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Numerical experiments on coefficients of instanton partition functions

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We analyze the coefficients of partition functions of Vafa-Witten (VW) theory on a four-manifold. These partition functions factorize into a product of a function enumerating pointlike instantons and a function enumerating smooth instantons. For gauge groups $SU(2)$ and $SU(3)$ and four-manifold the complex projective plane $\mathbb{CP}^2$, we experimentally study the latter functions, which are examples of mock modular forms of depth 1, weight 3/2, and depth 2, weight 3 respectively. We also introduce the notion of "mock cusp form", and study an example of weight 3 related to the $SU(3)$ partition function. Numerical experiments on the first 200 coefficients of these mock modular forms suggest that the coefficients of these functions grow as $O(n^{k-1})$ for the respective weights $k = 3/2$ and 3. This growth is similar to that of a modular form of weight k. On the other hand the coefficients of the mock cusp form of weight 3 appear to grow as $O(n^{3/2})$, which exceeds the growth of classical cusp forms of weight 3. We provide bounds using saddle point analysis, which however largely exceed the experimental observation.

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1. Introduction

Instantons are (anti-)self-dual solutions of the Yang-Mills equations of motion. These non-perturbative solutions are central in theoretical physics [43, 6, 50, 29, 65, 70], and connect to many subjects including geometry [68, 27, 28, 47, 75, 48] and analytic number theory [36, 69, 40, 13, 18]. Of particular interest are moduli spaces of instanton solutions and their topological invariants. This article considers generating functions of Euler numbers of instanton moduli spaces for four-manifolds, in particular $\mathbb{CP}^2$. Rather strikingly, these partition functions are examples of (mock) modular

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forms as a consequence of electromagnetic duality [60, 69, 71], and even give rise to new functions of this type [52, 55, 2, 56].

Physically, these generating functions are the partition functions of a specific topological twist of $\mathcal{N} = 4$ Yang-Mills theory [74, 69] on a four-manifold X. The supersymmetry makes an explicit analysis feasible for many physical quantities of interest. For manifolds with the topological property that $b_2^+(X) > 1$, such as $X = K3$, the partition functions of the theory are known to give rise to classical modular forms as a consequence of electric-magnetic duality. On the other hand for four-manifolds with $b_2^+(X) = 1$, such as the rational surfaces, the duality leads to more intricate partition functions, namely instances of mock modular forms [69, 53, 22] and even mock modular forms of higher depth [2, 56]. More precisely, the functions are examples of mixed mock modular forms and their coefficients grow exponentially. An exact formula of Rademacher type was derived for these coefficients for the gauge group $SU(2)$ in [13], and the gauge group $SU(3)$ in [15].

The partition functions can be expressed as a product of two functions by a result of Göttsche [37, Prop. 3.1]. The coefficients of the first function enumerate smooth instanton solutions, or locally free sheaves in algebraic geometry. Assuming that X is simply connected, the second function is a negative power of the Dedekind eta function, and its coefficients enumerate pointlike instantons. These objects compactify the moduli space of smooth instanton solutions, and correspond within algebraic geometry to sheaves whose ranks jumps at these points. This function gives rise to the exponential growth mentioned above, which is of crucial importance for physical questions related to the quantum-mechanical entropy [19, 20].

On the other hand, much of the challenge in the determination of the VW partition functions is in the first function [47, 75, 48, 55, 73]. This article is mostly concerned with this function for gauge groups $SU(2)$ and $SU(3)$ and $X = \mathbb{CP}^2$. These capture interesting arithmetic information and appear to be pure mock modular forms. For the gauge group $SU(2)$ these coefficients are famously Hurwitz class numbers [47, 77].[1] This article explores the coefficients for the gauge group $SU(3)$. We find intriguing patterns for the coefficients of these partition functions, while we furthermore determine an upper bound on the asymptotic growth of these coefficients. In our analysis, we introduce the notion of a "mock cusp form". For our specific example of a mock cusp form of weight $k = 3$, we find experimentally that the coefficients of the function appear to grow as $O(n^{3/2})$. We are however unable

[1]See for the definition of Hurwitz class number for example [42].

to prove this growth. Using the saddle point method, we are able to put an upper bound to the growth of the coefficients by $O(n^{5/2})$, and using a more heuristic argument based on lattice sums by $O(n^2)$. To put these bounds into context, we note that the saddle point method applied to classical cusp forms of weight k (or Hecke bound) gives $O(n^{k/2})$, which is notably larger than the sharper Deligne-Petersson-Ramanujan bound $O(n^{(k-1)/2})$. It would be interesting to explore the growth of similar functions, such as the coefficients of VW partition functions for $N > 3$, and those of generating functions of bound states of black holes [17].

We find it intriguing that the growth of the Euler numbers of non-compact moduli spaces of smooth instantons is only polynomial as function of the instanton number, whereas the Euler numbers of the compactified moduli spaces, which include pointlike instantons, is exponential. It would be interesting to understand the growth of the coefficients, as well as the analytic properties of the partition functions, from a more geometric and physical perspective.

The outline of this article is as follows. We briefly review VW theory in Section 2.1, followed by a discussion of the partition functions for the gauge groups $SU(2)$ and $SU(3)$ in Section 2.2. In Section 3.1, we recall aspects of modular forms and their transformation properties. In Section 3.2, we review aspects of the mock modular forms, state their transformation properties and extend the definition to mock cusp forms. Section 4 provides detailed numerical results and plots obtained for the partition functions of $SU(3)$ VW theory and how these functions grow for p-th coefficients for prime p. Finally in Section 5, we analyse the asymptotics of the Fourier coefficients of the partition functions of VW theory for gauge group $SU(3)$. In Section 5.1, we review the rough bound on the growth of coefficients of modular and cusp forms. We extend this in Section 5.2 to the growth of coefficients of mock cusp forms associated with the partition functions of VW theory. Section 5.3 provides an heuristic argument for the growth of the coefficients of theta series.

2. Yang-Mills theory and mock modularity

This section briefly reviews $\mathcal{N} = 4$ super Yang-Mills theory and its topologically twisted version, Vafa-Witten theory [69]. For more detailed expositions in mathematics and physics we suggest [47, 62, 63, 75, 76], and [31, 32, 30, 29].

2.1. Instanton solutions

For a simply connected gauge group G, we let F_{ij} be the field strength for the gauge potential A_i with $i, j \in \{1, \dots, 4\}$,

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]. \tag{1}$$

The Yang-Mills action reads

$$S = \frac{1}{2g^2} \int_X d^4x \, \mathrm{Tr}\, F_{ij} F^{ij}, \tag{2}$$

where the trace is over the indices of the representation of the Lie algebra of G, and g is the gauge coupling. The action S is left invariant by the gauge symmetry, which acts on the covariant derivative $D_i = \partial_i + A_i$ as,

$$D_i \to h^{-1} D_i h, \text{ for } h(\vec{x}) \in G. \tag{3}$$

The action is bounded below by the instanton number

$$k = -\frac{1}{8\pi^2} \int_X \mathrm{Tr}\, F \wedge F, \tag{4}$$

which is a topological invariant of a solution to the Yang-Mills equations of motion. The instanton number gives a lower bound on the action,

$$\begin{aligned} S &= \frac{1}{2g^2} \int_X d^4x \, \mathrm{Tr}\, F_{ij} F^{ij} = \frac{1}{g^2} \int_X \mathrm{Tr}\, F \wedge *F \\ &= \frac{2}{g^2} \int_X d^4x \, \mathrm{Tr} \left((F^+)^2 - \frac{1}{2} F \wedge F \right) \\ &\geq \frac{8\pi^2}{g^2} k, \end{aligned} \tag{5}$$

where $F = \frac{1}{2} F_{ij} dx^i \wedge dx^j$, and $F^{\pm}_{ij}$ are the self dual and anti-self dual components of F_{ij}:

$$2F^+_{ij} = F_{ij} + \frac{1}{2} \epsilon_{ijkl} F^{kl}, \quad 2F^-_{ij} = F_{ij} - \frac{1}{2} \epsilon_{ijkl} F^{kl}. \tag{6}$$

Solutions which saturate the bound (5), those with with $F^+ = 0$, are called “instantons”.

Instanton solutions were studied by several physicists as well as mathematicians since the 1970s. On the physics side, instantons are related to magnetic monopole solutions by dimensional reduction [43, 44, 45, 46]. A complete construction using linear algebra alone for the Yang-Mills self dual instantons in Euclidean S^4 was first given in [5, 33].

The moduli space $\mathcal{M}_k$ of instanton solutions modulo gauge transformations with instanton number k is finite dimensional. The dimension corresponds physically to the number of fermionic zero modes (ground states), or more precisely the index of the Dirac operator. The sequence of topological invariants of instanton moduli spaces as function of the Euler number is naturally combined to a generating function. For example, the generating function of Euler numbers of moduli spaces of instantons on $\mathbb{R}^4$ with boundary conditions for the gauge potential, $A|_{r\to\infty} \sim \frac{1}{r^2}$ and $F|_{r\to\infty} \sim \frac{1}{r^3}$ for $r \to \infty$ were computed in [62, 63]. Such generating functions are often realized physically as a statistical partition function of a topologically twisted supersymmetric Yang-Mills theory. Depending on the field content, different topological observables can be realized [50]. The subject of this paper is the Vafa-Witten twist of $\mathcal{N} = 4$ Yang-Mills theory, whose partition function is a generating function of Euler numbers of instanton and monopole moduli spaces. The precise definition of the Euler numbers of these potentially singular spaces is an important aspect, which requires sophisticated techniques within algebraic geometry. See for example [35, 67] for generic algebraic surfaces, and [58] for Fano surfaces. For the purpose of this paper, it is not necessary to further review these techniques.

We recall a few aspects from [69] in what follows. The action of the bosonic fields of $\mathcal{N} = 4$ super Yang-Mills theory reads

$$S_b(A_i, v_a) = \frac{1}{g^2}\int_X d^4x\, \mathrm{Tr}\left(\frac{1}{2}F_{ij}F^{ij} + \sum_{a<b}^{6}[v_a, v_b]^2 + \sum_{a=1}^{6}(D_i v_a)^2 - \frac{ig^2\theta}{8\pi^2}F\wedge F\right), \tag{7}$$

where v_a, $a = 1, \ldots, 6$, are scalar fields, and θ is the theta angle. The theory has a global R-symmetry group $SU(4)$. The four scalars v_a transform under the 6-dimensional representation of $SU(4)$, whereas the gauge field A_i is a singlet. The four supercharges transform under the four-dimensional representation **4** of $SU(4)$.

