^2E_4^2+17A_2E_4^3+10A_2E_6^2-15B_2E_4E_6\right),\\
P_2 &= \frac{1}{72}(24A_1^2E_4^2-14A_2E_4^3+5A_2E_6^2-15B_2E_4E_6),\\
P_3 &= \frac{7}{18}(-27A_1A_2E_4^2-45A_1B_2E_6+37A_3E_4^3+35A_3E_6^2),\\
P_4 &= \frac{1}{864}(126A_1^3E_4^4-414A_1^3E_4E_6^2+675A_1A_2E_4^2E_6^2-243A_1A_2E_4^5\\
&\qquad -1440A_1B_2E_6^3-251A_3E_4^3E_6^2+122A_3E_4^6+465A_3E_6^4\\
&\qquad -20B_3E_4^4E_6+980B_3E_4E_6^3).
\end{aligned}$$

The other polynomials are very long and we omit their expressions here. When the index is greater than 2 the above polynomial is not unique because the associated space $J^{\mathrm{w},W(E_8)}_{-8,E_8,t}$ is non-trivial.

Proof of Theorem 1.1. (1) This is a direct consequence of Lemma 3.1.

(2) It follows from Lemma 3.4.

(3) Let φ_t be a $W(E_8)$-invariant weak Jacobi form of index t. By Lemma 3.4, there exists an integer N such that

$$\varphi_t = \frac{\sum_{j=0}^{t_1} P_j E_4^j P_{16,5}^{t_1-j}}{\Delta^N E_4^{t_1}}. \tag{3.5}$$

To prove the third assertion, it suffices to show that one can choose N satisfying $N \leq N_t$. Recall that the space of all $W(E_8)$-invariant weak Jacobi forms of fixed index t is a free module of rank $r(t)$ over $M_*(\mathrm{SL}_2(\mathbb{Z}))$ (see [22, Theorem 4.1]). It is enough to prove that one can always choose $N \leq N_t$ such that (3.5) holds for every generator of the free module. Let ϕ_j, $1 \leq j \leq r(t)$, be the generators of the free module $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$. Clearly, every ϕ_j can be expressed in the form (3.5). We denote by M_j the smallest integer N such that (3.5) holds for ϕ_j. Assume that M_1 is the largest of all M_j. Obviously, every generator ϕ_j has non-zero q^0-term. Moreover, the q^0-term of every ϕ_j is a polynomial in the eight fundamental Weyl orbits satisfying the restriction defined in (2.6). By [22, Lemma 4.2] and the construction of forms in Lemma 3.5, for any sufficiently large integer D there exists a $W(E_8)$-invariant weak Jacobi form ψ_t of index t in the ring

$$\mathbb{C}[E_4, E_6, A_1, \varphi_{4a,2}, \varphi_{4b,2}, \varphi_{4a,3}, \varphi_{4b,3}, \varphi_{4a,4}, \varphi_{4b,4}, \varphi_{4,5}, \varphi_{4,6}]$$

such that the q^0-term of the difference $E_4^D\phi_1 - \psi_t$ is zero. Therefore, $(E_4^D\phi_1 - \psi_t)/\Delta$ is a $W(E_8)$-invariant weak Jacobi form of index t and then a $\mathbb{C}[E_4, E_6]$-linear combination of the generators ϕ_j. It follows that the number M_1 defined above is exactly the smallest integer N appearing in the expression of type (3.5) for ψ_t. We then derive the formula of N_t from the powers of Δ in the construction of basic forms in Lemma 3.5. □

4. Free modules of weak Jacobi forms of given index

It was proved in [22, Theorem 4.1] that the space

$$J^{\mathrm{w},W(E_8)}_{*,E_8,t} := \bigoplus_{k\in\mathbb{Z}} J^{\mathrm{w},W(E_8)}_{k,E_8,t}$$

of $W(E_8)$-invariant weak Jacobi forms of integral weight and given index t is a free module over $M_*(\mathrm{SL}_2(\mathbb{Z}))$ and the rank $r(t)$ is given by the generating series

$$\frac{1}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)^2(1-x^5)(1-x^6)} = \sum_{t\geq 0} r(t)x^t. \tag{4.1}$$

We formulate the first values of $r(t)$ in Table 1.

In [22, §5] the generators were determined and constructed when the index is less than 5 by an approach based on the weight raising differential operators of Jacobi forms. In this section, as an application of Theorem 1.1,

Table 1: The rank of $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ over $M_*(\mathrm{SL}_2(\mathbb{Z}))$

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$r(t)$	1	3	5	10	15	27	39	63	90	135	187	270	364	505	670

we determine the weights of generators and construct the generators in terms of Sakai's forms A_i and B_j when the index is less than 14. To this aim, we introduce the following algorithm.

Algorithm 4.1.

1. We determine a basis of $J^{\mathrm{w},W(E_8)}_{k,E_8,t}$ for any even weight k. The space $J^{\mathrm{w},W(E_8)}_{k,E_8,t}$ is a finite-dimensional vector space over $\mathbb{C}$. Every form in this space has an expression of form (1.2) which corresponds to a solution of the system of linear equations defined by the vanishing of q^n-terms $(0 \le n \le N_t - 1)$ in the Fourier expansion

$$\sum_{j=0}^{t_1} P_j E_4^j P_{16,5}^{t_1-j} = O(q^{N_t}) \tag{4.2}$$

   in weight $k + 12N_t + 4t_1$. There are only finitely many linearly independent solutions in any fixed weight, because the weights and indices of these P_j (polynomials in E_4, E_6, A_i, B_j) are bounded from above.
2. We determine the minimal weight of non-zero $W(E_8)$-invariant weak Jacobi forms of given index t. If the equation (4.2) has only zero solution in weight $K+12N_t+4t_1$ and in weight $K+12N_t+4t_1-2$, then its solution is always trivial in any lower weight and thus $J^{\mathrm{w},W(E_8)}_{k,E_8,t} = \{0\}$ when $k \le K$.
3. We collect all $r(t)$ generators of the free module $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ from the minimal weight K to the larger weight.

We first calculate the Fourier expansions of Sakai's forms A_i and B_j up to q^9-terms. The Fourier expansions involve 268 Weyl orbits of vectors of norm $\frac{1}{2}(v,v) \le 54$. We then express these Weyl orbits as polynomials in the eight fundamental Weyl orbits. Using the data and Algorithm 4.1, we successfully determine all generators of $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ for $1 \le t \le 13$.

