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Rankin-Cohen brackets for Calabi-Yau modular forms

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For any positive integer n, we introduce a modular vector field R on a moduli space T of enhanced Calabi-Yau n-folds arising from the Dwork family. By Calabi-Yau quasi-modular forms associated to R we mean the elements of the graded $\mathbb{C}$-algebra $\widetilde{\mathscr{M}}$ generated by solutions of R, which are provided with natural weights. The modular vector field R induces the derivation $\mathscr{R}$ and the Ramanujan-Serre type derivation ∂ on $\widetilde{\mathscr{M}}$. We show that they are degree 2 differential operators and there exists a proper subspace $\mathscr{M} \subset \widetilde{\mathscr{M}}$, called the space of Calabi-Yau modular forms associated to R, which is closed under ∂. Using the derivation $\mathscr{R}$, we define the Rankin-Cohen brackets for $\widetilde{\mathscr{M}}$ and prove that the subspace generated by the positive weight elements of $\mathscr{M}$ is closed under the Rankin-Cohen brackets. We find the mirror map of the Dwork family in terms of the Calabi-Yau modular forms.

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1. Introduction

The proof of Fermat's last theorem led to the celebrated modularity theorem, which states that elliptic curves over the field of rational numbers $\mathbb{Q}$ are related with modular forms. Elliptic curves are 1-dimensional Calabi-Yau (CY) varieties, which makes it natural to ask whether a similar statement of modularity holds for higher dimensional CY varieties. This question persuaded mathematicians and theoretical physicists to the subject of *modularity of CY manifolds* which is one of the considerable present challenges of the modern algebraic number theory. Some relevant results can be found, for instance, in [28] and the references therein. Yui in [28] divides the modularity of CY varieties in arithmetic modularity and geometric modularity

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including (1) the modularity (automorphy) of Galois representations of CY varieties (or motives) defined over $\mathbb{Q}$ or number fields, (2) the modularity of solutions of Picard-Fuchs differential equations of families of CY varieties, and mirror maps (mirror moonshine), (3) the modularity of generating functions of invariants counting certain quantities on CY varieties, and (4) the modularity of moduli for families of CY varieties. But so far, in a general context, even there is no unified formulation or statement of the modularity of CY varieties. Yamaguchi and Yau [27] in 2004 showed that the partition functions of topological string theory for the mirror quintic, which provide generating functions of higher genus Gromov–Witten invariants, can be expressed in terms of finitely many generators of a differential ring, and somehow they play the role of quasi-modular forms; then Alim and Lange [2] in 2007 generalized their results for arbitrary CY 3-folds. Movasati in [18] says:

> All the attempts to find an arithmetic modularity for mirror quintic have failed, and this might be an indication that maybe such varieties need a new kind of modular forms.

Because of this, he introduced CY (quasi-)modular forms which somehow can be considered as a modern generalization of the classical quasi-modular forms (automorphic forms) theory. The present paper provides some evidence in favor of this generalization; namely, we introduce the space of CY quasi-modular forms $\widetilde{\mathscr{M}}$ for the Dwork family and furnish it with a Rankin-Cohen algebra structure. Then we find a proper subspace of $\widetilde{\mathscr{M}}$ which is closed under the Rankin-Cohen brackets. This can be considered as a generalization of the work of Zagier [29] for the space of classical (quasi-)modular forms.

Movasati in [16] used an algebraic method, called *Gauss-Manin connection in disguise* (GMCD), in a geometric framework and reencountered the Ramanujan [25] vector field (system) Ra (see (2.2)) on certain moduli of a family of enhanced elliptic curves (see (3.27) and (3.28)). It is known that the triple of Eisenstein series (E_2, E_4, E_6) gives a solution of the Ramanujan system Ra, and the space of modular forms $\mathcal{M}$ and quasi-modular forms $\widetilde{\mathcal{M}}$ for $\mathrm{SL}_2(\mathbb{Z})$ are respectively graded $\mathbb{C}$-algebras $\mathcal{M} = \mathbb{C}[E_4, E_6]$ and $\widetilde{\mathcal{M}} = \mathbb{C}[E_2, E_4, E_6]$. Note that E_4, E_6 are modular forms of weight 4 and 6, respectively, and E_2 is a quasi-modular form of weight 2 which is not modular. After this work, in the paper [17] he applied GMCD to the family of mirror quintic 3-fold and a few years later expanded it to the book [18], where he introduced CY modular forms for the mirror quintic 3-fold. In particular, he reencountered the so-called *Yukawa coupling* of Candelas et al. [7] and expressed it in terms of CY modular forms, and also by considering [5] and [27], he and his coauthors in [3] wrote the topological string partition functions for the mirror quintic 3-fold in terms of CY modular forms. The mirror quintic 3-fold is the particular case $n = 3$ of families of mirror n-folds, $n \in \mathbb{Z}_{>0}$, arising from the so-called Dwork family (see [11]). The author and Movasati in [19] applied GMCD to the families of the mirror n-folds arising from the Dwork family, for all positive integers n, which briefly is as follows. We considered the moduli space $\mathsf{T} = \mathsf{T}_n$ of the pairs $(X, [\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}])$, where X is a mirror n-fold arising from the Dwork family and $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\}$ refers to a basis of the n-th algebraic de Rham cohomology $H^n_{\rm dR}(X)$ which is compatible with the Hodge filtration of $H^n_{\rm dR}(X)$ (see (3.21)) and its intersection form matrix is constant (see (3.22)). We showed that there exist a unique vector field $\mathsf{R} = \mathsf{R}_n$, called *modular vector field*, and regular functions Y_i, $1 \leq i \leq n-2$, that satisfy certain equation involving the Gauss-Manin connection of the universal family of T (see Theorem 3.1 and also [20, Theorem 1.1] in a more general context). Due to [16] we can say that the modular vector field R is a generalization of the Ramanujan vector field Ra. For $n = 1, 2, 3, 4$ we found the q-expansion of solution components of the modular vector field R whose coefficients are surprisingly integers. Actually, for $n = 1, 2$, where T is the moduli of enhanced elliptic curves and K3-surfaces, respectively, the solution components, as it was expected, are quasi-modular forms (see (3.39) and (3.40)) and generate the space of (quasi-)modular forms for $\Gamma_0(3)$ and $\Gamma_0(2)$, respectively (see [22]). See also [1] for similar computations. In the case $n = 3$, R_3 is explicitly computed in [17] and it is verified that Y_1 is the Yukawa coupling introduced in [7], which predicts the numbers of rational curves of various degrees on a general quintic three-fold. For $n = 4$, we computed the modular vector field R_4 explicitly in [19] and we observed that $\mathsf{Y}_1^2 = \mathsf{Y}_2^2$ is the same as the 4-point function presented in [11, Table 1, $d = 4$], and we computed the mirror map z given in [12, §6.1] in terms of solutions of R_4 (see Section 7 for a more complete discussion). Unlike the cases $n = 1, 2$, for $n = 3, 4$ we believe that it is not possible to write the solution components of R in terms of the classical quasi-modular forms, since the coefficients of their q-expansions increase very rapidly. This leads us to think of another theory which generalizes the theory of quasi-modular forms, where the space generated by solution components of R is the adequate candidate of the desired generalization.

One of the initial steps in the above-mentioned generalization is the correct assignment of weights to the components of a solution of R. In order to do this, motivated by an $\mathfrak{sl}_2(\mathbb{C})$-Lie algebra arising from the Ramanujan vector field Ra (see Section 4.1), we use the results given in [21], where we proved that for any n there are vector fields H and F on $\mathsf{T} = \mathsf{T}_n$ which along with the modular vector field R generate a copy of $\mathfrak{sl}_2(\mathbb{C})$ in $\mathfrak{X}(\mathsf{T})$ (see Theorem 3.4)(the notations H and F in the whole manuscript are used for the same vector fields given in Theorem 3.4). Furthermore, we observe in (3.53) that the vector field H can be written in the form $\mathsf{H} = \sum_{j=1}^{\mathsf{d}} w_j t_j \frac{\partial}{\partial t_j}$, where $\mathsf{d} := \dim \mathsf{T}$, $(t_1, t_2, \ldots, t_{\mathsf{d}})$ is a chart of T, which will be constructed in Subsection 3.1, and $w_j \in \mathbb{Z}_{\geq 0},\ j = 1, 2, \ldots, \mathsf{d}$. To avoid a technical problem in Section 4.1 we change the coordinate chart t_{d} for odd $n \geq 3$, and, by abuse of notation, we denote it again by t_{d} (to solve the same problem we also introduce another vector field D in Section 6 which is not very appropriate, but it is interesting). By this change, the vector field H remains the same. Then, we define $\deg(t_j) := w_j,\ j = 1, 2, \ldots, \mathsf{d}$. By applying these weights, in Proposition 4.1 we show that for any positive integer n the modular vector field $\mathsf{R} = \mathsf{R}_n$ is a quasi-homogeneous vector field of degree 2. Now, suppose that $\mathsf{t}_j,\ j = 1, 2, \ldots, \mathsf{d}$, is the component of a particular solution of R associated with the coordinate chart t_j carrying the same weight, i.e., $\deg(\mathsf{t}_j) = w_j$. We define the space of *CY quasi-modular forms associated to* R for Dwork family as $\widetilde{\mathscr{M}} := \mathbb{C}[\mathsf{t}_1, \mathsf{t}_2, \mathsf{t}_3, \ldots, \mathsf{t}_{\mathsf{d}}, \frac{1}{\mathsf{t}_{n+2}(\mathsf{t}_{n+2} - \mathsf{t}_1^{n+2})\check{\mathsf{t}}}]$ and the space of *CY modular forms associated to* R for the Dwork family as $\mathscr{M} := \mathbb{C}[\mathsf{t}_1, \widehat{\mathsf{t}_2}, \mathsf{t}_3, \mathsf{t}_4, \ldots, \mathsf{t}_{\mathsf{d}}, \frac{1}{\mathsf{t}_{n+2}(\mathsf{t}_{n+2} - \mathsf{t}_1^{n+2})\check{\mathsf{t}}}]$, where $\check{\mathsf{t}}$ is a product of a few number of t_j's (see (3.32)) and the symbol $\widehat{\mathsf{t}_2}$ means that the component t_2 is omitted, i.e., $\mathsf{t}_2 \notin \mathscr{M}$; indeed $\mathscr{M}$ is a subspace of $\widetilde{\mathscr{M}}$, and $\widetilde{\mathscr{M}} = \mathscr{M}[\mathsf{t}_2]$. For any n, Remark 3.3 yields that $\deg(\mathsf{t}_2) = 2$. In our approach t_2 plays the same role of the quasi-modular form E_2 in the theory of quasi-modular forms for $\mathrm{SL}_2(\mathbb{Z})$, which gives sense to the definition of $\mathscr{M}$ (recall that $\widetilde{\mathcal{M}} = \mathcal{M}[E_2]$). Throughout by CY quasi-modular forms or CY modular forms we mean CY quasi-modular forms associated to R or CY modular forms associated to R for the Dwork family.

To motivate and explain better our main results, we recall again some known facts of the classical theory of quasi-modular forms. It is well known that the derivative of a modular form is not necessarily a modular form. More precisely, for any positive integer r and any modular form $f \in \mathcal{M}_r$ of weight r for $\mathrm{SL}_2(\mathbb{Z})$, we know that $f' \in \widetilde{\mathcal{M}}_{r+2}$ is a quasi-modular form of weight $r+2$ which is not necessarily modular. But the derivative f' can be corrected using the Ramanujan-Serre derivation $\partial f = f' - \frac{1}{12} r E_2 f$ which yields $\partial f \in \mathcal{M}_{r+2}$ (see (2.4) and (2.5)). Rankin in [26] described some sufficient conditions under which a polynomial in a given modular form and its derivatives is again a modular form. Cohen [8] generalized the result of Rankin and for any non-negative integer k, defined a bilinear operator $F_k(\cdot,\cdot)$ and proved that for all $f \in \mathcal{M}_r,\ g \in \mathcal{M}_s$ one gets $F_k(f,g) \in \mathcal{M}_{r+s+2k}$. Later, Zagier in [29] called these bilinear forms *Rankin-Cohen brackets* and denoted them by $[\cdot,\cdot]_k$ (see (2.6)). Furthermore, he developed the theory of *Rankin-Cohen algebras*, which are briefly described in Section 2. The principal objective of this paper is to endow $\widetilde{\mathscr{M}}$ and $\mathscr{M}$ with standard Rankin-Cohen and canonical Rankin-Cohen algebra structures, respectively. In order to do this we will need a degree 2 differential operator and a Ramanujan-Serre–type derivation on $\widetilde{\mathscr{M}}$ and $\mathscr{M}$, respectively. To this end, we observe that R induces a differential operator on $\widetilde{\mathscr{M}}$ which is denoted by $\mathscr{R}$ (see (6.8)). It is not difficult to observe that the space of CY modular forms $\mathscr{M}$ is not closed under $\mathscr{R}$, but by correcting the derivation $\mathscr{R}$ we can define the *Ramanujan-Serre–type derivation* ∂ (see (5.5)). In the following theorem we state the first main result of this work, in which δ^n_m denotes the Kronecker delta function.

Theorem 1.1. *The following hold.*

1. *The derivation $\mathscr{R}$ induced by the modular vector field R is a degree 2 differential operator on $\widetilde{\mathscr{M}}$.*
2. *The Ramanujan-Serre–type derivation ∂, which is defined on the generators of $\widetilde{\mathscr{M}}$ as follows:*

$$\partial f := \mathscr{R} f + (1 - \frac{1}{2}\delta^n_2) r \mathsf{t}_2 f\,, \quad \forall f \in \widetilde{\mathscr{M}_r} \text{ and } \forall r \in \mathbb{Z},$$

is a degree 2 differential operator on $\mathscr{M}$.

We emphasize that, due to Theorem 1.1, $\mathscr{M}$ is closed under ∂, and in particular for all integers r we have $\partial : \mathscr{M}_r \to \mathscr{M}_{r+2}$. Using the derivation $\mathscr{R}$, for any non-negative integers k, s, r and any $f \in \widetilde{\mathscr{M}_r}, g \in \widetilde{\mathscr{M}_s}$, we define the k-th Rankin-Cohen bracket $[f,g]_{\mathscr{R},k}$ of CY quasi-modular forms in (5.15) and observe that $[f,g]_{\mathscr{R},k} \in \widetilde{\mathscr{M}_{r+s+2}}$. Indeed, $[\cdot,\cdot]_{\mathscr{R},k}$ provides $\widetilde{\mathscr{M}}$ with a standard Rankin-Cohen algebra structure. Finally, in the following theorem we establish the second main, and more important, result of the present paper.

Theorem 1.2. *For all positive integers r, s, k and for any $f \in \mathscr{M}_r$, $g \in \mathscr{M}_s$ we have:*

$$[f,g]_{\mathscr{R},k} \in \mathscr{M}_{r+s+2k}\,.$$

In the other words, Theorem 1.2 says that the space of CY modular forms of positive weight is closed under the Rankin-Cohen brackets of the CY quasi-modular forms, and hence we provide this space with a canonical

Rankin-Cohen algebra structure. We prove Theorem 1.1 and Theorem 1.2 in Section 5. It is worth mentioning that for various examples of CY modular forms of negative weight we used the computer and observed that their Rankin-Cohen brackets are again CY modular forms. Thus, we conjecture that the whole space of the CY modular forms $\mathscr{M}$ is closed under the Rankin-Cohen brackets.

Remark 1.1. *The proof of Theorem 1.2 is based on Proposition 2.1 [29, Proposition 1]. After releasing the present paper, we proved an extended version of this proposition (see* **Conjecture 1** *given in Section 8), which consequently implies that Theorem 1.2 holds for all integers r and s. The proofs, which are technical and long, will appear in [23].*

This manuscript is organized as follows. In Section 2 we briefly review the relevant definitions and facts of [29] which will be used in the rest of the text. Section 3 gives a short summary of [19] and [21] which constructs the foundation of the present research and also lets us have a self contained manuscript. In Section 4 we prove that the modular vector field R is a quasi-homogeneous vector field of degree 2 and we present a new chart for odd positive integers $n \geq 3$. In Section 5 our main results are stated and proved. Namely, we define the concepts of: spaces of CY quasi-modular forms and CY modular forms, derivation $\mathscr{R}$, Ramanujan-Serre–type derivation ∂ and Rankin-Cohen brackets of the CY quasi-modular forms. We provide the proofs of Theorem 1.1 and Theorem 1.2 in this section. In various examples of the same section, for $n = 1, 2, 3, 4$, the derivations $\mathscr{R}$, ∂ and Rankin-Cohen brackets of a few CY modular forms are explicitly calculated. In Section 6 we introduce another derivation $\mathscr{D}$ (vector field D) for which Theorem 1.1 and Theorem 1.2 hold. In Section 7 we discuss the mirror map of the CY n-folds arising from the Dwork family and observe that it can be written in terms of the CY modular forms. Section 8 deals with the final remarks and a few open questions.

