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# Whittaker Fourier type solutions to differential equations arising from string theory*

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In this article, we find the full Fourier expansion for solutions of $(\Delta-\lambda)f(z) = -E_k(z)E_\ell(z)$ for $z = x+iy \in \mathfrak{H}$ for certain values of parameters $k, \ell$ and $\lambda$. When such an $f$ is fully automorphic these functions are referred to as generalized non-holomorphic Eisenstein series. We give a connection of the boundary condition on such Fourier series with convolution formulas on the divisor functions. Additionally, we discuss a possible relation with the differential Galois theory.

## 1. Introduction

The goal of this paper is to examine the Fourier expansion of the solutions to inhomogeneous eigenvalue equations involving of a product of two non-holomorphic Eisenstein series. Explicitly, for certain $k, \ell \in \mathbb{Z}_{>0} + 1/2$, we find a Whittaker Fourier expansion for a solution $f(z)$ to equations of the form

$$(\Delta - \lambda)f(z) = -E_k(z)E_\ell(z), \quad z = x + iy \in \mathfrak{H}, \tag{1}$$

where the Eisenstein series, $E_s(z)$, is defined as

$$E_s(z) := \sum_{\gamma \in P\cap\Gamma\backslash\Gamma} \mathrm{Im}(\gamma z)^s \tag{2}$$

for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $P$ the subgroup of upper triangular matrices. We recall the non-holomorphic Eisenstein series as $E_s(z)$ converge absolutely for $\mathrm{Re}(s) > 1$ and are eigenfunctions for the Laplace operator $-\Delta := -y^2(\partial_x^2 + \partial_y^2)$ with eigenvalue $s(1-s)$. There is currently no universal method for finding explicit solutions to equations of the form (1) and the method we propose gives a general form for $k, \ell \in \mathbb{Z}_{>0} + 1/2$ and certain $\lambda$.

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Solutions to equations of the form (1) have been studied in [1, 2, 5, 6, 9, 13, 14, 17]. If they satisfy the $\mathrm{SL}_2(\mathbb{Z})$-automorphicity condition, these solutions are sometimes referred to as *generalised non-holomorphic Eisenstein series* [2]. Such functions arise in the maximally supersymmetric $\mathcal{N}=4$ super-Yang-Mills (SYM) theory when studying the duality properties of certain correlation functions in the $1/N$ expansion. For $k,\ell \in \mathbb{Z}_{>0}+1/2$ with $k+\ell = q+2, q, \dots,$ the generalized non-holomorphic Eisenstein series appear in the the order $\frac{1}{N^q}$ contributions with $q \in \mathbb{Z}_{>0}$ to the SYM free energy $F = -\log Z$ [2, p. 6]. At low orders there is an explicit connection between the correlator functions of the $SU(N)$ $\mathcal{N}=4$ super Yang-Mills theory in the $1/N$ expansion [2] and the 10-dimensional type IIB superstring scattering amplitude of gravitons. The $D^6R^4$ interactions in the low energy expansion of the 4-loop supergraviton is given by the the solution to (1) where $\lambda = 12$ and $k=\ell=3/2$ [10, 9] and an explicit Fourier solution has been given this case in [9]. More generally, solutions to (1) for $k,\ell \in \mathbb{Z}_{>0}$ are examples of modular graph functions and solutions have been found in [4, 5, 6].

We use the following method to investigate solutions of (1): for $z = x+iy$ we start by assuming $f(z)$ is periodic in $x$ and expand it in corresponding Fourier series. From (1), we deduce an ordinary differential equation on every Fourier mode of $f$. Each of these differential equations is an inhomogeneous differential equation of the second order; the homogeneous part coincides with the modified Bessel equation, while the inhomogeneous part comprises an infinite sum involving polynomials and $K$-Bessel functions. The indices of the $K$-Bessel function in the inhomogeneous part are independent on the parameters in the homogeneous part. Assuming that the solution has this form, we introduce a system of linear equations on parameters upon which this special form depends. For certain physically relevant $(\lambda, k, \ell)$ we solve this system of linear equations using a symbolic algebra system to obtain the searched parameters. In addition to finding all Fourier modes for such solutions, we are able to determine conditions on $(\lambda, k, \ell)$ when solutions of this form do not exist. Experimentally, we are able to find explicit Fourier solutions in many instances outside those contained here; however, we have chosen to only include these for brevity.

The method we present in this paper is motivated by the exact expression of the solution in [9]. In the former article, Green, Miller, and Vanhove found the explicit expressions for the Fourier modes of the function, satisfying (1) with $\lambda = 12$ and $k=\ell=3/2$. The Fourier modes of the solution in [9] are exactly of the form Theorem A. We also note that the full spectral solution for $f$ in terms of $L^2(\Gamma\backslash\mathfrak{H})$-eigenfunctions is given in [13]. The Fourier expansion of the solution $f$ to (1) for $k=\ell=3/2$ and $\lambda=12$ was also explicitly computed [14] using the Poincaré series solution found in [9, Appendix A]. The method used in [14] is different from that used in [9] and outlined below. Importantly, in [14], the authors are not able to extend their method outside of the case where $k = \ell$ in (1); however, the method outlined in this paper does not require such a dependence.

In [2], Chester, Green, Pufu, Wang, and Wen generalized Eisenstein series were studied for certain values of $k, \ell, \lambda$. Although each full Fourier expansion was not obtained in [2], the authors provided many important properties to the solution. We would like to note that they have expressed the solution to the zeroth Fourier term not in terms of $K_0$ and $K_1$ as we did, but rather in terms of modified Bessel functions of integer index. These representations are related to the ones found below via a recurrence relation of $K$-Bessel functions.

Inhomogeneous differential equations of Bessel type with inhomogeneous parts involving Bessel functions appear not only in string theory, but also in the theory of vector-valued automorphic functions. More precisely, in [7], Fedosova, Pohl, and Rowlett considered functions $\varphi : \mathfrak{H} \to V$ for some complex finite-dimensional vector space $V$ that are Laplace eigenfunctions with eigenvalue $s(1-s)$ for $s \in \mathbb{C}$, thus

$$\big(\Delta - s(1-s)\big)\varphi = 0. \tag{3}$$

Additionally, they required that $\varphi$ satisfies the *twist-periodicity* condition

$$\varphi(z+1) = A\varphi(z) \tag{4}$$

for all $z \in \mathfrak{H}$ for some $A \in \mathrm{GL}(V)$. When $A$ is a unitary matrix, one obtains that the Fourier coefficients of $\varphi$ satisfy a modified Bessel equation, depending on $A$. For diagonalizable $A$, this modified Bessel equation is a homogeneous differential equation. Interestingly enough, if we allow a non-diagonalizable matrix, then some entries of the Fourier coefficient of $\varphi$ satisfy the differential equation

$$\big(y^2\partial_y^2 - \lambda - 4\pi^2 n^2 y^2\big) f(y) = g(y), \quad n \in \mathbb{Z},$$

where $g$ is a combination of the modified Bessel function of the second kind and a certain polynomial in $y$.

### 1.1. Discussion of main results

We denote by $\mathscr{S}$ the set containing all $(\lambda, k, \ell)$ such that either

(i) $\lambda = 12, 30, 56$ and $k = \ell = \frac{3}{2}$, or
(ii) $\lambda = 20$ and $k = \frac{3}{2}, \ell = \frac{5}{2}$, or
(iii) $\lambda = 30$ and $k = \ell = \frac{5}{2}$, or
(iv) $\lambda = 30$ and $k = \frac{3}{2}, \ell = \frac{7}{2}$.

Of these cases, in Appendix C of [2] Chester, Green, Pufu, Wang and Wen examine the zero mode of solutions (i) for $\lambda = 12, 30, 56, 90$ and (ii) for $\lambda = 20, 42$ and (iii) and (iv) for $\lambda = 30, 56, 90$. However, for the nonzero modes the full Fourier coefficients were not explicitly given. The method outlined in this paper gives all Fourier modes in these cases as well.[1]

**Theorem A.** *Let $(\lambda = r(r+1), k, \ell) \in \mathscr{S}$ and let $f : \mathfrak{H} \to \mathbb{C}$ be a 1-periodic function in the $x$-variable that satisfies*

$$(\Delta - \lambda) f(z) = -E_k(z) E_\ell(z), \quad z = x + iy \in \mathfrak{H}$$

*for $E_\bullet(z)$, $\bullet \in \{k, \ell\}$ as in (2). Then $f(z) = \sum_{n\in\mathbb{Z}} \hat{f}_n(y) e^{2\pi i n x}$ and there exist $\alpha_n, \beta_n \in \mathbb{C}$ such that for $n \neq 0$,*

$$\begin{aligned}\hat{f}_n(y) &= \alpha_n \sqrt{y} K_{r+1/2}(2\pi|n|y) + \beta_n \sqrt{y} I_{r+1/2}(2\pi|n|y) \\ &\quad + \sum_{\substack{n_1,n_2\in\mathbb{Z} \\ n_1+n_2=n}} \sum_{i,j\in\{0,1\}} q^{i,j}(y) K_i(2\pi|n_1|y) K_j(2\pi|n_2|y),\end{aligned}$$

*and for $n = 0$,*

$$\hat{f}_0(y) = \alpha_0 y^{-r} + \beta_0 y^{r+1} + \sum_{\substack{n_1,n_2\in\mathbb{Z} \\ n_1+n_2=0}} \sum_{i,j\in\{0,1\}} \mu^{i,j}(y) K_i(2\pi|n_1|y) K_j(2\pi|n_2|y),$$

*where for $\eta \in \mathbb{C}$, $I_\eta$ and $K_\eta$ denote the modified Bessel function of the first and second kind of index $\eta$, respectively, and where $q^{i,j} = q^{i,j}_{n_1,n_2,\lambda,k,\ell}$ and $\mu^{i,j} = \mu^{i,j}_{n_1,n_2,\lambda,k,\ell}$ are Laurent polynomials in $y$. In the case $n_1 n_2 = 0$, the modified Bessel functions have to be replaced by appropriate limits, see Sections 3.2 and 3.3. If we impose the requirement*

$$|\hat{f}_n(y)| = o\big(e^y\big), \quad y \to \infty, \tag{5}$$

*then $\beta_n$ with $n \neq 0$ vanishes.*

[1] In fact, this method gives a solution to (1) for hundreds of other triples as well (see Section 4). We note that we did not check these solutions for convergence and we leave them out of this paper for the interest of space.

More precisely, we obtain the degrees of polynomials $q^{i,j}$ in Figure 1. We denote by $m_{i,j}$ and $M_{i,j}$ the lowest and highest power of $y$ present in $q^{i,j}(y)$, respectively.

| $(k, \ell)$ | $m_{0,0}$ | $M_{0,0}$ | $m_{0,1}$ | $M_{0,1}$ | $m_{1,1}$ | $M_{1,1}$ |
|---|---|---|---|---|---|---|
| $(3/2, 3/2)$ | $-r+2$ | 1 | $-r+1$ | 0 | $-r+2$ | 1 |
| $(3/2, 5/2)$ | $-r+2$ | 0 | $-r+1$ | 1 | $-r+2$ | 0 |
| $(5/2, 5/2)$ | $-r+2$ | 1 | $-r+1$ | 0 | $\min\{-r+1, -1\}$ | 1 |
| $(3/2, 7/2)$ | $-r+2$ | 1 | $-r+1$ | 0 | $-r+2$ | 1 |

Figure 1: For $\lambda = r(r+1)$, let $m_{i,j}$ and $M_{i,j}$ be the lowest and highest power of $y$ present in $q^{i,j}(y)$, respectively. Note that $m_{0,1} = m_{1,0}$ and $M_{0,1} = M_{1,0}$.

### 1.2. Automorphy of the solution

Theorem A does not require the function $f$ to be automorphic. However, any function that is automorphic is also 1-periodic in the $x$-variable; thus, the Fourier expansion of an automorphic solution to (1) has the form as in Theorem A. We note that such an automorphic function always exists for $(\lambda, k, \ell) \in \mathscr{S}$ as established in [13] with the help of spectral methods. Additionally, in [14], the existence of an automorphic solution with Fourier expansion of the form in Theorem A was verified in many cases where $k = \ell$.

If an automorphic function exhibits a certain large-$y$ growth behavior, its Fourier coefficients must obey a small-$y$ boundary condition [9, Lemma 2.9]. More precisely, the Fourier expansion of every automorphic solution to (1) with $O(y^s)$ for $s > 1$ as $y \to \infty$ must satisfy $O(y^{1-s})$ as $y \to 0$. Hence, if we apriori know the large-$y$ behavior of the solution, it is natural to impose a small-$y$ boundary condition on its Fourier coefficients.

In the course of the proof of Theorem A, we split $\hat{f}_n(y) = \sum_{n_1+n_2=n} \hat{f}_{n_1,n_2}(y)$ where each of $\hat{f}_{n_1,n_2}(y)$ satisfies an ordinary differential equation involving modified Bessel functions. The inhomogeneous part of this differential equation corresponds to the product of the $n_1$-th Fourier coefficient of $E_k(z)$ and the $n_2$-th Fourier coefficient of $E_\ell(z)$ (for more details on the splitting, see (13) and further). The space of solutions of each ODE is two-dimensional. Specifically, for $n \neq 0$, this two-dimensional space is parameterized by $\alpha_{n_1,n_2}$ and $\beta_{n_1,n_2}$ so that $\sum_{n_1+n_2=n} \alpha_{n_1,n_2} = \alpha_n$, $\sum_{n_1+n_2=n} \beta_{n_1,n_2} = \beta_n$, and for $n_1 n_2 \neq 0$,

$$\hat{f}_{n_1,n_2}(y) = \alpha_{n_1,n_2} K_{r+1/2}(2\pi|n|y) + \beta_{n_1,n_2} I_{r+1/2}(2\pi|n|y)$$

$$+\sum_{i,j\in\{0,1\}} q^{i,j}(y)K_i(2\pi|n_1|y)K_j(2\pi|n_2|y).$$

For each $n_1, n_2$ there is a unique choice of solution $\hat{f}_{n_1,n_2}(y)$ from this 2-dimensional space so that $\hat{f}_{n_1,n_2}(y)$ satisfies $O(y^{-r})$ as $y \to 0$ and $o(e^y)$ as $y \to \infty$. We note that this does not guarantee that $\hat{f}_n(y)$ itself satisfies both of these growth conditions (though we suspect that it does). As seen from [9, Lemma 2.9], $f_n(y)$ satisfying both of these conditions is necessary, but not sufficient, for the full solution to be automorphic.

### 1.3. Theorem A for $(\lambda, k, \ell) \notin \mathscr{S}$

We are also able to find solutions of the form in Theorem A for values $(\lambda, k, \ell) \notin \mathscr{S}$. Specifically, we find explicit solutions for $k = \ell = 3/2$ and $\lambda = 2$ (see Section A.1); $k = 3/2, \ell = 5/2$ and $\lambda = 6$ (see Section A.2); $k = \ell = 5/2$ and $\lambda = 2$ and 12 (see Sections A.3 and A.4); and $k = 3/2, \ell = 7/2$ and $\lambda = 12$ (see Section A.5). However, in these cases, it is not clear what an appropriate small-$y$ condition for $\hat{f}_{n_1,n_2}(y)$ means. More precisely, for each $n_1$ and $n_2$, there is a no choice of $\alpha_{n_1,n_2}$ so that $\hat{f}_{n_1,n_2}(y)$ is of necessary order of vanishing and $\sum_{n_1+n_2=n} \alpha_{n_1,n_2}$ converges. In Section A we make a unique choice based on the vanishing of the second term in the asymptotic expansion as $y \to 0$. We note that the divergence of the homogeneous sum $\sum_{n_1+n_2=n} \alpha_{n_1,n_2}$ seems to occur when $\lambda$ is relatively small depending on the size of $k$ and $\ell$.

### 1.4. Shifted divisor sums

In [2, Section C.1 (a)], Chester, Green, Pufu, Wang, and Wen conjectured, based on ideas from the AdS-CFT correspondence and Yang-Mills theory, that for an automorphic $f$, the total sum of the Fourier coefficients corresponding to the homogeneous solution vanishes, that is, if $n \neq 0$, then $\alpha_n = 0$. In [14], the authors provided an argument in support of this conjecture for every non-zero Fourier term for $\lambda = 12$, $k = \ell = 3/2$ (the zeroth term can be dealt with with the help of Ramanujan summation formulas). We do not prove this conjecture in this article. However, the special choices of $\hat{f}_{n_1,n_2}(y)$ made in each case in order to obtain a unique boundary condition imply that $\alpha_n$ can be evaluated with certain convolution series on divisor functions. Following methods similar to those of [9] and [2], we show at least for one choice of parameters and the zeroth coefficient (Section A.1.4), the formal vanishing of the homogeneous part follows from a certain derivative of the Ramanujan identity. If we want to deal with the non-zero Fourier coefficient, we would have to prove a more general version of the Ramanujan identity.

### 1.5. Application to large $N$ expansion of integrated correlators

In [2], the authors gave an evidence that generalised Eisenstein series that arise in coefficients $\mathcal{H}(q,\tau,\overline{\tau})$ of even terms in the $1/N$ up to order $1/N^3$ in [2, (2.11)]. To be more precise, [2, (2.11)] expresses $\partial_m^4 \log Z|_{m=0,b=1}$, that is a fourth derivative of the squashed sphere partition function of the $N=2$ theory with respect to the squashing parameter $b=1$ and mass parameter $m=0$. In Sections 5–8, we derive the Fourier coefficients for the generalised Eisenstein series appearing the first few terms of the large-$N$ expansion of $\partial_m^4 \log Z|_{m=0,b=1}$.

If we denote by $\mathcal{E}(r,k,\ell;z)$ the modular functions that satisfy the inhomogeneous Laplace equation

$$\big(\Delta - r(r+1)\big)\mathcal{E}(r,k,\ell;z) = -4\zeta(2k)\zeta(2\ell)E_k(z)E_\ell(z),$$

where $\zeta$ denotes the Riemann zeta function, then $1/N^2$ contribution from [2, (2.13)] is conjectured to be equal to

$$T_{-2}(y) = C_1 + \frac{14175}{704\pi^4}\mathcal{E}\big(6,\tfrac{5}{2},\tfrac{3}{2}\big) - \frac{1215}{88\pi^4}\mathcal{E}\big(4,\tfrac{5}{2},\tfrac{3}{2}\big)$$

for some constant $C_1$.