Theorem 4.2. *Let $d_{k,t}$ denote the number of generators of weight k of $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$. For $1 \le t \le 13$ the Laurent polynomials*

$$P^{\mathrm{w}}_t := \sum_{k\in\mathbb{Z}} d_{k,t} x^k$$

describing the weights of generators are determined as follows

$$\begin{aligned}
P^{\mathrm{w}}_1 &= x^4, \qquad P^{\mathrm{w}}_2 = x^{-4}+x^{-2}+1, \qquad P^{\mathrm{w}}_3 = x^{-8}+x^{-6}+x^{-4}+x^{-2}+1,\\
P^{\mathrm{w}}_4 &= x^{-16}+x^{-14}+x^{-12}+x^{-10}+2x^{-8}+x^{-6}+x^{-4}+x^{-2}+1,\\
P^{\mathrm{w}}_5 &= 2x^{-16}+2x^{-14}+3x^{-12}+2x^{-10}+2x^{-8}+x^{-6}+x^{-4}+x^{-2}+1,\\
P^{\mathrm{w}}_6 &= 2x^{-24}+2x^{-22}+3x^{-20}+3x^{-18}+3x^{-16}+3x^{-14}+3x^{-12}+2x^{-10}\\
&\quad +2x^{-8}+x^{-6}+x^{-4}+x^{-2}+1,\\
P^{\mathrm{w}}_7 &= x^{-26}+3x^{-24}+5x^{-22}+7x^{-20}+4x^{-18}+4x^{-16}+4x^{-14}+3x^{-12}\\
&\quad +2x^{-10}+2x^{-8}+x^{-6}+x^{-4}+x^{-2}+1,\\
P^{\mathrm{w}}_8 &= 2x^{-32}+4x^{-30}+7x^{-28}+6x^{-26}+7x^{-24}+6x^{-22}+6x^{-20}+5x^{-18}\\
&\quad +5x^{-16}+4x^{-14}+3x^{-12}+2x^{-10}+2x^{-8}+x^{-6}+x^{-4}+x^{-2}+1,\\
P^{\mathrm{w}}_9 &= x^{-36}+2x^{-34}+8x^{-32}+10x^{-30}+11x^{-28}+9x^{-26}+9x^{-24}+7x^{-22}\\
&\quad +7x^{-20}+6x^{-18}+5x^{-16}+4x^{-14}+3x^{-12}+2x^{-10}+2x^{-8}+x^{-6}\\
&\quad +x^{-4}+x^{-2}+1,\\
P^{\mathrm{w}}_{10} &= 4x^{-40}+7x^{-38}+11x^{-36}+12x^{-34}+14x^{-32}+12x^{-30}+12x^{-28}\\
&\quad +11x^{-26}+10x^{-24}+8x^{-22}+8x^{-20}+6x^{-18}+5x^{-16}+4x^{-14}\\
&\quad +3x^{-12}+2x^{-10}+2x^{-8}+x^{-6}+x^{-4}+x^{-2}+1,\\
P^{\mathrm{w}}_{11} &= 5x^{-42}+15x^{-40}+19x^{-38}+20x^{-36}+16x^{-34}+17x^{-32}+15x^{-30}\\
&\quad +14x^{-28}+12x^{-26}+11x^{-24}+9x^{-22}+8x^{-20}+6x^{-18}+5x^{-16}\\
&\quad +4x^{-14}+3x^{-12}+2x^{-10}+2x^{-8}+x^{-6}+x^{-4}+x^{-2}+1,\\
P^{\mathrm{w}}_{12} &= 8x^{-48}+13x^{-46}+21x^{-44}+22x^{-42}+22x^{-40}+22x^{-38}+22x^{-36}\\
&\quad +20x^{-34}+20x^{-32}+17x^{-30}+15x^{-28}+13x^{-26}+12x^{-24}+9x^{-22}\\
&\quad +8x^{-20}+6x^{-18}+5x^{-16}+4x^{-14}+3x^{-12}+2x^{-10}+2x^{-8}+x^{-6}\\
&\quad +x^{-4}+x^{-2}+1,\\
P^{\mathrm{w}}_{13} &= 2x^{-52}+10x^{-50}+24x^{-48}+32x^{-46}+37x^{-44}+28x^{-42}+29x^{-40}\\
&\quad +28x^{-38}+26x^{-36}+23x^{-34}+22x^{-32}+18x^{-30}+16x^{-28}+14x^{-26}\\
&\quad +12x^{-24}+9x^{-22}+8x^{-20}+6x^{-18}+5x^{-16}+4x^{-14}+3x^{-12}+2x^{-10}
\end{aligned}$$

$$+ 2x^{-8} + x^{-6} + x^{-4} + x^{-2} + 1.$$

Clearly, the Laurent expansion of the following rational function at $x = 0$ gives the dimension of the space of weak Jacobi forms of arbitrary weight and given index t

$$\frac{P_t^{\mathrm{w}}}{(1-x^4)(1-x^6)} = \frac{\sum_{k\in\mathbb{Z}} d_{k,t}x^k}{(1-x^4)(1-x^6)} = \sum_{k\in\mathbb{Z}} \dim J^{\mathrm{w},W(E_8)}_{k,E_8,t} x^k.$$

This series is called the generating series of weak Jacobi forms of given index. We will present these generating series separately in Appendix A.

At the end of this section, we explicitly show some generators. For $W(E_8)$-invariant weak Jacobi form of index $t = 2$, the three generators are

(4.3)
$$\phi_{-4,2} = \frac{A_1^2 - A_2E_4}{\Delta}, \quad \phi_{-2,2} = \frac{A_2E_6 - B_2E_4}{\Delta}, \quad \phi_{0,2} = \frac{A_1^2E_4 - B_2E_6}{\Delta}.$$

For index $t = 3$, the five generators are constructed as

(4.4)
$$\begin{aligned}
\phi_{-8,3} &= \frac{1}{\Delta^2}(6A_1^3E_4 - 9A_1A_2E_4^2 + A_3(3E_4^3 - 10E_6^2) \\
&\qquad + 30A_1B_2E_6 - 20B_3E_4E_6), \\
\phi_{-6,3} &= \frac{1}{\Delta^2}(6A_1^3E_6 + 3A_1E_4(10B_2E_4 - 3A_2E_6) \\
&\qquad - E_4^2(20B_3E_4 + 7A_3E_6)), \\
\phi_{-4,3} &= \frac{1}{\Delta}(A_1A_2 - A_3E_4), \\
\phi_{-2,3} &= \frac{1}{\Delta}(A_1B_2 - A_3E_6), \\
\phi_{0,3} &= \frac{1}{\Delta}(A_1^3 - B_3E_6).
\end{aligned}$$