2. Rankin-Cohen algebra

In this section we recall the important facts and terminologies of [29] which are necessary for the present paper. Let $\widetilde{\mathcal{M}} = \bigoplus_{r\geq 0} \widetilde{\mathcal{M}}_r$ and $\mathcal{M} = \bigoplus_{r\geq 0} \mathcal{M}_r$, respectively, be the graded algebras of quasi-modular forms and of modular forms, where $\widetilde{\mathcal{M}}_r := \widetilde{\mathcal{M}}_r(\mathrm{SL}_2(\mathbb{Z}))$ and $\mathcal{M}_r := \mathcal{M}_r(\mathrm{SL}_2(\mathbb{Z}))$, respectively, are the spaces of quasi-modular forms and of modular forms of weight r for $\mathrm{SL}_2(\mathbb{Z})$. It is well known that $\widetilde{\mathcal{M}} = \mathbb{C}[E_2, E_4, E_6]$ and $\mathcal{M} = \mathbb{C}[E_4, E_6]$, where

E_2, E_4, E_6 are Eisenstein series given as:

$$E_{2j}(q) = 1 + b_j \sum_{k=1}^{\infty} \sigma_{2j-1}(k) q^k \,, \tag{2.1}$$

where $\sigma_i(k) = \sum_{d|k} d^i$ and $(b_1, b_2, b_3) = (-24, 240, -504)$. Note that E_4 and E_6 are modular forms of weight 4 and 6, respectively, while E_2 is a quasi-modular form of weight 2 which is not modular. The triple (E_2, E_4, E_6) satisfies the system of ordinary differential equations

$$\mathsf{Ra} : \begin{cases} t_1' = \frac{1}{12}(t_1^2 - t_2) \\ t_2' = \frac{1}{3}(t_1 t_2 - t_3) \\ t_3' = \frac{1}{2}(t_1 t_3 - t_2^2) \end{cases} , \tag{2.2}$$

which is known as the *Ramanujan relations between Eisenstein series*, and from now on we call it the *Ramanujan vector field*. Note that here $t_j' = q\frac{\partial t_j}{\partial q} = \frac{1}{2\pi i}\frac{dt_j}{d\tau}$ where $q = e^{2\pi i \tau}$ and $\tau \in \mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. The Ramanujan vector field $\mathsf{Ra} = t_1' \frac{\partial}{\partial t_1} + t_2' \frac{\partial}{\partial t_2} + t_3' \frac{\partial}{\partial t_3}$ together with two vector fields $H = 2t_1 \frac{\partial}{\partial t_1} + 4t_2 \frac{\partial}{\partial t_2} + 6t_3 \frac{\partial}{\partial t_3}$ and $F = -12\frac{\partial}{\partial t_1}$ forms a copy of $\mathfrak{sl}_2(\mathbb{C})$; this follows from the fact that $[\mathsf{Ra}, F] = H$, $[H, \mathsf{Ra}] = 2\mathsf{Ra}$, $[H, \mathsf{F}] = -2F$, where $[\ ,\ ]$ refers to the Lie bracket of vector fields. We know that if $f \in \mathcal{M}_r$ is a modular form of weight r, then f' is not necessarily a modular form. If instead of the usual derivation, we use the so-called *Ramanujan-Serre derivation* ∂ given by

$$\partial f = f' - \frac{1}{12} r E_2 f, \tag{2.3}$$

then ∂f is a modular form of weight $r + 2$. After substituting (t_1, t_2, t_3) by (E_2, E_4, E_6) in the Ramanujan vector field (2.2), for any non-negative integer r and any $f \in \widetilde{\mathcal{M}}_r$, we get $f' = \mathcal{D} f$ where the differential operator $\mathcal{D} : \widetilde{\mathcal{M}}_r \to \widetilde{\mathcal{M}}_{r+2}$ is given as follows:

$$f' = \mathcal{D} f = \frac{E_2^2 - E_4}{12} \frac{\partial f}{\partial E_2} + \frac{E_2 E_4 - E_6}{3} \frac{\partial f}{\partial E_4} + \frac{E_2 E_6 - E_4^2}{2} \frac{\partial f}{\partial E_6} \,, \tag{2.4}$$

which is a degree 2 differential operator. Therefore, for any $f \in \mathcal{M}_r$ since $\frac{\partial f}{\partial E_2} = 0$, we can express the Ramanujan-Serre derivation (2.3) as follows:

$$\partial f = -\frac{E_6}{3}\frac{\partial f}{\partial E_4} - \frac{E_4^2}{2}\frac{\partial f}{\partial E_6}\,, \tag{2.5}$$

from which we get that the Ramanujan-Serre derivation ∂ kills the terms which include E_2. Zagier [29] in 1994, based on the works of Rankin [26] and Cohen [8], for any non-negative integer k defined the k-th Rankin-Cohen bracket $[f, g]_k$, for all $f \in \mathcal{M}_r$ and $g \in \mathcal{M}_s$, as follows:

$$[f, g]_k := \sum_{i+j=k} (-1)^j \binom{k+r-1}{i}\binom{k+s-1}{j} f^{(j)} g^{(i)}\,, \tag{2.6}$$

where $f^{(j)}$ and $g^{(j)}$ refer to the j-th derivative of f and g with respect to the derivation given in (2.4). It was proven by Cohen that $[f, g]_k \in \mathcal{M}_{r+s+2k}$. Note that the 0-th bracket is considered as usual multiplication, i.e. $[f, g]_0 = fg$. We list some algebraic properties of the Rankin-Cohen brackets given in [29] below, in which we assume $f \in \mathcal{M}_r,\ g \in \mathcal{M}_s$ and $h \in \mathcal{M}_l$:

$$[f, g]_k = (-1)^k [g, f]_k\,, \quad \forall\, k \geq 0\,, \tag{2.7}$$
$$[[f, g]_0, h]_0 = [f, [g, h]_0]_0\,, \tag{2.8}$$
$$[f, 1]_0 = [1, f]_0 = f\,, \quad [f, 1]_k = [1, f]_k = 0\,, \ \forall\, k > 0\,, \tag{2.9}$$
$$[[f, g]_1, h]_1 + [[g, h]_1, f]_1 + [[h, f]_1, g]_1 = 0\,, \tag{2.10}$$
$$[[f, g]_0, h]_1 + [[g, h]_0, f]_1 + [[h, f]_0, g]_1 = 0\,, \tag{2.11}$$
$$l[[f, g]_1, h]_0 + s[[g, h]_1, f]_0 + r[[h, f]_1, g]_0 = 0\,, \tag{2.12}$$
$$[[f, g]_0, h]_1 = [[g, h]_1, f]_0 - [[h, f]_1, g]_1\,, \tag{2.13}$$
$$(r+s+l)[[f, g]_1, h]_0 = r[[g, h]_0, f]_1 - s[[h, f]_0, g]_1\,, \tag{2.14}$$
$$(r+1)(s+1)[[f, g]_0, h]_2 = -l(l+1)[[f, g]_2, h]_0 + (r+1)(r+s+1)[[g, h]_2, f]_0 + (s+1)(r+s+1)[[h, f]_2, g]_0, \tag{2.15}$$
$$(r+s+l+1)(r+s+l+2)[[f, g]_2, h]_0 = (r+1)(s+1)[[f, g]_0, h]_2 - (r+1)(r+s+1)[[g, h]_0, f]_2 - (s+1)(r+s+1)[[h, f]_0, g]_2, \tag{2.16}$$
$$[[f, g]_1, h]_1 = [[g, h]_0, f]_2 - [[h, f]_0, g]_2 + [[g, h]_2, f]_0 - [[h, f]_2, g]_0\,. \tag{2.17}$$

Zagier defined a Rankin-Cohen algebra over a field k (of characteristic zero) as a graded k-vector space $M = \bigoplus_{r\geq 0} M_r$, with $M_0 = \mathsf{k}.1$ and $\dim_{\mathsf{k}} M_r$ finite for all r, together with bilinear operations $[\ ,\ ]_k : M_r \otimes M_s \to M_{r+s+2k},\ r, s, k \geq 0$, which satisfy (2.7)-(2.17) and all the other algebraic identities satisfied by the Rankin-Cohen brackets given in (2.6). A basic example of Rankin-Cohen algebras can be constructed as follows, and for future uses we state it as a remark.

Remark 2.1. *Let M be a commutative and associative graded algebra with unit over the field* k *together with a derivation D of degree 2, i.e. $D : M_r \to M_{r+2}$ for all integers $r \geq 0$. Given $f \in M_r$ and $g \in M_s$, for any non-negative integer k define the Rankin-Cohen bracket $[f, g]_{D,k}$ as follows:*

$$[f,g]_{D,k} = \sum_{i+j=k} (-1)^j \binom{k+r-1}{i} \binom{k+s-1}{j} f^{(j)} g^{(i)} \in M_{r+s+2k}, \tag{2.18}$$

where $f^{(j)} = D^j f$ and $g^{(j)} = D^j g$ are the j-th derivative of f and g with respect to the derivation D. Then $(M, [\cdot,\cdot]_{D,*})$ is a Rankin-Cohen algebra which is called the standard Rankin-Cohen algebra*.*

For example $(\widetilde{\mathcal{M}}, [\cdot,\cdot]_{\mathcal{D},*}) = (\widetilde{\mathcal{M}}, [\cdot,\cdot]_*)$, where $\mathcal{D}$ is given in (2.4), is a standard Rankin-Cohen algebras. Hence, $(\mathcal{M}, [\cdot,\cdot]_{\mathcal{D},*})$ is a sub Rankin-Cohen algebra of $(\widetilde{\mathcal{M}}, [\cdot,\cdot]_{\mathcal{D},*})$, but it is not a standard Rankin-Cohen algebras, since $\mathcal{M}$ is not closed under $\mathcal{D}$. We can relate $(\mathcal{M}, [\cdot,\cdot]_*)$ with another bilinear form which is defined using the Ramanujan-Serre derivation ∂. This fact, in a more general version, is given in the following proposition, and since a part of its proof will be needed, we summarize the proof and for more details the reader is referred to the given Ref.

Proposition 2.1. ([29, Proposition 1]) *Let M be a commutative and associative graded* k*-algebra with $M_0 = \mathsf{k} \cdot 1$ together with a derivation ∂ of degree 2 on M, and let $\Lambda \in M_4$. For any $k \geq 0$ define brackets $[\cdot,\cdot]_{\partial,\Lambda,k}$ by*

$$[f,g]_{\partial,\Lambda,k} = \sum_{i+j=k} (-1)^j \binom{k+r-1}{i} \binom{k+s-1}{j} f_{(j)} g_{(i)} \in M_{r+s+2k}\ , \tag{2.19}$$

where $f \in M_r$, $g \in M_s$, and $f_{(j)} \in M_{r+2j}$, $g_{(i)} \in M_{s+2i}$ are defined recursively as follows

$$f_{(j+1)} = \partial f_{(j)} + j(j+r-1)\Lambda f_{(j-1)},\ g_{(i+1)} = \partial g_{(i)} + i(i+s-1)\Lambda g_{(i-1)}, \tag{2.20}$$

with initial conditions $f_{(0)} = f$, $g_{(0)} = g$. Then $(M, [\cdot,\cdot]_{\partial,\Lambda,*})$ is a Rankin-Cohen algebra.

Sketch of proof. The only way is to embed $(M, [\cdot,\cdot]_{\partial,\Lambda,*})$ into a standard Rankin-Cohen algebra $(R, [\cdot,\cdot]_{D,*})$ for some larger R with derivation

D. Indeed, it is taken $R = M[\lambda] := M \otimes_{\mathsf{k}} \mathsf{k}[\lambda]$, where $\lambda \notin M_2$ has degree 2, and the derivation D is defined on the generators of R as follows:

$$D(f) = \partial(f) + k\lambda f \in R_{k+2}, \ \forall f \in M_k, \text{ and } D(\lambda) = \Lambda + \lambda^2 \in R_4, \tag{2.21}$$

which can be extended uniquely as a derivation on R. Then, for any $k \geq 0$ and any $f \in M_r,\ g \in M_s$, for all $r, s \in \mathbb{Z}_{\geq 0}$, we have:

$$[f, g]_{D,k} = [f, g]_{\partial,\Lambda,k} \text{ (see the proof of [29, Proposition 1]).} \tag{2.22}$$

This completes the proof, since M is obviously closed under the brackets $[\cdot,\cdot]_{\partial,\Lambda,k}$. □

A Rankin-Cohen algebra $(M, [\cdot,\cdot]_*)$ is called *canonical* if its brackets are given as in Proposition 2.1 for some derivation ∂ of degree 2 on M and some element $\Lambda \in M_4$, i.e., $[\cdot,\cdot]_k = [\cdot,\cdot]_{\partial,\Lambda,k}$. For example, $(\mathcal{M}, [\cdot,\cdot]_*)$ is a canonical Rankin-Cohen algebra with the Ramanujan-Serre derivation ∂ and $\Lambda = \frac{1}{12^2}E_4$.

3. GMCD for the Dwork family

In Sections 3.1 and 3.2 we recall some relevant facts and terminologies from [19, 21], and for more details one is referred to the same references. In this manuscript for any positive integer n we fix the notation $m := \frac{n+1}{2}$ if n is odd, and $m := \frac{n}{2}$ if n is even.

3.1. Moduli spaces and modular vector field R

This subsection is based on [19]. Let W_z, for $z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, be an n-dimensional hypersurface in $\mathbb{P}^{n+1}$ given by the so-called Dwork family:

$$W_z := \big\{(x_0, x_1, \ldots, x_{n+1}) \in \mathbb{P}^{n+1} \,|\, f_z(x_0, x_1, \ldots, x_{n+1}) = 0\big\}, \tag{3.1}$$

where

$$f_z := zx_0^{n+2} + x_1^{n+2} + x_2^{n+2} + \cdots + x_{n+1}^{n+2} - (n+2)x_0x_1x_2\cdots x_{n+1}. \tag{3.2}$$

W_z represents a family of CY n-folds. The group $G := \{(\zeta_0, \zeta_1, \ldots, \zeta_{n+1}) \mid \zeta_i^{n+2} = 1,\ \zeta_0\zeta_1\ldots\zeta_{n+1} = 1\}$, acts canonically on W_z as

$$(\zeta_0, \zeta_1, \ldots, \zeta_{n+1}).(x_0, x_1, \ldots, x_{n+1}) = (\zeta_0x_0, \zeta_1x_1, \ldots, \zeta_{n+1}x_{n+1}).$$

We obtain the variety $X = X_z,\ z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, by desingularization of the quotient space W_z/G (for more details see [11]). From now on, we call $X = X_z$ the *mirror variety*[1] which is also a CY n-fold. It is known that $\dim(H^n_{\rm dR}(X)) = n+1$ and all Hodge numbers $h^{ij},\ i+j=n$, of X are one.

We denote by S the moduli of the pairs (X, α_1), where X is an n-dimensional mirror variety and α_1 is a holomorphic n-form on X. We know that the family of mirror varieties X_z is a one parameter family and the n-form α_1 is unique, up to multiplication by a constant, therefore $\dim(\mathsf{S}) = 2$. Analogous to the construction of X_z, let $\mathsf{X}_{t_1,t_{n+2}}$, $(t_1, t_{n+2}) \in \mathbb{C}^2 \setminus \{(t_1^{n+2} - t_{n+2})t_{n+2} = 0\}$, be the mirror variety obtained by the quotient and desingularization of the CY n-folds given by

$$W_{t_1,t_{n+2}} := \big\{(x_0, \ldots, x_{n+1}) \in \mathbb{P}^{n+1} \,|\, f_{t_1,t_{n+2}}(x_0, x_1, \ldots, x_{n+1}) = 0\big\}, \tag{3.3}$$

where

$$f_{t_1,t_{n+2}} := t_{n+2}x_0^{n+2} + x_1^{n+2} + \cdots + x_{n+1}^{n+2} - (n+2)t_1x_0x_1x_2\cdots x_{n+1}. \tag{3.4}$$

We fix two n-forms η and ω_1 in the families X_z and $\mathsf{X}_{t_1,t_{n+2}}$, respectively, such that in the affine space $\{x_0 = 1\}$ are given as follows:

$$\eta := \frac{dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n+1}}{df_z}\ , \quad \omega_1 := \frac{dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n+1}}{df_{t_1,t_{n+2}}}\ . \tag{3.5}$$

Any element of S is in the form $(X_z, a\eta)$ where a is a non-zero constant. The pair $(X_z, a\eta)$ can be identified by $(\mathsf{X}_{t_1,t_{n+2}}, \omega_1)$ as follows:

$$(X_z, a\eta) \mapsto (\mathsf{X}_{t_1,t_{n+2}}, \omega_1)\,, \qquad (t_1, t_{n+2}) = (a^{-1}, za^{-(n+2)})\,, \tag{3.6}$$

$$(\mathsf{X}_{t_1,t_{n+2}}, \omega_1) \mapsto (X_z, t_1^{-1}\eta)\,, \qquad z = \frac{t_{n+2}}{t_1^{n+2}}\,. \tag{3.7}$$

Hence, (t_1, t_{n+2}) construct a chart for S; in the other words

$$\mathsf{S} = \mathrm{Spec}(\mathbb{C}[t_1, t_{n+2}, \frac{1}{(t_1^{n+2} - t_{n+2})t_{n+2}}])\,, \tag{3.8}$$

and the morphism $\mathsf{X} \to \mathsf{S}$ is the universal family of (X, α_1). Let $\nabla : H^n_{\rm dR}(\mathsf{X}/\mathsf{S}) \to \Omega^1_{\mathsf{S}} \otimes_{\mathscr{O}_{\mathsf{S}}} H^n_{\rm dR}(\mathsf{X}/\mathsf{S})$ be the Gauss-Manin connection of the

[1] The reason for this name is that due to argument given in [10], the family X_z belongs to the mirror parameter space.

two parameter family of varieties X/S. We define the n-forms $\omega_i,\ \ i = 1,2,\ldots,n+1$, as follows

$$\omega_i := (\nabla_{\frac{\partial}{\partial t_1}})^{i-1}(\omega_1), \tag{3.9}$$

in which $\frac{\partial}{\partial t_1}$ is considered as a vector field on the moduli space S. Then $\omega := \{\omega_1, \omega_2, \ldots, \omega_{n+1}\}$ forms a basis of $H^n_{\rm dR}(X)$ which is compatible with its Hodge filtration, i.e.,

$$\omega_i \in F^{n+1-i} \setminus F^{n+2-i}, i = 1,2,\ldots,n+1, \tag{3.10}$$

where F^i is the i-th piece of the Hodge filtration of $H^n_{\rm dR}(X)$. We can write the Gauss-Manin connection of X/S in the basis ω as follows

$$\nabla\omega = \mathsf{B}\omega\,,\ \text{with}\ \omega = \begin{pmatrix} \omega_1 & \omega_2 & \ldots & \omega_{n+1} \end{pmatrix}^{tr}. \tag{3.11}$$

If we denote by $\mathsf{B}[i,j]$ the (i,j)-th entry of the Gauss-Manin connection matrix B, then we obtain:

$$\mathsf{B}[i,i] = -\frac{i}{(n+2)t_{n+2}}dt_{n+2}\,,\ \ 1 \le i \le n\,, \tag{3.12}$$

$$\mathsf{B}[i,i+1] = dt_1 - \frac{t_1}{(n+2)t_{n+2}}dt_{n+2}\,,\ \ 1 \le i \le n\,, \tag{3.13}$$

$$\begin{aligned}\mathsf{B}[n+1,j] = {} & \frac{-S_2(n+2,j)t_1^j}{t_1^{n+2}-t_{n+2}}dt_1 \\ & + \frac{S_2(n+2,j)t_1^{j+1}}{(n+2)t_{n+2}(t_1^{n+2}-t_{n+2})}dt_{n+2}\,,\ \ 1 \le j \le n\,,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\mathsf{B}[n+1,n+1] = {} & \frac{-S_2(n+2,n+1)t_1^{n+1}}{t_1^{n+2}-t_{n+2}}dt_1 \\ & + \frac{\frac{n(n+1)}{2}t_1^{n+2} + (n+1)t_{n+2}}{(n+2)t_{n+2}(t_1^{n+2}-t_{n+2})}dt_{n+2}\,,\end{aligned} \tag{3.15}$$

where $S_2(r,s)$ is the Stirling number of the second kind defined by

$$S_2\left(r,s\right) := \frac{1}{s!}\sum_{i=0}^{s}(-1)^i\begin{pmatrix} s \\ i \end{pmatrix}(s-i)^r\,, \tag{3.16}$$

and the rest of the entries of B are zero. For any $\xi_1, \xi_2 \in H^n_{\rm dR}(X)$, in the context of the de Rham cohomology, the *intersection form* of ξ_1 and ξ_2, denoted by $\langle \xi_1, \xi_2 \rangle$, is given as

$$\langle \xi_1, \xi_2 \rangle := \frac{1}{(2\pi i)^n} \int_X \xi_1 \wedge \xi_2 \,,$$

which is a non-degenerate $(-1)^n$-symmetric form. We obtain

$$\langle \omega_i, \omega_j \rangle = 0, \ \text{if } i + j \leq n+1 \,, \tag{3.17}$$

$$\langle \omega_1, \omega_{n+1} \rangle = (-(n+2))^n \frac{c_n}{t_1^{n+2} - t_{n+2}}, \ \text{where } c_n \text{ is a constant}\,, \tag{3.18}$$

$$\langle \omega_j, \omega_{n+2-j} \rangle = (-1)^{j-1} \langle \omega_1, \omega_{n+1} \rangle, \ \text{for } j = 1, 2, \ldots, n+1\,. \tag{3.19}$$