A topological twist of the $\mathcal{N} = 4$ theory on a compact four-manifold X identifies a principal $SU(4)$ R-symmetry bundle. The identification follows by specifying the action of the space-time rotation group $Spin(4)$

on the bundle associated to the four-dimensional representation **4** of the $SU(4)$ R-symmetry. The Vafa-Witten twist is the twist for which the **4** is identified with the representation $(\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2})$ of the local frame group $Spin(4) \simeq SU(2)_+ \times SU(2)_-$. The identification of the principal $SU(4)$ R-symmetry bundle thus follows from the chiral spin bundle W^- on X, i.e. the bundle associated to the $(\mathbf{1}, \mathbf{2})$ representation of $Spin(4)$. Crucially, the supercharges of the topologically twisted theory include two scalar supercharges. The fermionic fields of the theory consists of two self-dual two-forms, two vectors and two scalars, whereas the bosonic field content of the twisted theory is:

1. Gauge field A_i,
2. Self-dual two-form B_{ij}. For $i = 0$, we abbreviate $B_{0j} = B_j$,
3. Real scalar C, and complex scalar ϕ.

The bosonic part of the action with $\phi, \bar{\phi}$ set to zero is given by,

$$S(A_i, B_i, C) = \frac{1}{g^2} \int_X d^4x \, \mathrm{Tr} \left(\frac{1}{2} F_{ij} F^{ij} + (D_i B_j)^2 + (D_i C)^2 \right. \\ \left. + \sum_{i<j} [B_i, B_j]^2 + \sum_i [C, B_i]^2 - \frac{i g^2 \theta}{8\pi^2} F \wedge F \right). \tag{8}$$

The instanton solutions are given by

$$B_{ij} = C = F^+_{ij} = 0 \quad \forall\, i, j. \tag{9}$$

Depending on the choice of X, there may be a monopole or Abelian branch [69]. This latter branch is absent on manifolds with positive constant scalar curvature such as $\mathbb{CP}^2$.

2.2. Partition functions of Vafa-Witten theory

We review the partition functions of the Vafa-Witten theory with gauge groups $U(1)$ and $SU(N)$, $N \geq 2$, before specializing to the four-manifolds $K3$ and $\mathbb{CP}^2$. The partition function for a four-manifold X is given schematically by the path integral formalism as

$$Z^X_N(\tau, \bar{\tau}) = \int \mathcal{D}\vec{\Phi} \, e^{-S(\vec{\Phi})}, \tag{10}$$

where the gauge group is $U(1)$ for $N = 1$ and $SU(N)$ for $N \geq 2$. $\vec{\Phi}$ represents the field content of the theory, which besides the bosonic fields in (8) includes the fermionic super-partners, ghost anti-ghost pairs and auxiliary fields. Moreover,

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \in \mathbb{H}, \tag{11}$$

is the complexified coupling constant, which takes values in the upperhalf of the complex plane, $\mathbb{H}$.

The partition function is formally independent of $\bar{\tau}$, and localizes to instanton solutions,

$$Z_N^X(\tau) = \sum_{n\geq 0} b_N(n)\, q^{n-N\chi(X)/24} \tag{12}$$

These partition functions are the generating functions of Euler numbers of compactified moduli spaces of semi-stable coherent sheaves $\mathcal{M}_n$. Thus we have essentially $b_N(n) = \chi(\mathcal{M}_{N,n})$. The moduli spaces $\mathcal{M}_{N,n}$ include singular or pointlike instantons.

For manifolds with $b_2(X)^+ > 1$, it is increasingly rigorous established that the partition functions are expressed in terms of Seiberg-Witten invariants and modular forms [26, 67, 38, 39]. More precisely the generating functions are weakly holomorphic modular forms of weight $-\chi(X)/2$, with $\chi(X)$ the Euler number of X. As a result, if X is simply connected, $b_1(X) = 0$ and thus $\chi(X) > 0$, the Euler numbers grow exponentially with the instanton number [4],

$$b_N(n) \sim \exp(\pi\sqrt{2nN\chi(X)/3}). \tag{13}$$

On the other hand, the Euler numbers of moduli spaces of smooth instantons related to these by an overall power of eta-functions [37, Prop. 3.1],

$$Z_N^X(\tau) = \frac{f_N^X(\tau)}{\eta(\tau)^{N\chi(X)}}. \tag{14}$$

The numerator $f_N^X(\tau)$ is the generating function of Euler numbers $\chi(\mathcal{N}_{N,n})$ of moduli spaces of smooth instantons, or locally free, semi-stable sheaves $\mathcal{N}_{N,n}$. Thus the coefficients $c_N(n)$ of its Fourier expansion,

$$f_N^X(\tau) = \sum_{n\geq 0} c_N(n)\, q^n, \tag{15}$$

equal $\chi(\mathcal{N}_{N,n})$. The $f_N^X(\tau)$ are expected to be modular forms of weight $(N-1)\chi(X)/2$. As a result the coefficients $c_N(n)$ of f_N^X grow much slower than the $c_N(n)$, at most polynomially in the exponent n. We discuss in the following sections suggest that

$$|c_N(n)| \leq C\, n^{(N-1)\chi(X)/2-1} \tag{16}$$

for some positive constant C.

Partition functions for $X = K3$

The partition function of Vafa-Witten theory on $K3$ and the gauge group $U(1)$ were given in [36, 69]

$$Z_1^{K3}(\tau) = \frac{1}{\eta(\tau)^{24}} = q^{-1} \sum_{k\geq 0} p_{24}(n) q^k \tag{17}$$

where $q = e^{2\pi i\tau}$, for $\tau \in \mathbb{H}$, $p_{24}(k)$ is the partition of a positive integer k in 24 colors, and $\eta(\tau)$ is the Dedekind eta function given by,

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n). \tag{18}$$

The function in (17) is also the partition function of 24 free scalar fields in two dimensions. The partition function for gauge group $SU(2)$ with vanishing 't Hooft flux is given by,

$$\begin{aligned} Z_2^{K3}(\tau) &= \frac{1}{4} Z_1^{K3}(2\tau) + \frac{1}{2} Z_1^{K3}\left(\frac{\tau}{2}\right) + \frac{1}{2} Z_1^{K3}\left(\frac{\tau+1}{2}\right) \\ &= \frac{1}{4} q^{-2} + 30 + 3200\, q + 176337\, q^2 + \cdots . \end{aligned} \tag{19}$$

For $f_2^{K3}(\tau)$, we find

$$f_2^{K3}(\tau) = \eta(\tau)^{48}\, Z_2^{K3}(\tau) = \frac{1}{4} - 12\, q + 300\, q^2 - 2000\, q^3 + \cdots . \tag{20}$$

Thus we see experimentally that the coefficients of f_2^{K3} grow much slower than those of Z_2^{K3}.

More generally, the partition function for gauge group $SU(N)$ with prime N and vanishing 't Hooft flux the partition function is argued to be [69],

$$Z_N^{K3} = \frac{1}{N^2} Z_1^{K3}(N\tau) + \frac{1}{N} \sum_{b=0}^{N-1} Z_1^{K3}\left(\frac{\tau + b}{N}\right). \tag{21}$$

The partition function for other 't Hooft fluxes can be expressed in a similar form [69].

Partition functions for $X = \mathbb{CP}^2$

We continue with the four-manifold $X = \mathbb{CP}^2$ which has $b_2(X) = b_2^+(X) = 1$. While the partition functions for K3 are given in terms of modular forms for any N, we will see that the partition functions for $\mathbb{CP}^2$ give rise to new functions,

$$Z_{N,\mu}^{\mathbb{CP}^2} = \frac{\widehat{f}_{N,\mu}^{\mathbb{CP}^2}(\tau, \bar{\tau})}{\eta(\tau)^{3N}}. \tag{22}$$

Here we have included the 't Hooft flux μ of the gauge bundle. As we will spell out in further detail, functions $\widehat{f}_{N,\mu}^{\mathbb{CP}^2} =: \widehat{f}_{N,\mu}$ in the numerator are non-holomorphic. Their holomorphic part enumerates smooth instanton solutions. Since $\mathbb{CP}^2$ is a toric four-manifold, these numbers also enumerate the number of fixed points of the toric action on the moduli space of locally free sheaves [47, 48, 73].

The result for the gauge group $U(1)$ is again given in terms of the Dedekind eta function [36],

$$Z_1^{\mathbb{CP}^2} = \frac{1}{\eta(\tau)^3}. \tag{23}$$

For $SU(2)$ with 't Hooft flux μ, a combination of physics [69], algebraic geometry [47, 75, 76] and analytic number theory [13] has demonstrated that the partition functions read:

$$Z_{2,\mu}^{\mathbb{CP}^2}(\tau, \bar{\tau}) = \frac{\widehat{f}_{2,\mu}(\tau, \bar{\tau})}{\eta(\tau)^6}, \qquad \mu = 0, 1, \tag{24}$$

with the $\widehat{f}_{2,\mu}$, explicitly given as

$$\widehat{f}_{2,\mu}(\tau,\bar\tau) = f_{2,\mu}(\tau) - \frac{3i}{4\sqrt{2}\pi}\int_{-\bar\tau}^{i\infty} \frac{\Theta_{\mu/2}(w)}{(-i(w+\tau))^{3/2}}dw. \tag{25}$$

The holomorphic parts $f_{2,\mu}$ are a multiple of the generating functions G_μ of Hurwitz class numbers $H(n)$ [78, 77],

$$f_{2,\mu}(\tau) = 3G_\mu(\tau), \qquad G_\mu(\tau) = \sum_{n=0}^{\infty} H(4n-\mu)\, q^{n-\mu/4}. \tag{26}$$

Moreover, Θ_α in (25) is the theta series defined by

$$\Theta_\alpha(\tau) = \sum_{n\in\mathbb{Z}+\alpha} q^{n^2/2}. \tag{27}$$

See equation (93) for explicit generating series for the $f_{2,\mu}(\tau)$. Table 1 in Section 4 lists the first few Hurwitz class numbers.