For index $t = 13$, the two lowest weight -52 generators are

$$\begin{aligned}
\phi_{-52a,13} = \frac{1}{\Delta^{10}E_4^2}(&281154281472A_1^{13}E_4^6 - 935841724416A_1^{11}A_2E_4^7 \\
&- 1468672041600A_1^9B_2^2E_4^7 + \dots)
\end{aligned}$$

and $\phi_{-52b,13} = \phi_{-16,4}\phi_{-36,9}$, where

(4.5)
$$\phi_{-16,4} = \frac{1}{\Delta^3}(192A_1^4E_4 + 75B_2^2E_4^2 + 9A_2^2E_4^3 - A_4E_4^4 + 90A_2B_2E_4E_6 + \dots),$$
$$\phi_{-36,9} = \frac{1}{\Delta^7 E_4}(1026432A_1^9E_4^4 - 2659392A_1^7A_2E_4^5 - 5891400A_1^5B_2^2E_4^5 + \dots),$$

are the unique generators of the lowest weight for index 4 and 9 respectively.

5. Weyl invariant holomorphic Jacobi forms

Similar to weak Jacobi forms, the space of $W(E_8)$-invariant holomorphic Jacobi forms of integral weight and given index t

$$J^{W(E_8)}_{*,E_8,t} := \bigoplus_{k=4}^{\infty} J^{W(E_8)}_{k,E_8,t}$$

is also a free module of rank $r(t)$ over $M_*(\mathrm{SL}_2(\mathbb{Z}))$. In this section we introduce some methods to compute the dimension of the space of holomorphic Jacobi forms. Firstly, the following lemma shows that the difference between the dimensions of the spaces of weak Jacobi forms and holomorphic Jacobi forms depends only on the index when the weight is greater than 4.

Proposition 5.1. *For any $t \geq 1$ and any even $k \geq 6$, the following identity holds*

$$\dim J^{\mathrm{w},W(E_8)}_{k,E_8,t} - \dim J^{W(E_8)}_{k,E_8,t} = \delta_t, \tag{5.1}$$

where

$$\delta_t = \sum_{a=1}^{\infty} \epsilon_t(a) \cdot \delta_t(a), \quad \epsilon_t(a) := \min\left\{x \in \mathbb{Z} : x \geq \frac{a}{t}\right\},$$

and $\delta_t(a)$ is the number of elements of the set $\mathbb{S}_t(a)$ defined by

$$\mathbb{S}_t(a) = \{x = (x_i)_{i=1}^8 \in \mathbb{N}^8 \setminus \{0\} : 2x_1 + 3x_2 + 4x_3 + 6x_4 + 5x_5 + 4x_6 + 3x_7 + 2x_8 \leq t, x^t S x = 2a\},$$

here S is the Gram matrix associated to the fundamental weights w_i of E_8 fixed in [22, §3.1]. The first values of δ_t are formulated in Table 2.

Table 2: The value of δ_t

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
δ_t	0	2	5	13	23	52	82	154	240	403	601	959	1373	2063	2911

Proof. On the one hand, for any weak Jacobi form $\phi_t \in J^{\mathrm{w},W(E_8)}_{k,E_8,t}$, if its Fourier expansion has no the following representatives of singular terms (i.e. $f(n,\ell)q^n \cdot \mathrm{orb}(\ell)$ satisfying $2nt-(\ell,\ell)<0$)

$$q^n \cdot \mathrm{orb}(x),\ 0 \le n < \epsilon_t(a),\ x \in \mathbb{S}_t(a),$$

then ϕ_t is a holomorphic Jacobi form. Recall that $\mathrm{orb}(x)$ stands for the Weyl orbit of the vector $\sum_{i=1}^8 x_i w_i \in \Lambda_+$. This yields that $\dim J^{\mathrm{w},W(E_8)}_{k,E_8,t} - \dim J^{W(E_8)}_{k,E_8,t} \le \delta_t$.

On the other hand, we can view Jacobi forms as vector-valued modular forms. By the theory of vector-valued modular forms for the Weil representation attached to the discriminant form of the rescaled lattice $E_8(t)$ (see [1, Theorem 3.1] or [3, Theorem 1.17]), the obstruction space, namely the space of cusp forms for the dual Weil representation, has weight $6-k$ and thus is trivial when $k \ge 6$. In the context of Jacobi forms, this implies that for each singular term above there exists a $W(E_8)$-invariant weak Jacobi form of weight k and index t whose Fourier expansion contains the given singular term but does not contain other singular terms. From this we conclude that $\dim J^{\mathrm{w},W(E_8)}_{k,E_8,t} - \dim J^{W(E_8)}_{k,E_8,t} \ge \delta_t$. We then prove the desired identity. □

By Proposition 5.1 and the generating series of $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$, we can determine immediately the dimension of the space of $W(E_8)$-invariant holomorphic Jacobi forms of weight k and index t when $1 \le t \le 13$ and $k \ge 6$. It remains to determine the space of holomorphic Jacobi forms of weight 4. We see from the above proof that

$$\dim J^{\mathrm{w},W(E_8)}_{4,E_8,t} - \dim J^{W(E_8)}_{4,E_8,t} \le \delta_t. \tag{5.2}$$

The value of $\dim J^{W(E_8)}_{4,E_8,t}$ has been determined in [22, Lemma 5.5] when $t \le 6$. By comparing the dimensions, we find that

$$\dim J^{\mathrm{w},W(E_8)}_{4,E_8,t} - \dim J^{W(E_8)}_{4,E_8,t} = \delta_t, \quad \text{when } t \le 6. \tag{5.3}$$

However, the identity of type (5.3) does not hold when $t \ge 7$. For example, $\dim J^{W(E_8)}_{4,E_8,8} \ge 2$, but $\dim J^{\mathrm{w},W(E_8)}_{4,E_8,8} = \delta_8 = 154$.