On account of these relations, we can determine all the rest of $\langle \omega_i, \omega_j \rangle$'s in a unique way. If we set $\Omega = \Omega_n := (\langle \omega_i, \omega_j \rangle)_{1 \leq i,j \leq n+1}$ to be the intersection form matrix in the basis ω, then we have

$$d\Omega = \mathsf{B}\Omega + \Omega \mathsf{B}^{\mathsf{tr}}. \tag{3.20}$$

For any positive integer n by *moduli space* $\mathsf{T} = \mathsf{T}_n$ *of enhanced mirror varieties* we mean the moduli of the pairs $(X, [\alpha_1, \cdots, \alpha_n, \alpha_{n+1}])$, where X is an n-dimensional mirror variety and $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\}$ constructs a basis of $H^n_{\mathrm{dR}}(X)$ satisfying the properties

$$\alpha_i \in F^{n+1-i} \setminus F^{n+2-i}, \quad i = 1, \cdots, n, n+1, \tag{3.21}$$

and

$$[\langle \alpha_i, \alpha_j \rangle]_{1 \leq i,j \leq n+1} = \Phi_n. \tag{3.22}$$

Here $\Phi = \Phi_n$ is the following constant $(n+1) \times (n+1)$ matrix:

$$\Phi_n := \begin{pmatrix} 0_m & J_m \\ -J_m & 0_m \end{pmatrix} \text{ if } n \text{ is odd, and } \Phi_n := J_{n+1} \text{ if } n \text{ is even,} \tag{3.23}$$

where by $0_k, k \in \mathbb{N}$, we mean a $k \times k$ block of zeros, $J_1 = 1$ and

$$J_k := \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}, \text{ for } k > 1. \tag{3.24}$$

In [19] the universal family $\pi : \mathsf{X} \to \mathsf{T}$ together with the global sections $\alpha_i,\ i = 1, \cdots, n{+}1$, of the relative algebraic de Rham cohomology $H^n_{\rm dR}(\mathsf{X}/\mathsf{T})$ was constructed, and in its main theorem we observed that:

Theorem 3.1. ([19, Theorem 1.1]) *There exist a unique vector field* $\mathsf{R} = \mathsf{R}_n \in \mathfrak{X}(\mathsf{T})$*, and unique regular functions* $\mathsf{Y}_i \in \mathscr{O}_\mathsf{T},\ 1 \leq i \leq n-2$*, such that:*

$$\nabla_{\mathsf{R}} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \\ \alpha_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathsf{Y}_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \mathsf{Y}_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mathsf{Y}_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}}_{\mathsf{Y}} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \\ \alpha_{n+1} \end{pmatrix}, \tag{3.25}$$

and $\mathsf{Y}\Phi + \Phi\mathsf{Y}^{\rm tr} = 0$*.*

Here $\mathscr{O}_\mathsf{T}$ refers to the $\mathbb{C}$-algebra of regular functions on T, and ∇_R stands for the algebraic Gauss-Manin connection

$$\nabla : H^n_{\rm dR}(\mathsf{X}/\mathsf{T}) \to \Omega^1_\mathsf{T} \otimes_{\mathscr{O}_\mathsf{T}} H^n_{\rm dR}(\mathsf{X}/\mathsf{T}),$$

composed with the vector field $\mathsf{R} \in \mathfrak{X}(\mathsf{T})$, in which Ω^1_T refers to the $\mathscr{O}_\mathsf{T}$-module of differential 1-forms on T. We call R as *modular vector field* attached to the Dwork family. Moreover, we found that:

$$\mathsf{d} = \mathsf{d}_n := \dim(\mathsf{T}) = \begin{cases} \frac{(n+1)(n+3)}{4} + 1, & \text{if } n \text{ is odd} \\ \frac{n(n+2)}{4} + 1, & \text{if } n \text{ is even} \end{cases}. \tag{3.26}$$

The above theorem is the key tool of GMCD. In the GMCD viewpoint, the vector field Ra given in (2.2), up to multiplying the coordinates by constants $(t_1, t_2, t_3) = (12t_1, 12t_2, \frac{12^3}{8}t_3)$, is the unique vector field that satisfies

$$\nabla_{\mathsf{Ra}}\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \alpha\ , \tag{3.27}$$

where $\alpha = (\ \alpha_1\ \ \alpha_2\ )^{\rm tr}$, with $\alpha_1 = [\frac{dx}{y}],\ \alpha_2 = [\frac{xdx}{y}]$, and ∇ is the Gauss-Manin connection of the universal family of the elliptic curves

$$y^2 = 4(x-t_1)^3 - t_2(x-t_1) - t_3\ , \quad \text{with}\ \ 27t_3^2 - t_2^3 \neq 0\ . \tag{3.28}$$

We can generalize the notion of the Ramanujan-Serre derivation (2.5) and the Rankin-Cohen bracket (2.6) for the modular vector fields $\mathsf{R} = \mathsf{R}_n$ using an analogous procedure explained for the Ramanujan vector field Ra, which will be treated in Section 5.

Next we are going to present a chart for the moduli space T. In order to do this, let $S = \left(\ s_{ij} \ \right)_{1\leq i,j\leq n+1}$ be a lower triangular matrix, whose entries are indeterminates $s_{ij}, \quad i \geq j$ and $s_{11} = 1$. We define

$$\underbrace{\left(\ \alpha_1 \quad \alpha_2 \quad \ldots \quad \alpha_{n+1} \ \right)^{tr}}_{\alpha} = S \underbrace{\left(\ \omega_1 \quad \omega_2 \quad \ldots \quad \omega_{n+1} \ \right)^{tr}}_{\omega},$$

which implies that α forms a basis of $H^n_{\mathrm{dR}}(X)$ compatible with its Hodge filtration. We would like that $(X, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}])$ be a member of T, hence it has to satisfy $\left(\ \langle \alpha_i, \alpha_j \rangle \ \right)_{1\leq i,j\leq n+1} = \Phi$, from what we get the following equation

$$S\Omega S^{\mathsf{tr}} = \Phi. \tag{3.29}$$

Using this equation we can express $d_0 := \frac{(n+2)(n+1)}{2} - \mathsf{d} - 2$ numbers of parameters s_{ij}'s in terms of other $\mathsf{d} - 2$ parameters that we fix them as *independent parameters*. For simplicity we write the first class of parameters as $\check{t}_1, \check{t}_2, \cdots, \check{t}_{d_0}$ and the second class as $t_2, t_3, \ldots, t_{n+1}, t_{n+3}, \ldots, t_{\mathsf{d}}$. We put the independent parameters t_i inside S according to the following rule which is not canonical: t_i's are written in S from left to right and top to bottom in the entries (i,j) for $i + j < n + 2$ if n is even and $i + j \leq n + 2$ if n is odd. The position of $\check{t}_i$'s inside S can be chosen arbitrarily. For instance, for $n = 1, 2, 3, 4, 5$ we have:

$$\begin{pmatrix} 1 & 0 \\ t_2 & \check{t}_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ t_2 & \check{t}_2 & 0 \\ \check{t}_4 & \check{t}_3 & \check{t}_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_2 & t_3 & 0 & 0 \\ t_4 & t_6 & \check{t}_2 & 0 \\ t_7 & \check{t}_4 & \check{t}_3 & \check{t}_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ t_2 & t_3 & 0 & 0 & 0 \\ t_4 & t_5 & \check{t}_3 & 0 & 0 \\ t_7 & \check{t}_7 & \check{t}_5 & \check{t}_2 & 0 \\ \check{t}_9 & \check{t}_8 & \check{t}_6 & \check{t}_4 & \check{t}_1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ t_2 & t_3 & 0 & 0 & 0 & 0 \\ t_4 & t_5 & t_6 & 0 & 0 & 0 \\ t_8 & t_9 & t_{10} & \check{t}_3 & 0 & 0 \\ t_{11} & t_{12} & \check{t}_7 & \check{t}_5 & \check{t}_2 & 0 \\ t_{13} & \check{t}_9 & \check{t}_8 & \check{t}_6 & \check{t}_4 & \check{t}_1 \end{pmatrix}.$$

Note that we have already used t_1, t_{n+2} as the coordinate system of S. In particular we find:

$$s_{(n+2-i)(n+2-i)} = \frac{(-1)^{n+i+1}}{c_n(n+2)^n}\frac{t_1^{n+2}-t_{n+2}}{s_{ii}},\ 1\leq i\leq m\,. \tag{3.30}$$

In this way, $\mathsf{t} := (t_1, t_2, \ldots, t_d)$ forms a chart for the moduli space T, and in fact

$$\mathsf{T} = \mathrm{Spec}(\mathbb{C}[t_1, t_2, \ldots, t_{\mathsf{d}}, \frac{1}{t_{n+2}(t_{n+2}-t_1^{n+2})\check{t}}])\,, \tag{3.31}$$

$$\mathscr{O}_{\mathsf{T}} = \mathbb{C}[t_1, t_2, \ldots, t_{\mathsf{d}}, \frac{1}{t_{n+2}(t_{n+2}-t_1^{n+2})\check{t}}]\,. \tag{3.32}$$

Here, $\check{t}$ is the product of $m-1$ independent parameters which are located in the main diagonal of S. From now on, we alternately use either s_{ij}'s, or t_i's and $\check{t}_j$'s to refer to the entries of S. If we denote by A the Gauss-Manin connection matrix of the family X/T written in the basis α, i.e., $\nabla\alpha = \mathsf{A}\alpha$, then we calculate A as follows:

$$\mathsf{A} = (dS + S\cdot\mathsf{B})\ S^{-1}\,. \tag{3.33}$$

If for any vector field $\mathsf{E}\in\mathfrak{X}(\mathsf{T})$ we define the *Gauss-Manin connection matrix* attached to E as $(n+1)\times(n+1)$ matrix A_{E} given by:

$$\nabla_{\mathsf{E}}\alpha = \mathsf{A}_{\mathsf{E}}\alpha, \tag{3.34}$$

then from (3.33) we obtain:

$$\dot{S}_{\mathsf{E}} = \mathsf{A}_{\mathsf{E}}S - S\,\mathsf{B}(\mathsf{E})\,, \tag{3.35}$$

where $\dot{S}_{\mathsf{E}} = dS(\mathsf{E})$ and $\dot{x} := dx(\mathsf{E})$ is the derivative of the function x along the vector field E in T. Note that equalities corresponding to $(1,1)$-th and $(1,2)$-th entries of (3.35) give us respectively $\dot{t}_1$ and $\dot{t}_{n+2}$, and any $\dot{t}_i$, $1\leq i\leq \mathsf{d}$, $i\neq 1, n+2$, corresponds to only one $\dot{s}_{jk}$, $1\leq j,k\leq n+1$. In the following remarks we recall some useful results deduced from the proof of Theorem 3.1 in [19, §7].

Remark 3.1. *We obtain the functions* Y_i*'s given in* (3.25) *as follows: if* n *is odd, then*

$$\mathsf{Y}_i = -\mathsf{Y}_{n-(i+1)} = \frac{s_{22}\,s_{(i+1)(i+1)}}{s_{(i+2)(i+2)}},\quad i = 1, 2, \ldots, \frac{n-3}{2}\,, \tag{3.36}$$

$$\mathsf{Y}_{\frac{n-1}{2}} = (-1)^{\frac{3n+3}{2}} c_n (n+2)^n \frac{s_{22}\, s^2_{\frac{n+1}{2}\frac{n+1}{2}}}{t_1^{n+2} - t_{n+2}}, \tag{3.37}$$

and if n is even, then

$$\mathsf{Y}_i = -\mathsf{Y}_{n-(i+1)} = \frac{s_{22}\, s_{(i+1)(i+1)}}{s_{(i+2)(i+2)}}, \quad i = 1, 2, \ldots, \frac{n-2}{2}. \tag{3.38}$$

Remark 3.2. *Let $\mathsf{E} \in \mathfrak{X}(\mathsf{T})$. If $\nabla_{\mathsf{E}}\alpha = 0$ for any $(X, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}]) \in \mathsf{T}$, then $\mathsf{E} = 0$.*

We finish this subsection with the following example.

Example 3.1. In [19] for $n = 1, 2$ we found the modular vector fields $\mathsf{R}_1, \mathsf{R}_2$, respectively, as follows:

$$\mathsf{R}_1 : \begin{cases} \dot{t}_1 = -t_1 t_2 - 9(t_1^3 - t_3) \\ \dot{t}_2 = 81 t_1 (t_1^3 - t_3) - t_2^2 \\ \dot{t}_3 = -3 t_2 t_3 \end{cases}, \qquad \mathsf{R}_2 : \begin{cases} \dot{t}_1 = t_3 - t_1 t_2 \\ \dot{t}_2 = 2t_1^2 - \frac{1}{2} t_2^2 \\ \dot{t}_3 = -2 t_2 t_3 + 8 t_1^3 \\ \dot{t}_4 = -4 t_2 t_4 \end{cases}, \tag{3.39}$$

where by $\dot{t}_j$ in R_1 we mean $\dot{t}_j = 3 \cdot q \cdot \frac{\partial t_j}{\partial q}$ and in R_2 we mean $\dot{t}_j = -\frac{1}{5} \cdot q \cdot \frac{\partial t_j}{\partial q}$, and furthermore in R_2 we have the polynomial equation $t_3^2 = 4(t_1^4 - t_4)$. For a complex number τ with $\mathrm{Im}\tau > 0$, if we set $q = e^{2\pi i \tau}$, then we obtained the following solutions of R_1 and R_2 respectively:

$$\begin{cases} \mathsf{t}_1(q) = \frac{1}{3}(2\theta_3(q^2)\theta_3(q^6) \\ \qquad\qquad -\theta_3(-q^2)\theta_3(-q^6)), \\ \mathsf{t}_2(q) = \frac{1}{8}(E_2(q^2) - 9E_2(q^6)), \\ \mathsf{t}_3(q) = \frac{\eta^9(q^3)}{\eta^3(q)}, \end{cases}, \quad \begin{cases} \frac{10}{6}\mathsf{t}_1(\frac{q}{10}) = \frac{1}{24}(\theta_3^4(q^2) + \theta_2^4(q^2)), \\ \frac{10}{4}\mathsf{t}_2(\frac{q}{10}) = \frac{1}{24}(E_2(q^2) + 2E_2(q^4)), \\ 10^4 \mathsf{t}_4(\frac{q}{10}) = \eta^8(q)\eta^8(q^2), \end{cases} \tag{3.40}$$

in which η and θ_i's are the classical eta and theta series given as follows:

$$\eta(q) = q^{\frac{1}{24}} \prod_{k=1}^{\infty}(1 - q^k), \;\; \theta_2(q) = \sum_{k=-\infty}^{\infty} q^{\frac{1}{2}(\frac{k+1}{2})^2}, \;\; \theta_3(q) = 1 + 2\sum_{k=1}^{\infty} q^{\frac{1}{2}k^2}.$$

3.2. AMSY-Lie algebra and $\mathfrak{sl}_2(\mathbb{C})$ Lie algebra

In this subsection we give a summary of the main results of [21]. For any positive integer n the algebraic group:

$$\mathsf{G} = \mathsf{G}_n := \{\mathsf{g} \in \mathrm{GL}(n+1, \mathbb{C}) \mid \mathsf{g} \text{ is upper triangular and } \mathsf{g}^{\mathsf{tr}}\Phi\mathsf{g} = \Phi\},$$

acts on the moduli space T from the right, and its Lie algebra:

$$\mathrm{Lie}(\mathsf{G}) = \{\mathfrak{g} \in \mathrm{Mat}(n+1,\mathbb{C}) \mid \mathfrak{g} \text{ is upper triangular and } \mathfrak{g}^{\mathsf{tr}}\Phi + \Phi\mathfrak{g} = 0\},$$

is a $\mathsf{d}-1$ dimensional Lie algebra with the canonical basis consisting of $\mathfrak{g}_{\mathfrak{a}\mathfrak{b}} = (g_{kl})_{(n+1)\times(n+1)}$, $1 \le \mathfrak{a} \le m,\ \mathfrak{a} \le \mathfrak{b} \le 2m+1-\mathfrak{a}$, given as follows: if n is odd, then

$$\begin{cases} g_{\mathfrak{a}\mathfrak{b}} = 1,\ g_{(n+2-\mathfrak{b})(n+2-\mathfrak{a})} = -1, \text{when } \mathfrak{b} \le m, \\ g_{\mathfrak{a}\mathfrak{b}} = g_{(n+2-\mathfrak{b})(n+2-\mathfrak{a})} = 1, \text{when } \mathfrak{b} \ge m+1, \\ \text{and the rest of the entries of } \mathfrak{g}_{\mathfrak{a}\mathfrak{b}} \text{are zero,} \end{cases}$$

and if n is even, then:

$$\begin{cases} g_{\mathfrak{a}\mathfrak{b}} = 1,\ g_{(n+2-\mathfrak{b})(n+2-\mathfrak{a})} = -1, \\ \text{and the rest of the entries of } \mathfrak{g}_{\mathfrak{a}\mathfrak{b}} \text{ are zero.} \end{cases}$$

The following theorem was proved in [21].

Theorem 3.2. ([21, Theorem 1.2]) *For any* $\mathfrak{g} \in \mathrm{Lie}(\mathsf{G})$*, there exists a unique vector field* $\mathsf{R}_{\mathfrak{g}} \in \mathfrak{X}(\mathsf{T})$ *such that:*

$$\mathsf{A}_{\mathsf{R}_{\mathfrak{g}}} = \mathfrak{g}^{\mathsf{tr}}, \tag{3.41}$$

i.e., $\nabla_{\mathsf{R}_{\mathfrak{g}}}\alpha = \mathfrak{g}^{\mathsf{tr}}\alpha$.

This theorem yields that the Lie algebra generated by $\mathsf{R}_{\mathfrak{g}_{\mathfrak{a}\mathfrak{b}}}$'s, $1 \le \mathfrak{a} \le m,\ \mathfrak{a} \le \mathfrak{b} \le 2m+1-\mathfrak{a}$, in $\mathfrak{X}(\mathsf{T})$ with the Lie bracket of the vector fields is isomorphic to $\mathrm{Lie}(\mathsf{G})$ with the Lie bracket of the matrices. Hence, we use $\mathrm{Lie}(\mathsf{G})$ alternately either as a Lie subalgebra of $\mathfrak{X}(\mathsf{T})$ or as a Lie subalgebra of $\mathrm{Mat}(n+1,\mathbb{C})$.

By *AMSY-Lie algebra*[2] $\mathfrak{G}$ we mean the $\mathscr{O}_{\mathsf{T}}$-module generated by $\mathrm{Lie}(\mathsf{G})$ and the modular vector field R in $\mathfrak{X}(\mathsf{T})$. In what follows, δ^k_j denotes the Kronecker delta, $\varrho(n)=1$ if n is an odd integer, and $\varrho(n)=0$ if n is an even integer, Y_j's, $1 \le j \le n-2$, are the functions given in Theorem 3.1, and besides them we let $\mathsf{Y}_0 = -\mathsf{Y}_{n-1} := 1$. The following theorem determines the Lie bracket of $\mathfrak{G}$, which was demonstrated in [21].