With the help of the method described in the article it is possible to show that

$$C_1 + \frac{14175}{704\pi^4}\mathcal{E}\big(6,\tfrac{5}{2},\tfrac{3}{2}\big) - \frac{1215}{88\pi^4}\mathcal{E}\big(4,\tfrac{5}{2},\tfrac{3}{2}\big) = C_1 + \sum_{n\in\mathbb{Z}} T_{-2,n}(y)e^{2\pi i n x},$$

where for $n \neq 0$,

$$\begin{aligned} T_{-2,n}(y) &= C_{2,n}\sqrt{y}K_{9/2}(2\pi|n|y) + C_{3,n}\sqrt{y}K_{13/2}(2\pi|n|y) \\ &\quad + \sum_{n_1+n_2=n} T_{-2,n_1,n_2}(y), \end{aligned}$$

for some $C_{2,n}$ and $C_{3,n}$, where for $n_1+n_2 \neq 0$ and $n_1n_2 \neq 0$,

$$T_{-2,n_1,n_2}(y) = -|n_1|^2|n_2|\sigma_{-4}\big(|n_1|\big)\sigma_{-2}\big(|n_2|\big) \tag{6}$$

$$\times \sum_{i,j=0}^{1} w_{i,j}(y) K_i(2\pi|n_1|y) K_j(2\pi|n_2|y)$$

for

$$\begin{aligned}
w_{0,0}(y) &= \operatorname{sgn}(n_1) \frac{27n_2}{\pi^6(n_1+n_2)^{12}y^4} \\
&\quad \times \big(25200n_1^6 - 201600n_1^5n_2 + 441000n_1^4n_2^2 - 352800n_1^3n_2^3 + 88200n_1^2n_2^4 \\
&\quad + \big(7630\pi^2n_1^8 - 70840\pi^2n_2n_1^7 + 72460\pi^2n_2^2n_1^6 + 210200\pi^2n_2^3n_1^5 \\
&\qquad - 89240\pi^2n_2^4n_1^4 - 119080\pi^2n_2^5n_1^3 + 29620\pi^2n_2^6n_1^2 + 200\pi^2n_2^7n_1 \\
&\qquad + 10\pi^2n_2^8\big)y^2 \\
&\quad + \big(235\pi^4n_1^{10} - 4180\pi^4n_2n_1^9 - 1899\pi^4n_2^2n_1^8 + 26848\pi^4n_2^3n_1^7 \\
&\qquad + 35838\pi^4n_2^4n_1^6 - 3624\pi^4n_2^5n_1^5 - 23270\pi^4n_2^6n_1^4 - 6784\pi^4n_2^7n_1^3 \\
&\qquad + 1383\pi^4n_2^8n_1^2 + 28\pi^4n_2^9n_1 + \pi^4n_2^{10}\big)y^4\big), \\
w_{0,1}(y) &= \operatorname{sgn}(n_1)\operatorname{sgn}(n_2) \frac{9}{\pi^7(n_1+n_2)^{13}y^5} \\
&\quad \times \big(-75600n_2^2n_1^5 + 529200n_2^3n_1^4 - 793800n_2^4n_1^3 + 264600n_2^5n_1^2 \\
&\quad + \big(210\pi^2n_1^9 + 9450\pi^2n_2n_1^8 - 351720\pi^2n_2^2n_1^7 + 587160\pi^2n_2^3n_1^6 \\
&\qquad + 1159200\pi^2n_2^4n_1^5 - 675360\pi^2n_2^5n_1^4 - 665280\pi^2n_2^6n_1^3 \\
&\qquad + 221760\pi^2n_2^7n_1^2 + 630\pi^2n_2^8n_1 + 30\pi^2n_2^9\big)y^2 \\
&\quad + \big(45\pi^4n_1^{11} + 2385\pi^4n_2n_1^{10} - 63642\pi^4n_2^2n_1^9 - 28218\pi^4n_2^3n_1^8 \\
&\qquad + 381438\pi^4n_2^4n_1^7 + 479622\pi^4n_2^5n_1^6 - 102672\pi^4n_2^6n_1^5 \\
&\qquad - 349392\pi^4n_2^7n_1^4 - 84243\pi^4n_2^8n_1^3 + 26913\pi^4n_2^9n_1^2 \\
&\qquad + 402\pi^4n_2^{10}n_1 + 18\pi^4n_2^{11}\big)y^4 \\
&\quad + \big(20\pi^6n_2n_1^{12} - 1140\pi^6n_2^2n_1^{11} - 1940\pi^6n_2^3n_1^{10} + 10004\pi^6n_2^4n_1^9 \\
&\qquad + 34632\pi^6n_2^5n_1^8 + 37752\pi^6n_2^6n_1^7 + 7384\pi^6n_2^7n_1^6 - 15960\pi^6n_2^8n_1^5 \\
&\qquad - 11676\pi^6n_2^9n_1^4 - 1988\pi^6n_2^{10}n_1^3 + 252\pi^6n_2^{11}n_1^2 + 4\pi^6n_2^{12}n_1\big)y^6\big), \\
w_{1,0}(y) &= \frac{9n_2}{\pi^7n_1(n_1+n_2)^{13}y^5}\big(75600n_1^7 - 529200n_1^6n_2 + 793800n_1^5n_2^2 \\
&\quad - 264600n_1^4n_2^3 + \big(60690\pi^2n_1^9 - 454230\pi^2n_2n_1^8 + 311040\pi^2n_2^2n_1^7 \\
&\qquad + 1310400\pi^2n_2^3n_1^6 - 357840\pi^2n_2^4n_1^5 - 761040\pi^2n_2^5n_1^4 \\
&\qquad + 88200\pi^2n_2^6n_1^3 + 7560\pi^2n_2^7n_1^2 + 630\pi^2n_2^8n_1 + 30\pi^2n_2^9\big)y^2 \\
&\quad + \big(5850\pi^4n_1^{11} - 62550\pi^4n_2n_1^{10} - 53577\pi^4n_2^2n_1^9 + 352947\pi^4n_2^3n_1^8
\end{aligned}$$

$$
\begin{aligned}
&+ 508248\pi^4 n_2^4 n_1^7 - 50088\pi^4 n_2^5 n_1^6 - 342822\pi^4 n_2^6 n_1^5 + 3\pi^4 n_2^{11} \\
&- 102702\pi^4 n_2^7 n_1^4 + 21222\pi^4 n_2^8 n_1^3 + 1398\pi^4 n_2^9 n_1^2 + 87\pi^4 n_2^{10} n_1\big) y^4 \\
&+ \big(20\pi^6 n_1^{13} - 1140\pi^6 n_2 n_1^{12} - 1940\pi^6 n_2^2 n_1^{11} + 10004\pi^6 n_2^3 n_1^{10} \\
&+ 34632\pi^6 n_2^4 n_1^9 + 37752\pi^6 n_2^5 n_1^8 + 7384\pi^6 n_2^6 n_1^7 - 15960\pi^6 n_2^7 n_1^6 \\
&- 11676\pi^6 n_2^8 n_1^5 - 1988\pi^6 n_2^9 n_1^4 + 252\pi^6 n_2^{10} n_1^3 + 4\pi^6 n_2^{11} n_1^2\big) y^6\big)
\end{aligned}
$$

$$
\begin{aligned}
w_{1,1}(y) = {}& \frac{9\,\mathrm{sgn}(n_2)}{\pi^6 n_1 (n_1+n_2)^{12} y^4} \big(30 n_2^8 + 210 n_1^8 + 9240 n_1^7 n_2 - 451680 n_1^6 n_2^2 \\
&+ 1633560 n_1^5 n_2^3 - 1157280 n_1^4 n_2^4 + 81240 n_1^3 n_2^5 + 6960 n_1^2 n_2^6 + 600 n_1 n_2^7 \\
&+ \big(150\pi^2 n_1^{10} + 6960\pi^2 n_2 n_1^9 - 218082\pi^2 n_2^2 n_1^8 + 315504\pi^2 n_2^3 n_1^7 \\
&+ 757824\pi^2 n_2^4 n_1^6 - 285072\pi^2 n_2^5 n_1^5 - 446400\pi^2 n_2^6 n_1^4 + 60528\pi^2 n_2^7 n_1^3 \\
&+ 4794\pi^2 n_2^8 n_1^2 + 384\pi^2 n_2^9 n_1 + 18\pi^2 n_2^{10}\big) y^2 \\
&+ \big(5\pi^4 n_1^{12} + 420\pi^4 n_2 n_1^{11} - 13025\pi^4 n_2^2 n_1^{10} - 3196\pi^4 n_2^3 n_1^9 \\
&+ 89202\pi^4 n_2^4 n_1^8 + 116952\pi^4 n_2^5 n_1^7 - 9026\pi^4 n_2^6 n_1^6 - 73800\pi^4 n_2^7 n_1^5 \\
&- 23271\pi^4 n_2^8 n_1^4 + 3652\pi^4 n_2^9 n_1^3 + 147\pi^4 n_2^{10} n_1^2 + 4\pi^4 n_2^{11} n_1\big) y^4\big).
\end{aligned}
$$

The cases $n_1 + n_2 = 0$ or $n_1 n_2 = 0$ can be obtained by a certain limiting procedure.

The $1/N^3$ contribution from [2, (2.13)] is conjectured to be equal to

$$
\alpha_r \mathcal{E}\big(3, \tfrac{3}{2}, \tfrac{3}{2}\big) + \sum_{r=5,7,9} \alpha_r \mathcal{E}\big(r, \tfrac{3}{2}, \tfrac{3}{2}\big) + \beta_r \mathcal{E}\big(r, \tfrac{5}{2}, \tfrac{5}{2}\big) + \gamma_r \mathcal{E}\big(r, \tfrac{7}{2}, \tfrac{3}{2}\big),
$$

where $\alpha_r, \beta_r, \gamma_r$ are not the same as in Theorem A but are defined in [2, (2.14)]. We can write the expression above as

$$
\sum_{n\in\mathbb{Z}} T_{-3,n}(y) e^{2\pi i n x},
$$

where for $n \neq 0$,

$$
\begin{aligned}
T_{-3,n}(y) = {}& C_{4,n}\sqrt{y} K_{7/2}(2\pi|n|y) + C_{5,n}\sqrt{y} K_{11/2}(2\pi|n|y) \\
&+ C_{6,n}\sqrt{y} K_{15/2}(2\pi|n|y) + C_{7,n}\sqrt{y} K_{19/2}(2\pi|n|y) \\
&+ \sum_{n_1+n_2=n} T_{-3,n_1,n_2}(y),
\end{aligned}
$$

for some $C_{j,n}$, $j = 4, 5, 6, 7$, where for $n_1 + n_2 \neq 0$ and $n_1 n_2 \neq 0$,

$$T_{-3,n_1,n_2}(y) = \sum_{i,j=0}^{1} v_{i,j}(y) K_i(2\pi|n_1|y) K_j(2\pi|n_2|y),$$

where $v_{i,j}(y)$ is some rational function on $y$.

## 2. Method of solution

In this section, we outline a method for finding the Fourier expansions of solutions $f(z)$ to equations of the form

$$(\Delta - \lambda) f(z) = c_{k,\ell}\, \zeta(2k)\zeta(2\ell) E_k(z) E_\ell(z) \tag{7}$$

where $\lambda = r(r+1)$ for $r \in \mathbb{N}$ and $k, \ell \in \mathbb{Z}_{>0} + 1/2$, and $c_{k,\ell}$ is some constant depending on $k$ and $\ell$. The constants $c_{k,\ell} \in \mathbb{C}$ are chosen for the convenience purpose and to shorten the outcome. For the particular $k, \ell$ for which we write down the exact solutions, we let

$$c_{k,\ell} := \begin{cases} -4, & k = \ell = 3/2, \\ -6, & k = 3/2, \ell = 5/2, \\ -9, & k = \ell = 5/2, \\ -30, & k = 3/2, \ell = 7/2. \end{cases}$$

We start with recalling that for $\mathrm{Re}(s) > 1$,

$$E_s(z) = \sum_{n \in \mathbb{Z}} a_{n,s}(y) e^{2\pi i n x},$$

where

$$a_{0,s}(y) = y^s + \frac{\sqrt{\pi}\,\Gamma(s - \frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)} y^{1-s}$$

and for $n \neq 0$,

$$a_{n,s}(y) = \frac{2\pi^s}{\Gamma(s)\zeta(2s)} |n|^{s-\frac{1}{2}} \sigma_{1-2s}(|n|) \sqrt{y} K_{s-\frac{1}{2}}(2\pi|n|y), \tag{8}$$

where for $z \in \mathbb{C}$ and $n \in \mathbb{N}$, $\sigma_z(n) := \sum_{d|n} d^z$ is the divisor function [18, p. 278]. We note that in the notations of [2, (2.10)], $E_s(z) = \frac{1}{2\zeta(2s)} E(s, z, \overline{z})$, and thus

$$4\zeta(2k)\zeta(2\ell)E_k(z)E_\ell(z) = E(k,z,\overline{z})E(\ell,z,\overline{z}). \tag{9}$$

This expansion implies that for $k, \ell > 1$

$$c_{k,\ell}\,\zeta(2k)\zeta(2\ell)E_k(z)E_\ell(z) = \sum_{n\in\mathbb{Z}} S_n(y)e^{2\pi inx}, \tag{10}$$

where

$$S_n(y) = \sum_{\substack{n_1,n_2\in\mathbb{Z}\\ n_1+n_2=n}} s_{n_1,n_2}(y) \tag{11}$$

for $s_{n_1,n_2}(y) = c_{k,\ell}\,\zeta(2k)\zeta(2\ell)a_{n_1,k}(y)a_{n_2,\ell}(y)$. Explicitly, each $s_{n_1,n_2}(y)$ can be written as follows:

(a) For $n_1 = n_2 = 0$:

$$\begin{aligned} s_{0,0}(y) = {} & c_{k,\ell}\,\zeta(2k)\zeta(2\ell)y^{k+\ell} + c_{k,\ell}\,\zeta(2\ell)y^{1-k+\ell}\frac{\sqrt{\pi}\Gamma(k-\frac{1}{2})\zeta(2k-1)}{\Gamma(k)} \\ & + c_{k,\ell}\,\zeta(2k)y^{1+k-\ell}\frac{\sqrt{\pi}\Gamma(\ell-\frac{1}{2})\zeta(2\ell-1)}{\Gamma(\ell)} \\ & + c_{k,\ell}\,y^{2-k-\ell}\frac{\pi\Gamma(k-\frac{1}{2})\Gamma(\ell-\frac{1}{2})\zeta(2k-1)\zeta(2\ell-1)}{\Gamma(k)\Gamma(\ell)}. \end{aligned}$$

(b) For $n_1 = 0$, $n_2 \neq 0$:

$$\begin{aligned} s_{0,n}(y) = {} & \frac{2c_{k,\ell}\pi^\ell\zeta(2k)}{\Gamma(\ell)\zeta(2\ell)}|n|^{\ell-\frac{1}{2}}\sigma_{1-2\ell}(|n|)y^{k+1/2}K_{\ell-\frac{1}{2}}(2\pi|n|y) \\ & + \frac{2c_{k,\ell}\pi^{\ell+1/2}\Gamma(k-\frac{1}{2})\zeta(2k-1)}{\Gamma(k)\Gamma(\ell)\zeta(2\ell)}|n|^{\ell-\frac{1}{2}}\sigma_{1-2\ell}(|n|) \\ & \times y^{3/2-k}K_{\ell-\frac{1}{2}}(2\pi|n|y). \end{aligned}$$

(c) For $n_1 \neq 0$, $n_2 = 0$:

$$\begin{aligned} s_{n,0}(y) = {} & \frac{2c_{k,\ell}\pi^k\zeta(2\ell)}{\Gamma(k)\zeta(2k)}|n|^{k-\frac{1}{2}}\sigma_{1-2k}(|n|)y^{\ell+1/2}K_{k-\frac{1}{2}}(2\pi|n|y) \\ & + \frac{2c_{k,\ell}\pi^{k+1/2}\Gamma(\ell-\frac{1}{2})\zeta(2\ell-1)}{\Gamma(\ell)\Gamma(k)\zeta(2k)}|n|^{k-\frac{1}{2}}\sigma_{1-2k}(|n|) \end{aligned}$$

$$\times\, y^{3/2-\ell} K_{k-\frac{1}{2}}(2\pi|n|y).$$

(d) For $n_1 n_2 \neq 0$:

$$s_{n_1,n_2}(y) = \frac{4c_{k,\ell}\pi^{k+\ell}}{\Gamma(k)\Gamma(\ell)} |n_1|^{k-\frac{1}{2}} |n_2|^{\ell-\frac{1}{2}} \sigma_{1-2k}(|n_1|)\sigma_{1-2\ell}(|n_2|) \times y K_{k-\frac{1}{2}}(2\pi|n_1|y) K_{\ell-\frac{1}{2}}(2\pi|n_2|y). \tag{12}$$

To solve (7), note that the Fourier expansion of the right is given by (10). Although we do not assume the $SL_2(\mathbb{Z})$-invariance of $f$, we do require that, for $z = x + iy$, $f(z)$ is periodic in the $x$-direction with period 1. Given this assumption, the differential equation, (7), can be equivalently stated as the simultaneous differential equations on the Fourier coefficients, $\hat{f}_n(y)$, of $f(z)$:

$$\left(y^2\partial_y^2 - \lambda - 4\pi^2 n^2 y^2\right)\hat{f}_n(y) = S_n(y), \quad n \in \mathbb{Z}. \tag{13}$$

We express

$$\hat{f}_n(y) = \sum_{\substack{n_1,n_2\in\mathbb{Z} \\ n_1+n_2=n}} \hat{f}_{n_1,n_2}(y), \tag{14}$$

for $\hat{f}_{n_1,n_2}(y)$ satisfying[2]

$$\left(y^2\partial_y^2 - \lambda - 4\pi^2(n_1+n_2)^2 y^2\right)\hat{f}_{n_1,n_2}(y) = s_{n_1,n_2}(y). \tag{15}$$

Each solution of (15) can be written as a sum of a solution, $\hat{f}^H_{n_1,n_2}(y)$, of the homogeneous equation

$$\left(y^2\partial_y^2 - \lambda - 4\pi^2(n_1+n_2)^2 y^2\right)\hat{f}^H_{n_1,n_2}(y) = 0, \tag{16}$$

[2] There is an ambiguity in the decomposition (14) of $\hat{f}_n(y)$ into a sum of solutions of (15) since $\hat{f}_{n_1,n_2}(y)$ is not uniquely defined by the property that it solves (15). However, by (16), any $\hat{f}^H_{n_1,n_2}(y)$ depends on the sum $n_1+n_2$ but not on the individual values of $n_1$ and $n_2$. Hence $\hat{f}^H_{n_1,n_2}(y)$ can be simultaneously added to $f_{n_1,n_2}(y)$ and subtracted from $f_{n_1',n_2'}(y)$ for any $n_1', n_2'$ if $n_1'+n_2' = n_1+n_2$. Below, we make the decomposition (14) unique by demanding certain boundary conditions on $\hat{f}_{n_1,n_2}(y)$ for $y \to \infty$ and $y \to 0$ (for the motivation for imposing boundary conditions, see Section 1.2).

and a particular solution, $\hat{f}^P_{n_1,n_2}(y)$, of (15). Thus,

$$\hat{f}_{n_1,n_2}(y) = \hat{f}^P_{n_1,n_2}(y) + \hat{f}^H_{n_1,n_2}(y). \tag{17}$$

We note that for $\lambda = r(r+1)$, $r \in \mathbb{R}$ and $n_1 + n_2 \neq 0$,

$$\hat{f}^H_{n_1,n_2}(y) = \alpha_{n_1,n_2}\sqrt{y}K_{r+1/2}(2\pi|n_1+n_2|y) + \beta_{n_1,n_2}\sqrt{y}I_{r+1/2}(2\pi|n_1+n_2|y)$$

for some $\alpha_{n_1,n_2}, \beta_{n_1,n_2} \in \mathbb{C}$. However, for $\beta_{n_1,n_2} \neq 0$, the function $\hat{f}^H_{n_1,n_2}(y)$ grows exponentially as $y \to \infty$, that contradicts (5). From this we deduce that $\beta_{n_1,n_2} = 0$ and thus

$$\hat{f}^H_{n_1,n_2}(y) = \alpha_{n_1,n_2}\sqrt{y}K_{r+1/2}(2\pi|n_1+n_2|y) \tag{18}$$

for some $\alpha_{n_1,n_2} \in \mathbb{C}$. We note that for $\mathrm{Re}(r) > -1/2$ and $y \to 0$, [3, 10.30.2] implies

$$\hat{f}^H_{n_1,n_2}(y) = \alpha_{n_1,n_2}y^{-r}\big(\tfrac{1}{2}|\pi n|^{-r-\frac{1}{2}}\Gamma\big(r+\tfrac{1}{2}\big) + O\big(y^2\big)\big). \tag{19}$$

In the case where $n_1 + n_2 = 0$, for $\lambda = r(r+1)$, $r \in \mathbb{R}$, $r \neq 1/2$,

$$\hat{f}^H_{n_1,n_2}(y) = \alpha_{n_1,n_2}y^{-r} + \beta_{n_1,n_2}y^{r+1}$$

for some $\alpha_{n_1,n_2}, \beta_{n_1,n_2} \in \mathbb{C}$. If we demand $\hat{f}^H_{n_1,n_2}(y) = o(y^{r+1})$ as $y \to \infty$, we would have to take $\beta_{n_1,n_2} = 0$ and thus

$$\hat{f}^H_{n_1,n_2}(y) = \alpha_{n_1,n_2}y^{-r}. \tag{20}$$

It remains to find a particular solution, $\hat{f}^P_{n_1,n_2}(y)$. In what follows we assume that the solution is a linear combination of special functions multiplied by rational functions. We find the explicit constants which appear in front of these special functions by solving systems of certain linear equations. In Section 3, we describe in more details which system of linear equations need to be solved depending on the values of $(n_1, n_2)$. More precisely, we will consider the following cases:

1. In Section 3.1, we will consider the case $n_1n_2 \neq 0$.
2. In Section 3.2, we will consider the case when exactly one of $n_i$, $i = 1, 2$, is equal to zero.
3. In Section 3.3, we will consider the case $n_1 = n_2 = 0$.

Finally, in Section 9, we will show that the sums $\sum_{n_1+n_2=n} \hat{f}_{n_1,n_2}(y)$ converge for each $n$ (at least in the physically relevance cases we have considered in Sections 5–8).

After we outline the strategy of finding solutions, we provide in further sections explicit examples of such for some physically relevant $\lambda, k$ and $\ell$. We stress that we are able to find the solutions as functions of $n_1$ and $n_2$ without restricting ourselves to any particular values of $n_1$ and $n_2$. More precisely, we write down the explicit solutions for $f(z)$ in the following cases:

I. $\lambda = 30, 56$ and $k = \ell = 3/2$ in Section 5 (we omit $\lambda = 12$, because it has been treated in [9]),

II. $\lambda = 20$ and $k = 3/2$, $\ell = 5/2$ in Section 6,

III. $\lambda = 30$ and $k = \ell = 5/2$ in Section 7,

IV. $\lambda = 30$ and $k = 3/2$, $\ell = 7/2$ in Section 8,

V. $\lambda = 2$ and $k = \ell = 3/2$; $\lambda = 6$ and $k = 3/2, \ell = 5/2$; $\lambda = 2, 12$ and $k = \ell = 5/2$; and $\lambda = 12$ and $k = 3/2, \ell = 7/2$ in Section A.

We have not included the solutions of the differential equations that would cover the full expansion of $\partial_m^4 \log Z|_{m=0,b=1}$ up to the order $1/N^3$ to keep the length of the article reasonable.

## 3. Particular solutions

In this section, we explicitly describe the system of linear equations that finds a particular solution of (15), depending on the values of $(n_1, n_2)$. As noted in Section 1.1, the convergence of the solution will depend on the explicit form of $f^P_{n_1,n_2}(y)$ and $f^H_{n_1,n_2}(y)$ (as given in Sections 5–8). Thus the proof that these solutions converge will be given in Section 9.