As another application of Theorem 1.1, we compute the dimension of the space of $W(E_8)$-invariant holomorphic Jacobi forms of weight 4 and small index. These so-called holomorphic Jacobi forms of singular (i.e. possible minimal positive) weight are usually difficult to determine and construct in the theory of modular forms.

Proposition 5.2. *The dimension of the space* $J^{W(E_8)}_{4,E_8,t}$ *for* $t \leq 11$ *is formulated in Table 3.*

Table 3: The dimension of $J^{W(E_8)}_{4,E_8,t}$

t	1	2	3	4	5	6	7	8	9	10	11
dim.	1	1	1	2	1	1	2	2	2	2	2

Proof. There exist $W(E_8)$-invariant holomorphic Jacobi forms of weight 4 and arbitrary index with Fourier expansion $1 + O(q)$ (e.x. $X_t = 1 + O(q)$, see (2.1)). Thus we only need to determine holomorphic Jacobi forms of weight 4 and index t whose q^0-term is zero. By Theorem 1.1, such forms can be expressed as

$$\frac{\sum_{j=0}^{t_1} P_j E_4^j P_{16,5}^{t_1-j}}{\Delta^{N_t-1} E_4^{t_1}}.$$

Since these holomorphic forms have singular weight 4, their Fourier expansion only involves Fourier coefficients $f(n,\ell)e^{2\pi i(n\tau+(\ell,\mathfrak{z}))}$ satisfying $(\ell,\ell) = 2nt$ (see [9]). Moreover, these forms are completely determined by coefficients of the following terms in their Fourier expansion:

$$q^{\frac{1}{2t}(m,m)}\,\mathrm{orb}(m)$$

where m are non-zero vectors satisfying $(m,m) \in 2t\mathbb{Z}$ in the set (see (2.4) for $T(m)$)

$$\mathcal{A}_t := \left\{ m = \sum_{i=1}^{8} m_i w_i \in \Lambda_+ : T(m) \leq t \right\}.$$

We define M_t as the smallest integer greater than or equal to the number

$$\max\left\{ \frac{1}{2t}(m,m) : m \in \mathcal{A}_t \right\}.$$

A $W(E_8)$-invariant holomorphic Jacobi form of weight 4 and index t whose q^0-term is zero corresponds to a solution of the system of linear equations

defined by the Fourier expansion

$$\frac{\sum_{j=0}^{t_1} P_j E_4^j P_{16,5}^{t_1-j}}{\Delta^{N_t-1} E_4^{t_1}} = \sum_{n=1}^{M_t-1} q^n \sum_{\substack{m\in\Lambda_+ \\ (m,m)=2nt}} \mathrm{orb}(m) + O(q^{M_t}). \tag{5.4}$$

(By the proof of Proposition 5.1, there is no singular Fourier coefficient in q^n-term when $n \geq M_t$. Hence the above expression with given Fourier expansion must be a holomorphic Jacobi form.) We then prove the proposition by solving these systems of linear equations. When $t \geq 9$, our data on the Fourier coefficients of A_i and B_j is not sufficient to solve directly (5.4). In this case, we first solve (5.4) up to q^{10-N_t}-terms. Fortunately, we find that all solutions have an expression of type (5.4), replacing $N_t - 1$ with a smaller power. We then prove the result for $t = 9, 10, 11$ by solving an analogue of (5.4) with a smaller power of Δ. □

We give some direct constructions of generators of $J^{W(E_8)}_{4,E_8,t}$. When the space is one-dimensional, it is generated by the form $X_t = 1 + O(q)$ constructed in (2.1). When $t = 4, 8, 9$, we construct the second generator as $A_4 = A_1(\tau, 2\mathfrak{z})$, $A_2(\tau, 2\mathfrak{z})$ and $A_1(\tau, 3\mathfrak{z})$ respectively. We do not know how to construct the second generator of $J^{W(E_8)}_{4,E_8,t}$ in a direct way for $t = 7, 10, 11$. However, they can be constructed in terms of Sakai's forms. We present this nice construction in Appendix B. Combining Proposition 5.1 and Proposition 5.2, it is easy to derive the generating series of $J^{W(E_8)}_{*,E_8,t}$ from the generating series of weak Jacobi forms. We omit these series here.

6. Some conjectures

In this section we formulate some conjectures related to our work. The pullback to $W(E_7)$-invariant Jacobi forms implies that the minimal weight of $W(E_8)$-invariant weak Jacobi forms of index t is not less than $-5t$ (see [22, Proposition 6.1]). Here we propose a conjecture about the exact minimal weight.

Conjecture 6.1. The weight of non-zero $W(E_8)$-invariant weak Jacobi forms of index t is not less than $-4t$.

By our results in §4, there exist $W(E_8)$-invariant weak Jacobi forms of weight $-4t$ and index t if t is even and greater than 2, or if $t = 9$, or if t is odd and greater than 11. These forms can be constructed as monomials in our generators of index less than 10. We give some evidence for this conjecture in the following lemma.

Lemma 6.2. *Let $t \geq 2$ and $\varphi_{k,t} \in J^{\mathrm{w},W(E_8)}_{k,E_8,t}$. We assume that the coefficient of $\mathrm{orb}(\frac{t}{2}w_1)$ (resp. $\mathrm{orb}(\frac{t-3}{2}w_1+w_2)$) in the q^0-term of $\varphi_{k,t}$ is non-zero when t is even (resp. odd). Then $k \geq -4t$ if t is even, and $k \geq -4(t-1)$ if t is odd.*

Proof. We use the pull-back trick built in [22, §5.6]. Let v_4 be a vector of E_8 satisfying $(v_4, v_4) = 4$. If $\varphi_{k,t} \in J^{\mathrm{w},W(E_8)}_{k,E_8,t}$, then $\varphi_{k,t}(\tau, zv_4) \in J^{\mathrm{w}}_{k,2t}$ which is the space of weak Jacobi forms of weight k and index $2t$ in the sense of Eichler and Zagier. Recall that the ring of classical weak Jacobi forms of integral index and even weight is freely generated by forms of index 1 and weight -2 and 0 denoted $\phi_{-2,1}$ and $\phi_{0,1}$. We calculate

$$\begin{aligned}\max(\mathrm{orb}(x), v_4) &:= \max\{(y, v_4) : y \in W(E_8)x\}\\ &= \max\{(x, u_4) : u_4 \in W(E_8)v_4\}.\end{aligned}$$