[2]The AMSY-Lie algebra was discussed for the first time in [3] for non-rigid compact CY 3-folds, and in [4] it is established for mirror elliptic K3 surfaces. Note that the AMSY-Lie algebra is called Gauss-Manin Lie algebra by the authors of [4].

Theorem 3.3. ([21, Theorem 1.3]) *The following hold:*

$$[\mathsf{R}, \mathsf{R}_{\mathfrak{g}_{11}}] = \mathsf{R}, \tag{3.42}$$

$$[\mathsf{R}, \mathsf{R}_{\mathfrak{g}_{22}}] = -\mathsf{R}, \tag{3.43}$$

$$[\mathsf{R}, \mathsf{R}_{\mathfrak{g}_{\mathfrak{a}\mathfrak{a}}}] = 0, \ 3 \leq \mathfrak{a} \leq m, \tag{3.44}$$

$$[\mathsf{R}, \mathsf{R}_{\mathfrak{g}_{\mathfrak{a}\mathfrak{b}}}] = \Psi_1^{\mathfrak{a}\mathfrak{b}}(\mathsf{Y})\, \mathsf{R}_{\mathfrak{g}_{(\mathfrak{a}+1)\mathfrak{b}}} + \Psi_2^{\mathfrak{a}\mathfrak{b}}(\mathsf{Y})\, \mathsf{R}_{\mathfrak{g}_{\mathfrak{a}(\mathfrak{b}-1)}}, \quad 1 \leq \mathfrak{a} \leq m, \ \mathfrak{a}+1 \leq \mathfrak{b} \leq 2m+1-\mathfrak{a}, \tag{3.45}$$

where

$$\Psi_1^{\mathfrak{a}\mathfrak{b}}(\mathsf{Y}) := (1 + \varrho(n)\delta_{\mathfrak{a}+\mathfrak{b}}^{2m} - \delta_{\mathfrak{a}+\mathfrak{b}}^{2m+1})\, \mathsf{Y}_{\mathfrak{a}-1}, \tag{3.46}$$

$$\Psi_2^{\mathfrak{a}\mathfrak{b}}(\mathsf{Y}) := (1 - 2\varrho(n)\delta_{\mathfrak{b}}^{m+1})\, \mathsf{Y}_{n+1-\mathfrak{b}}\,. \tag{3.47}$$

If $n = 1, 2$, then we see that $\mathfrak{G}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. In general, for $n \geq 3$ we have a copy of $\mathfrak{sl}_2(\mathbb{C})$ as a Lie subalgebra of $\mathfrak{G}$ which contains the modular vector field R and we state it in the following theorem from Ref. [21].

Theorem 3.4. ([21, Theorem 1.4]) *Let us define the vector fields* H *and* F *as follows:*

1. *if* $n = 1$*, then* $\mathsf{H} := -\mathsf{R}_{\mathfrak{g}_{11}}$ *and* $\mathsf{F} := \mathsf{R}_{\mathfrak{g}_{12}}$*,*
2. *if* $n = 2$*, then* $\mathsf{H} := -2\mathsf{R}_{\mathfrak{g}_{11}}$ *and* $\mathsf{F} := 2\mathsf{R}_{\mathfrak{g}_{12}}$*,*
3. *if* $n \geq 3$*, then* $\mathsf{H} := \mathsf{R}_{\mathfrak{g}_{22}} - \mathsf{R}_{\mathfrak{g}_{11}}$ *and* $\mathsf{F} := \mathsf{R}_{\mathfrak{g}_{12}}$*.*

Then the Lie algebra generated by the vector fields $\mathsf{R}, \mathsf{H}, \mathsf{F}$ *in* $\mathfrak{G} \subset \mathfrak{X}(\mathsf{T})$ *is isomorphic to* $\mathfrak{sl}_2(\mathbb{C})$*; indeed we get:*

$$[\mathsf{R}, \mathsf{F}] = \mathsf{H}\ , \quad [\mathsf{H}, \mathsf{R}] = 2\mathsf{R}\ , \quad [\mathsf{H}, \mathsf{F}] = -2\mathsf{F}\,.$$

According to Theorem 3.4, if $n = 1, 2$, then $\mathfrak{G}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ (see Example 5.1), and for $n \geq 3$ the Lie subalgebra of $\mathfrak{G}$ generated by R, $\mathsf{H} := \mathsf{R}_{\mathfrak{g}_{22}} - \mathsf{R}_{\mathfrak{g}_{11}}$ and $\mathsf{F} := \mathsf{R}_{\mathfrak{g}_{12}}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Using the equalities corresponding to $(1,1)$-th and $(1,2)$-th entries of (3.35) for the vector fields $\mathsf{R}_{\mathfrak{g}_{\mathfrak{a}\mathfrak{b}}}$'s we obtain the diagonal matrix $\mathsf{B}(\mathsf{R}_{\mathfrak{g}_{11}}) = \mathrm{diag}(1, 2, \ldots, n+1)$ and the null matrices $\mathsf{B}(\mathsf{R}_{\mathfrak{g}_{\mathfrak{a}\mathfrak{b}}}) = 0$, for $1 \leq \mathfrak{a} \leq m$, $\mathfrak{a} \leq \mathfrak{b} \leq 2m+1-\mathfrak{a}$, $\mathfrak{b} \neq 1$ (see [21, § 4.4]). Due to these facts and again (3.35), we can find $\dot{S}_{\mathsf{R}_{\mathfrak{g}_{\mathfrak{a}\mathfrak{b}}}}$'s, and consequently we obtain $\mathsf{R}_{\mathfrak{g}_{\mathfrak{a}\mathfrak{b}}}$'s. In particular, knowing that $\dot{S}_{\mathsf{H}} = \dot{S}_{\mathsf{R}_{\mathfrak{g}_{22}}} - \dot{S}_{\mathsf{R}_{\mathfrak{g}_{11}}}$, we get $dt_1(\mathsf{H}) = t_1$, $dt_{n+2}(\mathsf{H}) = (n+2)t_{n+2}$, and hence

(3.48)

$$\dot{S}_{\mathsf{H}} = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 2s_{21} & 3s_{22} & 0 & 0 & \ldots & 0 & 0 & 0 \\ s_{31} & 2s_{32} & 3s_{33} & 0 & \ldots & 0 & 0 & 0 \\ s_{41} & 2s_{42} & 3s_{43} & 4s_{44} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ s_{(n-1)1} & 2s_{(n-1)2} & 3s_{(n-1)3} & 4s_{(n-1)4} & \ldots & (n-1)s_{(n-1)(n-1)} & 0 & 0 \\ 0 & s_{n2} & 2s_{n3} & 3s_{n4} & \ldots & (n-2)s_{n(n-1)} & (n-1)s_{nn} & 0 \\ 2s_{(n+1)1} & 3s_{(n+1)2} & 4s_{(n+1)3} & 5s_{(n+1)4} & \ldots & ns_{(n+1)(n-1)} & (n+1)s_{(n+1)n} & (n+2)s_{(n+1)(n+1)} \end{pmatrix}.$$

Thus, for an even integer $n \geq 5$ we get:

$$\mathsf{H} = t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2} + 3t_3\frac{\partial}{\partial t_3} + \sum_{\substack{i=4 \\ i\neq n+2}}^{\mathsf{d}-1} w_i t_i\frac{\partial}{\partial t_i} + (n+2)t_{n+2}\frac{\partial}{\partial t_{n+2}} + \frac{n+2}{2}t_{\mathsf{d}+1}\frac{\partial}{\partial t_{\mathsf{d}+1}}, \tag{3.49}$$

$$\mathsf{F} = \frac{\partial}{\partial t_2}, \tag{3.50}$$

with $t_{\mathsf{d}+1}^2 = s^2_{\frac{n+2}{2}\frac{n+2}{2}} = \frac{(-1)^{\frac{n}{2}}}{c_n(n+2)^n}(t_1^{n+2} - t_{n+2})$ (see (3.30)), and for an odd integer $n \geq 5$ we obtain:

$$\mathsf{H} = t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2} + 3t_3\frac{\partial}{\partial t_3} + \sum_{\substack{i=4 \\ i\neq n+2}}^{\mathsf{d}-3} w_i t_i\frac{\partial}{\partial t_i} + (n+2)t_{n+2}\frac{\partial}{\partial t_{n+2}} + t_{\mathsf{d}-1}\frac{\partial}{\partial t_{\mathsf{d}-1}} + 2t_{\mathsf{d}}\frac{\partial}{\partial t_{\mathsf{d}}}, \tag{3.51}$$

$$\mathsf{F} = \frac{\partial}{\partial t_2} - t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}}}. \tag{3.52}$$

In both equations (3.49) and (3.51) we have $w_i = k$ if $t_i = s_{jk}$ for some $1 \le j, k \le n+1$, i.e., w_i is the number of the column of the entry t_i. Note that H and F have been computed explicitly for $n = 1, 2, 3, 4$ in Example 5.1, which are similar to the H and F founded above for the cases $n \ge 5$. Hence, in general we can write H as:

$$\mathsf{H} = \sum_{i=1}^{\mathsf{d}} w_i t_i \frac{\partial}{\partial t_i}\,, \tag{3.53}$$

where w_i's are non-negative integers.

Remark 3.3. *1. If $n = 1$, then $w_1 = 1$, $w_2 = 2$, $w_3 = 3$.*

2. If $n = 2$, then $w_1 = 2$, $w_2 = 2$, $w_4 = 8$.

3. If $n = 3$, then $w_1 = 1$, $w_2 = 2$, $w_3 = 3$, $w_4 = 0$, $w_5 = 5$, $w_6 = 1$, $w_7 = 2$.

4. If $n \ge 4$ is an even integer, then $w_1 = 1$, $w_2 = 2$, $w_3 = 3$, $w_{n+2} = n+2$, $w_{\mathsf{d}} = 0$.

5. If $n \ge 5$ is an odd integer, then $w_1 = 1$, $w_2 = 2$, $w_3 = 3$, $w_{n+2} = n+2$, $w_{\mathsf{d}-2} = 0$, $w_{\mathsf{d}-1} = 1$, $w_{\mathsf{d}} = 2$.

4. R as a quasi-homogeneous vector field

Let us attach to any t_i in $\mathscr{O}_{\mathsf{T}}$ the weight $\deg(t_i) = w_i$, in which the non-negative integers w_i's are given in (3.53). Recall that a vector field $\mathsf{E} = \sum_{j=1}^{\mathsf{d}} \mathsf{E}^j \frac{\partial}{\partial t_j} \in \mathfrak{X}(\mathsf{T})$, with $\mathsf{E}^j \in \mathscr{O}_{\mathsf{T}}$, is said to be *quasi-homogeneous of degree* d if for any $1 \le j \le \mathsf{d}$ we have $\deg(\mathsf{E}^j) = w_j + d$. Hence, on account of (3.49), (3.50), (3.51), (3.52) and Remark 3.3 the vector fields H and F are quasi-homogeneous of degree 0 and -2, respectively. The vector field H is also known as the radial vector field. Moreover, in the following proposition we show that R is a quasi-homogeneous vector field as well.

Proposition 4.1. *The modular vector field R is a quasi-homogeneous vector field of degree 2 on T.*

Proof. Due to Example 5.1 the affirmation is valid for $n = 1, 2, 3, 4$. Hence we suppose that $n \ge 5$. First note that in the proof of Theorem 3.2 (see [21, § 4.1]) it is verified that the equations $S\Omega S^{\mathsf{tr}} = \Phi$ and $\dot{S}_{\mathfrak{g}} = \mathsf{A}_{\mathfrak{g}} S - S\,\mathsf{B}(\mathfrak{g})$ are compatible for any $\mathfrak{g} \in \mathrm{Lie}(\mathsf{G})$. In particular, it holds for $\mathfrak{g} = \mathsf{H}$. This implies that the degree of any entry s_{jk} of S, $2 \le j \le n+1$, $1 \le k \le j$, is equal to the integer multiple of s_{jk} in the matrix $\dot{S}_{\mathsf{H}}$, which is stated in (3.48). If we set $\mathsf{R} = \sum_{i=1}^{\mathsf{d}} \dot{t}_i \frac{\partial}{\partial t_i}$, then $\dot{t}_i$'s follow from

$$\dot{S}_{\mathsf{R}} = \mathsf{Y} S - S\,\mathsf{B}(\mathsf{R})\,. \tag{4.1}$$

More precisely, from the equalities corresponding to $(1,1)$-th and $(1,2)$-th entries of (4.1) we obtain:

$$\dot{t}_1 = s_{22} - t_1 s_{12} \ \ \& \ \ \dot{t}_{n+2} = -(n+2)s_{21}t_{n+2}\,. \tag{4.2}$$

These equalities and (3.12)-(3.15) imply:

$$\begin{aligned}
&\left(-\frac{k}{(n+2)t_{n+2}}dt_{n+2}\right)(\mathsf{R}) = ks_{21},\ 1\leq k\leq n\,,\\
&\left(dt_1 - \frac{t_1}{(n+2)t_{n+2}}dt_{n+2}\right)(\mathsf{R}) = s_{22}\,,\\
&\left(\frac{-S_2(n+2,j)t_1^j}{t_1^{n+2}-t_{n+2}}dt_1 + \frac{S_2(n+2,j)t_1^{j+1}}{(n+2)t_{n+2}(t_1^{n+2}-t_{n+2})}dt_{n+2}\right)(\mathsf{R})\\
&\qquad\qquad\qquad = \frac{-S_2(n+2,j)t_1^j s_{22}}{t_1^{n+2}-t_{n+2}}\,,\\
&\left(\frac{-S_2(n+2,n+1)t_1^{n+1}}{t_1^{n+2}-t_{n+2}}dt_1 + \frac{\frac{n(n+1)}{2}t_1^{n+2}+(n+1)t_{n+2}}{(n+2)t_{n+2}(t_1^{n+2}-t_{n+2})}dt_{n+2}\right)(\mathsf{R})\\
&\qquad\qquad\qquad = (n+1)s_{21} - \frac{(n+1)(n+2)}{2}\frac{t_1^{n+1}s_{22}}{t_1^{n+2}-t_{n+2}}\,.
\end{aligned}$$

Note that in the above last equality we used the fact that $S_2(n+2,n+1) = \frac{(n+1)(n+2)}{2}$. Therefore:

$$\mathsf{B}(\mathsf{R}) = \begin{pmatrix}
s_{21} & s_{22} & 0 & \dots & 0\\
0 & 2s_{21} & s_{22} & \dots & 0\\
\vdots & \vdots & \ddots & \ddots & \vdots\\
0 & 0 & \dots & ns_{21} & s_{22}\\
\frac{-S_2(n+2,1)t_1 s_{22}}{t_1^{n+2}-t_{n+2}} & \frac{-S_2(n+2,2)t_1^2 s_{22}}{t_1^{n+2}-t_{n+2}} & \dots & \frac{-S_2(n+2,n)t_1^n s_{22}}{t_1^{n+2}-t_{n+2}} & (n+1)s_{21} - \frac{(n+1)(n+2)}{2}\frac{t_1^{n+1}s_{22}}{t_1^{n+2}-t_{n+2}}
\end{pmatrix},$$

hence

(4.3)

$$S\,\mathsf{B}(\mathsf{R})=\begin{pmatrix} s_{21} & s_{22} & 0 & 0 & \dots & 0 & 0\\ s_{21}s_{21} & s_{21}s_{22}+2s_{22}s_{21} & s_{22}s_{22} & 0 & \dots & 0 & 0\\ s_{31}s_{21} & s_{31}s_{22}+2s_{32}s_{21} & s_{32}s_{22}+3s_{33}s_{21} & s_{33}s_{22} & \dots & 0 & 0\\ s_{41}s_{21} & s_{41}s_{22}+2s_{42}s_{21} & s_{42}s_{22}+3s_{43}s_{21} & s_{43}s_{22}+4s_{44}s_{21} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ s_{n1}s_{21} & s_{n1}s_{22}+2s_{n2}s_{21} & s_{n2}s_{22}+3s_{n3}s_{21} & s_{n3}s_{22}+4s_{n4}s_{21} & \dots & s_{n(n-1)}s_{22}+ns_{nn}s_{21} & s_{nn}s_{22}\\ S\mathsf{B}(\mathsf{R})[n+1,1] & S\mathsf{B}(\mathsf{R})[n+1,2] & S\mathsf{B}(\mathsf{R})[n+1,3] & S\mathsf{B}(\mathsf{R})[n+1,4] & \dots & S\mathsf{B}(\mathsf{R})[n+1,n] & S\mathsf{B}(\mathsf{R})[n+1,n+1] \end{pmatrix},$$

in which:

$$S\mathsf{B}(\mathsf{R})[n+1,1]=s_{(n+1)1}s_{21}-\frac{S_2(n+2,1)t_1s_{22}s_{(n+1)(n+1)}}{t_1^{n+2}-t_{n+2}}\,,$$

$$S\mathsf{B}(\mathsf{R})[n+1,j]=s_{(n+1)(j-1)}s_{22}+js_{(n+1)j}s_{21}-\frac{S_2(n+2,j)t_1^js_{22}s_{(n+1)(n+1)}}{t_1^{n+2}-t_{n+2}}\,,\quad 2\le j\le n\,,$$

$$S\mathsf{B}(\mathsf{R})[n+1,n+1]=s_{(n+1)n}s_{22}+s_{(n+1)(n+1)}\Bigg((n+1)s_{21}-\frac{(n+1)(n+2)}{2}\,\frac{t_1^{n+1}s_{22}}{t_1^{n+2}-t_{n+2}}\Bigg)\,.$$

Observe that

(4.4)

$$\mathsf{Y}S=\begin{pmatrix} s_{21} & s_{22} & 0 & 0 & \dots & 0 & 0\\ \mathsf{Y}_1s_{31} & \mathsf{Y}_1s_{32} & \mathsf{Y}_1s_{33} & 0 & \dots & 0 & 0\\ \mathsf{Y}_2s_{41} & \mathsf{Y}_2s_{42} & \mathsf{Y}_2s_{43} & \mathsf{Y}_2s_{44} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathsf{Y}_{n-2}s_{n1} & \mathsf{Y}_{n-2}s_{n2} & \mathsf{Y}_{n-2}s_{n3} & \mathsf{Y}_{n-2}s_{n4} & \dots & \mathsf{Y}_{n-2}s_{nn} & 0\\ -s_{(n+1)1} & -s_{(n+1)2} & -s_{(n+1)3} & -s_{(n+1)4} & \dots & -s_{(n+1)n} & -s_{(n+1)(n+1)}\\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

and (3.36)-(3.38) imply that $\deg(\mathsf{Y}_1) = \deg(\mathsf{Y}_{n-2}) = 3$ and $\deg(\mathsf{Y}_j) = 2$, $2 \leq j \leq n-3$. If we denote the (i,j)-th entry of $\dot{S}_{\mathsf{R}}$ by $\dot{S}_{\mathsf{R}}[i,j]$, then (4.1), (4.3) and (4.4) yield $\deg(\dot{S}_{\mathsf{R}}[i,j]) = \deg(s_{ij}) + 2$, $2 \leq i \leq n+1$, $1 \leq j \leq i$, which complete the proof. □

Remark 4.1. *Using the matrix $\dot{S}_{\mathsf{R}} = \mathsf{Y}S - S\,\mathsf{B}(\mathsf{R})$ computed in the proof of the above proposition we can encounter the modular vector field R explicitly for any $n \geq 5$.*

We can prove Proposition 4.1 in a simpler way, but we can not compute R explicitly in this case. Below we give this proof as well.