The holomorphic part $f_{2,\mu}$ transforms as

(28)

$$f_{2,\mu}\left(-\frac{1}{\tau}\right) = -\frac{\tau^{3/2}}{\sqrt{2}}\sum_{\nu=0}^{1}(-1)^{\mu\nu}\left(f_{2,\nu}(\tau) - \frac{3i}{4\sqrt{2}\pi}\int_0^{i\infty}\frac{\Theta_{\nu/2}(w)}{(-i(w+\tau))^{3/2}}\,dw\right),$$

$$f_{2,\mu}(\tau+1) = e^{-\pi i\mu^2/2} f_{2,\mu}(\tau).$$

Next we consider the transformations of the non-holomorphic function $\widehat{f}_{2,\mu}$ (25). For $\tau \to -1/\tau$, the shift by the period integral in (28) is absorbed by the non-holomorphic integral in (25). As a result, $\widehat{f}_{2,\mu}$ transforms as a modular form of weight 3/2. The full partition function is thus a non-holomorphic modular form of weight $-3/2$. The non-holomorphic $\widehat{f}_{N,\mu}$ satisfy a compact holomorphic anomaly equation, which can be derived using localization techniques on the Coulomb branch of the effective field theory [7, 22, 57], or in string theory [59, 1, 3].

For $X = \mathbb{CP}^2$, the partition functions of $SU(N)$ Vafa-Witten theory are determined for arbitrary N in [55]. These expressions give rise to higher dimensional analogues of Appell functions. For $SU(3)$, the complete non-holomorphic partition function has a similar form to (24), it reads [56]:

$$Z_{3,\mu}^{\mathbb{CP}^2}(\tau,\bar\tau) = \frac{\widehat{f}_{3,\mu}(\tau,\bar\tau)}{\eta(\tau)^9}, \tag{29}$$

where $\widehat{f}_{3,\mu}$ reads

(30) $$\widehat{f}_{3,\mu}(\tau,\bar\tau) = f_{3,\mu}(\tau) - \frac{i\,(3/2)^{3/2}}{\pi}\sum_{\nu=0,1}\int_{-\bar\tau}^{i\infty}\frac{\widehat{f}_{2,\nu}(\tau,-v)\,\Theta_{\frac{\mu}{3}+\frac{\nu}{2}}(3v)}{(-i(v+\tau))^{3/2}}\,dv,$$

with $\widehat{f}_{2,\nu}$ as in (25). The holomorphic part $f_{3,1}$ was determined in [48, 73, 52] and $f_{3,0}$ in [54]. The first few terms of their q-expansions are:

(31) $$\begin{aligned} f_{3,0}(\tau) &= \frac{1}{9} - q + 3\,q^2 + \cdots\,, \\ f_{3,1}(\tau) &= f_{3,2}(\tau) = 3\,q^{5/3} + 15\,q^{8/3} + 36\,q^{11/3} + \cdots\,. \end{aligned}$$

Exact expressions for $f_{3,\mu}$ are given in [56] and reproduced in Appendix A. Table 2 in Section 4 provides a longer list of the first coefficients. The holomorphic parts of $\widehat{f}_{3,\mu}$ transform as [56]:

(32)
$$\begin{aligned} f_{3,\mu}(-1/\tau) &= \frac{i\tau^3}{\sqrt{3}}\sum_{\nu=0}^{2} e^{-2\pi i\mu\nu/3} \\ &\qquad\times\left(f_{3,\nu}(\tau) - \frac{i(3/2)^{3/2}}{\pi}\sum_{\alpha=0,1}\int_0^{i\infty}\frac{\widehat{f}_{2,\alpha}(\tau,-w)\,\Theta_{\frac{2\nu+3\alpha}{6}}(3w)}{(-i(w+\tau))^{3/2}}dw\right), \\ f_{3,\mu}(\tau+1) &= (-1)^{\mu}e^{\pi i\mu^2/3}f_{3,\mu}(\tau). \end{aligned}$$

In the next section, we will introduce various notions of modular forms and mock modular forms to characterize the various functions, which appeared above.

3. Modular forms and mock modular forms

In this section, we review the definitions of modular and mock modular forms for $SL_2(\mathbb{Z})$ and their vector-valued counterparts. We also introduce the notion of mock cusp forms.

3.1. Modular, mock modular and mock cusp forms

We start with basic definitions of modular forms. We refer for further aspects to textbooks such as [4, 25, 16]

Definition. A modular form of weight k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$, which

1. transforms under all $SL_2(\mathbb{Z})$ matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ as follows:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \tag{33}$$

2. and whose growth as $\tau \to i\infty$ is such that

$$\lim_{\tau\to i\infty} (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) \tag{34}$$

is bounded for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$.

There are two generators of $SL_2(\mathbb{Z})$, namely $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Under these transformations

$$f(\tau+1) = f(\tau), \quad f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau). \tag{35}$$

Due to the T-transformation, $f(\tau)$ can be expanded as a Fourier series. This series starts with a constant term $a(0)$ as a result of the growth condition,

$$f(\tau) = \sum_{n=0}^{\infty} a(n)\, q^n, \qquad q = e^{2\pi i\tau}.$$

One can allow for a non-trivial character $\varepsilon(\gamma)$ for any $\gamma \in SL_2(\mathbb{Z})$ with $|\varepsilon(\gamma)| = 1$. The transformations for the generators then read,

$$\begin{aligned} f(\tau+1) &= \varepsilon(T)\, f(\tau), \\ f\left(\frac{-1}{\tau}\right) &= \varepsilon(S)\, \tau^k f(\tau). \end{aligned}$$

Since $S^2 = (ST)^3 = -I$ we have $\varepsilon(S) = \varepsilon(T)^{-3}$.

A weakly holomorphic modular form is a weaker notion than a modular form. It is a function f which satisfies Eq. (35), but is allowed to have poles at the cusps $i\infty \cup \mathbb{Q}$.

We next recall the notion of a cusp form:

Definition. A modular form $f(\tau)$ for $SL_2(\mathbb{Z})$ is a cusp form for which the limit (34) vanishes for $\tau \to i\infty$.

As a result, the constant term $a(0)$ of its Fourier series vanishes.

The space of modular forms is well-known to be finite dimensional for fixed weight k. The Petersson inner product [66]

$$\langle f, g\rangle = \int_{\mathbb{H}/SL_2(\mathbb{Z})} \frac{d\tau \wedge d\bar{\tau}}{y^{2-k}} f(\tau)\,\overline{g(\tau)}, \tag{36}$$

forms a natural inner product on the space of cusp forms. Moreover, $\langle f, g\rangle$ vanishes for f an Eisenstein series and g a cusp form.

The definitions of a modular form and cusp are readily extended from the full modular group $SL_2(\mathbb{Z})$ to a congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$ [4, 25]. Another useful notion is a vector-valued modular form, as for example discussed in [34].

Definition. With $d \geq 1$, a d-dimensional vector-valued modular form of weight k under $SL_2(\mathbb{Z})$ is a vector of holomorphic functions

$$\vec{f} = \begin{bmatrix} f_0 \\ \vdots \\ f_{d-1} \end{bmatrix} : \mathbb{H} \to \mathbb{C}^d, \tag{37}$$

with the following properties:

1. The elements of the vector transform under the S and T transformations as

$$S: \qquad \vec{f}\left(\frac{-1}{\tau}\right) = \tau^k\, \mathbf{M}(S)\vec{f}(\tau), \tag{38}$$

$$T: \qquad \vec{f}(\tau+1) = \mathbf{M}(T)\vec{f}(\tau), \tag{39}$$

   with $\mathbf{M}(S)$ and $\mathbf{M}(T) \in GL(d,\mathbb{C})$, with norm of their determinant equal to 1, $|\det(\mathbf{M}(T))| = |\det(\mathbf{M}(S))| = 1$.
2. For each element f_μ, $\mu = 0, \ldots, d-1$, the limit (34) is bounded for all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$.

We note that $\mathbf{M}(T)$ is diagonal in many cases. This notion is readily extended to a vector-valued cusp form. The elements f_μ of the vector $\vec{f}$ are modular forms for a congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$.

We next continue with the definition of mock modular form [77, 79, 78]. We first introduce a map on non-holomorphic functions. For a function g :

$\mathbb{H} \times \bar{\mathbb{H}} \to \mathbb{C}$ and $l \in \mathbb{R}$, we define the function $g_l^* : \mathbb{H} \times \bar{\mathbb{H}} \to \mathbb{C}$ by:

$$g_l^*(\tau, \bar\tau) = -2^{1-k} i \int_{-\bar\tau}^{i\infty} \frac{g(\tau, -v)}{(-i(v+\tau))^l}\, dv. \tag{40}$$

For the right hand side to be well defined, we require that g satisfies the following growth condition,

$$\lim_{\bar v \to -i\infty} |g(\tau, \bar v)| \le |C(\tau)|\, e^{-K\, |\mathrm{Im}(v)|}, \tag{41}$$

for some weakly holomorphic function $C : \mathbb{H} \to \mathbb{C}$, and K, l satisfying

$$\{K > 0 \,|\, l \in \mathbb{R}\}, \qquad \text{or} \qquad \{K = 0 \,|\, l > 1\}. \tag{42}$$

Assuming that g transforms under elements of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ as

$$g\left(\frac{a\tau+b}{c\tau+d}, \frac{au+b}{cu+d}\right) = (c\tau+d)^k (cu+d)^{2-k+\ell} g(\tau, u), \tag{43}$$

for some k and ℓ, the function $g_{k-\ell}^*$ transforms as,

$$g_{k-\ell}^*\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar\tau+b}{c\bar\tau+d}\right) = (c\tau+d)^k \left(g_{k-\ell}^*(\tau,\bar\tau) + 2^{1-k} i \int_{d/c}^{i\infty} \frac{g(\tau,-v)}{(-i(v+\tau))^{k-\ell}} dv \right). \tag{44}$$

We then define following [79, 78]:

Definition. A mock modular form of weight k is a holomorphic q-series $f : \mathbb{H} \to \mathbb{C}$, such that its completion,

$$\widehat{f}(\tau, \bar\tau) := f(\tau) + g_k^*(\tau, \bar\tau), \tag{45}$$

1. transforms as a modular form of weight k,
2. g_k^* is the image under the map (40) of the complex conjugate of a modular form with weight $2-k$,
3. The limit (34) is bounded for f and all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$.

Since $g_k^*(\tau+1, \bar\tau+1) = g_k^*(\tau, \bar\tau)$ by (44), the holomorphic part is also periodic $f(\tau+1) = f(\tau)$, such that f has a Fourier expansion.