We find that

$$\begin{array}{lll}\max(\mathrm{orb}(w_1), v_4) = 4 & \max(\mathrm{orb}(w_2), v_4) = 5 & \max(\mathrm{orb}(w_3), v_4) = 7\\ \max(\mathrm{orb}(w_4), v_4) = 10 & \max(\mathrm{orb}(w_5), v_4) = 8 & \max(\mathrm{orb}(w_6), v_4) = 6\\ \max(\mathrm{orb}(w_7), v_4) = 4 & \max(\mathrm{orb}(w_8), v_4) = 2. & \end{array}$$

Since $\max(\mathrm{orb}(x+y), v_4) = \max(\mathrm{orb}(x), v_4) + \max(\mathrm{orb}(y), v_4)$ and the average contributions of the eight fundamental Weyl orbits (with respect to index one) are respectively 4/2, 5/3, 7/4, 10/6, 8/5, 6/4, 4/3, 2/2, we conclude the following:

1. When $t = 2m$, $\max(\mathrm{orb}(mw_1), v_4) = 4m$ and $\max(\mathrm{orb}(x), v_4) < 4m$ for all other $x \in \mathcal{A}_t$;
2. When $t = 2m+1$, $\max(\mathrm{orb}((m-1)w_1 + w_2), v_4) = 4m+1$ and $\max(\mathrm{orb}(x), v_4) < 4m+1$ for all other $x \in \mathcal{A}_t$.

We note that the set $\mathcal{A}_t$ is defined in the proof of Proposition 5.2.

When $t = 2m$, the pull-back $\varphi_{k,t}(\tau, zv_4) = (\zeta^{\pm 4m} + \cdots) + O(q)$ is obviously non-zero, where $\zeta = e^{2\pi i z}$. If $k < -8m$, then $\varphi_{k,t}(\tau, zv_4) \in \phi^{4m}_{-2,1} \cdot J^{\mathrm{w}}_{k+8m,0}$, which leads to a contradiction because there is no non-zero $\mathrm{SL}_2(\mathbb{Z})$ modular form of negative weight.

When $t = 2m+1$, the pull-back $\varphi_{k,t}(\tau, zv_4) = (\zeta^{\pm(4m+1)} + \cdots) + O(q)$ is also non-zero. If $k < -8m$, then $\varphi_{k,t}(\tau, zv_4) \in \phi^{4m}_{-2,1} \cdot J^{\mathrm{w}}_{k+8m,2}$. In this case, $k+8m < 0$. The only classical weak Jacobi forms of negative even weight and index 2 are $\phi^2_{-2,1}$ and $\phi_{-2,1}\phi_{0,1}$, which all have leading Fourier coefficients $\zeta^{\pm 2}$ in their q^0-terms. This contradicts the q^0-term of $\varphi_{k,t}(\tau, zv_4)$. □

We have checked that the q^0-term of the unique $W(E_8)$-invariant weak Jacobi form of weight -36 and index 9 has no the Weyl orbit $\mathrm{orb}(3w_1+w_2)$, which is consistent with the above result.

Let $d_{k,t}$ be the number of weight k generators of the free module $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$. By observing the generating series, we have the following conjecture on the stability.

Conjecture 6.3. For any even $K \leq 0$, there exist positive integer $L(K)$ such that for any fixed k satisfying $K \leq k \leq 0$, the number $d_{k,t}$ is constant for all $t \geq L(K)$.

We list the $L(K)$ and the relevant stable constants for weight $K \geq -24$ in Table 4.

Table 4: The values of $L(K)$ and $d_{K,L(K)}$

K	0	-2	-4	-6	-8	-10	-12	-14	-16	-18	-20	-22	-24
$L(K)$	2	2	2	3	4	5	5	7	8	9	10	11	12
$d_{K,L(K)}$	1	1	1	1	2	2	3	4	5	6	8	9	12

For any irreducible root system not of type E_8, the ring of Weyl invariant weak Jacobi forms is a polynomial algebra. In such case, it is easy to prove the analogue of the above conjecture. We now prove the above conjecture for some special weights.

Proposition 6.4. *For any $t \geq 2$, the free module $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ is generated by forms of non-positive weight. Moreover, the number of generators of weight 0, -2 and -4 are all one.*

Proof. Let t be a positive integer greater than 9. We first fix some weak Jacobi forms. Let $\varphi_{-4,2}$, $\varphi_{-2,2}$ and $\varphi_{0,2}$ be the generators of $J^{\mathrm{w},W(E_8)}_{*,E_8,2}$. Let $\varphi_{-8,3}$ and $\varphi_{-6,3}$ be the generators of weight -8 and -6 of $J^{\mathrm{w},W(E_8)}_{*,E_8,3}$. Let $\varphi_{-16,4}$ and $\varphi_{-14,4}$ be the generators of weight -16 and -14 of $J^{\mathrm{w},W(E_8)}_{*,E_8,4}$. Let $\varphi_{-16,5}$ be the generator of weight -16 of $J^{\mathrm{w},W(E_8)}_{*,E_8,5}$ whose q^0-term involves the fundamental Weyl orbit $\mathrm{orb}(w_5)$. Let $\varphi_{-24,6}$ be the generator of weight -24 of $J^{\mathrm{w},W(E_8)}_{*,E_8,6}$ whose q^0-term involves $\mathrm{orb}(w_4)$. Since the q^0-terms of the eight negative-weight forms $\varphi_{-,-}$ involve the eight fundamental Weyl orbits respectively, the monomials of the eight forms in index t have q^0-terms involving all monomials of fundamental Weyl orbits $\prod_{i=1}^{8}\mathrm{orb}(w_i)^{m_i}$ with $T(m)=t$. Thus they are linearly independent over $M_*(\mathrm{SL}_2(\mathbb{Z}))$. Moreover, their weights are not greater than $-t$. Similarly, the products of their