Another proof of Proposition 4.1. Since $\mathsf{H} = \sum_{j=1}^{\mathsf{d}} w_j \frac{\partial}{\partial t_j}$ and $\deg(t_j) = w_j$, we can easily observe that for a given $f \in \mathscr{O}_{\mathsf{T}}$ we have $\mathsf{H}(f) = kf$, for a $k \in \mathbb{Z}$, if and only if f is a quasi-homogeneous element of degree k of $\mathscr{O}_{\mathsf{T}}$. According to $[\mathsf{H}, \mathsf{R}] = 2\mathsf{R}$, for any quasi-homogeneous element $f \in \mathsf{T}$ of degree $k \in \mathbb{Z}$ we have:

$$[\mathsf{H}, \mathsf{R}](f) = 2\mathsf{R}(f) \Rightarrow \mathsf{H}(\mathsf{R}(f)) - \mathsf{R}(\mathsf{H}(f)) = 2\mathsf{R}(f) \Rightarrow \mathsf{H}(\mathsf{R}(f)) = (k+2)\mathsf{R}(f),$$

which implies $\mathsf{R}(f)$ is a quasi-homogeneous element of degree $k+2$. This is equivalent to say that R is a quasi-homogeneous vector field of degree 2. □

The following lemma is useful for future use.

Lemma 4.1. *If we write*

$$\mathsf{R} = \sum_{j=1}^{\mathsf{d}} \mathsf{R}^j(t_1, t_2, \ldots, t_{\mathsf{d}}) \frac{\partial}{\partial t_j}, \ \textit{with} \ \ \mathsf{R}^j \in \mathscr{O}_{\mathsf{T}}\,,$$

and define

$$\Lambda(t_1, t_2, \ldots, t_{\mathsf{d}}) := \begin{cases} -\frac{1}{2}\mathsf{R}^2(t_1, t_2, \ldots, t_{\mathsf{d}}) - \frac{1}{4}t_2^2, & \textit{if } n = 2; \\ -\mathsf{R}^2(t_1, t_2, \ldots, t_{\mathsf{d}}) - t_2^2, & \textit{if } n \neq 2, \end{cases} \tag{4.5}$$

then $\deg(\Lambda) = 4$ *and* $\frac{\partial \Lambda}{\partial t_2} = 0$.

Proof. For $n = 1, 2, 3, 4$ the modular vector field R has been explicitly stated in Example 5.1 and one can easily check the truth of the statement. For $n \geq 5$ the component R^2 of the modular vector field R corresponds to the

$(2,1)$-th entry of the matrix $\dot{S}_{\mathsf{R}} = \mathsf{Y}S - S\,\mathsf{B}(\mathsf{R})$ computed in the proof of Proposition 4.1 that yields:

$$\mathsf{R}^2(t_1, t_2, \ldots, t_{\mathsf{d}}) = \mathsf{Y}_1 t_4 - t_2^2, \quad \text{(note that } t_2 = s_{21} \text{ and } t_4 = s_{31}\text{)}.$$

From (3.36) and (3.38) we get $\mathsf{Y}_1 = \frac{s_{22}^2}{s_{33}} = \frac{t_3^2}{t_6}$, which implies:

$$\mathsf{R}^2(t_1, t_2, \ldots, t_{\mathsf{d}}) = \frac{t_3^2 t_4}{t_6} - t_2^2 .$$

Hence, for $n \geq 5$ we obtain $\Lambda = -\frac{t_3^2 t_4}{t_6}$ and the proof is complete. □

4.1. A new chart

Remember from Section 2 that the Ramanujan vector field $\mathsf{Ra} = \frac{1}{12}(t_1^2 - t_2)\frac{\partial}{\partial t_1} + \frac{1}{3}(t_1 t_2 - t_3)\frac{\partial}{\partial t_2} + \frac{1}{2}(t_1 t_3 - t_2^2)\frac{\partial}{\partial t_3}$ along with $H = 2t_1\frac{\partial}{\partial t_1} + 4t_2\frac{\partial}{\partial t_2} + 6t_3\frac{\partial}{\partial t_3}$ and $F = -12\frac{\partial}{\partial t_1}$ generates a copy of $\mathfrak{sl}_2(\mathbb{C})$, and $(t_1, t_2, t_3) = (E_2, E_4, E_6)$ is a solution of Ra. Note that $\deg(E_2) = 2,\ \deg(E_4) = 4,\ \deg(E_6) = 6$ and these integers appear as coefficients of the components of the vector field H. Moreover, we have $F(t_1) = -12 \neq 0,\ F(t_2) = F(t_3) = 0$; indeed, if we consider F as a derivation on $\widetilde{\mathcal{M}}$, then $\ker F = \mathcal{M}$. In the case of modular vector fields, we observed in Theorem 3.4 that the Lie algebra generated by $\mathsf{R}, \mathsf{H}, \mathsf{F}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. According to Example 5.1, (3.50) and (3.52), we find the vector field F for any positive integer n as follows:

$$\mathsf{F} = (1 + \delta_2^n)\frac{\partial}{\partial t_2}, \quad \text{if } n = 1 \text{ or } n \text{ is even}, \tag{4.6}$$

$$\mathsf{F} = \frac{\partial}{\partial t_2} - t_4\frac{\partial}{\partial t_7}, \quad \text{if } n = 3, \tag{4.7}$$

$$\mathsf{F} = \frac{\partial}{\partial t_2} - t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}}}, \quad \text{if } n \geq 5 \text{ is odd}. \tag{4.8}$$

For any n we have $\mathsf{F}(t_2) = 1 + \delta_2^n \neq 0$. If $n = 1$ or n is even, then $\mathsf{F}(t_j) = 0$ for all $1 \leq j \leq \mathsf{d}$ and $j \neq 2$. But if $n \geq 3$ is odd, then, besides $\mathsf{F}(t_2) \neq 0$, we also have $\mathsf{F}(t_7) = -t_4 \neq 0$, when $n = 3$, and $\mathsf{F}(t_{\mathsf{d}}) = -t_{\mathsf{d}-2} \neq 0$, when $n \geq 5$. These will cause problems for our purposes in Section 5, when $n \geq 3$ is odd. To overcome these problems, we need to change one of the coordinate charts of T. If for any positive odd integer $n \geq 3$ we set:

$$\tilde{t}_{\mathsf{d}} := \begin{cases} t_7 + t_2 t_4, & \text{if } \ n = 3; \\ t_{\mathsf{d}} + t_2 t_{\mathsf{d}-2}, & \text{if } \ n \geq 5 \ \text{ is } \ \text{odd}; \end{cases} ,$$

then it is easy to observe that $\mathsf{F}(\tilde{t}_{\mathsf{d}})=0$. According to (3.51), we get that $\mathsf{H}(\tilde{t}_{\mathsf{d}})=2$, which means $\deg(\tilde{t}_{\mathsf{d}})=2$. If instead the chart $(t_1,t_2,\ldots,t_{\mathsf{d}})$ for T, we use the chart $(t_1,t_2,\ldots,t_{\mathsf{d}-1},\tilde{t}_{\mathsf{d}})$, then we have:

$$\mathsf{R}=\sum_{j=1}^{\mathsf{d}-1}\mathsf{R}^j\frac{\partial}{\partial t_j}+\tilde{\mathsf{R}}^{\mathsf{d}}\frac{\partial}{\partial\tilde{t}_{\mathsf{d}}},\ \text{where } \tilde{\mathsf{R}}^{\mathsf{d}}=\mathsf{R}(\tilde{t}_{\mathsf{d}}), \tag{4.9}$$

$$\mathsf{H}=\sum_{j=1}^{\mathsf{d}-1}w_jt_j\frac{\partial}{\partial t_j}+2\tilde{t}_{\mathsf{d}}\frac{\partial}{\partial\tilde{t}_{\mathsf{d}}},\ \text{where } w_j,\ j=1,2,\ldots,\mathsf{d}-1, \text{ are as before}, \tag{4.10}$$

$$\mathsf{F}=\frac{\partial}{\partial t_2}\,. \tag{4.11}$$

For simplicity, from now on, we denote $\tilde{t}_{\mathsf{d}}$ also by t_{d}, but we remember that whenever $n\geq 3$ is an odd positive integer, then we consider the chart $(t_1,t_2,\ldots,t_{\mathsf{d}-1},\tilde{t}_{\mathsf{d}})$, and $\mathsf{R},\mathsf{H},\mathsf{F}$ are the ones given in (4.9), (4.10), (4.11), respectively.

5. Rankin-Cohen algebras for CY (quasi-)modular forms

Let us suppose that a solution of the vector field R is given by $(\mathsf{t}_1,\mathsf{t}_2,\ldots,\mathsf{t}_{\mathsf{d}})$. The reader should take care to differ the notations $\mathsf{t}_1,\mathsf{t}_2,\ldots,\mathsf{t}_{\mathsf{d}}$ which stand for solution components of R from the notations $t_1,t_2,\ldots,t_{\mathsf{d}}$ which are used for the coordinate charts of T (also note that for the positive odd integers $n\geq 3$ we have $t_{\mathsf{d}}=\tilde{t}_{\mathsf{d}}$). Nevertheless, any solution component t_i is associated with the coordinate chart t_i. We define the *space of CY quasi-modular forms* $\widetilde{\mathscr{M}}$, and the *space of CY modular forms* $\mathscr{M}$*, associated to the modular vector field* R, respectively, as follows (from now on, if there is no danger of confusion, we drop the part "associated to the modular vector field R"):

$$\widetilde{\mathscr{M}}:=\mathbb{C}[\mathsf{t}_1,\mathsf{t}_2,\mathsf{t}_3,\ldots,\mathsf{t}_{\mathsf{d}},\frac{1}{\mathsf{t}_{n+2}(\mathsf{t}_{n+2}-\mathsf{t}_1^{n+2})\check{\mathsf{t}}}]\,, \tag{5.1}$$

$$\mathscr{M}:=\mathbb{C}[\mathsf{t}_1,\widehat{\mathsf{t}_2},\mathsf{t}_3,\mathsf{t}_4,\ldots,\mathsf{t}_{\mathsf{d}},\frac{1}{\mathsf{t}_{n+2}(\mathsf{t}_{n+2}-\mathsf{t}_1^{n+2})\check{\mathsf{t}}}]\,, \tag{5.2}$$

in which $\check{\mathsf{t}}$ is associated with $\check{t}$ given in (3.31) or (3.32) and the symbol $\widehat{\mathsf{t}_2}$ means that the component t_2 is omitted, i.e., $\mathsf{t}_2\notin\mathscr{M}$. Indeed, we have $\widetilde{\mathscr{M}}=\mathscr{M}[\mathsf{t}_2]$ and, in our generalization, the CY quasi-modular form t_2 has the role of the quasi-modular form E_2 in the theory of quasi-modular forms. Let us attach to any solution component t_i, $1\leq i\leq\mathsf{d}$, the weight $\deg(\mathsf{t}_i)=$

w_i, in which the non-negative integers w_i's are given in (3.53). For any integer $r \in \mathbb{Z}$ we define $\widetilde{\mathscr{M}_r}$ and $\mathscr{M}_r$ to be the $\mathbb{C}$-vector spaces generated by $\{ f \in \widetilde{\mathscr{M}} \mid \deg(f) = r \}$ and $\{ f \in \mathscr{M} \mid \deg(f) = r \}$, respectively. Note that any constant in $\mathbb{C}$ is considered as a weight zero CY (quasi-)modular form. Therefore, elements of $\widetilde{\mathscr{M}_r}$ and $\mathscr{M}_r$ are CY quasi-modular forms and CY modular forms of weight r, respectively. In particular, t_2 is a CY quasi-modular form of weight 2 which is not modular, see Remark 3.3, and the other t_j's, $1 \leq j \leq \mathsf{d}$ and $j \neq 2$, are CY modular forms of weight w_j. In particular we have:

$$\widetilde{\mathscr{M}} = \bigoplus_{r\in\mathbb{Z}} \widetilde{\mathscr{M}_r} \quad \text{and} \quad \mathscr{M} = \bigoplus_{r\in\mathbb{Z}} \mathscr{M}_r \,. \tag{5.3}$$

Thus, $\widetilde{\mathscr{M}}$ and $\mathscr{M}$ are commutative and associative graded algebras on $\mathbb{C}$.

Notation 5.1. *From now on $\mathscr{R}$, $\mathscr{H}$ and $\mathscr{F}$ refer to the differential operators on $\widetilde{\mathscr{M}}$ induced by the vector fields R, H and F, respectively, in which we substitute the coordinate chart t_j, $1 \leq j \leq \mathsf{d}$, by the solution component t_j and $\frac{\partial}{\partial t_j}$ by the partial derivation $\frac{\partial}{\partial \mathsf{t}_j}$. For example, if we write $\mathsf{R} = \sum_{j=1}^{\mathsf{d}} \mathsf{R}^j(t_1, t_2, \ldots, t_d)\frac{\partial}{\partial t_j}$, with $\mathsf{R}^j(t_1, t_2, \ldots, t_d) \in \mathscr{O}_{\mathsf{T}}$, then the corresponding differential operator is $\mathscr{R} = \sum_{j=1}^{\mathsf{d}} \mathsf{R}^j(\mathsf{t}_1, \mathsf{t}_2, \ldots, \mathsf{t}_d)\frac{\partial}{\partial \mathsf{t}_j}$. We consider the Lie bracket of the such obtained differential operators the same as the Lie bracket of the associated vector fields. Hence, due to Theorem 3.4 we get:*

$$[\mathscr{R}, \mathscr{F}] = \mathscr{H} \;,\quad [\mathscr{H}, \mathscr{R}] = 2\mathscr{R} \;,\quad [\mathscr{H}, \mathscr{F}] = -2\mathscr{F} \,. \tag{5.4}$$

We recall that, for an integer d, a degree d differential operator D on $\widetilde{\mathscr{M}}$, denoted by $D : \widetilde{\mathscr{M}_*} \to \widetilde{\mathscr{M}_{*+d}}$, is a differential operator that satisfies $D(\widetilde{\mathscr{M}_r}) \subseteq \widetilde{\mathscr{M}_{r+d}}$ for any positive integer r. Indeed, if we can write $D = \sum_{j=1}^{\mathsf{d}} D^j \frac{\partial}{\partial \mathsf{t}_j}$, with $D^j \in \widetilde{\mathscr{M}}$, then D has degree d provided $\deg(D^j) - w_j = d$ for any $1 \leq j \leq \mathsf{d}$. A degree d differential operator on $\mathscr{M}$ is defined analogously. Hence, due to Proposition 4.1, $\mathscr{R}$ is a degree 2 differential operator on $\widetilde{\mathscr{M}}$.

Definition 5.1. *By the* Ramanujan-Serre–type derivation ∂ *on $\widetilde{\mathscr{M}}$ we mean the differential operator that on the generators of $\widetilde{\mathscr{M}}$ is defined as follows:*

$$\partial f := \mathscr{R} f + (1 - \frac{1}{2}\delta_2^n) r \mathsf{t}_2 f \,,\quad \forall f \in \widetilde{\mathscr{M}_r} \text{ and } \forall r \in \mathbb{Z}. \tag{5.5}$$

We would like that the derivation $\mathscr{R}$ and the Ramanujan-Serre–type derivation ∂ behave the same as the usual derivation (2.4) and the Ramanujan-Serre derivation (2.3) of the classical quasi-modular form theory, respectively. In the following example we state the derivations $\mathscr{R}$ and ∂ explicitly for $n = 1, 2, 3, 4$.

Example 5.1. In [21] we found $\mathsf{R}, \mathsf{H}, \mathsf{F}$ explicitly for $n = 1, 2, 3, 4$. In these cases, we obtain the derivation Ramanujan-Serre–type derivation ∂ as follows:

- $n = 1$.

$$\mathsf{R} = (-t_1t_2 - 9(t_1^3 - t_3))\frac{\partial}{\partial t_1} + (81t_1(t_1^3 - t_3) - t_2^2)\frac{\partial}{\partial t_2} + (-3t_2t_3)\frac{\partial}{\partial t_3}\,,$$

$$\mathsf{H} = t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2} + 3t_3\frac{\partial}{\partial t_3}, \tag{5.6}$$

$$\mathsf{F} = \frac{\partial}{\partial t_2}\,. \tag{5.7}$$

By definition, the vector field (5.6) implies $\deg(\mathsf{t}_1) = 1$, $\deg(\mathsf{t}_2) = 2$ and $\deg(\mathsf{t}_3) = 3$. Hence, we observe that:

$$\partial = -9(\mathsf{t}_1^3 - \mathsf{t}_3)\frac{\partial}{\partial \mathsf{t}_1} + (81\mathsf{t}_1(\mathsf{t}_1^3 - \mathsf{t}_3) + \mathsf{t}_2^2)\frac{\partial}{\partial \mathsf{t}_2}\,. \tag{5.8}$$

If we let ∂ acts just on $\mathscr{M}$, then we get:

$$\partial = -9(\mathsf{t}_1^3 - \mathsf{t}_3)\frac{\partial}{\partial \mathsf{t}_1}\,.$$

- $n = 2$.

$$\mathsf{R} = (t_3 - t_1t_2)\frac{\partial}{\partial t_1} + (2t_1^2 - \frac{1}{2}t_2^2)\frac{\partial}{\partial t_2} + (-2t_2t_3 + 8t_1^3)\frac{\partial}{\partial t_3} + (-4t_2t_4)\frac{\partial}{\partial t_4}\,, \tag{5.9}$$

$$\mathsf{H} = 2t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2} + 4t_3\frac{\partial}{\partial t_3} + 8t_4\frac{\partial}{\partial t_4}, \tag{5.10}$$

$$\mathsf{F} = 2\frac{\partial}{\partial t_2}\,, \tag{5.11}$$

where the polynomial equation $t_3^2 = 4(t_1^4 - t_4)$ holds among t_i's. From (5.10) we get $\deg(\mathsf{t}_1) = 2$, $\deg(\mathsf{t}_2) = 2$, $\deg(\mathsf{t}_3) = 4$ and $\deg(\mathsf{t}_4) = 8$. Thus, we find:

$$\partial = \mathsf{t}_3 \frac{\partial}{\partial \mathsf{t}_1} + (2\mathsf{t}_1^2 + \frac{1}{2}\mathsf{t}_2^2)\frac{\partial}{\partial \mathsf{t}_2} + 8\mathsf{t}_1^3 \frac{\partial}{\partial \mathsf{t}_3}\,. \tag{5.12}$$