### 3.1. Solutions to (15) for $n_1 n_2 \neq 0$

Substituting (12) into (15) and denoting

$$\hat{f}^P_{n_1,n_2}(y) = \frac{4\pi^{k+\ell}}{\Gamma(k)\zeta(2k)\Gamma(\ell)\zeta(2\ell)} |n_1|^{k-\frac{1}{2}} |n_2|^{\ell-\frac{1}{2}} \sigma_{1-2k}(n_1)\sigma_{1-2\ell}(n_2) g(y),$$

we obtain a differential equation on $g$:

(21)
$$\left(-4\pi^2 y^2 (n_1+n_2)^2 + y^2 \partial_y^2 - \lambda\right) g(y) = y K_{k-1/2}(2\pi |n_1| y) K_{\ell-1/2}(2\pi |n_2| y),$$

or

$$(22)\qquad \big(-4\pi^2y^2\big(|n_1|+\operatorname{sgn}(n_1n_2)|n_2|\big)^2+y^2\partial_y^2-\lambda\big)g(y) = yK_{k-1/2}(2\pi|n_1|y)K_{\ell-1/2}(2\pi|n_2|y).$$

We introduce the notation

$$P_\lambda := -4\pi^2y^2\big(|n_1|+\operatorname{sgn}(n_1n_2)|n_2|\big)^2+y^2\partial_y^2-\lambda.$$

In this notation, (22) reads

$$(23)\qquad P_\lambda g(y) = yK_{k-1/2}(2\pi|n_1|y)K_{\ell-1/2}(2\pi|n_2|y).$$

If $k,\ell\in\mathbb{Z}+\frac{1}{2}$, then, using recursive formulas for the modified Bessel functions as in [3, 10.29(i)] – or, for the particular choices of $k$ and $\ell$, as in Section B.2 – we can rewrite the right hand side for (23) as

$$(24)\qquad \sum_{i,j=0}^{1} h^{i,j}(y)K_i(2\pi|n_1|y)K_j(2\pi|n_2|y),$$

where $h^{i,j}$ for each $i,j\in\{0,1\}$ is a polynomial in $y$ and $y^{-1}$.

We note that for any $a,b,c,d\in\mathbb{Z}$,

$$\begin{aligned}
(25)\quad & P_\lambda\big(y^aK_0(2\pi|n_1|y)K_0(2\pi|n_2|y)\big)\\
&= \big(-\operatorname{sgn}(n_1n_2)8\pi^2|n_1||n_2|y^{a+2}+a^2y^a-\lambda y^a-ay^a\big)\\
&\qquad\times K_0(2\pi|n_1|y)K_0(2\pi|n_2|y)\\
&\quad+\big(2\pi|n_2|y^{a+1}-4\pi|n_2|ay^{a+1}\big)K_0(2\pi|n_1|y)K_1(2\pi|n_2|y)\\
&\quad+\big(2\pi|n_1|y^{a+1}-4\pi|n_1|ay^{a+1}\big)K_1(2\pi|n_1|y)K_0(2\pi|n_2|y)\\
&\quad+\big(8\pi^2|n_1||n_2|y^{a+2}\big)K_1(2\pi|n_1|y)K_1(2\pi|n_2|y),\\
(26)\quad & P_\lambda\big(y^bK_0(2\pi|n_1|y)K_1(2\pi|n_2|y)\big)\\
&= \big(2\pi|n_2|y^{b+1}-4\pi b|n_2|y^{b+1}\big)K_0(2\pi|n_1|y)K_0(2\pi|n_2|y)\\
&\quad+\big(b^2y^b-\operatorname{sgn}(n_1n_2)8\pi^2|n_1||n_2|y^{b+2}-\lambda y^b-3by^b+2y^b\big)\\
&\qquad\times K_0(2\pi|n_1|y)K_1(2\pi|n_2|y)\\
&\quad+\big(8\pi^2|n_1||n_2|y^{b+2}\big)K_1(2\pi|n_1|y)K_0(2\pi|n_2|y)\\
&\quad+\big(6\pi|n_1|y^{b+1}-6\pi b|n_1|y^{b+1}\big)K_1(2\pi|n_1|y)K_1(2\pi|n_2|y),\\
(27)\quad & P_\lambda\big(y^cK_1(2\pi_1|n_1|y)K_0(2\pi|n_2|y)\big)
\end{aligned}$$

$$
\begin{aligned}
&= \big(2\pi|n_1|y^{c+1} - 4\pi|n_1|cy^{c+1}\big)K_0(2\pi|n_1|y)K_0(2\pi|n_2|y)\\
&\quad + \big(8\pi^2|n_1||n_2|y^{c+2}\big)K_0(2\pi|n_1|y)K_1(2\pi|n_2|y)\\
&\quad - \big(\operatorname{sgn}(n_1n_2)28\pi^2|n_1||n_2|y^{c+2} - c^2y^c + \lambda y^c + 3cy^c - 2y^c\big)\\
&\qquad\qquad \times K_1(2\pi|n_1|y)K_0(2\pi|n_2|y)\\
&\quad + \big(6\pi|n_2|y^{c+1} - 4\pi|n_2|cy^{c+1}\big)K_1(2\pi|n_1|y)K_1(2\pi|n_2|y),
\end{aligned}
$$

$$
\begin{aligned}
(28)\quad & P_\lambda\big(y^dK_1(2\pi|n_1|y)K_1(2\pi|n_2|y)\big)\\
&= \big(8\pi^2|n_1||n_2|y^{d+2}\big)K_0(2\pi|n_1|y)K_0(2\pi|n_2|y)\\
&\quad + \big(6\pi|n_1|y^{d+1} - 4\pi|n_1|dy^{d+1}\big)K_0(2\pi|n_1|y)K_1(2\pi|n_2|y)\\
&\quad + \big(6\pi|n_2|y^{d+1} - 4\pi|n_2|dy^{d+1}\big)K_1(2\pi|n_1|y)K_0(2\pi|n_2|y)\\
&\quad - \big(\operatorname{sgn}(n_1n_2)8\pi^2|n_1||n_2|y^{d+2} - d^2y^d + \lambda y^d + 5dy^d - 6y^d\big)\\
&\qquad\qquad \times K_1(2\pi|n_1|y)K_1(2\pi|n_2|y).
\end{aligned}
$$

If we assume that a solution of (22) can be expressed as a sum

$$
g(y) = \sum_{i,j=0}^{1} q^{i,j}(y)K_i(2\pi|n_1|y)K_j(2\pi|n_2|y), \tag{29}
$$

where $q^{i,j}$ are some polynomials in $y$ and $y^{-1}$, then $q^{i,j}$ can be found by solving a system of linear equations on the coefficients of $q^{i,j}$. More precisely, assume that

$$
\max_{i,j} \deg q^{i,j} = M, \quad \min_{i,j} \deg q^{i,j} = m.
$$

Then each of $q^{i,j}$ for $i, j \in \{0, 1\}$ is parametrized by $(M - m + 1)$ complex coefficients.

On the other hand,

$$
P_\lambda(g) = \sum_{i,j=0}^{1} \tilde{q}^{i,j}(y)K_i(2\pi|n_1|y)K_j(2\pi|n_2|y)
$$

for some polynomials $\tilde{q}^{i,j}$ such that

$$
\max_{i,j} \deg \tilde{q}^{i,j} = M + 2, \quad \min_{i,j} \deg \tilde{q}^{i,j} = m.
$$

Thus, (23) can be equivalently written as $4(M+3-m)$ linear equations with $4(M - m + 1)$ variables: the variables are exactly the coefficients of $q^{i,j}$, and the linear equations come from the requirement that $h^{i,j} = \tilde{q}^{i,j}$.

We note that as one can see from (25)–(27), the corresponding matrix of the system of linear equation is a band matrix, that simplifies the calculations.

The possibility of writing $g$ in such form depends on $k$, $\ell$, $\lambda$, $M$ and $m$. Below, we write down some elementary limitations on the set of parameters that are needed in order for a solution of such form to exist. Further in the article, we speculate on possible connection of restriction with the differential Galois theory.

**Proposition 3.1.** *A solution to* (1) *of the form* (29) *with the condition*

$$\min_{i,j\in\{0,1\}} \deg h^{i,j} > \min_{i,j\in\{0,1\}} \deg q^{i,j}$$

*does not exist unless* $\lambda$ *is of the form* $r(r+1)$ *for* $r \in \mathbb{Z}_{>0}$.

*Proof.* We note that from $h^{i,j} = \tilde{q}^{i,j}$, the equality

$$\min_{i,j\in\{0,1\}} \deg h^{i,j} = \min_{i,j\in\{0,1\}} \deg \tilde{q}^{i,j}$$

must hold, that implies

$$\min_{i,j\in\{0,1\}} \deg \tilde{q}^{i,j} < \min_{i,j\in\{0,1\}} \deg q^{i,j}.$$

Together with (25)–(28), the inequality above implies that there exist $a, b, c, d \in \mathbb{Z}$ such that at least one of the following equalities holds:

$$\begin{aligned} a^2 - \lambda - a &= 0, \\ b^2 - \lambda - 3b + 2 &= 0, \\ c^2 - \lambda - 3c + 2 &= 0, \\ d^2 - \lambda - 5d + 6 &= 0. \end{aligned}$$

That implies the statement of the proposition. □

Additionally, we prove the following elementary proposition:

**Proposition 3.2.** *A solution to* (1) *of the form* (29) *does not exist unless* $k \in \frac{1}{2} + \mathbb{Z}$ *and* $\ell \in \frac{1}{2} + \mathbb{Z}$.

*Proof.* We give the proof for $k$ by contradiction; the proof for $\ell$ is similar. Consider the right hand sides of (25)–(28). Although we have assumed $n_1, n_2$ to be non-zero integers, the formulas above would hold if we let $n_1, n_2$ be non-zero real numbers. Keeping that in mind, we let $n_2 = n_2(y) = \frac{1}{y}$ depend on $y$. Having fixed the product of $n_2$ and $y$ and keeping in mind asymptotic expansions of the modified Bessel function of the second kind, we consider the corresponding asymptotic expansions of $P_{\lambda,\pm}g(y)$ for $y \to 0$ only to find integer powers of $y$ and $\log(y)$.

On the other hand, if $k \neq 0$, the asymptotic expansion of $yK_{k-1/2}(2\pi|n_1|y)K_{\ell-1/2}(2\pi)$ as $y \to 0$ contains only terms of the type $y^{k+1/2+\star}$ for $\star \in \mathbb{Z}$. Thus, $k \in \mathbb{Z} + 1/2$. □

In what follows, we give explicit solutions to (22) for some physically relevant combinations of $k, \ell$ and $\lambda$.

### 3.2. Solutions to (15) for $n_1 n_2 = 0$, but not both zero

Without loss of generality we assume $n_1 = 0, n_2 \neq 0$. We note that if we find $g_1$ and $g_2$ that satisfy

$$\big(-4\pi^2 y^2|n_2|^2 + y^2\partial_y^2 - \lambda\big)g_1(y) = y^{\frac{1}{2}+k}K_{\ell-1/2}(2\pi|n_2|y)$$

or

$$\big(-4\pi^2 y^2|n_2|^2 + y^2\partial_y^2 - \lambda\big)g_2(y) = y^{\frac{3}{2}-k}K_{\ell-1/2}(2\pi|n_2|y),$$

then the function

$$\frac{2\pi^\ell}{\Gamma(\ell)\zeta(2\ell)}|n|^{\ell-\frac{1}{2}}\sigma_{1-2\ell}(n)g_1(y) + \frac{2\pi^{\ell+1/2}\Gamma(k-\frac{1}{2})\zeta(2k-1)}{\Gamma(k)\zeta(2k)\Gamma(\ell)\zeta(2\ell)}|n|^{\ell-\frac{1}{2}}\sigma_{1-2\ell}(n)g_2(y)$$

solves (15) for $n_1 = 0$ and $|n_2| = |n|$.

We assume that each of $g_i$ with $i = 1,2$ can be represented as the following sum:

$$\sum_{j=0}^{1} p^j(y)K_j(2\pi|n_2|y),$$

where $p^j$ is a polynomial in $y$ and $y^{-1}$. We note that for $g, h \in \mathbb{R}$,

$$L_\lambda\big(x^g K_0(2\pi|n_2|y)\big) = \big((g-1)g - \lambda\big)x^g K_0(2\pi|n_2|x) + 2\pi|n_2|(1-2g)x^{g+1}K_1(2\pi|n_2|x),$$

$$L_\lambda\big(x^h K_1(2\pi|n_2|y)\big) = \big(-\lambda + (h-3)h + 2\big)x^h K_1(2\pi|n_2|x) \\ + 2\pi|n_2|(1-2h)x^{h+1}K_0(2\pi|n_2|x),$$

for $L_\lambda := -4\pi^2 y^2 |n_2|^2 + y^2\partial_y^2 - \lambda$. Let

$$\max_{j\in\{0,1\}} \deg p^j = M, \qquad \min_{j\in\{0,1\}} \deg p^j = m.$$

Then, in order to find coefficients of $p^j$ for $j = 0, 1$, we have to solve a system of linear equations with $2(M-m+1)$ variables, that are coefficients of the polynomials $p^j$ for $j = 0, 1$, and $2(M-m+1)+2$ equalities on coefficients at

$$\bigcup_{\ell=m}^{M+2}\bigcup_{j=0}^{1}\big\{y^\ell K_j(2\pi|n_2|y)\big\}.$$

### 3.3. Solutions to (15) for $n_1 = n_2 = 0$

We note that in order to solve (15) for $n_1 = n_2 = 0$ it is sufficient to find solutions of

$$\big(y^2\partial_y^2 - \lambda\big)g(y) = y^{j_1+j_2}, \quad j_1 \in \{k, 1-k\} \text{ and } j_2 \in \{\ell, 1-\ell\}.$$

A particular solution can be easily constructed as products and sums of $\log(y)$ and polynomials in half-powers of $y$ and $y^{-1}$.

## 4. Differential Galois theory

This article gives explicit solutions for specific combination of $k$, $\ell$ and $\lambda = r(r+1)$ with $r > 0$ listed in Section 1.1; however, experimentally we were able to compute solutions for other combinations of $k$, $\ell$ and $\lambda$ using the same methods outlined above.

Finding solutions for large $k, \ell$ and $\lambda$ involves solving a systems of linear equation for a large number of variables. This becomes computationally challenging, even though the corresponding matrices are band matrices. We obtained that a particular solution of (15) is of the form (24), at least, in the cases where

$$k, \ell \in \mathbb{Z} + \tfrac{1}{2}, \quad k + \ell + r \in 2\mathbb{Z}, \quad |k-\ell| < r \tag{30}$$

and

$$1 < k < 30, \quad 1 < \ell < 30, \quad 0 < r < 15. \tag{31}$$

Note that the functions that appear in [2, (2.11)] satisfy the condition above. We note that we did not check the convergence of the solution in each of these cases.

Below, we make a conjecture that the solution are "nice" if $(k, \ell, \gamma)$ belongs to (30), regardless of how large each parameter may be. Discussing in which way they are "nice" would require some basic facts from the differential Galois theory, that we outline as follows.

The fundamental system of the homogeneous solution of (16) is well-known for any values of $r \in \mathbb{R}$ and involves the modified Bessel functions (see (17)). Moreover, it is possible to show[3] that modified Bessel equations, $K_\eta$, can be expressed via elementary functions if and only if their index, $\eta$, belongs to $\frac{1}{2} + \mathbb{Z}$. In our notations, this corresponds to demanding $r \in \mathbb{Z}$.

Recall [12, Chapter 3] that a *differential field*, $K$, is a field together with a *derivation* (i.e. an additive map that satisfies the Leibniz rule, $\partial(ab) = \partial(a)b + a\partial(b)$). An elementary example of a differential field would be the field $\mathbb{C}(t)$ of rational functions over $\mathbb{C}$ together with a usual operation of differentiation. Solutions of the type (24) belong to a particular object in differential Galois theory – namely, they belong to a certain Picard-Vessiot extension of a differential field. A differential field $P$ is called a *Picard-Vessiot extension of the field* $K$, if there exists a linear differential equation with coefficients in $K$ such that $P$ is obtained from $K$ by adjoining a fundamental system of solutions of this differential equation.

When $n_1, n_2 \in \mathbb{Z} \setminus \{0\}$ with $n := n_1 + n_2 \neq 0$, the differential field, $P$, that we are interested in can be obtained by adjoining to $\mathbb{C}(t)$ solutions of the equations

$$\big(y^2\partial_y^2 - 4\pi^2 n_1^2 y^2 + 1/4\big) f(y) = 0, \tag{32}$$

$$\big(y^2\partial_y^2 - 4\pi^2 n_2^2 y^2 + 1/4\big) f(y) = 0, \tag{33}$$

and

$$\big(y^2\partial_y^2 - r(r+1) - 4\pi^2 (n_1+n_2)^2 y^2\big) f(y) = 0. \tag{34}$$

[3]For $J$-Bessel functions, the proof can be found in [15, Appendix]; we can obtain the same statement for $K$ and $I$-Bessel functions by exploiting formulas relating Bessel functions to each other.

We note that $P$ contains $\sqrt{y}\,K_0(2\pi|n_1|y)$ and $\sqrt{y}\,K_0(2\pi|n_2|y)$ by construction. Since it is an extension of $\mathbb{C}(t)$, it also contains any sum of the type

$$\sum_{j=0}^{n} a_j y^{j+1/2} K_0(2\pi|n_1|y), \quad n \in \mathbb{N}, a_j \in \mathbb{C}.$$

Moreover, the recurrence relation between $K_0$ and $K_1$ implies, that $P$ contains any sums of type

$$\sum_{j=0}^{n} b_j y^{j+1/2} K_1(2\pi|n_1|y), \quad n \in \mathbb{N}, b_j \in \mathbb{C}.$$

Thus, we obtain solutions of the type (24) belong to the field $P$ that we have just constructed.

On the other hand, we can reformulate the inhomogeneous differential equation (15) as the following homogeneous differential equation of the third order on $g_{n_1,n_2}(y)$:

$$\frac{\partial}{\partial y}\left(\frac{(y^2\partial_y^2 - \lambda - 4\pi^2(n_1+n_2)^2y^2)g_{n_1,n_2}(y)}{s_{n_1,n_2}(y)}\right) = 0. \tag{35}$$

We note that for any solution (35) there is a constant $c$ such that $g_{n_1,n_2}(y)$ is a solution of

$$\big(y^2\partial_y^2 - \lambda - 4\pi^2(n_1+n_2)^2y^2\big)g_{n_1,n_2}(y) = cs_{n_1,n_2}(y).$$

And, on the other hand, every $\hat{f}_{n_1,n_2}(y)$ is also solution of (35).

We will prove in this article, that for certain $k,\ell,\lambda$, the function $g$ belongs to $P$ simply by providing an explicit solution. Moreover, every solution of (35) is of the form $c_1g_{n_1,n_2}(y) + c_2f(y)$ where $f(y)$ is a solution of (34) and $c_1, c_2 \in \mathbb{C}$. Note that $f(y) \in P$, thus the Galois group of (35) is trivial. On the other hand, calculations with the help of system of computer algebra suggest that it is also trivial for all $k,\ell,\lambda$ satisfying (30) and (31).

**Conjecture 4.1.** *Let $P$ be a Picard-Vessiot extension of the field of rational functions over $\mathbb{C}$, obtained by adjoining solutions of* (32)*–*(34)*. Then the Galois group of* (35) *is trivial in the category of algebraic groups if $k,\ell,r$, however big they are, satisfy* (30)*.*

However, proving or disproving this conjecture is beyond the scope of this paper. There are certain related results, see [16], where the authors present the algorithm that calculates the differential Galois group of a third-order homogeneous linear differential equation. Specifically, in the case where $k, \ell \in \mathbb{Z}$, the product of modified Bessel function in the inhomogeneous differential equation will become a product rational functions and exponentials. It appears that methods of differential Galois theory may be more directly applicable to this context for $k, \ell \in \mathbb{Z}$.

## 5. $k = \ell = 3/2$

In this section, we solve

$$(\Delta - \lambda)f(z) = -\big(2\zeta(3)E_{3/2}(z)\big)^2, \quad z = x + iy \in \mathfrak{H}$$

for

$$f(z) = \sum_{n\in\mathbb{Z}} \sum_{n_1+n_2=n} \hat{f}_{n_1,n_2}(y)e^{2\pi inx}$$

in terms of $\hat{f}_{n_1,n_2}(y) = \hat{f}^P_{n_1,n_2}(y) + \hat{f}^H_{n_1,n_2}(y)$.