monomials in index $t-1$ with A_1 have weight $\leq 5-t$ and have q^0-terms involving all monomials $\prod_{i=1}^{8}\mathrm{orb}(w_i)^{m_i}$ satisfying $T(m)=t-1$. We prove the proposition by induction on t. Suppose that $J^{\mathrm{w},W(E_8)}_{*,E_8,t-2}$ is generated by forms of non-positive weight and has only one generator of weight 0, -2 and -4 respectively. Then the products of the generators of $J^{\mathrm{w},W(E_8)}_{*,E_8,t-2}$ with $\varphi_{0,2}$ have q^0-terms involving all $\prod_{i=1}^{8}\mathrm{orb}(w_i)^{m_i}$ with $T(m)\leq t-2$. We have constructed $r(t)$ $W(E_8)$-invariant weak Jacobi forms of index t which are linearly independent over $M_*(\mathrm{SL}_2(\mathbb{Z}))$. All of them have non-positive weight and the numbers of forms of weight 0, -2, -4 are all one. Therefore, $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ is generated by forms of non-positive weight and there is at most one generator of weight k for $k=0,-2,-4$. It remains to prove that there do exist generators of weight $0,-2,-4$. The reduction of any weak Jacobi form of negative weight is identically zero when $\mathfrak{z}=0$. However, there are weak Jacobi forms of weight 0 whose reduction is not zero. Therefore, there are generators of weight 0. If there is no generator of weight -2, then there are $r(t)$ weak Jacobi forms of weight 0 whose q^0-terms are linearly independent. It follows that there exists a weak Jacobi form of weight 0 whose q^0-term is non-zero constant, which contradicts [22, Lemma 3.5]. If there is no generator of weight -4, then there are $r(t)-1$ weak Jacobi forms of weight -2 whose q^0-terms are linearly independent. Hence there exists a weak Jacobi form of weight -2 whose q^0-term is $\mathrm{orb}(w_8)-240$. Acting the differential operator on this form (see [22, Lemma 3.4]), we can construct a weak Jacobi form of weight 0 whose q^0-term is $(\frac{1}{2}-\frac{1}{t})\,\mathrm{orb}(w_8)-120$, which contradicts [22, Lemma 3.5] again. We have thus proved the desired result. □

In principle, the proof above should be able to extend to the cases of lower weights.

It is known that the ring of weak Jacobi forms of integral weight and index one for a unimodular lattice L is generated over $M_*(\mathrm{SL}_2(\mathbb{Z}))$ by the Jacobi theta function associated to L which has positive weight $\frac{1}{2}\mathrm{rank}(L)$. As tE_8 lattice is no longer unimodular for $t\geq 2$, inspired by the above result, we formulate a similar conjecture for general Jacobi forms on non-unimodular lattices.

Conjecture 6.5. Let L be an even positive definite lattice. Assume that L is irreducible and is not unimodular. Then the free module of weak Jacobi forms of integral weight and index one associated to L is generated by forms of non-positive weight.

We also make a conjecture on $W(E_8)$-invariant holomorphic Jacobi forms of singular weight.

Conjecture 6.6. Let $H(t)$ be the dimension of the space of $W(E_8)$-invariant holomorphic Jacobi forms of weight 4 and positive index t. Let $N(t)$ denote the number of distinct Weyl orbits of vectors of norm t (i.e. $\frac{1}{2}(v,v) = t$). Then

$$H(t) = N(t).$$

Equivalently, for any Weyl orbit $\mathrm{orb}(m)$ of norm t, there exists a unique $W(E_8)$-invariant holomorphic Jacobi form of weight 4 and index t which has the Fourier expansion

$$\Phi_{t,m} = 1 + \frac{240}{|\operatorname{orb}(m)|} q \cdot \operatorname{orb}(m) + O(q^2).$$

This conjecture has been proved for index $t \leq 11$ in Proposition 5.2. Because we only calculated the Fourier expansions of A_i and B_j up to q^9-terms, it is not sufficient to extend Proposition 5.2 to index 12 and 13. However, by solving Jacobi forms of weight 4 which can be expressed as $P(E_4, E_6, A_i, B_j)/\Delta^5 E_4^2$ and have Fourier expansion of type (5.4) up to q^4-terms, we find that the dimension of the solution space (including the form X_t) is 2 when $t = 12$ and 3 when $t = 13$. These are consistent with the above conjecture. We formulate some values of $N(t)$ in Table 5 below.

Table 5: The value of $N(t)$

t	1	2	3	4	5	6	7	8	9	10	11	12
$N(t)$	1	1	1	2	1	1	2	2	2	2	2	2
t	13	14	15	16	17	18	19	20	21	22	23	
$N(t)$	3	2	2	4	3	3	4	3	3	4	4	

Finally, we formulate a conjecture on the global structure of $J^{\mathrm{w},W(E_8)}_{*,E_8,*}$. This is motivated by Conjecture 6.1 and Conjecture 6.3. In addition, it was proved in [24] that the algebra of weak Jacobi forms for arbitrary rank-two lattice is finitely generated, which also motivates our conjecture.

Conjecture 6.7. The algebra of all $W(E_8)$-invariant weak Jacobi forms of integral weight and integral index is finitely generated over $M_*(\mathrm{SL}_2(\mathbb{Z}))$.

By §4, the form $\phi_{-52a,13}$, which is one of the two generators of weight -52 for $J^{\mathrm{w},W(E_8)}_{*,E_8,13}$, has to be a generator of the bigraded algebra $J^{\mathrm{w},W(E_8)}_{*,E_8,*}$. This means that $J^{\mathrm{w},W(E_8)}_{*,E_8,*}$ has generators of large index and its structure is extremely complicated.

Appendix A

In this appendix we present the generating series of $J^{\mathrm{w},W(E_8)}_{*,E_8,t}$ for index $t \leq 13$. Let us define

$$\mathcal{J}^{\mathrm{w}}_t := \frac{P^{\mathrm{w}}_t}{(1-x^4)(1-x^6)} = \frac{\sum_{k\in\mathbb{Z}} d_{k,t}x^k}{(1-x^4)(1-x^6)} = \sum_{k\in\mathbb{Z}} \dim J^{\mathrm{w},W(E_8)}_{k,E_8,t}\, x^k.$$

We here present the series $\mathcal{J}^{\mathrm{w}}_t$ up to $O(x^{22})$. It is known that the free module $J^{W(E_8)}_{*,E_8,t}$ of holomorphic Jacobi forms is generated by forms of weight not greater than 16. Thus our data is sufficient to deduce the generating series of $J^{W(E_8)}_{*,E_8,t}$ from the results in §5.