There are many variations to the above definitions. One variation is a mixed mock modular form as for example studied in [21, 11]. A mixed mock modular form is a function f as above but with g a product of a modular form of weight k and the complex conjugate of a modular form of weight $2-k+\ell$ for some k and ℓ, such that

$$\widehat{f} = f + g^*_{k-\ell}. \tag{46}$$

Similarly, one can consider g which are sums of products.

Clearly, the function g is crucial information to characterize the mock modular form f. It is called the *shadow* of f, and can be obtained by taking a non-holomorphic derivative of $\widehat{f}$,

$$g := y^{k-\ell}\partial_{\bar{\tau}}\widehat{f}, \tag{47}$$

with $y = \mathrm{Im}(\tau)$. Thus the shadow is an element of $\overline{M}_{2-k+\ell}(\Gamma)$ for some group $\Gamma \subset SL_2(\mathbb{Z})$, that is to say the shadow is the complex conjugate of a modular form of weight $2-k+\ell$. For $\ell = 0$, Eq. (47) demonstrates that $\widehat{f}$ is a harmonic Maass form [11], which are functions annihilated by the hyperbolic Laplacian Δ_k for weight k,

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{48}$$

As a result, f is the holomorphic part of a harmonic Maass form in this case.

A further interesting extension of mock modular forms are those of depth $r \geq 1$. Mock modular forms of depth 1 are the functions defined in the previous definition and equation (45). The notion of mock modular forms of depth r is defined iteratively for $r \geq 1$ as follows [64, 11, 56]. To this end, we introduce the vector spaces $\mathbb{M}^r_k(\Gamma)$ of mock modular forms of depth $r \geq 1$, with weight k for the group $\Gamma \subset SL_2(\mathbb{Z})$. We furthermore let $\mathbb{M}^{-1}_k(\Gamma) = \emptyset$, and $\mathbb{M}^0_k(\Gamma) = M_k(\Gamma)$, i.e. the space of standard modular forms of weight k introduced in the beginning of the section. We furthermore introduce the vector spaces $\widehat{\mathbb{M}}^r_k$ of the non-holomorphic completions of the functions in $\mathbb{M}^r_k(\Gamma)$. We then have the following definition:

Definition. A mock modular form of depth r and with weight k for the group $\Gamma \subset SL_2(\mathbb{Z})$ is defined by the property that its shadow is an element of $\widehat{\mathbb{M}}^{r-1}_k \otimes \overline{M}_{2-k+\ell}$.

For $r > 1$, these functions are typically not annihilated by Δ_k and are therefore not examples of harmonic Maass forms.

We introduce next the notion of a mock cusp form.

Definition. A mock cusp form of depth r is a mock modular form $f(\tau)$ of depth r such that the limit (34) vanishes for all elements of $SL_2(\mathbb{Z})$.

Since f has a Fourier expansion, the constant term of the Fourier series for $\tau \to i\infty$ of a mock cusp form vanishes.

While for $r = 0$ this definition reduces to the classical notion of cusp form, for $r = 1$, this definition is slightly weaker than the notion of Maass cusp form, which requires that $\widehat{f}$ is square integrable on $\mathbb{H}/\Gamma$. Indeed, the function S_μ we will study in the next section is not square integrable on $\mathbb{H}/\Gamma$. A related property is that f may not have a Fourier expansion near other cusps of $\mathbb{H}/\Gamma$.

The notion of mock modular form and mock cusp form are readily extended to vector-valued mock modular forms and vector-valued mock cusp forms.

3.2. VW partition functions and (mock) modular forms

With the terminology developed in above, we can characterise the functions appearing Section 2.2. It is well-known that the Dedekind eta function (18) is a modular form of weight 1/2 with $\varepsilon(T) = e^{2\pi i/24}$ and $\varepsilon(S) = \varepsilon(T)^{-3} = e^{-2\pi i/8}$. Furthermore for $\mu = 0, 1$, $\Theta_{\mu/2}$ are modular forms of weight 1/2 for the congruence subgroup $\Gamma_0(4)$. The vector $\vec{\Theta}$ of the two functions,

$$\vec{\Theta}(\tau) = \begin{bmatrix} \Theta_0(\tau) \\ \Theta_{1/2}(\tau) \end{bmatrix} \tag{49}$$

is a vector-valued modular form for $SL_2(\mathbb{Z})$.

The vector $\vec{f_2}$ of the functions $f_{2,\mu}, \mu = 0, 1$ (26),

$$\vec{f_2}(\tau) = \begin{bmatrix} f_{2,0}(\tau) \\ f_{2,1}(\tau) \end{bmatrix}, \tag{50}$$

is a vector-valued mock modular form of weight 3/2 and depth 1.

Moreover, the vector $\vec{f_3}$ of the functions $f_{3,\mu}$, $\mu = 0, 1, 2$ (31) is a vector-valued mock modular form of weight 3 and depth 2 [56]. To further study these functions, we first introduce the cubic theta series [9]

$$b_{3,\mu}(\tau) = \sum_{k_1,k_2 \in \mathbb{Z}+\mu/3} q^{k_1^2+k_2^2+k_1k_2}. \tag{51}$$

The modular transformations of these functions are:

$$\begin{aligned} b_{3,\mu}\left(-\frac{1}{\tau}\right) &= -\frac{i\tau}{\sqrt{3}} \sum_{\nu=0}^{2} e^{-2\pi i\mu\nu/3} b_{3,\nu}(\tau), \\ b_{3,\mu}(\tau+1) &= e^{2\pi i\mu^2/3} b_{3,\mu}(\tau). \end{aligned} \tag{52}$$

We then form the 3-dimensional vector-valued modular form $\vec{m} : \mathbb{H} \to \mathbb{C}^3$,

$$\vec{m} = \begin{bmatrix} m_0 \\ m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} \frac{b_{3,0}^3 + 2b_{3,1}^3}{9} \\ \frac{b_{3,0} b_{3,1}^2}{3} \\ \frac{b_{3,0} b_{3,1}^2}{3} \end{bmatrix}, \tag{53}$$

The transformations of m_μ under the generators of $SL_2(\mathbb{Z})$ are given by,

$$\begin{aligned} m_\mu\left(-\frac{1}{\tau}\right) &= \frac{i\tau^3}{\sqrt{3}} \sum_{\nu=0}^{2} e^{-2\pi i \nu\mu/3} m_\nu(\tau), \\ m_\mu(\tau+1) &= (-1)^\mu e^{\pi i \mu^2/3} m_\mu(\tau). \end{aligned} \tag{54}$$

These transformations are identical to those of the completed mock modular form $\widehat{f}_{3,\mu}$ (30) obtained from $SU(3)$ Vafa-Witten theory. There is no analog of this type of purely holomorphic functions for $SU(2)$, since a holomorphic modular form with the same transformations as $\widehat{f}_{2,\mu}$ does not exist [51].

The first few terms in the series expansion of m_μ can be given by,

$$\begin{aligned} m_0(\tau) &= \frac{1}{9} + 8\,q + 30\,q^2 + \cdots \\ m_1(\tau) &= 3\,q^{2/3} + 24\,q^{5/3} + 51\,q^{8/3} + \cdots \end{aligned} \tag{55}$$

We can express m_μ in terms of weight 3 Eisenstein series for $\Gamma(3)$ [25], from which we can derive the q-series of $m_0(\tau)$ and $m_1(\tau)$ explicitly as:

$$\begin{aligned} m_0(\tau) &= \frac{1}{9} + \sum_{\substack{n=0 \bmod 3 \\ n>0}} \sum_{d|n} \chi_{n,d}\, d^2 q^{n/3}, \\ m_1(\tau) &= \sum_{n=2 \bmod 3} \sum_{\substack{\frac{n}{d}=1 \bmod 3 \\ d\in\mathbb{Z}}} \operatorname{sgn}(d)\, d^2 q^{n/3}, \end{aligned} \tag{56}$$

where, the character $\chi_{n,d}$ is given for $n > 0$ by,

$$\chi_{n,d} = \begin{cases} -1, & \text{if } d = 1 \bmod 3,\ n/d = 0 \bmod 3, \\ 1, & \text{if } d = 2 \bmod 3,\ n/d = 0 \bmod 3, \\ 1, & \text{if } d = 0 \bmod 3,\ n/d = 1 \bmod 3, \\ -1, & \text{if } d = 0 \bmod 3,\ n/d = 2 \bmod 3, \\ 0, & \text{otherwise.} \end{cases} \tag{57}$$

As a result, the coefficient of $q^{p/3}$ of $m_1(\tau)$ with p a prime is given by p^2-1. The coefficients $d_0(p)$ of q^p of $m_0(\tau)$ with p prime are of two types,

1. For $p = 2 \bmod 3$,

$$d_0(p) = 10p^2 - 10, \tag{58}$$

2. For $p = 1 \bmod 3$,

$$d_0(p) = 8p^2 + 8. \tag{59}$$

Having described $\vec{m}$, we can obtain an example $\vec{S}$ of a vector-valued mock cusp form of depth 2. Namely, we define $\vec{S}$ as

$$\vec{S} = \frac{1}{3}(\vec{m} - \vec{f}_3). \tag{60}$$

The first terms in the q-series of S_0 and S_1 are:

$$\begin{aligned} S_0(\tau) &= 3\,q + 9\,q^2 + 21\,q^3 + \cdots\,, \\ S_1(\tau) &= S_2(\tau) = q^{2/3} + 7\,q^{5/3} + \cdots\,. \end{aligned} \tag{61}$$

The constant terms of S_μ thus vanish as for the classical cusp forms. More terms are listed in Table 2 in Section 4.