In the case that ∂ is considered on $\mathscr{M}$ we have:

$$\partial = \mathsf{t}_3 \frac{\partial}{\partial \mathsf{t}_1} + 8\mathsf{t}_1^3 \frac{\partial}{\partial \mathsf{t}_3}\,.$$

- $n = 3$.

$$\begin{aligned}
\mathsf{R} &= (t_3 - t_1 t_2)\frac{\partial}{\partial t_1} + \frac{t_3^3 t_4 - 5^4 t_2^2 (t_1^5 - t_5)}{5^4 (t_1^5 - t_5)} \frac{\partial}{\partial t_2} \\
&\quad + \frac{t_3^3 t_6 - 3 \times 5^4 t_2 t_3 (t_1^5 - t_5)}{5^4 (t_1^5 - t_5)} \frac{\partial}{\partial t_3} + (-t_2 t_4 - t_7)\frac{\partial}{\partial t_4} \\
&\quad + (-5 t_2 t_5)\frac{\partial}{\partial t_5} + (5^5 t_1^3 - t_2 t_6 - 2 t_3 t_4)\frac{\partial}{\partial t_6} + (-5^4 t_1 t_3 - t_2 t_7)\frac{\partial}{\partial t_7}\,, \\
\mathsf{H} &= t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2} + 3t_3 \frac{\partial}{\partial t_3} + 5t_5 \frac{\partial}{\partial t_5} + t_6 \frac{\partial}{\partial t_6} + 2t_7 \frac{\partial}{\partial t_7}\,, \\
\mathsf{F} &= \frac{\partial}{\partial t_2} - t_4 \frac{\partial}{\partial t_7}\,.
\end{aligned}$$

We obtain $\deg(\mathsf{t}_1) = 1$, $\deg(\mathsf{t}_2) = 2$, $\deg(\mathsf{t}_3) = 3$, $\deg(\mathsf{t}_4) = 0$, $\deg(\mathsf{t}_5) = 5$, $\deg(\mathsf{t}_6) = 1$ and $\deg(\mathsf{t}_7) = 2$. Remember that in this case we substitute the coordinate t_7 by:

$$\tilde{t}_7 := t_7 + t_2 t_4.$$

from which we obtain:

$$\mathsf{R}(\tilde{t}_7) = -5^4 t_1 t_3 + \frac{t_3^3 t_4^2}{5^4 (t_1^5 - t_5)} - 2 t_2 \tilde{t}_7.$$

Note that here t_7 is the component of a solution of R associated with coordinate $\tilde{t}_7$. Hence, we get the derivation $\mathscr{R}$ on $\widetilde{\mathscr{M}}$ as follows:

$$\mathscr{R} = \Big(\mathsf{t}_3 - \mathsf{t}_2 \mathsf{t}_1\Big)\frac{\partial}{\partial \mathsf{t}_1} + \left(\frac{\mathsf{t}_3^3 \mathsf{t}_4}{5^4(\mathsf{t}_1^5 - \mathsf{t}_5)} - \mathsf{t}_2^2\right) \frac{\partial}{\partial \mathsf{t}_2}$$

$$+\left(\frac{\mathsf{t}_3^3\mathsf{t}_6}{5^4(\mathsf{t}_1^5-\mathsf{t}_5)}-3\mathsf{t}_2\mathsf{t}_3\right)\frac{\partial}{\partial \mathsf{t}_3}-\mathsf{t}_7\frac{\partial}{\partial \mathsf{t}_4}-5\mathsf{t}_2\mathsf{t}_5\frac{\partial}{\partial \mathsf{t}_5}$$
$$+\left(5^5\mathsf{t}_1^3-2\mathsf{t}_3\mathsf{t}_4-\mathsf{t}_2\mathsf{t}_6\right)\frac{\partial}{\partial \mathsf{t}_6}+\left(-5^4\mathsf{t}_1\mathsf{t}_3+\frac{\mathsf{t}_3^3\mathsf{t}_4^2}{5^4(\mathsf{t}_1^5-\mathsf{t}_5)}-2\mathsf{t}_2\mathsf{t}_7\right)\frac{\partial}{\partial \mathsf{t}_7},$$

and we obtain ∂ on $\mathscr{M}$ as follows:

$$\partial=\mathsf{t}_3\frac{\partial}{\partial \mathsf{t}_1}+\frac{\mathsf{t}_3^3\mathsf{t}_6}{5^4(\mathsf{t}_1^5-\mathsf{t}_5)}\frac{\partial}{\partial \mathsf{t}_3}-\mathsf{t}_7\frac{\partial}{\partial \mathsf{t}_4}+(5^5\mathsf{t}_1^3-2\mathsf{t}_3\mathsf{t}_4)\frac{\partial}{\partial \mathsf{t}_6}$$
$$-\left(5^4\mathsf{t}_1\mathsf{t}_3-\frac{\mathsf{t}_3^3\mathsf{t}_4^2}{5^4(\mathsf{t}_1^5-\mathsf{t}_5)}\right)\frac{\partial}{\partial \mathsf{t}_7}.$$

- $n=4$.

$$\mathsf{R}=(t_3-t_1t_2)\frac{\partial}{\partial t_1}+\frac{6^{-2}t_3^2t_4t_8-t_1^6t_2^2+t_2^2t_6}{t_1^6-t_6}\frac{\partial}{\partial t_2}$$
$$+\frac{6^{-2}t_3^2t_5t_8-3t_1^6t_2t_3+3t_2t_3t_6}{t_1^6-t_6}\frac{\partial}{\partial t_3}$$
$$+\frac{-6^{-2}t_3^2t_7t_8-t_1^6t_2t_4+t_2t_4t_6}{t_1^6-t_6}\frac{\partial}{\partial t_4}$$
$$+\frac{6^{-2}t_3t_5^2t_8-4t_1^6t_2t_5-2t_1^6t_3t_4+5t_1^4t_3t_8+4t_2t_5t_6+2t_3t_4t_6}{2(t_1^6-t_6)}\frac{\partial}{\partial t_5}$$
$$-6t_2t_6\frac{\partial}{\partial t_6}+\frac{6^{-2}t_4^2-t_1^2}{2\times 6^{-2}}\frac{\partial}{\partial t_7}+\frac{-3t_1^6t_2t_8+3t_1^5t_3t_8+3t_2t_6t_8}{t_1^6-t_6}\frac{\partial}{\partial t_8},$$
$$\mathsf{H}=t_1\frac{\partial}{\partial t_1}+2t_2\frac{\partial}{\partial t_2}+3t_3\frac{\partial}{\partial t_3}+t_4\frac{\partial}{\partial t_4}+2t_5\frac{\partial}{\partial t_5}+6t_6\frac{\partial}{\partial t_6}+3t_8\frac{\partial}{\partial t_8},$$
$$\mathsf{F}=\frac{\partial}{\partial t_2},$$

where the equation $t_8^2=36(t_1^6-t_6)$ holds among t_i's. Analogous to the previous cases we have $\deg(\mathsf{t}_1)=1$, $\deg(\mathsf{t}_2)=2$, $\deg(\mathsf{t}_3)=3$, $\deg(\mathsf{t}_4)=1$, $\deg(\mathsf{t}_5)=2$, $\deg(\mathsf{t}_6)=6$, $\deg(\mathsf{t}_7)=0$, $\deg(\mathsf{t}_8)=3$. Therefore, (5.5) yields the Ramanujan-Serre–type derivation on $\mathscr{M}$ as follows:

$$\partial=\mathsf{t}_3\frac{\partial}{\partial \mathsf{t}_1}+\frac{6^{-2}\mathsf{t}_3^2\mathsf{t}_5 t_8}{\mathsf{t}_1^6-\mathsf{t}_6}\frac{\partial}{\partial \mathsf{t}_3}-\frac{6^{-2}\mathsf{t}_3^2\mathsf{t}_7\mathsf{t}_8}{\mathsf{t}_1^6-\mathsf{t}_6}\frac{\partial}{\partial \mathsf{t}_4}$$
$$+\frac{6^{-2}\mathsf{t}_3\mathsf{t}_5^2\mathsf{t}_8-2\mathsf{t}_1^6\mathsf{t}_3\mathsf{t}_4+5\mathsf{t}_1^4\mathsf{t}_3\mathsf{t}_8+2\mathsf{t}_3\mathsf{t}_4\mathsf{t}_6}{2(\mathsf{t}_1^6-\mathsf{t}_6)}\frac{\partial}{\partial \mathsf{t}_5} \tag{5.13}$$

$$+\frac{6^{-2}\mathsf{t}_4^2-\mathsf{t}_1^2}{2\times 6^{-2}}\frac{\partial}{\partial \mathsf{t}_7}+\frac{3\mathsf{t}_1^5\mathsf{t}_3\mathsf{t}_8}{\mathsf{t}_1^6-\mathsf{t}_6}\frac{\partial}{\partial \mathsf{t}_8}\,.$$

Remark 5.1. *If we look closely to all cases stated in Example 5.1 we find out that the derivation $\mathscr{R}$ and the Ramanujan-Serre–type derivation ∂ have degree 2. Besides these, the Ramanujan-Serre–type derivation ∂ sends any element of $\mathscr{M}$ to another element of $\mathscr{M}$. More precisely, the same as what we mentioned for the Ramanujan-Serre derivation given in (2.5), in all the above cases we observe that for any $f\in\mathscr{M}_r$ the term $(1-\frac{1}{2}\delta_2^n)r\mathsf{t}_2 f$ in (5.5) kills all the terms including t_2 in $\mathscr{R}f$ which implies $\partial f\in\mathscr{M}_{r+2}$, and consequently $\mathscr{M}$ is closed under ∂. All these facts hold for any positive integer n which are stated in Theorem 1.1.*

Now we are in the situation that we can present the proof of Theorem 1.1.

Proof of Theorem 1.1.

1. This item follows immediately from Proposition 4.1.
2. First note that according to Remark 3.3 we always have $\deg(\mathsf{t}_2)=w_2=2$. Hence, from part 1 and (5.5) we deduce that ∂ is a degree 2 differential operator. To prove that for all $f\in\mathscr{M}$ we get $\partial f\in\mathscr{M}$, it is enough to observe that for all integers r and for all $f\in\mathscr{M}_r$ we have $\partial f\in\mathscr{M}_{r+2}$, which is equivalent to:

$$\begin{aligned}
\partial\mathsf{t}_j\in\mathscr{M}_{w_j+2}\,,\ \ \forall j\neq 2, &\Leftrightarrow (1+\delta_2^n)\frac{\partial}{\partial\mathsf{t}_2}(\partial\mathsf{t}_j)=0\,,\quad \forall j\neq 2,\\
&\Leftrightarrow (1+\delta_2^n)\frac{\partial}{\partial\mathsf{t}_2}(\mathscr{R}\mathsf{t}_j+(1-\frac{\delta_2^n}{2})w_j\mathsf{t}_2\mathsf{t}_j)=0\,,\quad \forall j\neq 2,\\
&\Leftrightarrow (1+\delta_2^n)\frac{\partial}{\partial\mathsf{t}_2}(\mathscr{R}\mathsf{t}_j)=-w_j\mathsf{t}_j\,,\quad \forall j\neq 2,\\
&\Leftarrow \sum_{j=1}^{\mathsf{d}}(1+\delta_2^n)\frac{\partial}{\partial\mathsf{t}_2}(\mathscr{R}\mathsf{t}_j)\frac{\partial}{\partial\mathsf{t}_j}=-\sum_{j=1}^{\mathsf{d}}w_j\mathsf{t}_j\frac{\partial}{\partial\mathsf{t}_j}=-\mathscr{H}\,,\\
&\Leftrightarrow [(1+\delta_2^n)\frac{\partial}{\partial\mathsf{t}_2},\mathscr{R}]=-\mathscr{H}\,,\\
&\Leftrightarrow [\mathscr{R},(1+\delta_2^n)\frac{\partial}{\partial\mathsf{t}_2}]=\mathscr{H}\,.
\end{aligned}$$

Note that $\mathscr{F}=(1+\delta_2^n)\frac{\partial}{\partial\mathsf{t}_2}$ and the last affirmation is valid due to (5.4). □

Next, to use Proposition 2.1, we need the CY quasi-modular forms of positive weight. Hence, we consider the spaces of CY quasi-modular forms $\widetilde{\mathscr{M}}^{>0}$ and CY modular forms $\mathscr{M}^{>0}$ of positive weight as follows:

$$\widetilde{\mathscr{M}}^{>0} := \bigoplus_{r\geq 0} \widetilde{\mathscr{M}_r} \quad , \quad \mathscr{M}^{>0} := \bigoplus_{r\geq 0} \mathscr{M}_r \,, \tag{5.14}$$

in which we suppose that $\widetilde{\mathscr{M}_0} = \mathscr{M}_0 = \mathbb{C}$. Thus, the space of CY quasi-modular forms of positive weight $\widetilde{\mathscr{M}}^{>0}$ is a commutative and associative graded algebra with unit over the field $\mathbb{C}$ together with the derivation $\mathscr{R} : \widetilde{\mathscr{M}}^{>0}_* \to \widetilde{\mathscr{M}}^{>0}_{*+2}$ of degree 2. Therefore, due to Remark 2.1, $(\widetilde{\mathscr{M}}^{>0}, [\cdot,\cdot]_{\mathscr{R},*})$ is a standard Rankin-Cohen, and hence a Rankin-Cohen algebra. We call $[\cdot,\cdot]_{\mathscr{R},*}$ the *Rankin-Cohen bracket for CY quasi-modular forms*, which for any non-negative integers k, r, s and any $f \in \widetilde{\mathscr{M}_r}\,,\ g \in \widetilde{\mathscr{M}_s}$ is defined:

$$[f,g]_{\mathscr{R},k} := \sum_{i+j=k} (-1)^j \binom{k+r-1}{i} \binom{k+s-1}{j} f^{(j)} g^{(i)} \,, \tag{5.15}$$

where $f^{(j)} = \mathscr{R}^j f$ and $g^{(j)} = \mathscr{R}^j g$ refer to the j-th derivative of f and g under $\mathscr{R}$, respectively. It is evident that $[f,g]_{\mathscr{R},k} \in \widetilde{\mathscr{M}_{r+s+2k}}$. Next, we demonstrate Theorem 1.2 which shows that the space of CY modular forms of positive weight $\mathscr{M}^{>0}$ is closed under the Rankin-Cohen bracket for CY quasi-modular forms given in (5.15).

Proof of Theorem 1.2.
The idea of the proof is to use Proposition 2.1 and its proof. To this end, first note that according to the part 2 of Theorem 1.1 the Ramanujan-Serre–type derivation $\partial : \mathscr{M}^{>0}_* \to \mathscr{M}^{>0}_{*+2}$ is a degree 2 differential operator. If we set $\Lambda = \Lambda(\mathsf{t}_1, \mathsf{t}_2, \ldots, \mathsf{t}_\mathsf{d})$, where Λ is given in Lemma 4.1, then the same lemma yields $\Lambda \in \mathscr{M}_4$. Therefore, from Proposition 2.1 we get that $(\mathscr{M}^{>0}, [\cdot,\cdot]_{\partial,\Lambda,*})$, where the k-th bracket $[\cdot,\cdot]_{\partial,\Lambda,k}\,,\ k \geq 0$, is given by (2.19), is a canonical Rankin-Cohen algebra. On the other hand, by letting $\lambda = (\frac{1}{2}\delta_2^n - 1)\mathsf{t}_2\,$, from (5.5) we obtain

$$\mathscr{R} f = \partial f + r\lambda f \,,\ \forall f \in \mathscr{M}_r \,. \tag{5.16}$$

Furthermore, if we write $\mathscr{R} = \sum_{j=1}^{\mathsf{d}} \mathsf{R}^j \frac{\partial}{\partial \mathsf{t}_j}$, with $\mathsf{R}^j \in \widetilde{\mathscr{M}}$, then

$$\mathscr{R}(\lambda) = (\frac{1}{2}\delta_2^n - 1)\mathscr{R}(\mathsf{t}_2) = (\frac{1}{2}\delta_2^n - 1)\mathsf{R}^2 \,, \tag{5.17}$$

which along with (4.5) implies:

$$\mathscr{R}(\lambda) = \Lambda + \lambda^2 . \tag{5.18}$$

The relations (5.16) and (5.18) show that (2.21) is satisfied. Hence, from the proof of Proposition 2.1 we obtain $[\cdot,\cdot]_{\partial,\Lambda,*} = [\cdot,\cdot]_{\mathscr{R},*}$ (see (2.22)). Finally, since $\mathscr{M}^{>0}$ is closed under $[\cdot,\cdot]_{\partial,\Lambda,*}$, we conclude that $\mathscr{M}^{>0}$ is closed under $[\cdot,\cdot]_{\mathscr{R},*}$, and this finishes the proof of the theorem. □

In particular, Theorem 1.2 implies that $(\mathscr{M}^{>0}, [\cdot,\cdot]_{\mathscr{R},*})$ is a sub Rankin-Cohen algebra of $(\widetilde{\mathscr{M}}^{>0}, [\cdot,\cdot]_{\mathscr{R},*})$.

Corollary 5.1. *The Rankin-Cohen bracket for CY quasi-modular forms* $[\cdot,\cdot]_{\mathscr{R},*}$ *endows* $\mathscr{M}^{>0}$ *with a canonical Rankin-Cohen algebra structure.*

5.1. Examples of Rankin-Cohen brackets of CY modular forms

We know that the modular discriminant is given by $\Delta = \frac{1}{1728}(E_4^3 - E_6^2)$, which is related with the discriminant $t_2^3 - 27t_3^2$ of the family of elliptic curves stated in (3.28). One can easily compute (or find in [29]) the following examples of Rankin-Cohen brackets (2.6) of modular forms:

$$\begin{aligned} &[E_4, E_6]_1 = -3456\Delta\,, \quad [E_4, E_6]_2 = 0\,, \quad [E_4, E_4]_2 = 4800\Delta, \\ &[E_6, E_6]_2 = -21168 E_4\Delta\,, \quad [\Delta,\Delta]_2 = -13E_4\Delta^2 . \end{aligned} \tag{5.19}$$

Note that for any (quasi-)modular form or any CY (quasi-)modular form f of non-negative weight r and any integer $k \geq 0$ it is evident by definition that:

$$[f,f]_{2k+1} = 0 \ \text{ or } \ [f,f]_{\mathscr{R},2k+1} = 0 . \tag{5.20}$$

For any positive integer n, the discriminant of the Dwork family (3.4) is given by the polynomial $t_{n+2}(t_1^{n+2} - t_{n+2})$. Hence, in the rest of this section for any n we fix the notation $\Delta := \mathsf{t}_{n+2}(\mathsf{t}_1^{n+2} - \mathsf{t}_{n+2})$. Next, we compute a few examples of Rankin-Cohen brackets (5.15) of CY modular forms for $n = 1,2,3,4$, which are motivated by examples given in (5.19).