$$\begin{aligned}
\mathcal{J}^{\mathrm{w}}_1 =& x^4 + x^8 + x^{10} + x^{12} + x^{14} + 2x^{16} + x^{18} + 2x^{20} + O(x^{22}).\\
\mathcal{J}^{\mathrm{w}}_2 =& x^{-4} + x^{-2} + 2 + 2x^2 + 3x^4 + 3x^6 + 4x^8 + 4x^{10} + 5x^{12} + 5x^{14} + 6x^{16}\\
&+ 6x^{18} + 7x^{20} + O(x^{22}).\\
\mathcal{J}^{\mathrm{w}}_3 =& x^{-8} + x^{-6} + 2x^{-4} + 3x^{-2} + 4 + 4x^2 + 6x^4 + 6x^6 + 7x^8 + 8x^{10} + 9x^{12}\\
&+ 9x^{14} + 11x^{16} + 11x^{18} + 12x^{20} + O(x^{22}).\\
\mathcal{J}^{\mathrm{w}}_4 =& x^{-16} + x^{-14} + 2x^{-12} + 3x^{-10} + 5x^{-8} + 5x^{-6} + 8x^{-4} + 9x^{-2} + 11\\
&+ 12x^2 + 15x^4 + 15x^6 + 18x^8 + 19x^{10} + 21x^{12} + 22x^{14} + 25x^{16}\\
&+ 25x^{18} + 28x^{20} + O(x^{22}).\\
\mathcal{J}^{\mathrm{w}}_5 =& 2x^{-16} + 2x^{-14} + 5x^{-12} + 6x^{-10} + 9x^{-8} + 10x^{-6} + 14x^{-4} + 15x^{-2}\\
&+ 19 + 20x^2 + 24x^4 + 25x^6 + 29x^8 + 30x^{10} + 34x^{12} + 35x^{14} + 39x^{16}\\
&+ 40x^{18} + 44x^{20} + O(x^{22}).\\
\mathcal{J}^{\mathrm{w}}_6 =& 2x^{-24} + 2x^{-22} + 5x^{-20} + 7x^{-18} + 10x^{-16} + 13x^{-14} + 18x^{-12}\\
&+ 20x^{-10} + 26x^{-8} + 29x^{-6} + 34x^{-4} + 38x^{-2} + 44 + 46x^2 + 53x^4\\
&+ 56x^6 + 61x^8 + 65x^{10} + 71x^{12} + 73x^{14} + 80x^{16} + 83x^{18} + 88x^{20}\\
&+ O(x^{22}).\\
\mathcal{J}^{\mathrm{w}}_7 =& x^{-26} + 3x^{-24} + 6x^{-22} + 11x^{-20} + 13x^{-18} + 20x^{-16} + 25x^{-14}\\
&+ 30x^{-12} + 36x^{-10} + 44x^{-8} + 47x^{-6} + 56x^{-4} + 62x^{-2} + 68 + 74x^2\\
&+ 83x^4 + 86x^6 + 95x^8 + 101x^{10} + 107x^{12} + 113x^{14} + 122x^{16} + 125x^{18}\\
&+ 134x^{20} + O(x^{22}).\\
\mathcal{J}^{\mathrm{w}}_8 =& 2x^{-32} + 4x^{-30} + 9x^{-28} + 12x^{-26} + 20x^{-24} + 25x^{-22} + 34x^{-20}
\end{aligned}$$

$$
\begin{aligned}
&+41x^{-18}+52x^{-16}+59x^{-14}+71x^{-12}+79x^{-10}+91x^{-8}+99x^{-6}\\
&+112x^{-4}+120x^{-2}+133+141x^{2}+154x^{4}+162x^{6}+175x^{8}+183x^{10}\\
&+196x^{12}+204x^{14}+217x^{16}+225x^{18}+238x^{20}+O(x^{22}).\\
\mathcal{J}_{9}^{\mathrm{w}}=&x^{-36}+2x^{-34}+9x^{-32}+13x^{-30}+22x^{-28}+30x^{-26}+42x^{-24}+50x^{-22}\\
&+66x^{-20}+76x^{-18}+91x^{-16}+104x^{-14}+120x^{-12}+131x^{-10}+150x^{-8}\\
&+161x^{-6}+178x^{-4}+192x^{-2}+209+220x^{2}+240x^{4}+251x^{6}+268x^{8}\\
&+282x^{10}+299x^{12}+310x^{14}+330x^{16}+341x^{18}+358x^{20}+O(x^{22}).\\
\mathcal{J}_{10}^{\mathrm{w}}=&4x^{-40}+7x^{-38}+15x^{-36}+23x^{-34}+36x^{-32}+46x^{-30}+64x^{-28}\\
&+78x^{-26}+97x^{-24}+114x^{-22}+137x^{-20}+153x^{-18}+178x^{-16}\\
&+197x^{-14}+220x^{-12}+240x^{-10}+266x^{-8}+283x^{-6}+310x^{-4}\\
&+330x^{-2}+354+374x^{2}+401x^{4}+418x^{6}+445x^{8}+465x^{10}+489x^{12}\\
&+509x^{14}+536x^{16}+553x^{18}+580x^{20}+O(x^{22}).\\
\mathcal{J}_{11}^{\mathrm{w}}=&5x^{-42}+15x^{-40}+24x^{-38}+40x^{-36}+55x^{-34}+76x^{-32}+95x^{-30}\\
&+121x^{-28}+143x^{-26}+172x^{-24}+197x^{-22}+228x^{-20}+254x^{-18}\\
&+287x^{-16}+314x^{-14}+347x^{-12}+375x^{-10}+409x^{-8}+436x^{-6}\\
&+471x^{-4}+499x^{-2}+533+561x^{2}+596x^{4}+623x^{6}+658x^{8}+686x^{10}\\
&+720x^{12}+748x^{14}+783x^{16}+810x^{18}+845x^{20}+O(x^{22}).\\
\mathcal{J}_{12}^{\mathrm{w}}=&8x^{-48}+13x^{-46}+29x^{-44}+43x^{-42}+64x^{-40}+86x^{-38}+116x^{-36}\\
&+141x^{-34}+179x^{-32}+210x^{-30}+249x^{-28}+286x^{-26}+330x^{-24}\\
&+365x^{-22}+414x^{-20}+452x^{-18}+498x^{-16}+540x^{-14}+588x^{-12}\\
&+626x^{-10}+678x^{-8}+717x^{-6}+765x^{-4}+808x^{-2}+857+895x^{2}\\
&+948x^{4}+987x^{6}+1035x^{8}+1078x^{10}+1127x^{12}+1165x^{14}+1218x^{16}\\
&+1257x^{18}+1305x^{20}+O(x^{22}).\\
\mathcal{J}_{13}^{\mathrm{w}}=&2x^{-52}+10x^{-50}+26x^{-48}+44x^{-46}+73x^{-44}+96x^{-42}+136x^{-40}\\
&+171x^{-38}+214x^{-36}+257x^{-34}+311x^{-32}+353x^{-30}+413x^{-28}\\
&+464x^{-26}+521x^{-24}+575x^{-22}+640x^{-20}+689x^{-18}+756x^{-16}\\
&+812x^{-14}+873x^{-12}+930x^{-10}+998x^{-8}+1048x^{-6}+1117x^{-4}\\
&+1174x^{-2}+1236+1293x^{2}+1362x^{4}+1412x^{6}+1481x^{8}+1538x^{10}\\
&+1600x^{12}+1657x^{14}+1726x^{16}+1776x^{18}+1845x^{20}+O(x^{22}).
\end{aligned}
$$