From Eq. (32) and (54), it follows that the elements S_μ of $\vec{S}$ transform as

(62)
$$\begin{aligned} S_\mu(-1/\tau) &= \frac{i\tau^3}{\sqrt{3}} \sum_{\nu=0}^{2} e^{-2\pi i\mu\nu/3} \\ &\quad \times \left(S_\nu(\tau) + \frac{\sqrt{3}i}{2\sqrt{2}\pi} \sum_{\alpha=0,1} \int_0^{i\infty} \frac{\widehat{f}_{2,\alpha}(\tau,-w)\,\Theta_{\frac{2\nu+3\alpha}{6}}(3w)}{(-i(w+\tau))^{3/2}}\, dw \right), \\ S_\mu(\tau+1) &= (-1)^\mu e^{\pi i\mu^2/3} S_\mu(\tau). \end{aligned}$$

Since

$$\lim_{\tau\to\infty} \int_0^{i\infty} \frac{\widehat{f}_{2,\alpha}(\tau,w)\,\Theta_{\frac{2\nu+3\alpha}{6}}(3w)}{(-i(w+\tau)^{3/2})}\, dw = 0, \tag{63}$$

the limit (34) vanishes for S_μ. We thus confirm that S_μ are mock cusp forms for the congruence subgroup $\Gamma(3) \subset SL_2(\mathbb{Z})$. It would be interesting to understand better the spaces of such functions, and for example define a suitable inner product. The standard inner product (36) diverges for functions such as $\widehat{S}_\mu(\tau,\bar{\tau})$, and a suitable regularization will need to be defined. For different types of functions, such regularizations have been developed in the literature before [41, 8, 10].

We have some interesting observations for the growth of the Fourier coefficients of these functions. These are shown as plots in the next section. In Section 5 we briefly discuss the behavior of the growth of coefficients of cusp forms and how they might change when the there is a non-holomorphic piece in the modular transformation.

4. Numerical experiments for the coefficients of VW partition functions

This section carries out varies numerical experiments on the coefficients of $f_{2,\mu}$, $f_{3,\mu}$, m_μ and S_μ. Especially for S_μ we find some intriguing pattern, namely the growth of the coefficients for prime powers of q appear to be well approximated by a constant times $p^{3/2}$.

4.1. Coefficients for $N = 2$

We study in this subsection the coefficients of $f_{2,\mu} = 3G_\mu$. Since the coefficients of G_μ are the well-known Hurwitz class numbers, we will focus on these. We define for $\mu = 0, 1$:

$$G_\mu(\tau) = \sum_{n\in\mathbb{Z}-\frac{\mu}{4}} a_\mu(n)q^n. \tag{64}$$

We first tabulate the coefficients for of G_μ up to $n = 45$ in Table 1.

We observe from the table that the behavior of the coefficients as function of n is not monotonic, but that the coefficients grow on average. To get a better impression, we plot the coefficients up to $n = 400$ in Figure 1. The distribution of the coefficients appears chaotic and highly scattered, but on average the coefficients appear to grow as a power law.

Now in many cases, such as Eisenstein series of integer weight, the growth of the coefficients is more regular for prime numbers. This we for example encountered for the functions m_μ (56). We therefore present plots for $a_\mu(p)$ as function of prime p in Figures 2 and 3.

Table 1: First few coefficients $a_\mu(n) = H(4n - \mu)$ of the class number generating functions G_μ (93)

n	$a_0(n)$	$a_1(n-1/4)$
0	$-\frac{1}{12}$	0
1	$\frac{1}{2}$	$\frac{1}{3}$
2	1	1
3	$\frac{4}{3}$	1
4	$\frac{3}{2}$	2
5	2	1
6	2	3
7	2	$\frac{4}{3}$
8	3	3
9	$\frac{5}{2}$	2
10	2	4
11	4	1
12	$\frac{10}{3}$	5
13	2	2
14	4	4
15	4	3
16	$\frac{7}{2}$	5
17	4	1
18	3	7
19	4	$\frac{7}{3}$
20	6	5
21	4	3
22	2	6
23	6	2
24	6	8
25	$\frac{5}{2}$	3
26	6	5
27	$\frac{16}{3}$	3
28	4	8
29	6	2
30	4	10
31	6	2
32	7	5
33	4	5
34	4	8
35	8	3
36	$\frac{15}{2}$	10
37	2	$\frac{7}{3}$
38	6	7
39	8	4
40	6	10
41	8	1
42	4	11
43	4	5
44	10	7
45	6	5

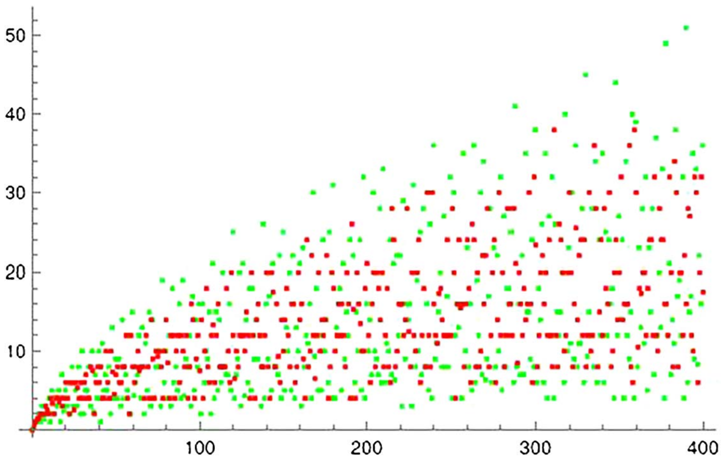

Figure 1: Plot of the coefficients $a_\mu(n) = H(4n - \mu)$ of G_μ as function of n. The red dots represent the coefficients $a_0(n)$, while the green ones represent the coefficients $a_1(n - 1/4)$.

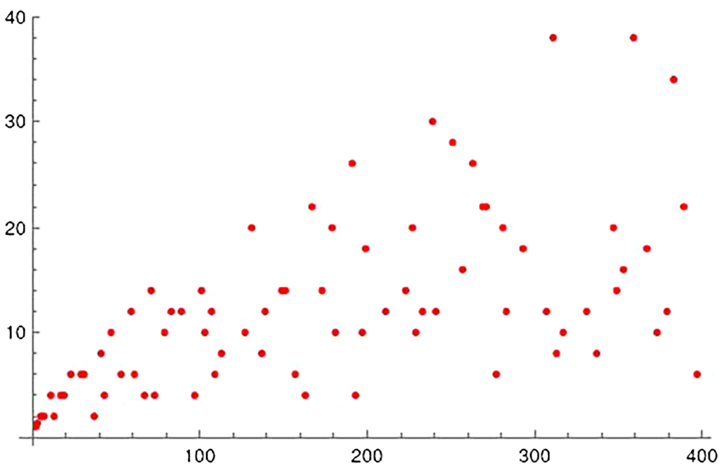

Figure 2: Diagram with the coefficients $a_0(p)$ as function of prime p.

4.2. Coefficients for $N = 3$

We proceed in this subsection with partition functions for $SU(3)$, and compare our findings with those for $N = 2$. We define the coefficients $c_\mu(n)$,

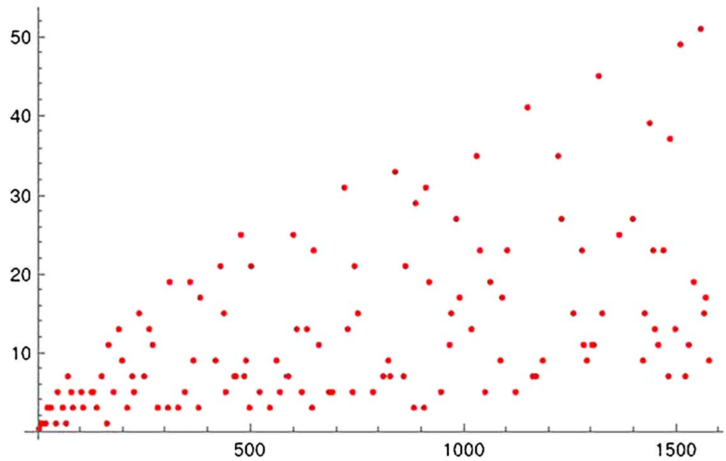

Figure 3: Diagram with the coefficients $a_1(p/4)$ as function for prime p of type $4n+3$.

$d_\mu(n)$ and $s_\mu(n)$ through

$$\begin{aligned} f_{3,\mu}(\tau) &= \sum_{n\in\mathbb{Z}-\frac{\mu}{3}} c_\mu(n)\, q^n, \\ m_\mu(\tau) &= \sum_{n\in\mathbb{Z}-\frac{\mu}{3}} d_\mu(n)\, q^n, \\ S_\mu(\tau) &= \sum_{n\in\mathbb{Z}-\frac{\mu}{3}} s_\mu(n)\, q^n. \end{aligned} \tag{65}$$

First we list the coefficients of $f_{3,\mu}$ and S_μ in Table 2. Since S_μ is a mock cusp form, we expect that the coefficients are smaller than those of $f_{3,\mu}$ for $n \gg 1$. We observe from the table that this is indeed the case.

To get a better impression of the growth, we plot the coefficients of the functions for a larger range, starting with f_0 in Figure 4. We observe from the table that for the vast majority of the coefficients their magnitude alternates between even and odd n. More precisely, for $\mu = 0$ and $16 \leq 2n \leq 300$,

$$c_0(2n) < c_0(2n-1), \qquad c_0(2n) < c_0(2n+1), \tag{66}$$

we find only two exceptions, namely $c(236) = 397\,644 > 393\,048 = c(235)$ and $c(296) = 629\,850 > 624\,582 = c(295)$.