When $n_1 = n_2 = 0$, $\hat{f}_{0,0}(y)$ contains no $K$-Bessel or divisor functions and is given by a a polynomial in $y$ and $1/y$ below. For $n_1 n_1 = 0$ but not both zero,

$$\hat{f}^P_{0,n}(y) = \hat{f}^P_{n,0}(y) = -16\pi \frac{\sigma_2(n)}{|n|} \sum_{i=0,1} \nu_i(n, y)K_i(2\pi|n|y), \tag{36}$$

for $n_1 n_2 \neq 0$ and $n_1 + n_2 \neq 0$,

$$\hat{f}^P_{n_1,n_2}(y) = -64\pi^2 \frac{\sigma_2(n_1)\sigma_2(n_2)}{|n_1 n_2|} \sum_{i,j=0,1} \eta_{i,j}(n_1, n_2, y)K_i(2\pi|n_1|y)K_j(2\pi|n_2|y), \tag{37}$$

and for $n_1 = -n_2$,

$$\begin{aligned}\hat{f}^P_{-n_2,n_2}(y) = -64\pi^2 &\frac{\sigma_2(n_2)\sigma_2(n_2)}{|n_2|^2} \\ &\times \sum_{(i,j)\in\{(0,0),(0,1),(1,1)\}} \mu_{i,j}(n_2, y)K_i(2\pi|n_2|y)K_j(2\pi|n_2|y),\end{aligned} \tag{38}$$

where $\nu_i, \eta_{i,j}$ and $\mu_{i,j}$ defined below depending on each value of $\lambda$.

## 5.1. $\lambda = 30$

This case corresponds to [2, Section C.3.1] with $r = 5$.

**5.1.1. $n_1 = 0$ and $n_2 = 0$** Any solution of (15) for $n_1 = n_2 = 0$ is equal to

$$\hat{f}_{0,0}(y) = \frac{105y^4\zeta(3)^2 + 56\pi^2y^2\zeta(3) + 10\pi^4}{630y} + \frac{\alpha_{0,0}}{y^5} + \beta_{0,0}\, y^6$$

for some $c_1, c_2 \in \mathbb{C}$. Its asymptotic behavior for $y \to 0$ can be described by

$$\hat{f}_{0,0}(y) = \alpha_{0,0}y^{-5}.$$

At this moment of time, we do not choose $\alpha_{0,0}$ – that will be reserved for Section 5.1.4. Our goal would be to choose $\alpha_{0,0}$ in such a way that

$$\sum_n \alpha_{n,-n} = 0.$$

In our notation and after the evaluation of the Riemann zeta function at even integers, the first three summands of the first line of [2, (C.27)] read

$$\frac{y^3\zeta(3)^2}{6} + \frac{4}{45}\pi^2 y\zeta(3) + \frac{\pi^4}{63y},$$

that coincides with our result.

**5.1.2. $n_1n_2 = 0$ but not both zero** Though this case of $\alpha = \ell = 3/2$ and $\lambda = 30$ is generally addressed in [2], we note that the term $\hat{f}^P_{0,n}(y)$ was not found explicitly. For $\hat{f}^P_{0,n}(y)$ as in (36), we have

$$\begin{aligned}\nu_0(n,y) = \operatorname{sgn}(n)\Bigg[&-\zeta(3)\left(\frac{126y^{-3}}{n^5\pi^5} + \frac{35y^{-1}}{n^3\pi^3} + \frac{y}{2n\pi}\right)\\ &+ 2\zeta(2)\left(\frac{3y^{-3}}{5n^3\pi^3} + \frac{y^{-1}}{6n\pi}\right)\Bigg],\\ \nu_1(n,y) = -\zeta(3)&\left(\frac{126y^{-4}}{n^6\pi^6} + \frac{98y^{-2}}{n^4\pi^4} + \frac{15}{2n^2\pi^2}\right) + 2\zeta(2)\left(\frac{3y^{-4}}{5n^4\pi^4} + \frac{7y^{-2}}{15n^2\pi^2}\right).\end{aligned}$$

Its asymptotic behavior for $y \to 0$ can be described as follows

$$\hat{f}^P_{0,n}(y) = -\frac{48\sigma_2(n)(\pi^2\zeta(2)n^2 - 105\zeta(3))}{5\pi^6 n^8 y^5} \tag{39}$$

$$+\frac{32(\pi^2\zeta(2)n^2-105\zeta(3))\sigma_2(|n|)}{15\pi^4n^6y^3}+O\left(\frac{1}{y^2}\right).$$

The unique choice of $\alpha_{n,0}=\alpha_{0,n}$ that gets rid of the $O(y^{-5})$-term in $\hat{f}^P_{0,n}(y)+\hat{f}^H_{0,n}(y)$ is

$$\alpha_{n,0}=\alpha_{0,n}=-\frac{1024(\pi^2\zeta(2)n^2-105\zeta(3))\sigma_2(n)}{1575\pi|n|^{5/2}}. \tag{40}$$

**5.1.3. $n_1n_2\neq 0$ and $n_1+n_2\neq 0$** In [2, p. 46], many terms[4] in the perturbative expansion $\hat{f}^P_{n_1,n_2}(y)$ were evaluated. However, these values were not explicitly written or evaluated in full in [2]. For $\hat{f}^P_{n_1,n_2}(y)$ as in (37), we have

$$\begin{aligned}\eta_{0,0}=&\operatorname{sgn}(n_1n_2)\Bigg[\frac{y^{-3}\,n_1n_2}{(n_1+n_2)^{10}}\frac{126}{\pi^4}\big(n_1^4-6n_1^3n_2+10n_1^2n_2^2-6n_1n_2^3+n_2^4\big)\\&+\frac{y^{-1}\,n_1n_2}{(n_1+n_2)^8}\frac{2}{5\pi^2}\big(89n_1^4-792n_1^3n_2+1598n_1^2n_2^2-792n_1n_2^3+89n_2^4\big)\\&+\frac{y\,n_1n_2}{(n_1+n_2)^6}\frac{2}{15}\big(5n_1^4-92n_1^3n_2+190n_1^2n_2^2-92n_1n_2^3+5n_2^4\big)\Bigg],\\\eta_{0,1}=&\operatorname{sgn}(n_1)\Bigg[\frac{y^{-4}\,n_1n_2^2}{(n_1+n_2)^{11}}\frac{126}{\pi^5}\big(-n_1^3+5n_1^2n_2-5n_1n_2^2+n_2^3\big)\\&+\frac{y^{-2}\,n_1}{5\pi^3(n_1+n_2)^9}\big(3n_1^5+99n_1^4n_2-2728n_1^3n_2^2\\&\qquad+6512n_1^2n_2^3-3611n_1n_2^4+493n_2^5\big)\\&+\frac{n_1}{30\pi(n_1+n_2)^7}\big(5n_1^5+147n_1^4n_2-2614n_1^3n_2^2\\&\qquad+5726n_1^2n_2^3-2799n_1n_2^4+239n_2^5\big)\Bigg],\\\eta_{1,0}=&\operatorname{sgn}(n_2)\Bigg[\frac{y^{-4}\,n_1^2n_2}{(n_1+n_2)^{11}}\frac{126}{\pi^5}\big(n_1^3-5n_1^2n_2+5n_1n_2^2-n_2^3\big)\\&+\frac{y^{-2}\,n_2}{5\pi^3(n_1+n_2)^9}\big(493n_1^5-3611n_1^4n_2+6512n_1^3n_2^2-2728n_1^2n_2^3\\&\qquad+99n_1n_2^4+3n_2^5\big)\end{aligned}$$

[4] Powers of around $D$-instanton contributions with $n\neq 0$, that include the instanton sectors of $(n_1,n_2)=(2,0),(1,1),(1,-2),(1,-3),(2,-3)$ were found.

$$+\frac{n_2}{30\pi(n_1+n_2)^7}\big(239n_1^5-2799n_1^4n_2+5726n_1^3n_2^2-2614n_1^2n_2^3+147n_1n_2^4+5n_2^5\big)\bigg],$$

$$\eta_{1,1}=\frac{y^{-3}}{5\pi^4(n_1+n_2)^{10}}\big(3n_1^6+102n_1^5n_2-3399n_1^4n_2^2+8124n_1^3n_2^3-3399n_1^2n_2^4+102n_1n_2^5+3n_2^6\big)$$
$$+\frac{y^{-1}}{15\pi^2(n_1+n_2)^8}\big(7n_1^6+220n_1^5n_2-4727n_1^4n_2^2+10280n_1^3n_2^3-4727n_1^2n_2^4+220n_1n_2^5+7n_2^6\big)$$
$$+\frac{y\,n_1n_2}{(n_1+n_2)^6}\frac{2}{15}\big(5n_1^4-92n_1^3n_2+190n_1^2n_2^2-92n_1n_2^3+5n_2^4\big).$$

Note that

(41)
$$\hat f^P_{n_1,n_2}(y)=-16y^{-5}\frac{\sigma_2(n_1)\sigma_2(n_2)}{5|n_1n_2|^2\pi^4(n_1+n_2)^{11}}\times\Big(\big(1260n_1^5n_2^2-6300n_1^4n_2^3+6300n_1^3n_2^4-1260n_1^2n_2^5\big)\log(|n_1/n_2|)+3n_1^7+105n_1^6n_2-3297n_1^2n_2^5+105n_1n_2^6+3n_2^7-3297n_1^5n_2^2+4725n_1^3n_2^4+4725n_1^4n_2^3\Big)+o\big(y^{-5}\big).$$

We recall that by (19),

$$\hat f^H_{n_1,n_2}(y)=\alpha_{n_1,n_2}\sqrt{y}K_{5+\frac12}(2\pi|n_1+n_2|y)=\alpha_{n_1,n_2}\frac{945}{64\pi^5|n_1+n_2|^{11/2}y^5}+o\big(y^{-5}\big).$$

Comparing the right hand sides of the two previous formulas, we obtain that there is a unique choice of $\alpha_{n_1,n_2}$ that guarantees that

$$\hat f_{n_1,n_2}(y)=\hat f^P_{n_1,n_2}(y)+\hat f^H_{n_1,n_2}(y)=o\big(y^{-5}\big),$$

as $y\to 0$ given by

$$\alpha_{n_1,n_2}=\frac{1024\pi\sigma_2(n_1)\sigma_2(n_2)\operatorname{sgn}(n_1+n_2)}{4725|n_1n_2|^2|n_1+n_2|^{11/2}}\times\big(3n_1^7+105n_1^6n_2-3297n_1^2n_2^5+105n_1n_2^6+3n_2^7-3297n_1^5n_2^2 \tag{42}$$

$$+ 4725 n_1^3 n_2^4 + 4725 n_1^4 n_2^3 + 1260 n_1^5 n_2^2 \log(|n_1/n_2|)$$
$$- 6300 n_1^4 n_2^3 \log(|n_1/n_2|) + 6300 n_1^3 n_2^4 \log(|n_1/n_2|)$$
$$- 1260 n_1^2 n_2^5 \log(|n_1/n_2|\pi)\Big).$$

Moreover, it is not complicated to check that for fixed values of $n$,

$$\alpha_{n-n_1,n_1} = O\big(|n_1|^{-4}\big), \quad |n_1| \to \infty. \tag{43}$$

**5.1.4. $n_1 = -n_2$** This case has been considered in [2, (C.29)].

For $\hat{f}^P_{-n_2,n_2}(y)$ as in (38) we have

$$\mu_{0,0} = \frac{y^{-1}}{110 n_2^2 \pi^2} + \frac{y}{110} - \frac{8 n_2^2 \pi^2 y^3}{1155} - \frac{512 n_2^4 \pi^4 y^5}{17325} + \frac{16384 n_2^6 \pi^6 y^7}{51975},$$
$$\mu_{0,1} = \operatorname{sgn}(n_2)\left(\frac{y^{-2}}{55 n_2^3 \pi^3} + \frac{3}{110 n_2 \pi} + \frac{4 n_2 \pi y^2}{385} + \frac{256 n_2^3 \pi^3 y^4}{17325} + \frac{8192 n_2^5 \pi^5 y^6}{51975}\right),$$
$$\mu_{1,1} = \frac{y^{-3}}{110 n_2^4 \pi^4} + \frac{y^{-1}}{55 n_2^2 \pi^2} - \frac{17y}{770} - \frac{8 n_2^2 \pi^2 y^3}{1925} - \frac{512 n_2^4 \pi^4 y^5}{51975} - \frac{16384 n_2^6 \pi^6 y^7}{51975}.$$

We note that

$$\hat{f}^P_{-n_2,n_2}(y) = -8 \frac{\sigma_2(n_2)^2}{|n_2|^2} \cdot \left(\frac{1}{55 \pi^4 n_2^6 y^5} + O\left(\frac{1}{y}\right)\right). \tag{44}$$

The unique choice of $\alpha_{-n_2,n_2}$ that gets rid of the $O(y^{-5})$ term in the expression above is

$$\alpha_{-n_2,n_2} = \frac{8 \sigma_2(n_2)^2}{55 \pi^4 |n_2|^8}. \tag{45}$$

Summing it up and using (99) for $a = 2, b = 2$ and $s = 8$, we obtain

$$\sum_{n_2=-\infty, n_2 \neq 0}^{\infty} \frac{\sigma_2(n_2)^2}{|n_2|^8} = 2 \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}\Bigg|_{a=2,b=2,s=8}$$
$$= \frac{143 \pi^{12}}{58769550},$$

and thus

$$\sum_{n_2=-\infty, n_2 \neq 0}^{\infty} \alpha_{-n_2,n_2} = \frac{52 \pi^8}{146923875} = \frac{104 \zeta(8)}{31095}. \tag{46}$$

This means, that $\sum_{n_2=-\infty, n_2\neq 0}^{\infty} \hat{f}^H_{-n_2,n_2}(y) = \frac{104\zeta(8)}{31095} y^{-5}$. Motivated by the desire to have the contribution from the homogeneous elements to be equal to zero, we obtain $\alpha_{0,0} = -\frac{104\zeta(8)}{31095}$. This matches[5] the last summand in the first line of [2, (C.27)].

### 5.2. $\lambda = 56$

This case corresponds to [2, Section C.3.1] with $r = 7$.

**5.2.1. $n_1 = 0$ and $n_2 = 0$** The solution of (15) for $n_1 = n_2 = 0$ is equal to

$$\hat{f}_{0,0}(y) = \frac{3402 y^4 \zeta(3)^2 + 2025\pi^2 y^2 \zeta(3) + 350\pi^4}{42525 y} + \frac{c_1}{y^7} + c_2 y^8$$

for some $c_1, c_2 \in \mathbb{C}$. Its asymptotic behavior for $y \to 0$ can be described by $\hat{f}_{0,0}(y) = \frac{c_1}{y^7}$. We do not specify the choice of $c_1$ for the moment, but we can set $c_2 = 0$ so that the $O(y^8)$-term vanishes. We note that the first three summands[6] of $\hat{f}_{0,0}(y)$ coincide with the first three terms of the second line of [2, (C.27)].

**5.2.2. $n_1 n_2 = 0$ but not both zero** Though this case of $k = \ell = 3/2$ and $\lambda = 56$ is generally addressed in [2], we note that the term $\hat{f}^P_{0,n}(y)$ was not found explicitly. This term is given by

$$\hat{f}^P_{0,n}(y) = \hat{f}^P_{n,0}(y) = -16\pi \frac{\sigma_2(n)}{|n|} \sum_{i,j=0,1} \nu_i(n,y) K_i(2\pi|n|y),$$

with

$$\begin{aligned}
\nu_0(n,y) = \operatorname{sgn}(n)\Bigg[ -\zeta(3)\left( \frac{30888 y^{-5}}{5n^7\pi^7} + \frac{10692 y^{-3}}{5n^5\pi^5} + \frac{126 y^{-1}}{n^3\pi^3} + \frac{y}{2n\pi} \right) \\
+ 2\zeta(2)\left( \frac{286 y^{-5}}{35 n^5\pi^5} + \frac{99 y^{-3}}{35 n^3\pi^3} + \frac{y^{-1}}{6n\pi} \right) \Bigg], \\
\nu_1(n,y) = -\zeta(3)\left( -\frac{30888 y^{-6}}{5n^8\pi^8} + \frac{26136 y^{-4}}{5n^6\pi^6} + \frac{3402 y^{-2}}{5n^4\pi^4} + \frac{14}{n^2\pi^2} \right) \\
+ 2\zeta(2)\left( \frac{286 y^{-6}}{35 n^6\pi^6} + \frac{242 y^{-4}}{35 n^4\pi^4} + \frac{9 y^{-2}}{10 n^2\pi^2} \right).
\end{aligned}$$

[5] Up to sign.

[6] However, our choice of $c_1$ does not coincide with [2, (C.27)] coefficient at $y^{-7}$.

The asymptotic expansion is

(47)
$$\hat{f}^P_{0,n}(y) = \hat{f}^P_{n,0}(y) = -16\frac{\sigma_2(n)}{|n|}\left(\frac{286(\pi^2\zeta(2)n^2 - 378\zeta(3))}{35\pi^8|n|^9y^7} + O\left(\frac{1}{y^5}\right)\right).$$

There exists a unique choice of $\alpha_{0,n} = \alpha_{n,0}$ such that $\hat{f}^P_{0,n}(y) + \hat{f}^H_{0,n}(y)$ and $\hat{f}^P_{n,0}(y) + \hat{f}^H_{n,0}(y)$ are of order $o(y^{-7})$. More precisely,

$$\hat{f}^H_{0,n}(y) = \sqrt{y}\alpha_{0,n}K_{7+\frac{1}{2}}(2\pi|n|y) = \frac{135135\alpha_{n,0}}{256\pi^7|n|^{15/2}y^7} + O(y^{-5}),$$

and thus we may set

$$\alpha_{0,n} = \alpha_{n,0} = \frac{8192(\pi^2\zeta(2)n^2 - 378\zeta(3))\sigma_2(|n|)}{33075\pi|n|^{5/2}}.$$

**5.2.3. $n_1n_2 \neq 0$ and $n_1 + n_2 \neq 0$** Though this case of $k = \ell = 3/2$ and $\lambda = 56$ is generally addressed in [2], we note that the term $\hat{f}^P_{0,n}(y)$ was not found explicitly. For $\hat{f}^P_{n_1,n_2}(y)$ as in (37), we have

$$\begin{aligned}
\eta_{0,0} = \operatorname{sgn}(n_1n_2)\Bigg[&\frac{y^{-5}\,n_1n_2}{(n_1+n_2)^{14}}\frac{10296}{5\pi^6}\big(3n_1^6 - 38n_1^5n_2 + 140n_1^4n_2^2 - 210n_1^3n_2^3 \\
&\qquad + 140n_1^2n_2^4 - 38n_1n_2^5 + 3n_2^6\big) \\
&+ \frac{y^{-3}\,n_1n_2}{(n_1+n_2)^{12}}\frac{22}{175\pi^4}\big(17075n_1^6 - 260700n_1^5n_2 + 1170813n_1^4n_2^2 \\
&\qquad -1907624n_1^3n_2^3 + 1170813n_1^2n_2^4 - 260700n_1n_2^5 + 17075n_2^6\big) \\
&+ \frac{y^{-1}\,n_1n_2}{175\pi^2(n_1+n_2)^{10}}\big(22545n_1^6 - 445070n_1^5n_2 + 2255599n_1^4n_2^2 \\
&\qquad -3778788n_1^3n_2^3 + 2255599n_1^2n_2^4 - 445070n_1n_2^5 + 22545n_2^6\big) \\
&+ \frac{2\,y\,n_1n_2}{525(n_1+n_2)^8}\big(175n_1^6 - 6510n_1^5n_2 + 35745n_1^4n_2^2 - 61572n_1^3n_2^3 \\
&\qquad + 35745n_1^2n_2^4 - 6510n_1n_2^5 + 175n_2^6\big)\big)\Bigg]
\end{aligned}$$

$$\begin{aligned}
\eta_{0,1} = \operatorname{sgn}(n_1)\Bigg[&\frac{y^{-6}\,n_1n_2^2}{(n_1+n_2)^{15}}\frac{10296}{5\pi^7}\big(-3n_1^5 + 35n_1^4n_2 - 105n_1^3n_2^2 + 105n_1^2n_2^3 \\
&\qquad - 35n_1n_2^4 + 3n_2^5\big)
\end{aligned}$$

$$+ \frac{y^{-4}\, n_1}{(n_1+n_2)^{13}} \frac{22}{175\pi^5}\big(65n_1^7+3965n_1^6n_2-283107n_1^5n_2^2+1865089n_1^4n_2^3 - 3515111n_1^3n_2^4+2360373n_1^2n_2^5-579415n_1n_2^6+41645n_2^7\big)$$

$$+ \frac{y^{-2}\, n_1}{175\pi^3(n_1+n_2)^{11}}\big(495n_1^7+28765n_1^6n_2-1501506n_1^5n_2^2 + 8955786n_1^4n_2^3-15910621n_1^3n_2^4+9855297n_1^2n_2^5 - 2066640n_1n_2^6+120280n_2^7\big)$$

$$+ \frac{n_1}{1050\pi(n_1+n_2)^9}\big(175n_1^7+9135n_1^6n_2-348825n_1^5n_2^2 + 1980687n_1^4n_2^3-3425931n_1^3n_2^4+2035749n_1^2n_2^5 - 384395n_1n_2^6+15645n_2^7\big)\Big]$$

$$\eta_{1,0} = \operatorname{sgn}(n_2)\Big[\frac{y^{-6}n_1^2n_2}{(n_1+n_2)^{15}}\frac{10296}{5\pi^7}\big(3n_1^5-35n_1^4n_2+105n_1^3n_2^2-105n_1^2n_2^3 + 35n_1n_2^4-3n_2^5\big)$$

$$+ \frac{y^{-4}\, n_2}{(n_1+n_2)^{13}}\frac{22}{175\pi^5}\big(41645n_1^7-579415n_1^6n_2+2360373n_1^5n_2^2 - 3515111n_1^4n_2^3+1865089n_1^3n_2^4-283107n_1^2n_2^5 + 3965n_1n_2^6+65n_2^7\big)$$

$$+ \frac{y^{-2}\, n_2}{175\pi^3(n_1+n_2)^{11}}\big(120280n_1^7-2066640n_1^6n_2+9855297n_1^5n_2^2 - 15910621n_1^4n_2^3+8955786n_1^3n_2^4-1501506n_1^2n_2^5 + 28765n_1n_2^6+495n_2^7\big)$$

$$+ \frac{n_2}{1050\pi(n_1+n_2)^9}\big(15645n_1^7-384395n_1^6n_2+2035749n_1^5n_2^2 - 3425931n_1^4n_2^3+1980687n_1^3n_2^4-348825n_1^2n_2^5 + 9135n_1n_2^6+175n_2^7\big)\Big]$$

$$\eta_{1,1} = \frac{y^{-5}}{175\pi^6(n_1+n_2)^{14}}\big(1430n_1^8+88660n_1^7n_2-7388524n_1^6n_2^2 + 50271364n_1^5n_2^3-90631684n_1^4n_2^4+50271364n_1^3n_2^5 - 7388524n_1^2n_2^6+88660n_1n_2^7+1430n_2^8\big)$$

$$+ \frac{y^{-3}}{175\pi^4(n_1+n_2)^{12}}\big(1210n_1^8+72160n_1^7n_2-4261268n_1^6n_2^2$$

$$+ 25948736n_1^5n_2^3 - 45143692n_1^4n_2^4 + 25948736n_1^3n_2^5$$
$$4261268n_1^2n_2^6 + 72160n_1n_2^7 + 1210n_2^8\big)$$
$$+ \frac{y^{-1}}{350\pi^2(n_1+n_2)^{10}}\big(315n_1^8 + 17550n_1^7n_2 - 785712n_1^6n_2^2$$
$$+ 4522866n_1^5n_2^3 - 7798806n_1^4n_2^4 + 4522866n_1^3n_2^5$$
$$- 785712n_1^2n_2^6 + 17550n_1n_2^7 + 315n_2^8\big)$$
$$+ \frac{2yn_1n_2}{525(n_1+n_2)^8}\big(175n_1^6 - 6510n_1^5n_2 + 35745n_1^4n_2^2 - 61572n_1^3n_2^3$$
$$+ 35745n_1^2n_2^4 - 6510n_1n_2^5 + 175n_2^6\big).$$