- $n = 1$. In this case we found $\mathsf{t}_1, \mathsf{t}_2, \mathsf{t}_3$ in the first list of (3.40) and we have $\Delta = \mathsf{t}_3(\mathsf{t}_1^3 - \mathsf{t}_3)$. The Rankin-Cohen brackets are calculated as follows:

$$[\mathsf{t}_1,\mathsf{t}_3]_{\mathscr{R},1} = 27\Delta\,, \quad [\mathsf{t}_1,\mathsf{t}_3]_{\mathscr{R},2} = 729\mathsf{t}_1^2\Delta\,, \quad [\mathsf{t}_1,\mathsf{t}_1]_{\mathscr{R},2} = 324\Delta, \tag{5.21}$$

$$[\mathsf{t}_3,\mathsf{t}_3]_{\mathscr{R},2}=-2916\mathsf{t}_1\Delta\,,\quad [\Delta,\Delta]_{\mathscr{R},2}=-5103\mathsf{t}_1^4\Delta^2\,.$$

Before passing to the next case, we express the combinations of $\mathsf{t}_1,\mathsf{t}_2,\mathsf{t}_3$ which appeared in the right hand side of the above relations in terms of eta and theta functions that seem to us interesting. These relations are obtained thanks to [24] and one can find out more about them by seeing the corresponding pages and references given there. By comparing the coefficients of t_1 with [24, A004016] we find:

$$\mathsf{t}_1=\frac{1}{3}(\theta_3(q)\theta_3(q^3)+\theta_2(q)\theta_2(q^3)), \tag{5.22}$$

and for t_1^2 and t_1^4 the reader is referred to [24, A008653] and [24, A008655], respectively. After computing the q-expansion of Δ, from [24, A007332] we get:

$$\Delta=\frac{1}{27}\eta^6(q)\eta^6(q^3)\,, \tag{5.23}$$

and on account of [24, A136747] we get:

$$\mathsf{t}_1^2\Delta=\frac{1}{243}\eta^6(q)\eta^4(q^3)\left(\eta^3(q)+9\eta^3(q^9)\right)^2. \tag{5.24}$$

The equations (5.22), (5.23) and (5.24) yield:

$$3\mathsf{t}_1=\theta_3(q)\theta_3(q^3)+\theta_2(q)\theta_2(q^3)=\frac{\eta^3(q)+9\eta^3(q^9)}{\eta(q^3)}. \tag{5.25}$$

- $n=2$. Here $\mathsf{t}_1,\mathsf{t}_2,\mathsf{t}_4$ are stated in the second list of (3.40). We know that $\Delta=\mathsf{t}_4(\mathsf{t}_1^4-\mathsf{t}_4)$, and we obtain:

$$\begin{aligned}&[\mathsf{t}_1,\mathsf{t}_4]_{\mathscr{R},1}=-8\mathsf{t}_3\mathsf{t}_4,\ [\mathsf{t}_1,\mathsf{t}_4]_{\mathscr{R},2}=192\mathsf{t}_1^3\mathsf{t}_4,\\ &[\mathsf{t}_1,\mathsf{t}_1]_{\mathscr{R},2}=36\mathsf{t}_1^4-9\mathsf{t}_3^2=36\mathsf{t}_4,\\ &[\mathsf{t}_4,\mathsf{t}_4]_{\mathscr{R},2}=-576\mathsf{t}_1^2\mathsf{t}_4^2\,,\quad [\Delta,\Delta]_{\mathscr{R},2}=-1088\mathsf{t}_1^2\mathsf{t}_4(\mathsf{t}_1^4+8\mathsf{t}_4)\Delta\,.\end{aligned} \tag{5.26}$$

Note that in (5.26) we used the fact that $\mathsf{t}_3^2=4(\mathsf{t}_1^4-\mathsf{t}_4)$, which also implies:

$$[\mathsf{t}_1,\mathsf{t}_4]^2_{\mathscr{R},1}=64\mathsf{t}_3^2\mathsf{t}_4^2=256\mathsf{t}_4\Delta\,. \tag{5.27}$$

• $n = 3$. In this case we have $\Delta = \mathsf{t}_5(\mathsf{t}_1^5 - \mathsf{t}_5)$, and we calculate the Rankin-Cohen brackets as follows (the q-expansion of $\mathsf{t}_1, \mathsf{t}_2, \dots, \mathsf{t}_7$ were computed in [17]):

$$(5.28)\quad [\mathsf{t}_1, \mathsf{t}_5]_{\mathscr{R},1} = -5\mathsf{t}_3\mathsf{t}_5\,, \quad [\mathsf{t}_1, \mathsf{t}_5]_{\mathscr{R},2} = \frac{-4\mathsf{t}_1\mathsf{t}_3^3\mathsf{t}_4\mathsf{t}_5 + 3\mathsf{t}_3^3\mathsf{t}_5\mathsf{t}_6}{125(\mathsf{t}_1^5 - \mathsf{t}_5)}\,,$$

$$[\mathsf{t}_1, \mathsf{t}_1]_{\mathscr{R},2} = \frac{-2500\mathsf{t}_3^2(\mathsf{t}_1^5 - \mathsf{t}_5) - 2\mathsf{t}_1\mathsf{t}_3^3(\mathsf{t}_1\mathsf{t}_4 - \mathsf{t}_6)}{625(\mathsf{t}_1^5 - \mathsf{t}_5)}\,,$$

$$[\mathsf{t}_5, \mathsf{t}_5]_{\mathscr{R},2} = \frac{-6\mathsf{t}_3^3\mathsf{t}_4\mathsf{t}_5^2}{25(\mathsf{t}_1^5 - \mathsf{t}_5)}\,,$$

$$[\Delta, \Delta]_{\mathscr{R},2} = \frac{\mathsf{t}_3^2\mathsf{t}_5^2}{25}\Big(\mathsf{t}_1^3(-20625\mathsf{t}_1^5 - 55000\mathsf{t}_5 + 22\mathsf{t}_1\mathsf{t}_3\mathsf{t}_6) - 44\mathsf{t}_3\mathsf{t}_4(\mathsf{t}_1^5 - \mathsf{t}_5)\Big)\,.$$

• $n = 4$. Here, the first 7 coefficients of the q-expansions of $\mathsf{t}_1, \mathsf{t}_2, \dots, \mathsf{t}_7, \mathsf{t}_8$ are given in [19, Table 2]. We get $\Delta = \mathsf{t}_6(\mathsf{t}_1^6 - \mathsf{t}_6)$ and hence:

$$(5.29)\quad [\mathsf{t}_1, \mathsf{t}_6]_{\mathscr{R},1} = -6\mathsf{t}_3\mathsf{t}_6\,, \quad [\mathsf{t}_1, \mathsf{t}_6]_{\mathscr{R},2} = \frac{-9\mathsf{t}_1\mathsf{t}_3^2\mathsf{t}_4\mathsf{t}_6\mathsf{t}_8 + 7\mathsf{t}_3^2\mathsf{t}_5\mathsf{t}_6\mathsf{t}_8}{12(\mathsf{t}_1^6 - \mathsf{t}_6)}\,,$$

$$[\mathsf{t}_1, \mathsf{t}_1]_{\mathscr{R},2} = \frac{-72\mathsf{t}_3^2(\mathsf{t}_1^6 - \mathsf{t}_6) - \mathsf{t}_1\mathsf{t}_3^2\mathsf{t}_8(\mathsf{t}_1\mathsf{t}_4 - \mathsf{t}_5)}{18(\mathsf{t}_1^6 - \mathsf{t}_6)}\,,$$

$$[\mathsf{t}_6, \mathsf{t}_6]_{\mathscr{R},2} = \frac{-7\mathsf{t}_3^2\mathsf{t}_4\mathsf{t}_6^2\mathsf{t}_8}{\mathsf{t}_1^6 - \mathsf{t}_6}\,,$$

$$[\Delta, \Delta]_{\mathscr{R},2} = \mathsf{t}_3^2\mathsf{t}_6^2\Big(\mathsf{t}_1^4(-1404\mathsf{t}_1^6 - 4680\mathsf{t}_6 + 26\mathsf{t}_1\mathsf{t}_5\mathsf{t}_8) - 52\mathsf{t}_4\mathsf{t}_8(\mathsf{t}_1^6 - \mathsf{t}_6)\Big)\,.$$

The relations given in (4.2) yield $\mathscr{R}\mathsf{t}_1 = \mathsf{t}_3 - \mathsf{t}_1\mathsf{t}_2$ and $\mathscr{R}\mathsf{t}_{n+2} = -(n+2)\mathsf{t}_2\mathsf{t}_{n+2}$ for any integer $n \geq 3$, from which we conclude the following expected result (see (5.28) and (5.29)):

$$(5.30)\qquad [\mathsf{t}_1, \mathsf{t}_{n+2}]_{\mathscr{R},1} = -(n+2)\mathsf{t}_3\mathsf{t}_{n+2}\,, \quad \forall n \geq 3\,.$$

Another interesting point that we observe in the above examples is that in all the cases $n = 1, 2, 3, 4$ the bracket $[\Delta, \Delta]_{\mathscr{R},2}$ is expressed as a polynomial in terms of $\mathsf{t}_1, \mathsf{t}_3, \mathsf{t}_4 \dots, \mathsf{t}_{\mathsf{d}}$, and we expect that this happens for higher dimensions as well.

It is also worth to point out that for any CY (quasi-)modular form f of weight r, the second Rankin-Cohen bracket $[f,f]_{\mathscr{R},2}$ provides a second order differential equation which is satisfied by f. More precisely, from (5.15) we obtain:

$$[f,f]_{\mathscr{R},2} = 6f\mathscr{R}^2 f - 9(\mathscr{R}f)^2, \tag{5.31}$$

which implies that f satisfies the second order ODE:

$$6y\mathscr{R}^2 y - 9(\mathscr{R}y)^2 = [f,f]_{\mathscr{R},2}\,. \tag{5.32}$$

For example, if $n=1$, then from the third bracket of (5.21) we get that the function

$$\begin{aligned}\mathsf{t}_1 &= \frac{1}{3}(2\theta_3(q^2)\theta_3(q^6) - \theta_3(-q^2)\theta_3(-q^6)) \\ &= \frac{1}{3}(\theta_3(q)\theta_3(q^3) + \theta_2(q)\theta_2(q^3)) = \frac{\eta^3(q)+9\eta^3(q^9)}{3\eta(q^3)},\end{aligned}$$

satisfies the following second order ODE:

$$2y\ddot{y} - 3\dot{y}^2 = 4\eta^6(q)\eta^6(q^3)\,, \tag{5.33}$$

in which $\dot{y} = 3q\frac{\partial y}{\partial q} = \frac{3}{2\pi i}\frac{dy}{d\tau}$.

6. Another derivation

Remember that in Section 4.1 to overcome the mentioned problems for odd integers $n \geq 3$ we changed the chart of T, but $\mathsf{R},\mathsf{H},\mathsf{F}$ stayed the same. Another way to solve the same problems is to change the modular vector field R, and consequently F, but let the chart of T and H to stay the same. In order to do this for any positive integer n we define:

$$\tilde{\mathsf{F}} := (1+\delta_2^n)\frac{\partial}{\partial t_2}, \tag{6.1}$$

$$\mathsf{D} := \mathsf{R} + t_2\left([\mathsf{R},\tilde{\mathsf{F}}] - \mathsf{H}\right)\,. \tag{6.2}$$

If $n=1$ or n is even, then $\tilde{\mathsf{F}} = \mathsf{F}$, which implies $\mathsf{D} = \mathsf{R}$. But, if $n \geq 3$ is odd, then D is different from R, whose disadvantages is that we do not know its solutions (in terms of q-expansions). In particular, the author tried to find the q-expansion of solutions of D for $n=3$, but he did not succeed and it seems that the solutions of D in these cases does not have q-expansion around ∞, because if we suppose that $\dot{t}_j = aq\frac{\partial t_j}{\partial q}$, for a constant $a \in \mathbb{C}$, then we find $a = 0$. Nevertheless, if we suppose that $\mathsf{t}_1, \mathsf{t}_2, \ldots, \mathsf{t}_\mathsf{d}$ are solutions of D and define $\widetilde{\mathscr{M}}$ and $\mathscr{M}$ the same as (5.1) and (5.2), respectively, then the derivation $\mathscr{D}$ and the Rankin-Cohen brackets $[\cdot,\cdot]_{\mathscr{D},*}$ induced by D satisfy Theorems 1.1 and 1.2. The proofs are exactly analogous, but before we need to prove the following lemma and the subsequent corollary. Recall that if we have two vector fields $V = \sum_{j=1}^{\mathsf{d}} V^j \frac{\partial}{\partial t_j}$ and $W = \sum_{j=1}^{\mathsf{d}} W^j \frac{\partial}{\partial t_j}$, then

$$[V, W] = VW - WV = \sum_{j=1}^{\mathsf{d}} \left(V(W^j) - W(V^j)\right) \frac{\partial}{\partial t_j}\,. \tag{6.3}$$

Lemma 6.1. *The vector field* D *is a quasi-homogeneous vector field of degree* 2 *in the AMSY-Lie algebra* $\mathfrak{G}$ *that satisfies:*

$$[\mathsf{D}, \tilde{\mathsf{F}}] = \mathsf{H}\,. \tag{6.4}$$

Proof. If $n = 1, 2, 3, 4$, then R, F, H are given explicitly in Example 5.1, and one can easily find that the affirmations hold. If $n \geq 5$ is even, then $\mathsf{D} = \mathsf{R}$ and $\tilde{\mathsf{F}} = \mathsf{F}$, which yield the results. Suppose that $n \geq 5$ is odd. Then $\tilde{\mathsf{F}} = \frac{\partial}{\partial t_2}$, and by applying (3.35) to $\mathsf{R}_{\mathfrak{g}_{1n}}$ and $\mathsf{R}_{\mathfrak{g}_{1(n+1)}}$ we obtain $\mathsf{R}_{\mathfrak{g}_{1n}} = \frac{\partial}{\partial t_{\mathsf{d}-2}} + t_2 \frac{\partial}{\partial t_\mathsf{d}}$ and $\mathsf{R}_{\mathfrak{g}_{1(n+1)}} = \frac{\partial}{\partial t_\mathsf{d}}$. Therefore, the relation (3.45) yields:

$$[\mathsf{R}, \frac{\partial}{\partial t_\mathsf{d}}] = [\mathsf{R}, \mathsf{R}_{\mathfrak{g}_{1(n+1)}}] = \mathsf{R}_{\mathfrak{g}_{1n}} = \frac{\partial}{\partial t_{\mathsf{d}-2}} + t_2 \frac{\partial}{\partial t_\mathsf{d}}\,. \tag{6.5}$$

If we write $\mathsf{R} = \sum_{j=1}^{\mathsf{d}} \mathsf{R}^j \frac{\partial}{\partial t_j}$, then Remark 4.1 yields $\mathsf{R}^{\mathsf{d}-2} = -t_\mathsf{d} - t_2 t_{\mathsf{d}-2}$, from which we get:

$$\begin{aligned} [\mathsf{R}, t_{\mathsf{d}-2}\frac{\partial}{\partial t_\mathsf{d}}] &= \mathsf{R}(t_{\mathsf{d}-2})\frac{\partial}{\partial t_\mathsf{d}} + t_{\mathsf{d}-2}[\mathsf{R}, \frac{\partial}{\partial t_\mathsf{d}}] \\ &\overset{(6.5)}{=} \mathsf{R}^{\mathsf{d}-2}\frac{\partial}{\partial t_\mathsf{d}} + t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}-2}} + t_2 t_{\mathsf{d}-2}\frac{\partial}{\partial t_\mathsf{d}} \\ &= t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}-2}} - t_\mathsf{d}\frac{\partial}{\partial t_\mathsf{d}}\,. \end{aligned} \tag{6.6}$$

Due to (3.52) we have $\tilde{\mathsf{F}} = \frac{\partial}{\partial t_2} = \mathsf{F} + t_{\mathsf{d}-2}\frac{\partial}{\partial t_\mathsf{d}}$, hence

$$\begin{aligned} \mathsf{D} = \mathsf{R} + t_2\left([\mathsf{R}, \frac{\partial}{\partial t_2}] - \mathsf{H}\right) &= \mathsf{R} + t_2\left([\mathsf{R}, \mathsf{F} + t_{\mathsf{d}-2}\frac{\partial}{\partial t_\mathsf{d}}] - \mathsf{H}\right) \\ &= \mathsf{R} + t_2 t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}-2}} - t_2 t_\mathsf{d}\frac{\partial}{\partial t_\mathsf{d}}\,. \end{aligned} \tag{6.7}$$

Note that in the last equality of the above equation we used (6.6) and the fact that $[\mathsf{R},\mathsf{F}]=\mathsf{H}$. Thus,

$$\begin{aligned}
[\mathsf{D},\tilde{\mathsf{F}}] &= [\mathsf{R},\frac{\partial}{\partial t_2}] + [t_2 t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}-2}},\frac{\partial}{\partial t_2}] - [t_2 t_{\mathsf{d}}\frac{\partial}{\partial t_{d}},\frac{\partial}{\partial t_2}] \\
&= [\mathsf{R},\frac{\partial}{\partial t_2}] - \frac{\partial}{\partial t_2}(t_2 t_{\mathsf{d}-2})\frac{\partial}{\partial t_{\mathsf{d}-2}} + \frac{\partial}{\partial t_2}(t_2 t_{\mathsf{d}})\frac{\partial}{\partial t_{\mathsf{d}}} \\
&= [\mathsf{R},\frac{\partial}{\partial t_2}] - t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}-2}} + t_{\mathsf{d}}\frac{\partial}{\partial t_{d}} \overset{(6.6)}{=} [\mathsf{R},\frac{\partial}{\partial t_2}] - [\mathsf{R}, t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}}}] \\
&= [\mathsf{R},\frac{\partial}{\partial t_2} - t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}}}] \overset{(3.52)}{=} [\mathsf{R},\mathsf{F}] = \mathsf{H}\,.
\end{aligned}$$

We know that R is quasi-homogeneous of degree 2 and $\deg(t_2)=2$, hence (6.7) implies that D is quasi-homogeneous of degree 2. In order to get $\mathsf{D}\in\mathfrak{G}$, first observe that $\frac{\partial}{\partial t_{\mathsf{d}}} = \mathsf{R}_{\mathfrak{g}_{1(n+1)}} \in \mathfrak{G}$. Hence,

$$\tilde{\mathsf{F}} = \frac{\partial}{\partial t_2} = \mathsf{F} + t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}}} \in \mathfrak{G}\,,$$

which yields $\mathsf{D}\in\mathfrak{G}$, and the proof is complete. □

Corollary 6.1. *The Lie algebra generated by the vector fields* D, H *and* $\tilde{\mathsf{F}}$ *in the AMSY-Lie algebra* $\mathfrak{G}\subset\mathfrak{X}(\mathsf{T})$ *is isomorphic to* $\mathfrak{sl}_2(\mathbb{C})$*.*

Proof. It suffices to show that $[\mathsf{D},\tilde{\mathsf{F}}]=\mathsf{H},\ [\mathsf{H},\mathsf{D}]=2\mathsf{D},\ [\mathsf{H},\tilde{\mathsf{F}}]=-2\tilde{\mathsf{F}}$. The truth of the first bracket is guaranteed by Lemma 6.1, and the last bracket follows from a simple computation after using (3.49) or (3.51), and (6.3). To demonstrate the second bracket $[\mathsf{H},\mathsf{D}]=2\mathsf{D}$, if $n=1,2,3,4$, then due to Example 5.1 one can easily find that the affirmation holds. For odd integers $n\geq 5$, we first use (6.7) to obtain:

$$[\mathsf{H},\mathsf{D}] = [\mathsf{H},\mathsf{R}] + [\mathsf{H}, t_2 t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}-2}} - t_2 t_{\mathsf{d}}\frac{\partial}{\partial t_{\mathsf{d}}}].$$