Table 2: First few coefficients of the mock modular forms $f_{3,\mu}$ and the mock cusp forms S_μ

n	$c_0(n)$	$c_1(n-\frac{1}{3})$	$s_0(n)$	$s_1(n-\frac{1}{3})$
0	$\frac{1}{9}$	0	0	0
1	-1	0	3	1
2	3	3	9	7
3	17	15	21	12
4	41	36	21	28
5	78	69	54	27
6	120	114	42	58
7	193	165	69	49
8	240	246	90	94
9	359	303	123	69
10	414	432	54	136
11	579	492	207	109
12	626	669	138	177
13	856	726	168	120
14	906	975	198	235
15	1194	999	258	187
16	1172	1332	156	292
17	1638	1338	414	155
18	1569	1743	207	355
19	1987	1716	303	278
20	2040	2226	360	418
21	2578	2130	474	252
22	2340	2775	180	435
23	3255	2625	675	373
24	2940	3354	414	562
25	3665	3129	381	327
26	3642	4041	486	653
27	4490	3735	690	395
28	3940	4752	420	712
29	5484	4317	972	411
30	4734	5532	342	796
31	5815	5070	627	598
32	5814	6393	792	765
33	7014	5694	942	553
34	5832	7317	360	961
35	8274	6582	1242	696
36	7115	8277	783	1057
37	8566	7272	798	456
38	8322	9345	846	1141
39	10018	8325	1194	865
40	8334	10425	486	1325
41	11778	9087	1674	693
42	9708	11541	864	1161
43	11785	10281	1005	942
44	11604	12855	1332	1435
45	13614	11058	1302	804
46	10998	14175	558	1531
47	15843	12327	2079	1091
48	13178	15486	1074	1638
49	15531	13263	1359	909
50	14817	16959	1071	1747

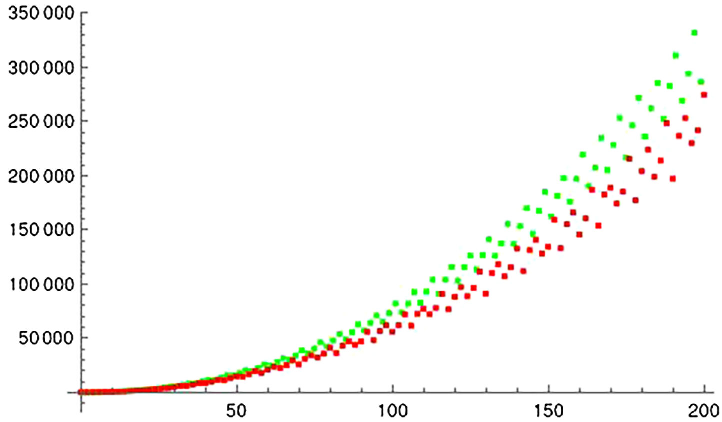

Figure 4: Green dots represent the coefficients $c_0(n)$ for odd n while the red ones represent the coefficients $c_0(n)$ for even n.

We therefore distinguish the even and odd coefficients with red and green respectively. We observe from the plot in Figure 4 that the even and odd coefficients remain separate. More precisely, it appears that for a sufficiently large n, there exists a monotic function $w(x)$ such that

$$c_0(2n) < w(n) < c_0(2n+1). \tag{67}$$

This behavior is not unique to $f_{3,0}$. Although not as strict, Figure 5 demonstrates a similar behavior for the coefficients of m_0.

We have seen in Section 3.2 that the growth of the coefficients of m_μ for prime powers of q has a simple form. See Eqs. (58) and (59). This is a general property of Eisenstein series. Let us therefore plot the prime coefficients $d_0(p)$ for m_0, and $c_0(p)$ for $f_{3,0}$. We observe that the plot is far less scattered than the original plots for both m_0 and $f_{3,0}$. We also observe that the growth in this range is roughly comparable. We will derive an upper bound for the growth in the next section, as well as discuss an average.

Next we consider the coefficients of the mock cusp form, $S_0(\tau) = \frac{1}{3}(m_0 - f_{3,0})$, which are possibly of the most interest. Its coefficients $s_\mu(n)$ are plotted in Figure 7, again distinguishing even and odd n. We observe that coefficients are more scattered than the coefficients of f_0 and m_0, and that the two sets of coefficients are not separated as was the case for $f_{3,0}$ or m_0. Still, the coefficients of odd powers of q are typically larger than those of even powers of q.

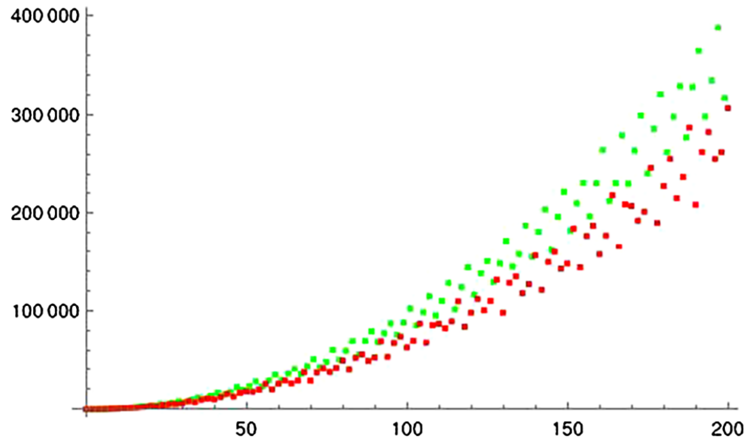

Figure 5: Green dots represent the coefficients $d_0(n)$ for odd n while the red ones represent the coefficients $d_0(n)$ for even n.

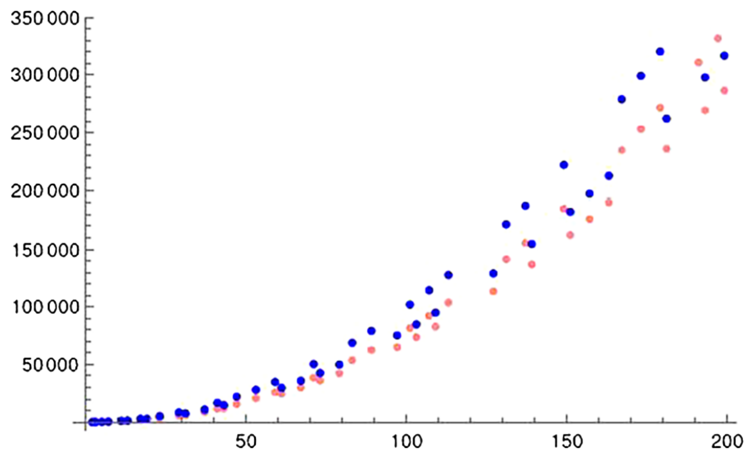

Figure 6: Blue dots represent the coefficients $d_0(p)$ of $m_0(\tau)$, and the pink ones represent the coefficients $c_0(p)$ of $f_{3,0}(\tau)$ for prime p.

As mentioned above, the behavior of the coefficients is in general better for the prime coefficients. We plot the coefficients $s_0(p)$ for prime p in Figure 8. Here we do see a striking behavior, namely that these points appear to lie on a smooth curve. Such a regular curve is not generally the case for classical cusp forms, such as η^{24}.

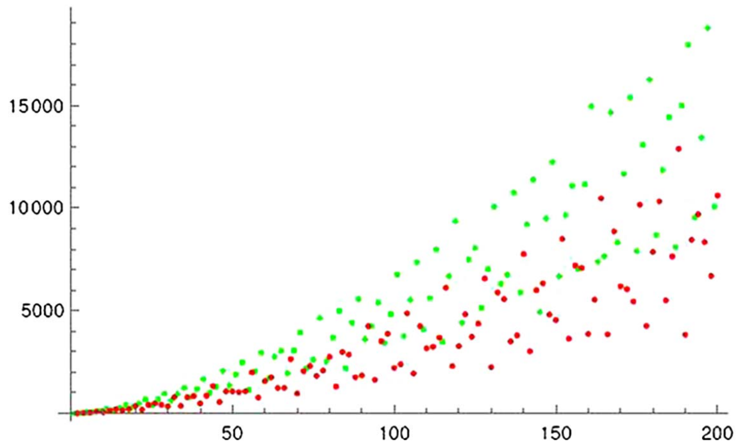

Figure 7: Green dots represent the coefficients $s_0(n)$ for odd n while the red ones represent the coefficients $s_0(n)$ for even n.

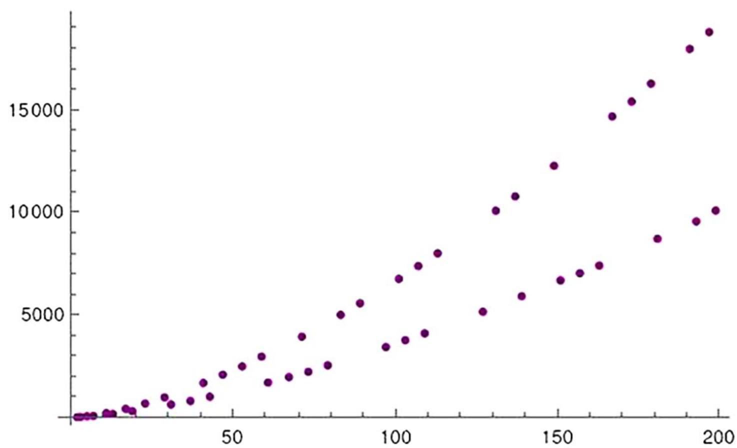

Figure 8: Plot of coefficients $s_0(p)$ for prime p. Note that from 11 up to 199 all twin primes show the following behavior, if p and $p+2$ are twin primes then $s_0(p) > s_0(p+2)$.

We make a curve fit in Figure 9. The least square fit in the last plot (except for $p = 3$), suggests that the coefficients grow as $s_0(p) \sim p^{3/2}$. Naturally, it is desirable to prove this growth. In Section 5, we give an upper bound, which is unfortunately much weaker.

Next we repeat the above plots for the functions with $\mu = 1$. Figure 10 plots the coefficient $c_1(n+2/3)$ of $f_{3,1}$. Similarly to Eq. (66), we checked

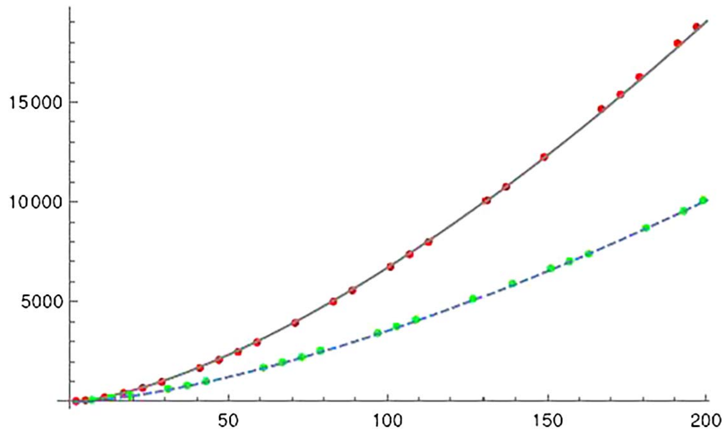

Figure 9: Least square fits for $s_0(p)$ with primes p of the form $3n \pm 1$. The red dots represent the coefficient $s_0(p)$ for prime $p = 3n-1$ and the grey line is given by $6.75467\, x^{3/2}$, the green dots represent $s_0(p)$ for prime $p = 3n+1$ and the dashed line is given by $3.57843\, x^{3/2}$.