The asymptotic expansion is

$$(48)\quad \hat{f}^P_{n_1,n_2}(y) = -\frac{4576\sigma_2(n_1)\sigma_2(n_2)}{175\pi^6n_1^2n_2^2(n_1+n_2)^{15}y^7}\big(5n_1^9 + 315n_2n_1^8 - 25524n_2^2n_1^7$$
$$+ 149940n_2^3n_1^6 - 141120n_2^4n_1^5 - 141120n_2^5n_1^4 + 149940n_2^6n_1^3$$
$$- 25524n_2^7n_1^2 + 315n_2^8n_1 + 5n_2^9$$
$$+ \big(7560n_2^2n_1^7 - 88200n_2^3n_1^6 + 264600n_2^4n_1^5 - 264600n_2^5n_1^4$$
$$+ 88200n_2^6n_1^3 - 7560n_2^7n_1^2\big)\log(|n_1/n_2|)\big) + O\big(y^{-5}\big).$$

We recall that by (19),

$$\hat{f}^H_{n_1,n_2}(y) = \alpha_{n_1,n_2}\sqrt{y}K_{7+\frac{1}{2}}(2\pi|n_1+n_2|y)$$
$$= \alpha_{n_1,n_2}\frac{135135}{256\pi^7|n_1+n_2|^{15/2}y^7} + o\big(y^{-7}\big).$$

Comparing the right hand sides of the two previous formulas, we obtain that there is a unique choice of $\alpha_{n_1,n_2}$ that guarantees that $\hat{f}_{n_1,n_2}(y) = \hat{f}^P_{n_1,n_2}(y) + \hat{f}^H_{n_1,n_2}(y) = o(y^{-7})$, $y \to 0$:

(49)
$$\alpha_{n_1,n_2} = \frac{1217536\,\mathrm{sgn}(n_1+n_2)\sigma_2(n_1)\sigma_2(n_2)}{23648625\pi n_1^2n_2^2|n_1+n_2|^{15/2}}\big(5n_1^9 + 315n_2n_1^8 - 25524n_2^2n_1^7$$
$$+ 149940n_2^3n_1^6 - 141120n_2^4n_1^5 - 141120n_2^5n_1^4 + 149940n_2^6n_1^3$$
$$- 25524n_2^7n_1^2 + 315n_2^8n_1 + 5n_2^9 + \big(7560n_2^2n_1^7 - 88200n_2^3n_1^6$$
$$+ 264600n_2^4n_1^5 + 88200n_2^6n_1^3 - 7560n_2^7n_1^2\big)\log(|n_1/n_2|)\big) + O\left(\frac{1}{y^5}\right).$$

Comparing the formula above with the leading terms in the asymptotic expansion of $\hat{f}^H_{n_1,n_2}(y)$ as $y \to 0$, we get that there exists a unique choice of $\alpha_{n_1,n_2}$ such that $\hat{f}_{n_1,n_2}(y) = \hat{f}^H_{n_1,n_2}(y) + \hat{f}^P_{n_1,n_2}(y) = o(y^{-7})$. Moreover, it follows that for fixed $n \neq 0$,

$$\alpha_{n-n_1,n_1} = O\big(|n_1|^{-8}\big), \quad |n_1| \to \infty. \tag{50}$$

**5.2.4.** $\boldsymbol{n_1 = -n_2}$ This case has been considered in [2, (C.30)]. For $\hat{f}^P_{-n_2,n_2}(y)$ defined in (38) we have

$$\begin{aligned}
\mu_{0,0} &= \frac{524288\pi^8 n_2^8 y^9}{7441875} - \frac{16384\pi^6 n^6 y^7}{2480625} - \frac{256\pi^4 n_2^4 y^5}{165375} - \frac{8\pi^2 n_2^2 y^3}{4725} + \frac{2}{175\pi^4 n_2^4 y^3} \\
&\quad + \frac{2}{175\pi^2 n_2^2 y} + \frac{y}{210}, \\
\mu_{0,1} &= \operatorname{sgn}(n_2)\left(\frac{262144\pi^7 n_2^7 y^8}{7441875} + \frac{8192\pi^5 n_2^5 y^6}{2480625} + \frac{128\pi^3 n_2^3 y^4}{55125}\right. \\
&\qquad\qquad \left. + \frac{6}{175\pi^3 n_2^3 y^2} + \frac{4}{175\pi^5 n_2^5 y^4} + \frac{4}{945}\pi n_2 y^2 + \frac{2}{105\pi n_2}\right), \\
\mu_{1,1} &= -\frac{524288\pi^8 n_2^8 y^9}{7441875} - \frac{16384\pi^6 n_2^6 y^7}{7441875} - \frac{256\pi^4 n_2^4 y^5}{275625} - \frac{8\pi^2 n_2^2 y^3}{6615} \\
&\quad + \frac{4}{175\pi^4 n_2^4 y^3} + \frac{2}{175\pi^6 n_2^6 y^5} + \frac{3}{175\pi^2 n_2^2 y} - \frac{23y}{1890}.
\end{aligned}$$

The asymptotic expansion is

$$\hat{f}^P_{-n_2,n_2}(y) = -\frac{32\sigma_2(n_2)^2}{175\pi^6 |n_2|^{10} y^7} + O\left(\frac{1}{y^4}\right). \tag{51}$$

Since $\hat{f}^H_{-n_2,n_2}(y) = \alpha_{n_1,n_2} y^{-7}$, in order to have

$$\hat{f}^P_{-n_2,n_2}(y) + \hat{f}^H_{-n_2,n_2}(y) = o\big(y^{-7}\big), \quad y \to 0,$$

we must set $\alpha_{-n_2,n_2} = \frac{32\sigma_2(n_2)^2}{175\pi^6|n_2|^{10}}$. Now it is time to decide on $\alpha_{0,0}$. We note from (99),

$$\sum_{n_2 \neq 0} \alpha_{n_2,-n_2} = \frac{32}{175\pi^6} \sum_{n_2 \neq 0} \frac{\sigma_2(n_2)^2}{n_2^{10}} = \frac{7072\pi^{16}}{1695787498125}, \tag{52}$$

and thus $\alpha_{0,0} = -\frac{7072\pi^{16}}{1695787498125}$.

## 6. $k = 3/2$ and $\ell = 5/2$

In this section, for $z = x + iy \in \mathfrak{H}$ we solve

$$(\Delta - \lambda) f(z) = -6\zeta(3)\zeta(5) E_{3/2}(z) E_{5/2}(z),$$

for

$$f(z) = \sum_{n\in\mathbb{Z}} \sum_{n_1+n_2=n} \hat{f}_{n_1,n_2}(y) e^{2\pi i n x}$$

in terms of $\hat{f}_{n_1,n_2}(y) = \hat{f}^P_{n_1,n_2}(y) + \hat{f}^H_{n_1,n_2}(y)$. In order to obtain $\mathcal{E}(\lambda, 5/2, 3/2, z, \overline{z})$ from [2, (2.13)], we use (9) to note that $\mathcal{E}(\lambda, 5/2, 3/2, z, \overline{z})$ can be obtained from $f$ solving the equation above by multiplying $f$ by $\frac{2}{3}$. Thus, instead of $-64$ in (55), we need to take $-\frac{128}{3}$.

When $n_1 = n_2 = 0$, $\hat{f}_{0,0}(y)$ contains no $K$-Bessel or divisor functions and is given by a a polynomial in $y$ and $1/y$ below. For $n_1 n_1 = 0$ but not both zero,

$$\hat{f}^P_{0,n}(y) = -8\pi^2 \sigma_{-4}(n) |n|^2 \sum_{i=0,1} \nu_i(n,y) K_i(2\pi|n|y) \tag{53}$$

and

$$\hat{f}^P_{n,0}(y) = -8\pi \sigma_{-2}(n) |n| \sum_{i,j=0,1} \nu_i(n,y) K_i(2\pi|n|y), \tag{54}$$

and for $n_1 n_2 \neq 0$ and $n_1 + n_2 \neq 0$,

$$\begin{aligned} \hat{f}^P_{n_1,n_2}(y) = &-64\pi^3 |n_1||n_2|^2 \sigma_{-2}(n_1)\sigma_{-4}(n_2) \\ &\times \sum_{i,j=0,1} \eta_{i,j}(n_1,n_2,y) K_i(2\pi|n_1|y) K_j(2\pi|n_2|y), \end{aligned} \tag{55}$$

and for $n_1 = -n_2$,

$$\begin{aligned} \hat{f}^P_{-n_2,n_2}(y) = &- 64\pi^3 |n_2|^3 \sigma_{-2}(n_2)\sigma_{-4}(n_2) \\ &\times \sum_{(i,j)\in\{(0,0),(0,1),(1,1)\}} \mu_{i,j}(n_2,y) K_i(2\pi|n_2|y) K_j(2\pi|n_2|y), \end{aligned} \tag{56}$$

where $\nu_i, \eta_{i,j}$ and $\mu_{i,j}$ defined below depending on each value of $\lambda$.

## 6.1. $\lambda = 20$

This subsection corresponds to [2, (C.19)] with $r = 4$.

In what follows we find $\hat{f}^P_{n_1,n_2}(y)$ for different values of $n_1$ and $n_2$.

**6.1.1. $n_1 = 0$ and $n_2 = 0$** The particular solution for $n_1 = n_2 = 0$ is equal to

$$\hat{f}_{0,0}(y) = \frac{14175y^6\zeta(3)\zeta(5) + 2100\pi^2y^4\zeta(5) + 84\pi^4y^2\zeta(3) + 40\pi^6}{18900y^2} + \frac{c_1}{y^4} + c_2y^5$$

for some $c_1, c_2 \in \mathbb{C}$. Its asymptotic behavior for $y \to 0$ can be described by $\hat{f}_{0,0}(y) = \frac{2\pi^6}{945y^2}$ and the leading term of the asymptotic behavior does not depend on $c_1$ and $c_2$. If we want to get rid of the $O(y^5)$-asymptotic, we need to set $c_2 = 0$. We refrain for a moment from defining $c_1$. However, we notice that the first four summands in the r.h.s. of the formula above coincide with the first four summands in the first line in [2, (C.21)].

**6.1.2. $n_1 = 0$ and $n_2 \neq 0$** For $\hat{f}^P_{0,n}(y)$ as in (53), we have

$$\nu_0(n,y) = \frac{\pi^2n^2(4\zeta(2) - 27\zeta(3)y^2) - 126\zeta(3)}{3\pi^4n^4y^2},$$

$$\nu_1(n,y) = \operatorname{sgn}(n) \cdot \left(\frac{-126\zeta(3) + \pi^4n^4y^2(2\zeta(2) - 3\zeta(3)y^2) + 2\pi^2n^2(2\zeta(2) - 45\zeta(3)y^2)}{3\pi^5n^5y^3}\right).$$

The asymptotic expansion of $\hat{f}^P_{0,n}(y)$ as $y \to 0$ is

$$-\frac{8\sigma_{-4}(n)(\pi^4n^2 - 189\zeta(3))}{9\pi^4n^4y^4} + O\left(\frac{1}{y^2}\right).$$

There is a unique choice of $\alpha_{0,n}$ that gets rid of the $y^{-4}$-term in the asymptotic expansion of $\hat{f}^P_{0,n}(y) + \hat{f}^H_{0,n}(y)$.

**6.1.3. $n_1 \neq 0$ and $n_2 = 0$** For $\hat{f}^P_{n,0}(y)$ as in (54), we have

$$\nu_0(n,y) = \operatorname{sgn}(n)\left(\frac{45\zeta(5)}{\pi^3n^3} + \frac{33600\zeta(5) + 64\pi^4\zeta(4)n^4}{160\pi^5n^5y^2} - \frac{\zeta(5)y^2}{2\pi n}\right),$$

$$\nu_1(n,y) = \frac{(128n^4\pi^4\zeta(4) + 67200\zeta(5))y^{-3}}{320n^6\pi^6} + \frac{150\zeta(5)y^{-1}}{n^4\pi^4} + \frac{9\zeta(5)y}{2n^2\pi^2}.$$

The asymptotic expansion of $\hat{f}^P_{n,0}(y)$ as $y \to 0$ is

$$-8\sigma_{-2}(n)\left(\frac{525\zeta(5)+\pi^4\zeta(4)n^4}{5\pi^6|n|^6y^4}+O\big(y^{-2}\big)\right).$$

There is a unique choice of $\alpha_{n,0}$ that gets rid of the $y^{-4}$-term in the asymptotic expansion of $\hat{f}^P_{n,0}(y)+\hat{f}^H_{n,0}(y)$.

**6.1.4. $n_1n_2 \neq 0$ and $n_1+n_2 \neq 0$** For $\hat{f}^P_{n_1,n_2}(y)$ as in (55), we have

$$\begin{aligned}
\eta_{0,0} =& \operatorname{sgn}(n_1)\bigg[\frac{y^{-2}\,7n_1n_2^2}{\pi^3(n_1+n_2)^8}\big(5n_1^2-8n_1n_2+3n_2^2\big)\\
&+\frac{n_1}{30\pi(n_1+n_2)^6}\big(3n_1^4+26n_1^3n_2+348n_1^2n_2^2-810n_1n_2^3+145n_2^4\big)\bigg],\\
\eta_{0,1} =& \operatorname{sgn}(n_1)\operatorname{sgn}(n_2)\bigg[\frac{y^{-3}\,7n_1n_2^3}{\pi^4(n_1+n_2)^9}(-5n_1+3n_2)\\
&+\frac{y^{-1}\,n_1}{30n_2\pi^2(n_1+n_2)^7}\big(3n_1^5+29n_1^4n_2+149n_1^3n_2^2+723n_1^2n_2^3\\
&-1820n_1n_2^4+460n_2^5\big)\\
&+\frac{y\,2n_1n_2}{15(n_1+n_2)^5}\big(n_1^3+15n_1^2n_2-45n_1n_2^2+5n_2^3\big)\bigg],\\
\eta_{1,0} =& \frac{y^{-3}\,n_1^2n_2^2}{\pi^4(n_1+n_2)^9}(35n_1-21n_2)\\
&+\frac{y^{-1}}{30\pi^2(n_1+n_2)^7}\big(3n_1^5+29n_1^4n_2+899n_1^3n_2^2-1827n_1^2n_2^3\\
&+190n_1n_2^4+10n_2^5\big)\\
&+\frac{y\,2n_1n_2}{15(n_1+n_2)^5}\big(n_1^3+15n_1^2n_2-45n_1n_2^2+5n_2^3\big),\\
\eta_{1,1} =& \operatorname{sgn}(n_2)\bigg[\frac{y^{-2}}{30n_2\pi^3(n_1+n_2)^8}\big(3n_1^6+32n_1^5n_2+178n_1^4n_2^2+872n_1^3n_2^3\\
&-2447n_1^2n_2^4+200n_1n_2^5+10n_2^6\big)\\
&+\frac{1}{30\pi(n_1+n_2)^6}\big(4n_1^5+43n_1^4n_2+334n_1^3n_2^2-880n_1^2n_2^3\\
&+110n_1n_2^4+5n_2^5\big)\bigg].
\end{aligned}$$

We note that

(57)
$$\hat{f}^P_{n_1,n_2}(y) = -y^{-4}\frac{8\sigma_{-2}(|n_1|)\sigma_{-4}(|n_2|)}{15\pi^2(n_1+n_2)^9}\big(3n_1^7 + 35n_1^6n_2 + 210n_1^5n_2^2 + 1050n_1^4n_2^3$$
$$+ 2100n_1^3n_2^4\log(|n_1|\pi) - 2100n_1^3n_2^4\log(|n_2|\pi) - 1575n_1^3n_2^4$$
$$- 1260n_1^2n_2^5\log(|n_1|\pi) + 1260n_1^2n_2^5\log(|n_2|\pi) - 2247n_1^2n_2^5$$
$$+ 210n_1n_2^6 + 10n_2^7\big) + o\big(y^{-4}\big).$$

Requiring $\hat{f}_{n_1,n_2}(y) = o(y^{-4})$ gives us a unique $\alpha_{n_1,n_2}$ that cancels with the $O(y^{-4})$-term. Comparing the formula above with the leading terms in the asymptotic expansion of $\hat{f}^H_{n_1,n_2}(y)$ as $y \to 0$, we get that there exists a unique choice of $\alpha_{n_1,n_2}$ such that $\hat{f}_{n_1,n_2}(y) = \hat{f}^H_{n_1,n_2}(y) + \hat{f}^P_{n_1,n_2}(y) = o(y^{-4})$. Moreover, it follows that for fixed $n \neq 0$,

$$\alpha_{n-n_1,n_1} = O\big(|n_1|^{-5}\big), \quad |n_1| \to \infty. \tag{58}$$

**6.1.5. $n_1 = -n_2$** For $\hat{f}^P_{-n_2,n_2}(y)$ as in (56), we have

$$\mu_{0,0} = \operatorname{sgn}(n_2)\left(\frac{1}{36n_2\pi} + \frac{2n_2\pi y^2}{315} + \frac{128n_2^3\pi^3y^4}{4725} - \frac{4096n_2^5\pi^5y^6}{14175}\right),$$
$$\mu_{0,1} = \frac{y^{-1}}{18n_2^2\pi^2} - \frac{y}{105} - \frac{64n_2^2\pi^2y^3}{4725} - \frac{2048n_2^4\pi^4y^5}{14175},$$
$$\mu_{1,1} = \operatorname{sgn}(n_2)\left(\frac{y^{-2}}{36n_2^3\pi^3} - \frac{1}{63n_2\pi} + \frac{2n_2\pi y^2}{525} + \frac{128n_2^3\pi^3y^4}{14175} + \frac{4096n_2^5\pi^5y^6}{14175}\right).$$

The asymptotic expansion is

$$\hat{f}^P_{-n_2,n_2}(y) = -64y\sigma_{-2}(n_2)\sigma_{-4}(n_2)\left(\frac{1}{144\pi^2|n_2|^2y^4} - \frac{1}{56y^2} + O(1)\right). \tag{59}$$

There is a unique choice of $k_{-n_2,n_2}$ such that

$$\hat{f}^P_{-n_2,n_2}(y) + \hat{f}^H_{-n_2,n_2}(y) = o\big(y^{-4}\big):$$
$$\alpha_{-n_2,n_2} = \frac{4y\sigma_{-2}(n_2)\sigma_{-4}(n_2)}{9\pi^2|n_2|^2}. \tag{60}$$

## 7. $k = \ell = 5/2$

In this section we solve

$$(\Delta - \lambda)f(z) = -\big(3\zeta(5)E_{5/2}(z)\big)^2, \quad z = x + iy \in \mathfrak{H}$$

for

$$f(z) = \sum_{n\in\mathbb{Z}} \sum_{n_1+n_2=n} \hat{f}_{n_1,n_2}(y)e^{2\pi inx}$$

in terms of $\hat{f}_{n_1,n_2}(y) = \hat{f}^P_{n_1,n_2}(y) + \hat{f}^H_{n_1,n_2}(y)$.