Appendix B

In this appendix we show the construction of the second generator Φ_t of $J^{W(E_8)}_{*,E_8,t}$ for $t = 7, 10, 11$ (the form X_t is the first generator). In the following, $P_{w,t}(E_4, E_6, A_i, B_j)$ are some very long polynomials of weight w and index t in E_4, E_6 and Sakai's nine forms A_i, B_j, of which we omit the explicit expression. The symbol $\mathcal{O}^{[m_1 \dots m_8]}_{a,d}$ stands for the Weyl orbit of the vector $\sum_{i=1}^{8} m_i w_i$ of norm a whose number of elements is d.

$$
\begin{aligned}
\Phi_7 &= \frac{P_{32,7}(E_4, E_6, A_i, B_j)}{\Delta^2 E_4} \\
&= 1 + \frac{q}{56}\mathcal{O}^{[00000011]}_{7,13440} + \frac{q^2}{280}\mathcal{O}^{[10000100]}_{14,604800} + \frac{q^3}{280}\left(5\mathcal{O}^{[00000013]}_{21,13440} + \mathcal{O}^{[10000101]}_{21,1814400}\right) \\
&\quad + \frac{q^4}{280}\left(5\mathcal{O}^{[00000022]}_{28,13440} + \mathcal{O}^{[10001001]}_{28,4838400}\right) + O(q^5).
\end{aligned}
$$

$$
\begin{aligned}
\Phi_{10} &= \frac{P_{56,10}(E_4, E_6, A_i, B_j)}{\Delta^4 E_4} \\
&= 1 + \frac{q}{126}\mathcal{O}^{[10000002]}_{10,30240} + \frac{q^2}{630}\left(5\mathcal{O}^{[20000002]}_{20,30240} + \mathcal{O}^{[10001000]}_{20,1209600}\right) \\
&\quad + \frac{q^3}{1260}\left(2\mathcal{O}^{[10000102]}_{30,1814400} + \mathcal{O}^{[00010010]}_{30,4838400}\right) \\
&\quad + \frac{q^4}{1260}\left(10\mathcal{O}^{[20000004]}_{40,30240} + 10\mathcal{O}^{[00002000]}_{40,241920} + 2\mathcal{O}^{[10100100]}_{40,4838400} + \mathcal{O}^{[00010011]}_{40,9676800}\right) \\
&\quad + \frac{q^5}{1260}\Big(140\mathcal{O}^{[50000000]}_{50,2160} + 10\mathcal{O}^{[10000006]}_{50,30240} + 2\mathcal{O}^{[10000120]}_{50,1814400} + 2\mathcal{O}^{[10001003]}_{50,4838400} \\
&\qquad\qquad + 2\mathcal{O}^{[10101000]}_{50,4838400} + \mathcal{O}^{[00010101]}_{50,14515200}\Big) + O(q^6).
\end{aligned}
$$

$$
\begin{aligned}
\Phi_{11} &= \frac{P_{60,11}(E_4, E_6, A_i, B_j)}{\Delta^4 E_4^2} \\
&= 1 + \frac{q}{756}\mathcal{O}^{[00000101]}_{11,181440} + \frac{q^2}{3780}\left(3\mathcal{O}^{[00010001]}_{22,2419200} + 5\mathcal{O}^{[10000020]}_{22,181440}\right) \\
&\quad + \frac{q^3}{7560}\Big(10\mathcal{O}^{[00000201]}_{33,181440} + 6\mathcal{O}^{[00101000]}_{33,2419200} + 10\mathcal{O}^{[10000013]}_{33,362880} + 5\mathcal{O}^{[11000011]}_{33,5806080} \\
&\quad + 10\mathcal{O}^{[30000010]}_{33,181440}\Big) + \frac{q^4}{7560}\Big(10\mathcal{O}^{[00000202]}_{44,181440} + 6\mathcal{O}^{[00100110]}_{44,7257600} + 6\mathcal{O}^{[01010001]}_{44,4838400} \\
&\quad + 5\mathcal{O}^{[11000012]}_{44,5806080} + 6\mathcal{O}^{[20001001]}_{44,4838400}\Big) + \frac{q^5}{7560}\Big(10\mathcal{O}^{[00000211]}_{55,362880} + 6\mathcal{O}^{[00010004]}_{55,2419200} \\
&\quad + 6\mathcal{O}^{[00100200]}_{55,2419200} + 6\mathcal{O}^{[00101002]}_{55,9676800} + 6\mathcal{O}^{[01010010]}_{55,9676800} + 10\mathcal{O}^{[10000023]}_{55,362880}
\end{aligned}
$$

$$+ 5\mathcal{O}_{55,5806080}^{[11000021]} + 6\mathcal{O}_{55,7257600}^{[20001010]} + 10\mathcal{O}_{55,362880}^{[30000012]}\Big) + O(q^6).$$

We remark that there are in total 11 Weyl orbits of norm 55 and nine of them are involved in the q^5-term of Φ_{11}. As explained in the proof of Proposition 5.2, the Fourier expansion of any $W(E_8)$-invariant holomorphic Jacobi form of singular weight 4 and index t is completely determined by the finitely-many Weyl orbits related to the action of $W(E_8)$ on the discriminant group E_8/tE_8. Therefore, in each Fourier expansion of Φ_t or X_t there are only a finite number of different coefficients for the Weyl orbits.

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