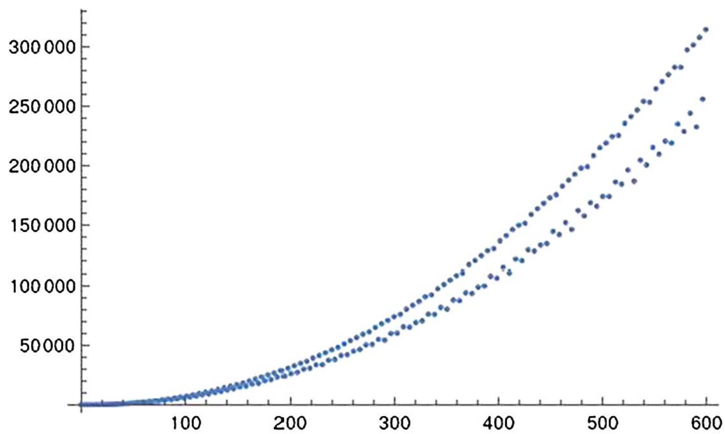

Figure 10: Plot of the coefficients $c_1(n+2/3)$ of $f_{3,1}$. The numbers on the horizontal axis are $3n+2$ for integer $n \leq 200$.

that in the window $18 \leq 2n \leq 300$,

$$c_1(2n+2/3) < c_1(2n-1/3), \qquad c_1(2n+2/3) < c_1(2n+5/3). \tag{68}$$

Similarly to the plot in Figure 4, we again observe that the points lie in

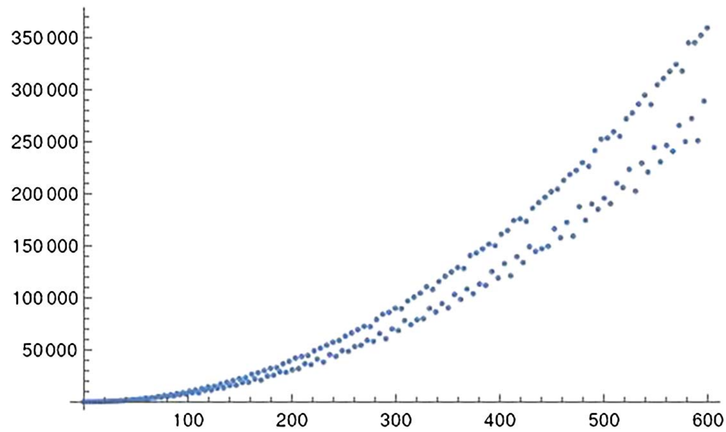

Figure 11: Plot of the coefficients $d_1(n+2/3)$ of m_1. The numbers on the horizontal axis are $3n+2$ for $n \leq 200$.

the neighborhood of a few smooth curves, three in this case. However, the coefficients for $f_{3,1}$ appear to follow the curves more closely than for $f_{3,0}$.

We continue with the plot for the coefficients $d_1(n+2/3)$ of m_1 in Figure 11. The plot is quite similar to Figures 5 and 10. The curves are quadratic since for $3n+2$ equal to a prime p, $d_1(p/3) = p^2 - 1$. The prime coefficients of $c_1(p/3)$ of $f_{3,1}$ are also well approximated by a parabolic curve.

Lastly, we consider the coefficients $s_1(n+2/3)$ of the mock cusp form S_1. We plot in Figure 12 the coefficients $s_1(n+2/3)$, and the least square fit for the prime coefficients $s_1(p/3)$. Similar to the coefficients of S_0, the least square fit for $p \leq 600$ suggests that the coefficients grow as $\sim p^{3/2}$. Based on this we conjecture that the leading term of $s_\mu(n)$ is $O(n^{3/2})$. This is notably larger than Deligne's bound $\sim n^1$ for the coefficients of proper cusp forms of weight 3. Thus the modified transformation of S_μ involving the period integral must have an important impact on the growth of the coefficients.

5. Asymptotics of Fourier coefficients

This section reviews the calculation of upper bounds on the growth of Fourier coefficients of modular forms and cusp forms using the saddle point method. We will then generalize this analysis to mock modular forms and cusp forms, in particular to the function $f_{2,\mu}$, $f_{3,\mu}$ and S_μ.

It is well known that the saddle point method (or Hecke bound) typically gives a very large upper bound, that is to say order n^k for modular forms

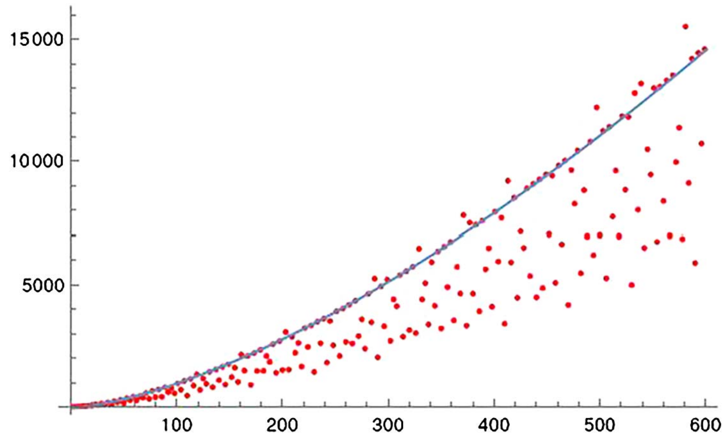

Figure 12: Plot of coefficients $s_1(n+2/3)$ with the numbers $3n+2$ on the horizontal axis. The least square fit for the prime coefficients $s_1(p/3)$ is given by $0.993519\,x^{3/2}$.

of weight k, whereas the correct bound is order n^{k-1}. For cusp forms the saddle point method gives $n^{k/2}$. On the other hand, the sharper Ramanujan-Petersson conjecture, proved by Deligne in [23, 24], states that the Fourier coefficients $a(n)$ of a normalized cuspidal Hecke eigenform with weight k for $SL_2(\mathbb{Z})$ are bounded by

$$|a(n)| < \sigma_0(n) n^{\frac{k-1}{2}}, \tag{69}$$

where $\sigma_0(n)$ is the number of divisors of n. Thus for prime numbers the bound is given by

$$|a(p)| < 2p^{\frac{k-1}{2}}.$$

Similarly, we do find that the saddle point analysis for mock modular and mock cusp forms gives rise to larger exponents than those observed in the previous section. We observe furthermore that the modified modular transformations involving the iterated period integral increases the exponent in the bound.

5.1. Bound for coefficients of modular forms and cusp forms

We review in this subsection the elementary saddle point analysis to determine a rough bound for modular forms. See for discussions in textbooks for example [4, Section 6.15] and [16, page 20]. Since we know the bound is not sharp, we will not be careful with errors to the bound.

Modular forms We first discuss the case of modular forms. This is relevant for the growth of the coefficients of f_N^X for (simply connected) algebraic surfaces with $b_2^+ > 1$ discussed in Section 2.2.

Let $f(\tau) = \sum_{n\geq 0} a(n)q^n$ with $n \in \mathbb{N}_0$ and $a(0) \neq 0$ be a modular form of weight k. The transformation of f under S is given by,

$$f\left(-\frac{1}{\tau}\right) = \varepsilon(S)\,\tau^k\, f(\tau), \tag{70}$$

with $\varepsilon(S)$ a phase as discussed in Section 3.

The n-th Fourier coefficient is by definition

$$a(n) = \int_0^1 f(\tau)\, e^{-2\pi i n\tau}\, d\tau. \tag{71}$$

Its magnitude is therefore bounded by

$$\begin{aligned} |a(n)| &\leq \int_0^1 |f(\tau)\, e^{-2\pi i n\tau}|\, d\tau \\ &= \int_0^1 \left| f\left(-\frac{1}{\tau}\right)\, \tau^{-k} e^{-2\pi i n\tau}\right|\, d\tau. \end{aligned} \tag{72}$$

We then apply the saddle point method to the integrand. We assume that $\mathrm{Im}(-1/\tau) \gg 0$ at the saddle point. Then the higher order terms in the Fourier expansion of f are exponentially suppressed compared to the constant term, $a(0)$. Neglecting these non-constant terms, one finds for the saddle point $\tau = \tau_s$,

$$-\frac{k}{\tau_s} = 2\pi i n, \qquad \tau_s = \frac{ik}{2\pi n}. \tag{73}$$

Thus the assumption is indeed satisfied for sufficiently large n.

Substitution of τ_s in (72) gives as upper bound for $a(n)$,

$$|a(n)| < c\, n^k. \tag{74}$$

for the constant $c = |a(0)|\left(\frac{2\pi e}{k}\right)^k$. This is clearly not a sharp bound, since we know from the discussion on Eisenstein series that the bound for modular forms is proportional to p^{k-1} for prime coefficients $a(p)$.

Cusp forms We use the same approach to bound the coefficients of a cusp form

$$h(\tau) = \sum_{n\geq\Delta>0} a(n)q^n,$$

of weight k, gives $c\,n^{k/2}$ for some constant c for the bound of its Fourier coefficients. We assume the following transformation property for $h(\tau)$ under S,

$$h\left(-\frac{1}{\tau}\right) = \varepsilon(S)\,\tau^k h(\tau), \tag{75}$$

for a phase $\varepsilon(S)$. Assuming again that $\mathrm{Im}(-1/\tau) \gg 0$ at the saddle point, we can approximate $S(-1/\tau)$ by its first non-vanishing term $a(\Delta)\,e^{-2\pi i\Delta/\tau}$. We then have

$$|a(n)| \leq \int_0^1 \left|f(\tau)e^{-2\pi i n\tau}\right|\,d\tau \approx \int_0^1 \left|\tau^{-k}a(\Delta)\,e^{-2\pi i\Delta/\tau}e^{-2\pi i n\tau}\right|\,d\tau. \tag{76}$$

Extremization of $-k\log(\tau) - 2\pi i\Delta/\tau - 2\pi i n\tau$, gives for the saddle point,

$$\tau_s = \frac{-k \pm \sqrt{k^2 - 16\pi^2\Delta n}}{4\pi i n} = \frac{-k \pm i\sqrt{\delta_n}}{4\pi i n}, \tag{77}$$

where we introduced $\delta_n = 16\pi^2\Delta n - k^2$. In the regime of interest i.e., large n, $\delta_n > 0$, and the assumption $\mathrm{Im}(-1/\tau_s) \gg 0$ is satisfied. We have for the magnitude

$$|\tau_s|^2 = \frac{\Delta}{n}. \tag{78}$$

Substitution of τ_s in the exponents gives,

$$e^{-2\pi i\Delta/\tau_s}e^{-2\pi i n\tau_s} = e^{\mp i\delta_n}. \tag{79}$$

This has unit magnitude and thus does not contribute to the magnitude of the integrand. We therefore find for the bound,

$$|a(n)| \leq c\,n^{k/2}, \tag{80}$$

with $c = \Delta^{-k/2}|a(\Delta)|$. This bound is known as the Hecke bound. We note that this bound is less accurate than the bound proven by Deligne (69).