When $n_1 = n_2 = 0$, $\hat{f}_{0,0}(y)$ contains no $K$-Bessel or divisor functions and is given by a polynomial in $y$ and $1/y$ below. For $n_1n_1 = 0$ but not both zero,

$$\hat{f}^P_{0,n}(y) = \hat{f}^P_{n,0}(y) = -8\pi^2\sigma_{-4}(n)|n|^2 \sum_{i=0} \nu_i(n,y)K_i(2\pi|n|y), \tag{61}$$

for $n_1n_2 \neq 0$ and $n_1 + n_2 \neq 0$,

$$\hat{f}^P_{n_1,n_2}(y) = -64\pi^4|n_1|^2|n_2|^2\sigma_{-4}(n_1)\sigma_{-4}(n_2) \times \sum_{i,j=0,1} \eta_{i,j}(n_1,n_2,y)K_i(2\pi|n_1|y)K_j(2\pi|n_2|y), \tag{62}$$

and for $n_1 = -n_2$,

$$\hat{f}^P_{-n_2,n_2}(y) = -64\pi^4|n_1|^2|n_2|^2\sigma_{-4}(n_2)\sigma_{-4}(n_2) \times \sum_{(i,j)\in\{(0,0),(0,1),(1,1)\}} \eta_{i,j}(n_2,y)K_i(2\pi|n_2|y)K_j(2\pi|n_2|y), \tag{63}$$

where $\nu_i, \eta_{i,j}$ and $\mu_{i,j}$ are defined below depending on each value of $\lambda$.

### 7.1. $\lambda = 30$

This case corresponds to [2, C.3.2] with $r = 5$.

**7.1.1. $n_1 = 0$ and $n_2 = 0$** It is not complicated to show that

$$\hat{f}_{0,0}(y) = \frac{80\zeta(4)^2 + 81\zeta(5)^2y^8 + 72\zeta(4)\zeta(5)y^4}{90y^3} + c_2y^6 + \frac{c_1}{y^5}$$

for some $c_1, c_2 \in \mathbb{C}$.

**7.1.2. $n_1 n_2 = 0$ but not both zero** For $\hat{f}^P_{0,n}(y)$ as in (61), we have

$$\nu_0 = \frac{4\zeta(4)\pi^4 n^4 + 7560\zeta(5)}{5\pi^6 n^6 y^3} + \frac{420\zeta(5)y^{-1}}{n^4\pi^4} + \frac{6\zeta(5)y}{n^2\pi^2}$$

$$\nu_1 = \operatorname{sgn}(n)\left(\frac{4\zeta(4)\pi^4 n^4 + 7560\zeta(5)}{5\pi^7 n^7 y^4} + \frac{4\zeta(4)\pi^4 n^4 + 11760\zeta(5)}{10\pi^5 n^5 y^2} + \frac{90\zeta(5)}{n^3\pi^3} - \frac{\zeta(5)y^2}{2n\pi}\right).$$

Its asymptotic behavior as $y \to 0$ can be described as

$$-8\sigma_{-4}(|n|)\left(\frac{2(1890\zeta(5) + \pi^4\zeta(4)n^4)}{5\pi^6 n^8 y^5} - \frac{840\zeta(5) + \pi^4\zeta(4)n^2}{5\pi^4 n^4 y^3} + O\left(\frac{1}{y^2}\right)\right). \tag{64}$$

**7.1.3. $n_1 n_2 \neq 0$ and $n_1 + n_2 \neq 0$** For $\hat{f}^P_{n_1,n_2}(y)$ as in (62) we have

$$\begin{aligned}
\eta_{0,0} =& \frac{y^{-3}}{\pi^4(n_1+n_2)^{10}}\left(252n_1^2n_2^2\left(n_1^2 - 2n_1n_2 + n_2^2\right)\right) \\
&+ \frac{y^{-1}}{5\pi^2(n_1+n_2)^8}\left(n_1^6 + 12n_1^5n_2 + 431n_1^4n_2^2 - 1400n_1^3n_2^3 \right.\\
&\qquad \left. + 431n_1^2n_2^4 + 12n_1n_2^5 + n_2^6\right) \\
&+ \frac{y}{15(n_1+n_2)^6}\left(2n_1n_2\left(n_1^4 + 20n_1^3n_2 - 90n_1^2n_2^2 + 20n_1n_2^3 + n_2^4\right)\right), \\
\eta_{0,1} =& \operatorname{sgn}(n_2)\left(\frac{y^{-4}}{\pi^5(n_1+n_2)^{11}}\left(252n_1^2n_2^3(-n_1+n_2)\right)\right. \\
&+ \frac{y^{-2}}{5n_2\pi^3(n_1+n_2)^9}\left(n_1^7 + 13n_1^6n_2 + 93n_1^5n_2^2 + 641n_1^4n_2^3\right. \\
&\qquad \left. - 2859n_1^3n_2^4 + 1073n_1^2n_2^5 + 13n_1n_2^6 + n_2^7\right) \\
&+ \frac{1}{30\pi(n_1+n_2)^7}\left(4n_1^6 + 57n_1^5n_2 + 623n_1^4n_2^2 - 2590n_1^3n_2^3\right. \\
&\qquad \left.\left. + 714n_1^2n_2^4 + 37n_1n_2^5 + 3n_2^6\right)\right), \\
\eta_{1,0} =& \operatorname{sgn}(n_1)\left(\frac{y^{-4}}{\pi^5(n_1+n_2)^{11}}\left(252n_1^3n_2^2(n_1-n_2)\right)\right. \\
&+ \frac{y^{-2}}{5n_1\pi^3(n_1+n_2)^9}\left(n_1^7 + 13n_1^6n_2 + 1073n_1^5n_2^2 - 2859n_1^4n_2^3 + 641n_1^3n_2^4\right.
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + 93n_1^2n_2^5 + 13n_1n_2^6 + n_2^7\big)\\
&+ \frac{1}{30\pi(n_1+n_2)^7}\big(3n_1^6 + 37n_1^5n_2 + 714n_1^4n_2^2 - 2590n_1^3n_2^3 + 623n_1^2n_2^4\\
&\qquad\qquad + 57n_1n_2^5 + 4n_2^6\big)\bigg),
\end{aligned}
$$

$$
\begin{aligned}
\eta_{1,1} =& \operatorname{sgn}(n_1n_2)\bigg(\frac{y^{-3}}{5n_1n_2\pi^4(n_1+n_2)^{10}}\big(n_1^8 + 14n_1^7n_2 + 106n_1^6n_2^2 + 734n_1^5n_2^3\\
&\qquad - 3758n_1^4n_2^4 + 734n_1^3n_2^5 + 106n_1^2n_2^6 + 14n_1n_2^7 + n_2^8\big)\\
&+ \frac{y^{-1}}{30n_1n_2\pi^2(n_1+n_2)^8}\big(3n_1^8 + 40n_1^7n_2 + 304n_1^6n_2^2 + 2360n_1^5n_2^3\\
&\qquad - 9254n_1^4n_2^4 + 2360n_1^3n_2^5 + 304n_1^2n_2^6 + 40n_1n_2^7 + 3n_2^8\big)\\
&+ \frac{y}{15(n_1+n_2)^6}\big(2n_1n_2\big(n_1^4 + 20n_1^3n_2 - 90n_1^2n_2^2 + 20n_1n_2^3 + n_2^4\big)\big)\bigg).
\end{aligned}
$$

We note that

(65)
$$
\begin{aligned}
\hat{f}^P_{n_1,n_2}(y) = &-\frac{y^{-5}16\sigma_{-4}(n_1)\sigma_{-4}(n_2)}{5\pi^2(n_1+n_2)^{11}}\big(n_1^9 + 15n_1^8n_2 + 120n_1^7n_2^2 + 840n_1^6n_2^3\\
&- 3024n_1^5n_2^4 - 3024n_1^4n_2^5 + 2520n_1^4n_2^4(n_1-n_2)\log\left(|n_1/n_2|\right)\\
&+ 840n_1^3n_2^6 + 120n_1^2n_2^7 + 15n_1n_2^8 + n_2^9\big) + o\big(y^{-5}\big).
\end{aligned}
$$

There exists a choice of $\alpha_{n_1,n_2}$ such that $\hat{f}_{n_1,n_2}(y) = o(y^{-5})$. Comparing the formula above with the leading terms in the asymptotic expansion of $\hat{f}^H_{n_1,n_2}(y)$ as $y \to 0$, we get that there exists a unique choice of $\alpha_{n_1,n_2}$ such that $\hat{f}_{n_1,n_2}(y) = \hat{f}^H_{n_1,n_2}(y) + \hat{f}^P_{n_1,n_2}(y) = o(y^{-5})$. Moreover, it follows that for fixed $n \neq 0$,

(66)
$$\alpha_{n-n_1,n_1} = O\big(|n_1|^{-6}\big), \quad |n_1| \to \infty.$$

**7.1.4. $\boldsymbol{n_1 = -n_2}$** For $\hat{f}^P_{-n_2,n_2}(y)$ as in (63), we have

$$
\begin{aligned}
\mu_{0,0} &= \frac{2y^{-1}}{55n_2^2\pi^2} - \frac{19y}{990} - \frac{8n_2^2\pi^2y^3}{3465} - \frac{512n_2^4\pi^4y^5}{51975} + \frac{16384n_2^6\pi^6y^7}{155925},\\
\mu_{0,1} &= \operatorname{sgn}(n_2)\left(\frac{4y^{-2}}{55n_2^3\pi^3} - \frac{1}{495n_2\pi} + \frac{4n_2\pi y^2}{1155} + \frac{256n_2^3\pi^3y^4}{51975} + \frac{8192n_2^5\pi^5y^6}{155925}\right),\\
\mu_{1,1} &= \frac{2y^{-3}}{55n_2^4\pi^4} + \frac{17y^{-1}}{990n_2^2\pi^2} + \frac{103y}{6930} - \frac{8n_2^2\pi^2y^3}{5775} - \frac{512n_2^4\pi^4y^5}{155925} - \frac{16384n_2^6\pi^6y^7}{155925}.
\end{aligned}
$$

Its asymptotic behavior for $y \to 0$ is

$$(67)\qquad -64\sigma_{-4}(n_2)^2\left(\frac{1}{110\pi^2 n_2^2 y^5} - \frac{1}{72y^3} + O\left(\frac{1}{y^2}\right)\right).$$

From this we obtain that

$$(68)\qquad \alpha_{-n_2,n_2} = O\big(|n_2|^{-6}\big), \quad |n_2| \to \infty.$$

## 8. $k = 3/2$ and $\ell = 7/2$

In this section we solve

$$(\Delta - \lambda)f(z) = -30\zeta(3)\zeta(7)E_{3/2}(z)E_{7/2}(z), \quad z = x + iy \in \mathfrak{H}$$

for

$$f(z) = \sum_{n\in\mathbb{Z}} \sum_{n_1+n_2=n} \hat{f}_{n_1,n_2}(y)e^{2\pi inx}$$

in terms of $\hat{f}_{n_1,n_2}(y) = \hat{f}^P_{n_1,n_2}(y) + \hat{f}^H_{n_1,n_2}(y)$.

For $n_1n_1 = 0$ but not both zero,

$$(69)\qquad \hat{f}^P_{0,n}(y) = -16\pi^3\sigma_{-6}(n)|n|^3 \sum_{i=0,1} \nu_i(n,y)K_i(2\pi|n|y),$$

and

$$(70)\qquad \hat{f}^P_{n,0}(y) = -8\pi\sigma_{-2}(n)|n| \sum_{i=0,1} \nu_i(n,y)K_i(2\pi|n|y),$$

for $n_1n_2 \neq 0$ and $n_1 + n_2 \neq 0$,

(71)

$$\hat{f}^P_{n_1,n_2}(y) = -128\pi^4|n_1||n_2|^3\sigma_{-2}(n_1)\sigma_{-6}(n_2) \times \sum_{(i,j)\in\{(0,0),(0,1),(1,1)\}} \eta_{i,j}(n_1,n_2,y)K_i(2\pi|n_1|y)K_j(2\pi|n_2|y),$$

and for $n_1 = -n_2$,

$$(72)\qquad \hat{f}^P_{-n_2,n_2}(y) = -128\pi^4|n_2|^4\sigma_{-2}(n_2)\sigma_{-6}(n_2) \times \sum_{(i,j)\in\{(0,0),(0,1),(1,1)\}} \mu_{i,j}(n_2,y)K_i(2\pi|n_2|y)K_j(2\pi|n_2|y),$$

where $\nu_i, \eta_{i,j}$ and $\mu_{i,j}$ defined below depending on each value of $\lambda$.

### 8.1. $\lambda = 30$

This case corresponds to [2, C.3.3] with $r = 5$.

**8.1.1. $n_1 = 0$ and $n_2 = 0$** It is not complicated to show that

$$\hat{f}_{0,0}(y) = \frac{2143260y^8\zeta(3)\zeta(7) + 297675\pi^2y^6\zeta(7) + 864\pi^6y^2\zeta(3) + 448\pi^8}{714420y^3} + \frac{c_1}{y^5} + c_2y^6$$

for some $c_1, c_2 \in \mathbb{C}$. Its asymptotic behavior for $y \to 0$ can be described by $\hat{f}_{0,0}(y) = \frac{16\pi^8}{25515y^3}$ and the leading term of the asymptotic behavior does not depend on $c_1$ and $c_2$.

**8.1.2. $n_1 = 0$ and $n_2 \neq 0$** For $n = n_2 \neq 0$,

$$\hat{f}^P_{0,n}(y) = -16\pi^3\sigma_{-6}(n)|n|^3 \sum_{i=0,1} \nu_i(n,y)K_i(2\pi|n|y),$$

where

$$\nu_0(n,y) = \operatorname{sgn}(n)\Bigg(4\zeta(2)\left(\frac{1}{(\pi^3n^3)y^3} + \frac{1}{(6\pi n)y}\right) - 2\zeta(3)\left(\frac{108}{(\pi^5n^5)y^3} + \frac{30}{(\pi^3n^3)y} + \frac{y}{2\pi n}\right)\Bigg)$$

$$\nu_1(n,y) = 4\zeta(2)\left(\frac{1}{(\pi^4n^4)y^4} + \frac{2}{(3\pi^2n^2)y^2}\right) - 2\zeta(3)\left(\frac{84}{(\pi^4n^4)y^2} + \frac{108}{(\pi^6n^6)y^4} + \frac{13}{2\pi^2n^2}\right).$$

Its asymptotic behavior is

$$-16\pi^3\sigma_{-6}(n)|n|^3\left(\frac{2(\pi^2\zeta(2)n^2 - 54\zeta(3))}{\pi^7n^7y^5} - \frac{2(\pi^2\zeta(2)n^2 - 36\zeta(3))}{3(\pi^5n^5)y^3} + O\left(\frac{1}{y^2}\right)\right).$$

**8.1.3. $n_1 \neq 0$ and $n_2 = 0$** For $n = n_1 \neq 0$,

$$\hat{f}^P_{n,0}(y) = -8\pi\sigma_{-2}(n)|n| \sum_{i=0,1} \nu_i(n,y)K_i(2\pi|n|y)$$

with

$$\nu_0 = \operatorname{sgn}(n)\left(\frac{8\pi\zeta(6)y^{-3}}{7n} - 15\zeta(7)\left(\frac{6y}{\pi^3n^3} + \frac{1512}{(\pi^7n^7)y^3} + \frac{420}{(\pi^5n^5)y} + \frac{y^3}{10\pi n}\right)\right),$$

$$\nu_1 = \frac{8\zeta(6)y^{-4}}{7n^2} - 15\zeta(7)\left(-\frac{2y^2}{5\pi^2n^2} + \frac{1176}{(\pi^6n^6)y^2} + \frac{1512}{(\pi^8n^8)y^4} + \frac{90}{\pi^4n^4}\right).$$

Its asymptotic behavior is

$$-8\sigma_{-2}(n)|n|\left(\frac{4(\pi^6\zeta(6)n^6 - 19845\zeta(7))}{7\pi^8n^9y^5} - \frac{4(\pi^6\zeta(6)n^6 - 4410\zeta(7))}{7(\pi^6n^7)y^3} + O\left(\frac{1}{y^2}\right)\right).$$

**8.1.4. $n_1n_2 \neq 0$ and $n_1 + n_2 \neq 0$** For $\hat{f}^P_{n_1,n_2}(y)$ as in (71), we have

$$\begin{aligned}
\eta_{0,0} = \operatorname{sgn}(n_1n_2)\Bigg(&\frac{y^{-3}}{\pi^4(n_1+n_2)^{10}}\big(36n_1n_2^3(7n_1^2 - 10n_1n_2 + 3n_2^2)\big)\\
&+ \frac{n_1\,y^{-1}}{35n_2\pi^2(n_1+n_2)^8}\big(5n_1^6 + 48n_1^5n_2 + 219n_1^4n_2^2 + 664n_1^3n_2^3\\
&\qquad + 4163n_1^2n_2^4 - 6440n_1n_2^5 + 1085n_2^6\big)\\
&+ \frac{2n_1n_2\,y}{105(n_1+n_2)^6}\big(3n_1^4 + 28n_1^3n_2 + 210n_1^2n_2^2 - 420n_1n_2^3 + 35n_2^4\big)\Bigg)\\
\eta_{0,1} = \operatorname{sgn}(n_1)\Bigg(&\frac{y^{-4}}{\pi^5(n_1+n_2)^{11}}\big(36n_1n_2^4(-7n_1 + 3n_2)\big)\\
&+ \frac{n_1\,y^{-2}}{35n_2^2\pi^3(n_1+n_2)^9}\big(5n_1^7 + 53n_1^6n_2 + 267n_1^5n_2^2 + 883n_1^4n_2^3\\
&\qquad + 2377n_1^3n_2^4 + 7593n_1^2n_2^5 - 13545n_1n_2^6 + 2975n_2^7\big)\\
&+ \frac{n_1}{210\pi(n_1+n_2)^7}\big(27n_1^5 + 253n_1^4n_2 + 1270n_1^3n_2^2 + 6594n_1^2n_2^3\\
&\qquad - 12145n_1n_2^4 + 1505n_2^5\big)\Bigg)\\
\eta_{1,0} = \operatorname{sgn}(n_2)\Bigg(&\frac{y^{-4}}{\pi^5(n_1+n_2)^{11}}\big(n_1^2n_2^3\cdot(252n_1 - 108n_2)\big)\\
&+ \frac{y^{-2}}{35n_2\pi^3(n_1+n_2)^9}\big(5n_1^7 + 53n_1^6n_2 + 267n_1^5n_2^2 + 883n_1^4n_2^3 + 9237n_1^3n_2^4\\
&\qquad - 12987n_1^2n_2^5 + 875n_1n_2^6 + 35n_2^7\big)\\
&+ \frac{1}{210\pi(n_1+n_2)^7}\big(24n_1^6 + 225n_1^5n_2 + 1063n_1^4n_2^2 + 7042n_1^3n_2^3\\
&\qquad - 11970n_1^2n_2^4 + 1085n_1n_2^5 + 35n_2^6\big)\Bigg)
\end{aligned}$$

$$\eta_{1,1} = \frac{y^{-3}}{35n_2^2\pi^4(n_1+n_2)^{10}}\big(5n_1^8 + 58n_1^7n_2 + 320n_1^6n_2^2 + 1150n_1^5n_2^3 + 3260n_1^4n_2^4 + 9970n_1^3n_2^5 - 16732n_1^2n_2^6 + 910n_1n_2^7 + 35n_2^8\big) + \frac{y^{-1}}{105n_2\pi^2(n_1+n_2)^8}\big(12n_1^7 + 132n_1^6n_2 + 716n_1^5n_2^2 + 2806n_1^4n_2^3 + 12332n_1^3n_2^4 - 21728n_1^2n_2^5 + 1820n_1n_2^6 + 70n_2^7\big) + \frac{y}{105(n_1+n_2)^6}\big(2n_1n_2\big(3n_1^4 + 28n_1^3n_2 + 210n_1^2n_2^2 - 420n_1n_2^3 + 35n_2^4\big)\big)$$

The leading term of the asymptotic expansion is

(73)
$$\hat{f}^P_{n_1,n_2}(y) = -\,y^{-5}\frac{32\pi^4\sigma_{-2}(n_1)\sigma_{-6}(n_2)}{35\pi^6(n_1+n_2)^{11}}\big(5n_1^9 + 63n_1^8n_2 + 378n_1^7n_2^2 + 1470n_1^6n_2^3 + 4410n_1^5n_2^4 + 13230n_1^4n_2^5 + 17640n_1^3n_2^6\log(|n_1/n_2|) - 6762n_1^3n_2^6 - 7560n_1^2n_2^7\log(|n_1/n_2|) - 15822n_1^2n_2^7 + 945n_1n_2^8 + 35n_2^9\big) + o\big(y^{-5}\big).$$

Comparing the formula above with the leading terms in the asymptotic expansion of $\hat{f}^H_{n_1,n_2}(y)$ as $y \to 0$, we get that there exists a unique choice of $\alpha_{n_1,n_2}$ such that $\hat{f}_{n_1,n_2}(y) = \hat{f}^H_{n_1,n_2}(y) + \hat{f}^P_{n_1,n_2}(y) = o(y^{-5})$. Moreover, it follows that for fixed $n \neq 0$,

$$\alpha_{n-n_1,n_1} = O\big(|n_1|^{-6}\big), \quad |n_1| \to \infty. \tag{74}$$

**8.1.5. $n_1 = -n_2$** For $\hat{f}^P_{-n_2,n_2}(y)$ as in (72), we have

$$\mu_{0,0} = \frac{3y^{-1}}{55n_2^2\pi^2} + \frac{103y}{6930} - \frac{8n_2^2\pi^2y^3}{4851} - \frac{512n_2^4\pi^4y^5}{72765} + \frac{16384n_2^6\pi^6y^7}{218295},$$
$$\mu_{0,1} = \operatorname{sgn}(n_2)\left(\frac{6y^{-2}}{55n_2^3\pi^3} + \frac{89}{6930n_2\pi} + \frac{4n_2\pi y^2}{1617} + \frac{256n_2^3\pi^3y^4}{72765} + \frac{8192n_2^5\pi^5y^6}{218295}\right),$$
$$\mu_{1,1} = \frac{3y^{-3}}{55n_2^4\pi^4} - \frac{y^{-1}}{495n_2^2\pi^2} - \frac{871y}{48510} - \frac{8n_2^2\pi^2y^3}{8085} - \frac{512n_2^4\pi^4y^5}{218295} - \frac{16384n_2^6\pi^6y^7}{218295}.$$

Its asymptotic behavior is

$$-\,128\sigma_{-2}(n_2)\sigma_{-6}(n_2)\left(\frac{3}{220\pi^2n_2^2y^5} - \frac{1}{36y^3} + O\left(\frac{1}{y^2}\right)\right), \tag{75}$$

and

$$\alpha_{n_2,-n_2} = O\big(|n_2|^{-6}\big), \quad |n_2| \to \infty. \tag{76}$$

## 9. Convergence of each Fourier mode

We now examine the $n$-th Fourier mode $\widehat{f}_n(y)$. We note that in all of our computations, the particular solution $\sum_{n_1+n_2=n} \hat{f}^P_{n_1,n_2}(y)$ converges. However, the homogeneous part of the solution only converges for large enough $\lambda$. Fortunately, these cases correspond to the physically relevant cases considered in [2] and here we provide an argument that the solutions for $(\lambda, k, \ell) \in \mathscr{S}$ converge.