Then the statement follows from the fact $[\mathsf{H},\mathsf{R}]=2\mathsf{R}$ given in Theorem 3.4 and using (6.3) for H stated in (3.51). □

Here we have $\mathscr{D}$ as follows:

$$\mathscr{D} := \begin{cases} \mathscr{R} + \mathsf{t}_2([\mathscr{R},\tilde{\mathscr{F}}]-\mathscr{H}), & \text{if } n\geq 3 \text{ is odd;} \\ \mathscr{R}, & \text{otherwise;} \end{cases} \,, \tag{6.8}$$

and the *Ramanujan-Serre–type derivation* ∂ on the generators of $\widetilde{\mathscr{M}}$ is given as follows:

$$\partial f := \mathscr{D} f + (1 - \frac{1}{2}\delta_2^n) r \mathsf{t}_2 f\,, \quad \forall f \in \widetilde{\mathscr{M}_r} \text{ and } \forall r \in \mathbb{Z}. \tag{6.9}$$

In particular, if $n = 3$, then we get $\mathscr{D} : \widetilde{\mathscr{M}} \to \widetilde{\mathscr{M}}$ as follows:

$$\begin{aligned}\mathscr{D} = &\Big(\mathsf{t}_3 - \mathsf{t}_2\mathsf{t}_1\Big)\frac{\partial}{\partial \mathsf{t}_1} + \left(\frac{\mathsf{t}_3^3\mathsf{t}_4}{5^4(\mathsf{t}_1^5 - \mathsf{t}_5)} - \mathsf{t}_2^2\right)\frac{\partial}{\partial \mathsf{t}_2}\\ &+ \left(\frac{\mathsf{t}_3^3\mathsf{t}_6}{5^4(\mathsf{t}_1^5 - \mathsf{t}_5)} - 3\mathsf{t}_2\mathsf{t}_3\right)\frac{\partial}{\partial \mathsf{t}_3} - \mathsf{t}_7\frac{\partial}{\partial \mathsf{t}_4} - 5\mathsf{t}_2\mathsf{t}_5\frac{\partial}{\partial \mathsf{t}_5}\\ &+ \Big(5^5\mathsf{t}_1^3 - 2\mathsf{t}_3\mathsf{t}_4 - \mathsf{t}_2\mathsf{t}_6\Big)\frac{\partial}{\partial \mathsf{t}_6} + \Big(-5^4\mathsf{t}_1\mathsf{t}_3 - 2\mathsf{t}_2\mathsf{t}_7\Big)\frac{\partial}{\partial \mathsf{t}_7}\,,\end{aligned}$$

and we obtain $\partial : \mathscr{M} \to \mathscr{M}$ as follows:

$$\partial = \mathsf{t}_3\frac{\partial}{\partial \mathsf{t}_1} + \frac{\mathsf{t}_3^3\mathsf{t}_6}{5^4(\mathsf{t}_1^5 - \mathsf{t}_5)}\frac{\partial}{\partial \mathsf{t}_3} - \mathsf{t}_7\frac{\partial}{\partial \mathsf{t}_4} + (5^5\mathsf{t}_1^3 - 2\mathsf{t}_3\mathsf{t}_4)\frac{\partial}{\partial \mathsf{t}_6} - 5^4\mathsf{t}_1\mathsf{t}_3\frac{\partial}{\partial \mathsf{t}_7}\,.$$

If $n \geq 5$ is odd, then $\mathscr{D} = \mathscr{R} + \mathsf{t}_2\mathsf{t}_{\mathsf{d}-2}\frac{\partial}{\partial \mathsf{t}_{\mathsf{d}-2}} - \mathsf{t}_2\mathsf{t}_{\mathsf{d}}\frac{\partial}{\partial \mathsf{t}_{\mathsf{d}}}$.

On account of (3.35), (3.25) and (6.7) we can compute the Gauss-Manin connection matrix A_{D} associated with D. For $n = 3$ we get:

$$\mathsf{A}_{\mathsf{D}} = \begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & \mathsf{Y}_1 & 0\\ t_2t_4 & 0 & 0 & -1\\ -t_2(t_2t_4 + t_7) & t_2t_4 & 0 & 0\end{pmatrix}, \tag{6.10}$$

in which $\mathsf{Y}_1 = \frac{t_3^3}{5^4(t_1^5 - t_5)}$; and for odd integers $n \geq 5$ we obtain:

$$\mathsf{A}_{\mathsf{D}} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \mathsf{Y}_1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \mathsf{Y}_2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & \mathsf{Y}_{n-2} & 0\\ t_2t_{\mathsf{d}-2} & 0 & 0 & 0 & \cdots & 0 & -1\\ -t_2(t_{\mathsf{d}-2} + t_{\mathsf{d}}) & t_2t_{\mathsf{d}-2} & 0 & 0 & \cdots & 0 & 0\end{pmatrix}, \tag{6.11}$$

in which Y_j's are the same regular functions given in Theorem 3.1. Due to Theorem 3.1, the Gauss-Manin connection matrix of R for $n = 3$ is as follows:

$$\mathsf{A}_\mathsf{R} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \mathsf{Y}_1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.12}$$

in which we also have $\mathsf{Y}_1 = \frac{t_3^3}{5^4(t_1^5 - t_5)}$. If we substitute the solutions of R in Y_1, then we get the Yukawa coupling. It would be very interesting if one can find out the (physical) interpretation of the non-zero part of the lower triangle of the matrix A_D stated in (6.10).

7. Mirror map

For more details, as well as more general discussions, of the facts stated in this section one can see any of [14, 15, 30, 13], [18, Appendix C] or references therein. We know that the Picard-Fuchs equation of the 1-parameter family of CY n-folds given by the Dwork family (3.1) is the generalized hypergeometric differential equation

$$\vartheta^{n+1} - z\left(\vartheta + \frac{1}{n+2}\right)\left(\vartheta + \frac{2}{n+2}\right)\cdots\left(\vartheta + \frac{n+1}{n+2}\right) = 0, \tag{7.1}$$

in which $\vartheta = z\frac{\partial}{\partial z}$. Through the Frobenius method we can find two linearly independent solutions $f_n(z)$ and $h_n(z) := f_n(z)\log z + g_n(z)$ of (7.1) around the regular singular point $z = 0$, where $f_n(z)$ is the generalized hypergeometric function

$$\begin{aligned} f_n(z) &= {}_{n+1}F_n\left(\begin{matrix} \frac{1}{n+2}, & \frac{2}{n+2}, & \ldots, & \frac{n}{n+2}, & \frac{n+1}{n+2} \\ 1, & 1, & \ldots, & 1 & \end{matrix} \,\middle|\, z \right) \\ &= \sum_{k=0}^{\infty} \frac{\big(k(n+2)\big)!}{(n+2)^{k(n+2)}(k!)^{n+2}} z^k, \end{aligned}$$

and $g_n(z)$ is given as follows:

$$g_n(z) = \sum_{k=1}^{\infty} \frac{\big(k(n+2)\big)!}{(n+2)^{k(n+2)}(k!)^{n+2}} \left(\sum_{j=1}^{n+1} \sum_{i=0}^{k-1} \left(\frac{n+2}{j+(n+2)i} - \frac{1}{1+i} \right) \right) z^k.$$

The *mirror map* $q(z) := q_n(z)$ is defined by:

$$q(z) = \exp\left(\frac{h_n(z)}{f_n(z)}\right) = z\exp\left(\frac{g_n(z)}{f_n(z)}\right),$$

which is a generalization of the Schwarz triangle function. We denote by $z(q) := z_n(q)$ the inverse of the mirror map $q(z)$, and by abuse of language, we call it also mirror map. Note that z is the parameter of the complex structure of the Dwork family. One of the important problems in this context is the integrality of the Taylor coefficients of the mirror map. It is worth noting that the coefficients a_n of the Taylor expansion $q(z) = \sum_{k=1}^{\infty} a_n z^n$, with $a_1 = 1$, are integers if and only if the Taylor coefficients of $z(q) = \sum_{k=1}^{\infty} b_n$, with $b_1 = 1$, are integers (one can observe it by induction, see e.g. [30, Lemma 2]). Lian and Yau in [15, Theorem 5.5] proved that if $n+2$ is a prime number, then the coefficients of the Taylor expansion of $z(q)$ are integers. Later in [30, Theorem 3] Zudilin showed that the integrality of the coefficients holds if $n+2$ is a power of a prime number. Finally, the result for any integer n follows from [30, Conjecture on p. 605] or [13, Theorem 5] proved by Krattenthaler and Rivoal in [13]. The main tool used in these works is Dwork's p-adic technique [9].

In our framework, to construct the moduli space S in (3.7) we observed that $z = \frac{t_{n+2}}{t_1^{n+2}}$. Hence, for any positive integer n we can express the mirror map in terms of CY modular forms as follows:

$$z(q) = \frac{\mathsf{t}_{n+2}}{\mathsf{t}_1^{n+2}}(q). \tag{7.2}$$

In the following we compute the q-expansion of the mirror map for $n = 1, 2, 3, 4$ and we observe that our results coincide with the ones computed in [14, 12]. We should mention that the parameter of the complex structure in our framework equals to the same in [14, 12] up to a constant multiplication; more precisely, if we denote the parameter of the complex structure in [14, 12] by $\hat{z}$, then $z = (n+2)^{n+2}\hat{z}$.

- $n = 1$. In this case we find:

$$\begin{aligned}\frac{1}{3^3}z(q) &= \frac{\mathsf{t}_3}{(3\mathsf{t}_1)^3}(q) = q - 15q^2 + 171q^3 - 1679q^4 + 15054q^5 - 126981q^6 \\ &\quad + 1024952q^7 - 8002191q^8 + 60868665q^9 - 453390354q^{10} + \dots \\ &= \frac{\eta^{12}(q^3)}{(\eta^4(q) + 9\eta(q)\eta^3(q^9))^3}.\end{aligned}$$

Indeed, the mirror map is a modular function for $\Gamma_0(3)$ with integer coefficients. Here we obtain $\frac{1}{3^3}z(q)=\hat{z}(3^3q)$, where $\hat{z}$ is the mirror map computed in [14, (4.17)] with the parameters $a=\frac{1}{3}$, $b=\frac{2}{3}$ and $\delta=3^3$. The appearance of the factor 3^3 in $\hat{z}(3^3q)$ is normal, see [14, §4.1] or Remark 2 in [14, §4.4].

- $n=2$. We obtain:

$$\begin{aligned}\frac{1}{4^4}z(q)=&\frac{\mathsf{t}_4}{(4\mathsf{t}_1)^4}(q)=q-104q^2+6444q^3-311744q^4+13018830q^5\\&-493025760q^6+17411253944q^7-583472867840q^8\\&+18770817643749q^9-584450497233840q^{10}+\dots\\=&\frac{\eta^8(q)\eta^8(q^2)}{\left(\theta_3^4(q^2)+\theta_2^4(q^2)\right)^4},\end{aligned}$$

which is a modular function for $\Gamma_0(2)$ with integer coefficients. The careful reader may wonder why in the solutions given for R_2 in (3.40) we found $\mathsf{t}_1(\frac{q}{10})$ and $\mathsf{t}_4(\frac{q}{10})$, but here we just use $\mathsf{t}_1(q)$ and $\mathsf{t}_4(q)$. Indeed, depending on the initial conditions we can do this type of manipulation, to be more precise, here we worked with the initial condition $\frac{d\mathsf{t}_3}{dq}(0)=-\frac{1}{10}$ instead $\frac{d\mathsf{t}_3}{dq}(0)=-1$. In this example we find that $\frac{1}{4^4}z(q)=\hat{z}(q)$, where $\hat{z}(=z_{X_{(4)}}(q))$ is the mirror map obtained in [14, (5.18)].

- $n=3$. In the case of the mirror quintic 3-fold we get:

$$\begin{aligned}\frac{1}{5^5}z(q)=&\frac{\mathsf{t}_5}{(5\mathsf{t}_1)^5}(q)=q-770q^2+171525q^3-81623000q^4\\&-35423171250q^5-54572818340154q^6\\&-71982448083391590q^7-102693620674349200800q^8\\&-152309618533468229185125q^9\\&-233336729173603108139387500q^{10}+\dots,\end{aligned}$$

which coincides with the mirror map given in [14, (4.23)]. Movasati and Shokri in [18, Appendix C] proved that the coefficients of the q-expansions of the CY (quasi-)modular forms for the mirror quintic 3-fold are integers (see [18, Chap. 2, Theorem 5] and for the proof see [18, §C.11]). For the proof they use the periods of the mirror quintic 3-fold and the fact that the q-expansion of the mirror map has integer coefficients.

- $n = 4$. For the sextic 4-fold we find:

$$\begin{aligned}\frac{1}{6^6}z(q) = &\frac{\mathsf{t}_6}{(6\mathsf{t}_1)^6}(q) = q - 6264q^2 - 8627796q^3 - 237290958144q^4 \\ &- 4523787606611250q^5 - 101677347076292728992q^6 \\ &- 2505516076689971667545160q^7 \\ &- 65911479451755065426649890304q^8 \\ &- 1819958282035176297707697612909339q^9 \\ &- 52163512644954752336845183286716410000q^{10} + \dots ,\end{aligned}$$

which is the same as the mirror map computed by Klemm and Pandharipande in [12, (44)]. We computed the first 100 coefficients of the q-expansions of t_j, $j = 1, 2, \dots 8$, which are integers, and we conjecture that all coefficients are integers (of course, up to multiplying t_j by a constant). For the first 7 coefficients see [19, Table 2].

In general, considering the evidences given in the above examples, by computing the CY modular forms t_1 and t_{n+2} we are offering a new method to find the mirror map through (7.2). Note that t_1 and t_{n+2} are CY modular forms of weight 1 and $n+2$, respectively. Hence, due to (7.2) the mirror map has weight 0, which can be considered as a generalization of the modular functions. We conjecture that the q-expansions of the CY (quasi-)modular forms t_j's have integer coefficients. If one proves this conjecture, at least for t_1 and t_{n+2}, without using the integrality of the mirror map of the CY n-folds arising from the Dwork family, then we have a new proof of integrality of the mirror map.

8. Final remarks and open questions

One of weak points of Theorem 1.2 is that we are just considering the CY modular forms of positive weight. If we look closely to the definition of $\widetilde{\mathscr{M}}$ and $\mathscr{M}$ given in (5.1) and (5.2), respectively, we observe that they contain non-constant elements of weight zero and elements of negative weight. For example for $n = 3$, the element $\mathsf{t}_4 \in \mathscr{M}$ is a non-constant element of weight zero and $\frac{1}{\mathsf{t}_5(\mathsf{t}_1^5 - \mathsf{t}_5)} \in \mathscr{M}$ is an element of weight -10. Thus, in general it is not necessarily valid that $\widetilde{\mathscr{M}_0} = \mathscr{M}_0 = \mathbb{C}$; indeed, $\widetilde{\mathscr{M}_0}$ and $\mathscr{M}_0$ are generated by $\mathbb{C} \cup \{ f \in \widetilde{\mathscr{M}} \mid \deg(f) = 0 \}$ and $\mathbb{C} \cup \{ f \in \mathscr{M} \mid \deg(f) = 0 \}$, respectively. We can consider the definition of the Rankin-Cohen bracket (5.15) for elements of negative weight as well, and hence we can endow $\widetilde{\mathscr{M}}$ with a Rankin-Cohen algebra structure. Using the computer we observed that the Rankin-Cohen brackets of all examined CY modular forms of negative weight are again CY modular forms, in the cases $n = 1, 2, 3, 4$, but we could not prove theoretically the assertion that the space of CY modular forms $\mathscr{M}$ is closed under the Rankin-Cohen bracket (5.15). We believe in the truth of this assertion, but our main difficulty in carrying out its proof is the use of Proposition 2.1, where the weight of non-constant elements of the graded algebra are considered positive. This led us to the following conjecture.

Conjecture 1. *The proposition 2.1 holds if the graded algebra M also contains elements of negative weight or non-constant elements of weight zero.*

In the above conjecture by constant elements we mean the elements of the field k. If we want to prove Conjecture 1 in an analogous way to the proof of Zagier given for [29, Proposition 1], the unsolved part is the equality (2.22). Once we prove Conjecture 1, we can prove that the space of CY modular forms $\mathscr{M}$ is closed under the Rankin-Cohen brackets (5.15).

Remark 8.1. *After releasing the present paper the author worked on the above conjecture and he could prove it. Since the proof is technical, and also since the present paper is big enough, this proof together with other results will be reported in [23].*

Comparing this modern theory of CY modular forms with the classical theory of the modular forms some natural questions may arise. Here we state a few of these open questions:

1. What is the domain of CY (quasi-)modular forms?
2. Do CY (quasi-)modular forms satisfy any functional equations? If yes, what are these functional equations?
3. Can we define L-functions for CY modular forms? Can we associate any algebraic variety to these possible L-functions?
4. Can we define Hecke-type operators for the space of CY modular forms?

To have an idea of how to answer these questions, it might be helpful to study the classical theory of modular forms through this approach, i.e., the space generated by solutions of certain systems of differential equations. In this direction, we studied these type of systems in more generality in [6]. Indeed, if we look closely at the system presented by R we realize that it has a specific form. This specific form and its relation with the Rankin-Cohen algebras led us to introduce and study certain systems called Ramanujan systems of Rankin-Cohen type in [6].

The present work enables us to do the correct choice of the CY quasi-modular forms and CY modular forms, and attribute to them the correct weights. For example, in the case $n = 3$, Movasati in [17, Theorem 2] found certain functional equations satisfied by t_j's. In his coordinate charts (which is different from ours) the functions t_1, t_2, t_3, t_6 do not satisfy the same functional as t_0, t_4, t_5, which Movasati expects to be satisfied by CY modular forms. The author believes that this is due to the non-adequate choice of the coordinate charts and the attached weights given in [17, Theorem 1]. As we observed in Section 4, to get our purpose for $n = 3$ we need to choose a specific coordinate charts. We conjecture that in our chosen chart the functions $\mathsf{t}_1, \mathsf{t}_3, \mathsf{t}_4, \mathsf{t}_5, \mathsf{t}_6, \mathsf{t}_7$, with the corresponding weights $w_i, i = 1, 3, \ldots, 7$, satisfy similar functionals to the ones given for t_0, t_4, t_5 in [17, Theorem 2].

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The initial inspiration of the present study came from a conversation between Hossein Movasati and Don Zagier, which was shared later with the author and others by Movasati. At that moment we did not succeed in solving the problem, because of the absence of some key points such as the correct weight of the CY (quasi-)modular forms and etc. After the work [21], the author could find the missing points of the research and completed the present work. Because of this, the author would like to thank both Movasati and Zagier, in particular he is very grateful to Movasati for his helpful discussions and comments. The author is also grateful to anonymous referees whose comments and suggestions improved the present paper.

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