5.2. Bound for coefficients of mock cusp forms

Next we proceed with the mock cusp form S_μ (60). From Eq. (62), we have the the following relations under the S-transformation

$$S_0(\tau) = \frac{-i\tau^{-3}}{\sqrt{3}}\left(S_0(-1/\tau) + 2S_1(-1/\tau) + \mathcal{J}_0(-1/\tau)\right), \tag{81}$$
$$S_1(\tau) = \frac{-i\tau^{-3}}{\sqrt{3}}\left(S_0(-1/\tau) - S_1(-1/\tau) + \mathcal{J}_1(-1/\tau)\right),$$

where,

$$\mathcal{J}_0(\tau) = \frac{\sqrt{3}i}{2\sqrt{2}\pi}\sum_{\nu=0}^{2}\sum_{\alpha=0}^{1}\int_0^{i\infty}\frac{\widehat{f}_{2,\alpha}(\tau,w)\Theta_{\frac{2\nu+3\alpha}{6}}(3w)}{(-i(w+\tau))^{3/2}}\,dw, \tag{82}$$
$$\mathcal{J}_1(\tau) = \frac{\sqrt{3}i}{2\sqrt{2}\pi}\sum_{\nu=0}^{2}\sum_{\alpha=0}^{1}e^{-2\pi i\nu/3}\int_0^{i\infty}\frac{\widehat{f}_{2,\alpha}(\tau,w)\Theta_{\frac{2\nu+3\alpha}{6}}(3w)}{(-i(w+\tau))^{3/2}}\,dw.$$

Similarly to the discussion above, we have

$$|s_\mu(n)| \leq \int_0^1 \left|S_\mu(\tau)\,e^{-2\pi i n\tau}\right|d\tau. \tag{83}$$

We then have to determine the leading contribution among the terms on the right hand side of (81). For large $\mathrm{Im}(-1/\tau)$, the leading terms of $S_\mu(-1/\tau)$ follow from (61). These are exponentially decreasing. To compare these with $\mathcal{J}_\mu$, we recall Lemma 3.1 in [13], (see also [12])

$$\int_0^\infty \frac{\Theta_{\alpha/2}(iz)}{(z+x)^{3/2}}dz = \frac{2}{\sqrt{x}}\delta_{\alpha,0} + O(x^{-3/2}), \tag{84}$$

with the leading term coming from the constant term of Θ_0. As a result, we have

$$\mathcal{J}_\mu(\tau) = -\frac{\sqrt{3}i}{2\sqrt{2}\pi}\frac{1}{4}\frac{2}{\sqrt{-i\tau}} + O(\tau^{-1}). \tag{85}$$

Thus the contribution from $\mathcal{J}_\mu(-1/\tau)$ provides the leading term to the right hand side in (81). We thus have for the coefficients

$$|s_\mu(n)| \leq \frac{1}{4\sqrt{2}\pi}\int_0^1 \left|\tau^{-5/2}e^{-2\pi i n\tau}\right|d\tau. \tag{86}$$

Using the saddle point method, we then arrive at

$$|s_\mu(n)| \le c\, n^{5/2}, \tag{87}$$

with $c = \frac{1}{4\sqrt{2}\pi}(4\pi e/5)^{5/2}$. This upper bound agrees with the numerical experiments, in the sense that the numerics suggest a growth proportional to $n^{3/2}$, which is clearly much smaller than $n^{5/2}$. We note that for a classical cusp form terms like $\mathcal{J}_\mu$ are absent in the modular transformation, and the saddle point method (87) gives $n^{3/2}$ for the growth.

5.3. Average growth for theta series

If more is known about the arithmetic nature of the coefficients, sharper bounds can often be obtained than the saddle point method. This is for example the case of Eisenstein series and cusp forms. Often the modular forms can also be expressed as lattice sum or theta series.

We give a heuristic argument (probably well-known to many) that the leading term of the average growth of the coefficients of a theta series equals that of Eisenstein series. This should of course be the case since all theta series which transform as modular forms can be expressed as linear combinations of Eisenstein series. To make the argument, let us consider a d-dimensional positive definite lattice L with integral quadratic form Q. The associated theta series $\Theta_L(\tau)$ and coefficients $d(n)$ are defined through,

$$\begin{aligned}\Theta_L(\tau) &= \sum_{k\in L} q^{Q(k)/2}\\ &= \sum_{n\ge 0} d(n)\, q^n.\end{aligned} \tag{88}$$

We also introduce the cumulative sum $D(N)$ as,

$$D(N) = \sum_{0\le n\le N} d(n), \tag{89}$$

such that

$$d(N) = D(N) - D(N-1). \tag{90}$$

For a theta series, $D(N)$ is a count of lattice points and thus scales as the volume of the domain in L, whose lattice points are enumerated by $D(N)$. The volume scales on average as $|k|^d = |Q(k)|^{d/2} = N^{d/2}$, such that

$d(N) = D(N) - D(N-1)$ scales as $N^{(d-2)/2}$. Since the weight k of Θ_L is $k = d/2$, we find for the leading term

$$d(n) \approx C\, n^{k-1}. \tag{91}$$

We can use the same approximation for indefinite theta series, which involve a sum over a positive definite cone in an indefinite lattice.[2] The class number generating function G_μ can be expressed in this form. Moreover, the expression of Kool [48] for $f_{3,1}$ are of this form. This matches our observation that the coefficients of $f_{3,\mu}$ grow as n^2, similar to the Eisenstein series of weight 3. This rough estimate for the coefficients of $f_{3,\mu}$ is even sharper than the bound for the coefficients of S_μ using the saddle point method (87).

Appendix A. Explicit expressions for $f_{N,\mu}$ for $N = 2, 3$

Explicit expressions for generating functions of class numbers have been determined starting with work of Kronecker, Mordell and Watson [49, 61, 72]. The generating function G reads:

$$G(\tau) = \sum_{n\geq 0} H(n)\, q^n = -\frac{1}{2\Theta_0(\tau+1/2)} \sum_{n\in\mathbb{Z}} \frac{n(-1)^n q^{n^2}}{1+q^{2n}} - \frac{1}{12}\Theta_0^3(\tau), \tag{92}$$

where Θ_α as in (27). Generating functions of the arithmetic progressions $0, 3$ mod 4 read

$$\begin{aligned} G_1(\tau) &= \sum_{n\geq 0} H(4n+3)\, q^{n+3/4} \\ &= -\frac{q^{-1/4}}{2\Theta_0(\tau)} \sum_{n\in\mathbb{Z}} \frac{(2n-1)q^{n^2}}{1-q^{2n-1}} + \frac{1}{6}\Theta_{1/2}^3(\tau), \\ G_0(\tau) &= \sum_{n\geq 0} H(4n)\, q^n \\ &= G(\tau/4) - G_1(\tau), \end{aligned}$$

The explicit expressions for $f_{3,\mu}$ as expansions in q series were given in [56], which are quoted as follows:

[2] It is important that the boundaries of the positive definite cone are strictly positive definite (except the tip at the origin of the lattice). Otherwise, the lattice sum may not converge absolutely [79].

(93)

$$
\begin{aligned}
b_{3,0}(\tau) f_{3,0}(\tau) = {}& \frac{13}{240} + \frac{E_2(\tau)}{24} + \frac{E_2(\tau)^2}{72} + \frac{E_4(\tau)}{720} \\
& - \frac{9}{2} \sum_{k\in\mathbb{Z}} k^2 q^{3k^2} + \sum_{k_1,k_2\in\mathbb{Z}} (k_1+2k_2)^2 q^{k_1^2+k_2^2+k_1k_2} \\
& + \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} S_{1,0}(k,q)\, q^{3k^2} \\
& + \sum_{\substack{k_1,k_2\in\mathbb{Z}\\ 2k_1+k_2\neq 0\\ k_2-k_1\neq 0}} S_2(2k_1+k_2, k_2-k_1, q)\, q^{k_1^2+k_2^2+k_1k_2+2k_1+k_2},
\end{aligned}
$$

(94)

$$
\begin{aligned}
b_{3,0}(\tau) f_{3,1}(\tau) = {}& \sum_{k\in\mathbb{Z}} S_{1,1}(k,q)\, q^{3k^2-1/3} \\
& + \sum_{\substack{k_1,k_2\in\mathbb{Z}\\ 2k_1+k_2\neq 1\\ k_2-k_1\neq 0}} S_2(2k_1+k_2-1, k_2-k_1, q)\, q^{k_1^2+k_2^2+k_1k_2+2k_1+k_2-1/3},
\end{aligned}
$$

where we have,

(95)

$$
\begin{aligned}
S_{1,\mu}(k,q) = {}& \frac{(E_2(\tau)-1)(k-\mu+1)}{2(1-q^{3k-\mu})} + \frac{9(k-\mu)^2+33(k-\mu)+31-E_2(\tau)}{2(1-q^{3k-\mu})^2} \\
& - \frac{15(k-\mu)+34}{(1-q^{k-\mu})^3} + \frac{19}{(1-q^{3k-\mu})^4}, \\
S_2(a,b,q) = {}& \frac{4q^b}{(1-q^a)(1-q^b)^3} + \frac{4q^a}{(1-q^b)(1-q^a)^3} + \frac{4}{(1-q^a)^2(1-q^b)^2} \\
& - \frac{2q^b(a+b+1)}{(1-q^a)(1-q^b)^2} - \frac{2q^a(a+b+1)}{(1-q^b)(1-q^a)^2} + \frac{(a+b-2)^2-8}{(1-q^a)(1-q^b)}.
\end{aligned}
$$

The blow-up formula provides relations among different representations for the q-series of $f_{3,\mu}$ [14].

The Eisenstein series E_2 and E_4 are given by,

(96) $$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n, \quad E_4(\tau) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n) q^n$$

with $\sigma_k(n) = \sum_{d|n} d^k$.

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