### 9.1. Zero Fourier modes

The zeroth Fourier mode is given by

$$\widehat{f}_0(y) = \widehat{f}_{0,0}(y) + \sum_{n_1 \neq 0} \widehat{f}_{n_1,-n_1}(y).$$

Furthermore, the sum above is given by

$$\begin{aligned} \sum_{n_1\neq 0} \widehat{f}_{n_1,-n_1}(y) &= \sum_{n_1\neq 0} \widehat{f}^P_{n_1,-n_1}(y) + \sum_{n_1\neq 0} \widehat{f}^H_{n_1,-n_1}(y) \\ &= \sum_{n_1\neq 0} \widehat{f}^P_{n_1,-n_1}(y) + y^{-r} \sum_{n_1\neq 0} \alpha_{n_1,-n_1} \end{aligned} \tag{77}$$

assuming both sums are convergent. Note that the second equality follows from (20).

From (46), (52), (60), (68), and (76) we see that each $\sum_{n_1\neq 0} \alpha_{n_1,-n_1}$ converges. Each expression for $\hat{f}^P_{n_1,-n_1}(y)$ is given by (38), (56), (63), and (72) and is exponentially suppressed as $y \to \infty$, as seen from the exponential decay of the modified Bessel functions of the second kind.

### 9.2. Non-zero Fourier modes

In order to show that the Fourier series is convergent, we first note that

$$\widehat{f}_n(y) = \widehat{f}_{n,0}(y) + \widehat{f}_{0,n}(y) + \sum_{n_1=1}^{n-1} \widehat{f}_{n_1,n-n_1}(y) + \Bigg(\sum_{n_1\geq n+1} + \sum_{n_1\leq -1}\Bigg)\widehat{f}_{n_1,n-n_1}(y).$$

We must verify that the last sum is convergent. Note that by (18),

(78)
$$\begin{aligned}\sum_{n_1\geq n+1}\widehat{f}_{n_1,n-n_1}(y) &= \sum_{n_1\geq n+1}\widehat{f}^P_{n_1,n-n_1}(y)+\sum_{n_1\geq n+1}\widehat{f}^H_{n_1,n-n_1}(y)\\ &= \sum_{n_1\geq n+1}\widehat{f}^P_{n_1,n-n_1}(y)+\sqrt{y}K_{r+1/2}(2\pi|n|y)\sum_{n_1\geq n+1}\alpha_{n_1,n-n_1}\end{aligned}$$

assuming both sums are convergent.

From (43), (50), (58), (66), (74) we see that

$$\alpha_{n_1,n-n_1}=o\big(|n_1|^{-2}\big),\quad |n_1|\to\infty.$$

Estimating the behavior of $\hat{f}^P_{n_1,n-n_1}(y)$ as $|n_1|\to\infty$ using (41), (48), (57), (65), and (73) (as well as (80), (85), (89), (92), and (96)), we see that the contribution from the modified Bessel functions, $K_i(2\pi|n_1|y)K_j(2\pi|n-n_1|y)$, exponentially suppresses these terms as $|n_1|$ gets large. We treat the term $\sum_{n_1\leq -1}$ in the same manner.

## Appendix A

In this section, we will provide the explicit solutions which do not appear in [2].

### A.1. $k=\ell=3/2$ and $\lambda=2$

**A.1.1. $n_1=0$ and $n_2=0$** Any solution of (15) for $n_1=n_2=0$ is equal to

$$\hat{f}_{0,0}(y)=\frac{16\zeta(2)^2-9\zeta(3)^2y^4+72\zeta(2)\zeta(3)y^2+48\zeta(2)^2\log(y)}{9y}+c_2y^2+\frac{c_1}{y}$$

for some $c_1,c_2\in\mathbb{C}$.

Its asymptotic behavior for $y\to 0$ can be described by

$$\hat{f}_{0,0}(y)=\frac{16\zeta(2)^2\log(y)}{3y}+y^{-1}\left(\frac{16\zeta(2)^2}{9}+c_1\right)+O(1),$$

and the leading term of the asymptotic behavior does not depend on $c_1$ and $c_2$. Taking $c_2=0$, the $O(y^2)$-term in the asymptotic expansion of $\hat{f}_{0,0}(y)$ vanishes. However, we refrain from choosing $c_1$ until Section A.1.4.

**A.1.2. $n_1 n_2 = 0$ but not both zero** For $\hat{f}^P_{n,0}$ as in (36), we have

$$\nu_0(n,y) = \mathrm{sgn}(n)\left(\frac{\zeta(2)}{3\pi n y} - \frac{\zeta(3)y}{2\pi n}\right),$$
$$\nu_1(n,y) = -\frac{\zeta(3)}{2\pi^2 n^2}.$$

We note that the asymptotic behavior of $\hat{f}^P_{0,n}(y)$ for $y \to 0$ is as follows:

$$y^{-1}\frac{4\sigma_2(n)(3\zeta(3) + 4\gamma\pi^2\zeta(2)n^2 + 4\pi^2\zeta(2)n^2\log(\pi|n|))}{3\pi^2 n^4} + \frac{16\zeta(2)\sigma_2(n)\log(y)}{3n^2 y} + O(y),$$

where $\gamma$ is the Euler-Mascheroni constant. We recall from (19) that

$$\hat{f}^H_{0,n}(y) = \frac{\alpha_{0,n}}{4\pi|n|^{3/2}y} - \frac{\pi\sqrt{|n|}\ \alpha_{0,n}}{2}y + O\big(y^2\big). \tag{79}$$

There is a natural choice of $\alpha_{0,n}$ that will get rid of the $O(y^{-1})$-term in $\hat{f}^H_{0,n}(y) + \hat{f}^P_{0,n}(y)$ (however, $O(y^{-1})$-term is not the leading term):

$$\alpha_{0,n} = -\frac{16\sigma_2(n)(3\zeta(3) + 4\gamma\pi^2\zeta(2)n^2 + 4\pi^2\zeta(2)n^2\log(\pi|n|))}{3n^{5/2}\pi}.$$

**A.1.3. $n_1 n_2 \neq 0$ and $n_1 + n_2 \neq 0$** For $\hat{f}^P_{n_1,n_2}(y)$ as in (37), we have

$$\eta_{0,0} = \mathrm{sgn}(n_1 n_2)\frac{2n_1 n_2 y}{3(n_1+n_2)^2}, \quad \eta_{0,1} = \mathrm{sgn}(n_1)\frac{n_1(n_1+3n_2)}{6\pi(n_1+n_2)^3},$$
$$\eta_{1,0} = \mathrm{sgn}(n_2)\frac{n_2(3n_1+n_2)}{6\pi(n_1+n_2)^3}, \quad \eta_{1,1} = \frac{2n_1 n_2 y}{3(n_1+n_2)^2}.$$

The asymptotic expansion of $\hat{f}^P_{n_1,n_2}(y)$ as $y \to 0$ is

$$\frac{16\sigma_2(n_1)\sigma_2(n_2)}{3n_1^2 n_2^2}\left(\frac{\log(y)}{y} + \frac{\gamma + \log(\pi)}{y} + \frac{n_1^3\log(|n_1|) + 3n_2 n_1^2\log(|n_1|) + 3n_2^2 n_1\log(|n_2|) + n_2^3\log(|n_2|)}{y(n_1+n_2)^3}\right). \tag{80}$$

We further note that

(81) $$\hat{f}^H_{n_1,n_2}(y) = \frac{\alpha_{n_1,n_2}}{4\pi|n_1+n_2|^{3/2}y} - \frac{\pi\sqrt{|n_1+n_2|}\ \alpha_{n_1,n_2}}{2}y + O\big(y^2\big).$$

We note that we cannot get rid of the $O(y^{-1}\log(y))$-term in $\hat{f}^H_{n_1,n_2}(y) + \hat{f}^P_{n_1,n_2}(y)$ by choosing an appropriate $\alpha_{n_1,n_2}$, but we can get rid of $O(y^{-1})$ by setting

(82)
$$\alpha_{n_1,n_2} = -4\pi|n_1+n_2|^{3/2}\cdot\frac{16\sigma_2(n_1)\sigma_2(n_2)}{3n_1^2n_2^2}\Bigg(\gamma+\log(\pi) + \frac{n_1^3\log(|n_1|)+3n_2n_1^2\log(|n_1|)+3n_2^2n_1\log(|n_2|)+n_2^3\log(|n_2|)}{(n_1+n_2)^3}\Bigg).$$

Choosing $n=1$ and investigating the asymptotic behavior of $\alpha_{n_1,1-n_1}$ as $n_1\to\infty$, we note that the sum $\sum_{n_1+n_2=1}\alpha_{n_1,n_2}$ does not converge. As we show in the next section, it might still be reasonable to make such a choice of $\alpha_{n_1,n_2}$.

**A.1.4. $n_1 = -n_2$** For $\hat{f}^P_{-n_2,n_2}(y)$ defined in (38) we have

$$\mu_{0,0} = \frac{y}{6}+\frac{8n_2^2\pi^2y^3}{9},\quad \mu_{0,1} = \operatorname{sgn}(n_2)\bigg(\frac{1}{6n_2\pi}+\frac{4n_2\pi y^2}{9}\bigg),$$
$$\mu_{1,1} = -\frac{5y}{18}-\frac{8n_2^2\pi^2y^3}{9}.$$

Its asymptotic behavior can be described as

(83) $$8\frac{\sigma_2(n_2)^2}{|n_2|^4}\bigg(\frac{6\log(\pi|n_2|)+6\log(y)+6\gamma+5}{9y}\bigg)+O(1).$$

Once again, we cannot get rid of the $O(y^{-1}\log(y))$-term by choosing $\alpha_{-n_2,n_2}$ appropriately, but we can get rid of the $O(y^{-1})$-term by setting

$$\alpha_{-n_2,n_2} := -\frac{8\sigma_2(n_2)\sigma_2(n_2)}{9|n_2|^4}\big(6\log(\pi|n_2|)+6\gamma-5\big).$$

We note that for such choice of $\alpha_{-n_2,n_2}$, the sum $\sum_{n_1+n_2=0,n_2\neq0}\alpha_{n_1,n_2}$ diverges, because the sum

$$\sum_{p\text{ prime}}\alpha_{-p,p} = -\frac{8}{9}\sum_{p\text{ prime}}\frac{(p^2+1)^2}{p^4}\big(6\log(\pi p)+6\gamma-5\big)$$

diverges. However, it is still possible to formally calculate the sum of $\alpha_{-n_2,n_2}$ using the Ramanujan summation (99) and its derivatives (100).

We note that we can choose $c_1$ from Section A.1.1 in such a way that at least formally

$$\sum_{n_2=-\infty}^{\infty} \alpha_{-n_2,n_2} = 0. \tag{84}$$

## A.2. $k = 3/2, \ell = 5/2$ and $\lambda = 6$

**A.2.1. $n_1 = 0$ and $n_2 = 0$** We can find a particular solution to be

$$\hat{f}_{0,0}(y) = \frac{c_1}{y^2} + c_2 y^3 + \frac{-6750y^6\zeta(3)\zeta(5) + 3375\pi^2 y^4\zeta(5) + 100\pi^4 y^2\zeta(3) + 40\pi^6\log(y) + 8\pi^6}{6750y^2}$$

for some $c_1, c_2 \in \mathbb{C}$. Its asymptotic behavior for $y \to 0$ can be described by

$$\hat{f}_{0,0}(y) = \frac{4\pi^6 \log(y)}{675y^2}$$

and the leading term of the asymptotic behavior doesn't depend on $c_1$ and $c_2$.

**A.2.2. $n_1 = 0$ and $n_2 \neq 0$** For $\hat{f}^P_{0,n}(y)$ as in (53), we have

$$\nu_0(n, y) = \frac{\zeta(2)y^{-2}}{5n^2\pi^2} - \frac{2\zeta(3)}{n^2\pi^2},$$
$$\nu_1(n, y) = \operatorname{sgn}(n)\left(\frac{n^2\pi^2\zeta(2) - 6\zeta(3)}{3n^3\pi^3 y} - \frac{\zeta(3)y}{n\pi}\right).$$

The asymptotic expansion of $\hat{f}^P_{0,n}(y)$ as $y \to 0$ is

$$-8\pi^2\sigma_{-4}(n)|n|^2\left(y^{-2}\left(-\frac{\gamma\zeta(2)}{5\pi^2 n^2} + \frac{\zeta(2)}{6\pi^2 n^2} - \frac{\zeta(3)}{\pi^4 n^4} - \frac{\zeta(2)\log(\pi|n|)}{5\pi^2 n^2}\right) - \frac{\zeta(2)\log(y)}{5\pi^2 n^2 y^2}\right) + O(1).$$

There is a unique choice of $\alpha_{0,n}$ that gets rid of the $y^{-2}$-term in the asymptotic expansion of $\hat{f}^P_{0,n}(y) + \hat{f}^H_{0,n}(y)$.

**A.2.3. $n_1 \neq 0$ and $n_2 = 0$** For $\hat{f}^P_{n,0}(y)$ as in (54), we have

$$\nu_0(n,y) = \operatorname{sgn}(n)\left(\frac{2\zeta(4)y^{-2}}{5n\pi} + \frac{3\zeta(5)}{n^3\pi^3} - \frac{\zeta(5)y^2}{2n\pi}\right),$$
$$\nu_1(n,y) = \frac{3\zeta(5)y^{-1}}{n^4\pi^4} + \frac{\zeta(5)y}{n^2\pi^2}.$$

The asymptotic expansion of $\hat{f}^P_{n,0}(y)$ as $y \to 0$ is

$$-8\pi\sigma_{-2}(n)|n|\left(y^{-2}\left(\frac{3\zeta(5)}{2\pi^5|n|^5} - \frac{2\gamma\zeta(4)}{5\pi|n|} - \frac{2\zeta(4)\log(\pi|n|)}{5\pi|n|}\right) - \frac{2\zeta(4)\log(y)}{5\pi|n|y^2}\right) + O(1).$$

There is a unique choice of $\alpha_{n,0}$ that gets rid of the $y^{-2}$-term in the asymptotic expansion of $\hat{f}^P_{n,0}(y) + \hat{f}^H_{n,0}(y)$.

**A.2.4. $n_1 n_2 \neq 0$ and $n_1 + n_2 \neq 0$** For $\hat{f}^P_{n_1,n_2}(y)$ as in (55), we have

$$\eta_{0,0} = \operatorname{sgn}(n_1)\left[\frac{1}{10\pi(n_1+n_2)^4}\left(n_1\left(n_1^2 + 4n_1n_2 + 11n_2^2\right)\right)\right],$$
$$\eta_{0,1} = \operatorname{sgn}(n_1n_2)\left[\frac{y^{-1}}{10n_2\pi^2(n_1+n_2)^5}\left(n_1\left(n_1^3 + 5n_1^2n_2 + 10n_1n_2^2 + 10n_2^3\right)\right) + \frac{y}{15(n_1+n_2)^3}\left(2n_1n_2(n_1+5n_2)\right)\right],$$
$$\eta_{1,0} = \frac{y^{-1}}{10\pi^2(n_1+n_2)^5}\left(n_2^2(5n_1+n_2)\right) + \frac{y}{15(n_1+n_2)^3}\left(2n_1n_2(n_1+5n_2)\right),$$
$$\eta_{1,1} = \frac{\operatorname{sgn}(n_2)}{30\pi(n_1+n_2)^4}\left(4n_1^3 + 19n_1^2n_2 + 44n_1n_2^2 + 5n_2^3\right).$$

We note that

(85)
$$\hat{f}^P_{n_1,n_2}(y) = \frac{8\sigma_{-4}(n_2)\sigma_{-2}(n_1)}{15n_1n_2^2(n_1+n_2)^5y^2}$$
$$\times \big(6n_1^5\log(y) + 30n_2n_1^4\log(y) + 60n_2^2n_1^3\log(y) + 60n_2^3n_1^2\log(y)$$
$$+ 30n_2^4n_1\log(y) + 6n_2^5\log(y) + 6\gamma n_1^5 + 30\gamma n_2n_1^4 - 4n_2n_1^4 + 60\gamma n_2^2n_1^3$$
$$- 23n_2^2n_1^3 + 60\gamma n_2^3n_1^2 - 63n_2^3n_1^2 + 30\gamma n_2^4n_1 - 49n_2^4n_1 + 6\gamma n_2^5 - 5n_2^5$$
$$+ 6n_1^5\log(\pi|n_1|) + 30n_2n_1^4\log(\pi|n_1|) + 60n_2^2n_1^3\log(\pi|n_1|)$$

$$+ 60n_2^3n_1^2\log(\pi|n_1|) + 30n_2^4n_1\log(\pi|n_2|) + 6n_2^5\log(\pi|n_2|))) + O(1).$$

We are not able to eliminate the highest term, $O(y^{-2}\log(y))$, by choosing appropriate $\alpha_{n_1,n_2}$. However, we are able to eliminate the $O(y^{-2})$-term.

**A.2.5. $n_1 = -n_2$** For $\hat{f}^P_{-n_2,n_2}(y)$ as in (56), we have

$$\mu_{0,0} = \operatorname{sgn}(n_2)\left(\frac{1}{10n_2\pi} + \frac{2n_2\pi y^2}{75} - \frac{64n_2^3\pi^3y^4}{225}\right),$$
$$\mu_{0,1} = \frac{y^{-1}}{10n_2^2\pi^2} - \frac{y}{75} - \frac{32n_2^2\pi^2y^3}{225},$$
$$\mu_{1,1} = \operatorname{sgn}(n_2)\left(-\frac{9}{100n_2\pi} + \frac{2n_2\pi y^2}{225} + \frac{64n_2^3\pi^3y^4}{225}\right).$$

The asymptotic expansion of $\hat{f}^P_{-n_2,n_2}(y)$ as $y \to 0$ is

$$\hat{f}^P_{-n_2,n_2}(y) = 4\sigma_{-2}(n_2)\sigma_{-4}(n_2) \times \left(\frac{20\log(\pi|n_2|) + 20\log(y) + 20\gamma + 9}{25y} + O(1)\right). \tag{86}$$

There is a choice of $\alpha_{-n_2,n_2}$ that gets rid of the $y^{-1}$-term in the asymptotic expansion of $\hat{f}^P_{-n_2,n_2}(y) + \hat{f}^H_{-n_2,n_2}(y)$. However, manipulating $\alpha_{-n_2,n_2}$ cannot help us get rid of the leading term. Thus, as in Section A.1.4, we can choose $c_1 = \alpha_{0,0}$ from Section A.1.1 so that the contribution from the homogeneous solutions vanishes, that is, at least formally

$$\sum_{n_2=-\infty}^{\infty} \alpha_{-n_2,n_2} = 0. \tag{87}$$

## A.3. $k = \ell = 5/2$ and $\lambda = 2$

**A.3.1. $n_1 = 0$ and $n_2 = 0$** We note that for $n_1 = n_2 = 0$

$$\hat{f}_{0,0}(y) = \frac{-10125y^8\zeta(5)^2 + 2700\pi^4y^4\zeta(5) - 4\pi^8}{20250y^3} + \frac{c_1}{y} + c_2y^2$$

for some $c_1, c_2 \in \mathbb{C}$. Its asymptotic behavior for $y \to 0$ can be described by $\hat{f}_{0,0}(y) = -\frac{2\pi^8}{10125y^3}$ and the leading term of the asymptotic behavior doesn't depend on $c_1$ and $c_2$.

**A.3.2. $n_1 n_2 = 0$ but not both zero** For $\hat{f}^P_{0,n}(y)$ as in (61), we have

$$\nu_0(n,y) = -\frac{\zeta(5)y}{n^2\pi^2},$$
$$\nu_1(n,y) = \operatorname{sgn}(n)\left(\frac{2\zeta(4)y^{-2}}{5n\pi} - \frac{\zeta(5)}{n^3\pi^3} - \frac{\zeta(5)y^2}{2n\pi}\right).$$

Its asymptotic behavior can be described by $y \to 0$ as

(88)
$$-8\pi^2\sigma_{-4}(n)|n|^2\left(\frac{\zeta(4)}{5\pi^2 n^2 y^3} + \frac{-5\zeta(5) + 4\gamma\pi^4\zeta(4)n^4 - 2\pi^4\zeta(4)n^4 + 4\pi^4\zeta(4)n^4\log(\pi|n|)}{10\pi^4 n^4 y} + \frac{4\pi^4\zeta(4)n^4\log(y)}{10\pi^4 n^4 y} + O(1)\right).$$

The leading term of the asymptotic expansion is $O(y^{-3})$, the second term is $O(y^{-1}\log(y))$. Manipulating homogeneous solution, we can get rid of the third term in the asymptotic expansion; that is, to get rid of $O(y^{-1})$.

**A.3.3. $n_1 n_2 \neq 0$ and $n_1 + n_2 \neq 0$** For $\hat{f}^P_{n_1,n_2}(y)$ as in (62) we have

$$\eta_{0,0} = \frac{y}{15(n_1+n_2)^2}(2n_1n_2),$$
$$\eta_{0,1} = \operatorname{sgn}(n_2)\frac{1}{30\pi(n_1+n_2)^3}\left(4n_1^2 + 9n_1n_2 + 3n_2^2\right),$$
$$\eta_{1,0} = \operatorname{sgn}(n_1)\frac{1}{30\pi(n_1+n_2)^3}\left(3n_1^2 + 9n_1n_2 + 4n_2^2\right),$$
$$\eta_{1,1} = \operatorname{sgn}(n_1n_2)\left(\frac{y^{-1}}{10n_1n_2\pi^2} + \frac{y}{15(n_1+n_2)^2}(2n_1n_2)\right).$$

We note that

$$\hat{f}^P_{n_1,n_2}(y) = -\frac{8y^{-3}\sigma_{-4}(n_1)\sigma_{-4}(n_2)}{5} + o\left(y^{-3}\right). \tag{89}$$

We are not capable to make the $O(y^{-3})$-term vanish by manipulating $\alpha_{n_1,n_2}$.

**A.3.4. $n_1 = -n_2$** For $\hat{f}^P_{-n_2,n_2}(y)$ as in (63), we have

$$\eta_{0,0} = -\frac{7y}{30} - \frac{8n_2^2\pi^2y^3}{45},$$
$$\eta_{0,1} = \operatorname{sgn}(n_2)\left(-\frac{2}{15n_2\pi} - \frac{4n_2\pi y^2}{45}\right),$$
$$\eta_{1,1} = \frac{y^{-1}}{10n_2^2\pi^2} + \frac{23y}{90} + \frac{8n_2^2\pi^2y^3}{45}.$$

The asymptotic expansion of $\hat{f}^P_{-n_2,n_2}(y)$ as $y \to 0$ is

(90)
$$-8\pi^2|n_2|^2\big(\sigma_{-4}(n_2)\big)^2 \times \left(\frac{1}{5\pi^2|n_2|^2y^3} + \frac{12\log(\pi|n_2|) + 12\log(y) + 12\gamma + 1}{9y} + O(1)\right).$$

As in Section A.1.4, we can choose $c_1 = \alpha_{0,0}$ from Section A.1.1 so that the contribution from the homogeneous solutions vanishes, that is, at least formally

$$\sum_{n_2=-\infty}^{\infty} \alpha_{-n_2,n_2} = 0. \tag{91}$$

## A.4. $k = \ell = 5/2$ and $\lambda = 12$

**A.4.1. $n_1 = 0$ and $n_2 = 0$** We note that

$$\hat{f}_{0,0}(y) = \frac{128\zeta(4)^2 - 441\zeta(5)^2y^8 + 784\zeta(4)\zeta(5)y^4 + 896\zeta(4)^2\log(y)}{392y^3} + c_2y^4 + \frac{c_1}{y^3}$$

for some $c_1, c_2 \in \mathbb{C}$. Its asymptotic behavior for $y \to 0$ can be described by $\hat{f}_{0,0}(y) =$ and the leading term of the asymptotic behavior does not depend on $c_1$ and $c_2$.

**A.4.2. $n_1n_2 = 0$ but not both zero** For $\hat{f}^P_{0,n}(y)$ as in (61), we have

$$\nu_0(n_0, y) = \frac{2\zeta(4)y^{-3}}{7n^2\pi^2} + \frac{15\zeta(5)y^{-1}}{n^4\pi^4} + \frac{3\zeta(5)y}{2n^2\pi^2},$$

$$\nu_1(n_0, y) = \operatorname{sgn}(n)\left(\frac{(4\zeta(4)\pi^4 n^4 + 150\zeta(5))y^{-2}}{10n^5\pi^5} + \frac{9\zeta(5)}{n^3\pi^3} - \frac{\zeta(5)y^2}{2n\pi}\right).$$

Its asymptotic expansion as $y \to 0$ is

$$-8\pi^2\sigma_{-4}(n)|n|^2 \times \left(\frac{525\zeta(5) - 20\gamma\pi^4\zeta(4)n^4 + 14\pi^4\zeta(4)n^4 - 20\pi^4\zeta(4)n^4\log(\pi|n|)}{70\pi^6 n^6 y^3} - \frac{20\pi^4\zeta(4)n^4\log(y)}{70\pi^6 n^6 y^3} + O\left(\frac{1}{y}\right)\right).$$

The leading asymptotic expansion as $y \to 0$ is $O(y^{-3}\log(y))$. The second leading asymptotic expansion is $O(y^{-3})$ – that one can be eliminated by manipulating the homogeneous solution.

**A.4.3. $n_1 n_2 \neq 0$, $n_1 + n_2 \neq 0$** For $\hat{f}^P_{n_1,n_2}(y)$ as in (62) we have

$$\begin{aligned}
\eta_{0,0} =& \frac{y^{-1}}{14\pi^2(n_1+n_2)^6}\left(n_1^4 + 6n_1^3 n_2 + 50n_1^2 n_2^2 + 6n_1 n_2^3 + n_2^4\right) \\
&+ \frac{y}{105(n_1+n_2)^4}\left(2n_1 n_2 \cdot \left(7n_1^2 + 54n_1 n_2 + 7n_2^2\right)\right), \\
\eta_{0,1} =& \operatorname{sgn}(n_2)\left[\frac{y^{-2}}{14n_2\pi^3(n_1+n_2)^7}\left(n_1^2\left(n_1^3 + 7n_1^2 n_2 + 21n_1 n_2^2 + 35n_2^3\right)\right)\right. \\
&\left.+ \frac{1}{210\pi(n_1+n_2)^5}\left(28n_1^4 + 199n_1^3 n_2 + 775n_1^2 n_2^2 + 145n_1 n_2^3 + 21n_2^4\right)\right], \\
\eta_{1,0} =& \operatorname{sgn}(n_1)\left[\frac{y^{-2}}{14n_1\pi^3(n_1+n_2)^7}\left(n_2^2 \cdot \left(35n_1^3 + 21n_1^2 n_2 + 7n_1 n_2^2 + n_2^3\right)\right)\right. \\
&\left.+ \frac{1}{210\pi(n_1+n_2)^5}\left(21n_1^4 + 145n_1^3 n_2 + 775n_1^2 n_2^2 + 199n_1 n_2^3 + 28n_2^4\right)\right], \\
\eta_{1,1} =& \operatorname{sgn}(n_1 n_2)\left[\frac{y^{-1}}{210n_1 n_2\pi^2(n_1+n_2)^6}\left(21n_1^6 + 166n_1^5 n_2 + 605n_1^4 n_2^2\right.\right. \\
&\left. + 1520n_1^3 n_2^3 + 605n_1^2 n_2^4 + 166n_1 n_2^5 + 21n_2^6\right) \\
&\left.+ \frac{2n_1 n_2\, y}{105(n_1+n_2)^4}\left(7n_1^2 + 54n_1 n_2 + 7n_2^2\right)\right].
\end{aligned}$$

We note that the asymptotic expansion is given by

$$\hat{f}^P_{n_1,n_2}(y) = \frac{16\sigma_{-4}(n_1)\sigma_{-4}(n_2)}{7} y^{-3}\log(y) + o\big(y^{-3}\log(y)\big). \tag{92}$$

**A.4.4. $n_1 = -n_2$** For $\hat{f}^P_{-n_2,n_2}(y)$ as in (63), we have

$$\mu_{0,0} = \frac{y^{-1}}{14n_2^2\pi^2} - \frac{13y}{294} - \frac{8n_2^2\pi^2y^3}{735} + \frac{256n_2^4\pi^4y^5}{2205},$$
$$\mu_{0,1} = \operatorname{sgn}(n_2)\left[\frac{y^{-2}}{14n_2^3\pi^3} - \frac{5}{294n_2\pi} + \frac{4n_2\pi y^2}{735} + \frac{128n_2^3\pi^3y^4}{2205}\right],$$
$$\mu_{1,1} = -\frac{5y^{-1}}{588n_2^2\pi^2} + \frac{59y}{1470} - \frac{8n_2^2\pi^2y^3}{2205} - \frac{256n_2^4\pi^4y^5}{2205}.$$

The asymptotic expansion is

$$(93)\quad 64\pi^4|n_2|^4\big(\sigma_{-4}(n_2)\big)^2\left(\frac{84\log(\pi|n_2|) + 84\log(y) + 84\gamma + 5}{2352\pi^4|n_2|^4y^3} + O\left(\frac{1}{y}\right)\right).$$

As in Section A.1.4, we can choose $c_1 = \alpha_{0,0}$ from Section A.1.1 so that the contribution from the homogeneous solutions vanishes, that is, at least formally

$$(94)\qquad \sum_{n_2=-\infty}^{\infty} \alpha_{-n_2,n_2} = 0.$$

## A.5. $k = 3/2, \ell = 7/2$ and $\lambda = 12$

**A.5.1. $n_1 = 0$ and $n_2 = 0$** It is not complicated to show that

$$\hat{f}_{0,0}(y) = \frac{-10418625y^8\zeta(3)\zeta(7) + 4630500\pi^2y^6\zeta(7) + 9408\pi^6y^2\zeta(3)}{2778300y^3} + \frac{4480\pi^8\log(y) + 640\pi^8}{2778300y^3} + c_2y^4 + \frac{c_1}{y^3}$$

for some $c_1, c_2 \in \mathbb{C}$. Its asymptotic behavior for $y \to 0$ can be described by

$$\hat{f}_{0,0}(y) = \frac{32\pi^8\log(y)}{19845y^3}$$

and the leading term of the asymptotic behavior doesn't depend on $c_1$ and $c_2$.

**A.5.2. $n_1 = 0$ and $n_2 \neq 0$** For $\hat{f}^P_{0,n}(y)$ as in (69), we have

$$\nu_0 = \operatorname{sgn}(n)\left(-2\zeta(3)\left(\frac{3y^{-1}}{n^3\pi^3} + \frac{y}{2n\pi}\right) + 4\zeta(2)\left(\frac{y^{-3}}{7n^3\pi^3} + \frac{y^{-1}}{6n\pi}\right)\right),$$
$$\nu_1 = -2\zeta(3)\left(\frac{3y^{-2}}{n^4\pi^4} + \frac{2}{n^2\pi^2}\right) + \frac{22\zeta(2)y^{-2}}{15n^2\pi^2}.$$

Its asymptotic behavior is

$$-16\sigma_{-6}(n)\left(\frac{-315\zeta(3) - 60\gamma\pi^2\zeta(2)n^2 + 77\pi^2\zeta(2)n^2}{105\pi^2n^2y^3} + \frac{-60\pi^2\zeta(2)n^2\log(\pi n) - 60\pi^2\zeta(2)n^2\log(y)}{105\pi^2n^2y^3}O\left(\frac{1}{y}\right)\right).$$

**A.5.3. $n_1 \neq 0$ and $n_2 = 0$** For $\hat{f}^P_{n,0}(y)$ as in (70), we have

$$\nu_0 = \operatorname{sgn}(n)\left(\frac{16\zeta(6)y^{-3}}{14n_2\pi} - 15\zeta(7)\left(\frac{6y^{-1}}{n_2^5\pi^5} + \frac{3y}{5n_2^3\pi^3} + \frac{y^3}{10n_2\pi}\right)\right),$$
$$\nu_1 = 15\zeta(7)\left(-\frac{6y^{-2}}{n_2^6\pi^6} - \frac{18}{5n_2^4\pi^4} + \frac{y^2}{10n_2^2\pi^2}\right).$$

Its asymptotic behavior is

$$-8\sigma_{-2}(n)\left(\frac{-315\zeta(7) - 8\gamma\pi^6\zeta(6)n^6 - 8\pi^6\zeta(6)n^6\log(\pi n)}{7\pi^6n^6y^3} - \frac{8\pi^6\zeta(6)n^6\log(y)}{7\pi^6n^6y^3} + O\left(\frac{1}{y}\right)\right). \tag{95}$$

**A.5.4. $n_1n_2 \neq 0$ and $n_1 + n_2 \neq 0$** For $\hat{f}^P_{n_1,n_2}(y)$ as in (71), we have

$$\eta_{0,0} = \operatorname{sgn}(n_1)\operatorname{sgn}(n_2)\Big(\frac{y^{-1}}{7n_2\pi^2(n_1+n_2)^6}\Big(n_1\big(n_1^4 + 6n_1^3n_2 + 15n_1^2n_2^2 + 20n_1n_2^3 + 22n_2^4\big) + \frac{y}{105(n_1+n_2)^4}\big(2n_1n_2\cdot\big(3n_1^2 + 14n_1n_2 + 35n_2^2\big)\big)\Big),$$
$$\eta_{0,1} = \operatorname{sgn}(n_1)\Big(\frac{y^{-2}}{7n_2^2\pi^3(n_1+n_2)^7}n_1\big(n_1^5 + 7n_1^4n_2 + 21n_1^3n_2^2 + 35n_1^2n_2^3 + 35n_1n_2^4 + 21n_2^5\big)$$

$$+\frac{1}{210\pi(n_1+n_2)^5}\left(n_1\cdot\left(27n_1^3+143n_1^2n_2+325n_1n_2^2+497n_2^3\right)\right)\Bigg),$$

$$\eta_{1,0}=\operatorname{sgn}(n_1)\Bigg(\frac{y^{-2}}{7\pi^3(n_1+n_2)^7}\left(n_2^3\cdot(7n_1+n_2)\right)$$
$$+\frac{1}{210\pi(n_1+n_2)^5}\left(24n_1^4+129n_1^3n_2+293n_1^2n_2^2+511n_1n_2^3+35n_2^4\right)\Bigg),$$

$$\eta_{1,1}=\frac{y^{-1}}{210n_2\pi^2(n_1+n_2)^6}\left(24n_1^5+153n_1^4n_2+422n_1^3n_2^2+678n_1^2n_2^3\right.$$
$$\left.+822n_1n_2^4+77n_2^5\right)$$
$$+\frac{y}{105(n_1+n_2)^4}\left(2n_1n_2\cdot\left(3n_1^2+14n_1n_2+35n_2^2\right)\right).$$

We note that

$$\hat{f}^P_{n_1,n_2}(y)=\frac{64\sigma_{-2}(n_1)\sigma_{-6}(n_2)\log(y)y^{-3}}{7}+o\left(y^{-3}\log(y)\right). \tag{96}$$

We cannot eliminate the leading term in the asymptotic expansion of $\hat{f}_{n_1,n_2}(y)$ by manipulating $\alpha_{n_1,n_2}$.

**A.5.5. $n_1=-n_2$** For $\hat{f}^P_{-n_2,n_2}(y)$ as in (72), we have

$$\mu_{0,0}=\frac{y^{-1}}{7n_2^2\pi^2}+\frac{59y}{1470}-\frac{8n_2^2\pi^2y^3}{1225}+\frac{256n_2^4\pi^4y^5}{3675},$$
$$\mu_{0,1}=\operatorname{sgn}(n_2)\left(\frac{y^{-2}}{7n_2^3\pi^3}+\frac{17}{735n_2\pi}+\frac{4n_2\pi y^2}{1225}+\frac{128n_2^3\pi^3y^4}{3675}\right),$$
$$\mu_{1,1}=-\frac{13y^{-1}}{147n_2^2\pi^2}-\frac{313y}{7350}-\frac{8n_2^2\pi^2y^3}{3675}-\frac{256n_2^4\pi^4y^5}{3675}.$$

The asymptotic behavior is

(97)
$$-32\sigma_{-2}(n_2)\sigma_{-6}(n_2)\left(\frac{-42\log(\pi|n_2|)-42\log(y)-42\gamma-13}{147y^3}+O\left(\frac{1}{y}\right)\right).$$

As in Section A.1.4, we can choose $c_1=\alpha_{0,0}$ from Section A.1.1 so that the contribution from the homogeneous solutions vanishes, that is, at least formally

$$\sum_{n_2=-\infty}^{\infty}\alpha_{-n_2,n_2}=0. \tag{98}$$

# Appendix B

## B.1. Convolution formulas for the divisor functions

We recall the two famous identities on divisor functions: by [11, Theorem 291], for $s > 1$ and $s - a > 1$,

$$\sum_{n=-\infty, n\neq 0}^{\infty} \frac{\sigma_a(n)}{|n|^s} = 2\zeta(s)\zeta(s-2),$$

and by [11, Theorem 305], for $s > 1$, $s - a > 1$, $s - b > 1$ and $s - a - b > 1$,

$$\sum_{n=-\infty, n\neq 0}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{|n|^s} = 2\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}. \tag{99}$$

We note that the latter identity is sometimes referred to as a Ramanujan identity. Differentiating (99) with respect to $s$, we obtain

$$\sum_{n=-\infty, n\neq 0}^{\infty} \frac{\sigma_a(n)\sigma_b(n)\log(|n|)}{|n|^s} = -2\frac{d}{ds}\left(\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}\right). \tag{100}$$

## B.2. Bessel functions and relations between them

By [8],

$$K_{n+1}(z) = K_{n-1}(z) + 2n\frac{K_n(z)}{z}, \tag{101}$$

thus

$$yK_1(2|n_1|\pi y)K_2(2|n_2|\pi y) = yK_1(2\pi|n_1|y)\left(K_0(2\pi|n_2|y) + \frac{1}{\pi|n_2|y}K_1(2\pi|n_2|y)\right).$$

Explicitly, we use the following relations to formulate $s_{n_1,n_2}$ in (12) in terms of $K_0$ and $K_1$. From [8, 8.486(12) and 8.486(13)] and (101),

$$K_3(y) = \frac{yK_1(y) + 4K_2(y)}{y} = \frac{(y^2+8)K_1(y) + 4yK_0(y)}{y^2}$$

$$= 8\frac{K_1(y)}{y^2} + K_1(y) + 4\frac{K_0(y)}{y},$$

thus

$$yK_1(2\pi|n_1|y)K_3(2\pi|n_2|y)$$
$$= K_1(2\pi|n_1|y)\left(2\frac{K_1(2\pi|n_2|y)}{\pi^2|n_2|^2y} + 2\frac{K_0(2\pi|n_2|y)}{\pi|n_2|} + yK_1(2\pi|n_2|y)\right).$$

By [8, 8.486(17)],

$$yK_2(2\pi|n_1|y)K_2(2\pi|n_2|y) = y\left(K_0(2\pi|n_1|y) + \frac{1}{\pi|n_1|y}K_1(2\pi|n_1|y)\right)$$
$$\times\left(K_0(2\pi|n_2|y) + \frac{1}{\pi|n_2|y}K_1(2\pi|n_2|y)\right).$$

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