Quantum geometry, stability and modularity

SERGEI ALEXANDROV, SOHEYLA FEYZBAKHSH, ALBRECHT KLEMM,
BORIS PIOLINE, AND THORSTEN SCHIMANNEK

By exploiting new mathematical relations between Pandharipande-Thomas (PT) invariants, closely related to Gopakumar-Vafa (GV) invariants, and rank 0 Donaldson-Thomas (DT) invariants counting D4-D2-D0 BPS bound states, we rigorously compute the first few terms in the generating series of Abelian D4-D2-D0 indices for compact one-parameter Calabi-Yau threefolds of hypergeometric type. In all cases where GV invariants can be computed to sufficiently high genus, we find striking confirmation that the generating series is modular, and predict infinite series of Abelian D4-D2-D0 indices. Conversely, we use these results to provide new constraints for the direct integration method, which allows to compute GV invariants (and therefore the topological string partition function) to higher genus than hitherto possible. The triangle of relations between GV/PT/DT invariants is powered by a new explicit formula relating PT and rank 0 DT invariants, which is proven in an Appendix by the second named author. As a corollary, we obtain rigorous Castelnuovo-type bounds for PT and GV invariants for CY threefolds with Picard rank one.

1. Introduction

More than 25 years after Strominger and Vafa's celebrated breakthrough [1], the precision counting of BPS black hole microstates in string vacua with $\mathcal{N} = 2$ supersymmetry in 4 dimensions remains an outstanding challenge at the frontier of theoretical physics and mathematics. Unlike in cases with higher supersymmetry, the index $\Omega_z(\gamma)$ counting BPS states with fixed electromagnetic charge γ has an intricate chamber structure with respect to the moduli z specifying the internal manifold, while that moduli space is itself

subject to complicated quantum corrections. As a result, the indices $\Omega_z(\gamma)$ are almost never known exactly.

For type IIA strings compactified on a Calabi-Yau (CY) threefold $\mathfrak{Y}$, the proper mathematical framework involves the derived category of coherent sheaves $\mathcal{C} = D^b \operatorname{Coh}\mathfrak{Y}$, the associated space of Bridgeland stability conditions $\mathcal{S} = \operatorname{Stab}\mathcal{C}$ and the Donaldson-Thomas (DT) invariants $\Omega_\sigma(\gamma)$ counting semi-stable objects in $\mathcal{C}$ with charge γ for a stability condition $\sigma = (Z, \mathcal{A}) \in \mathcal{S}$, where Z is a central charge function and $\mathcal{A}$ a certain Abelian subcategory of $\mathcal{C}$ locally determined by Z. While physics (or rather mirror symmetry) selects a particular slice $\Pi \subset \mathcal{S}$ where Z is a computable function of the (complexified) Kähler moduli $z \in \mathcal{M}_K$, the DT invariants $\Omega_\sigma(\gamma)$ are in principle well-defined in the larger space $\mathcal{S}$. In cases where $\Omega_\sigma(\gamma)$ can be shown to vanish at some particular point $\sigma \in \mathcal{S}$ (which need not belong to the physical slice Π), it then becomes possible to determine it elsewhere using the universal wall-crossing formulae of [2, 3].

This strategy has been pursued in a recent series of mathematical works [4, 5, 6, 7], which culminated in explicit formulae [8] relating rank 0 DT invariants, counting D4-D2-D0 bound states, to Pandharipande-Thomas (PT) invariants, counting D6-D2-D0 bound states with one unit of D6-brane charge. These rigorous results depend on a conjectural inequality which lies at the heart of the construction of stability conditions on CY threefolds [9, 10], and is widely believed to hold in general but proven only in a handful of cases. PT invariants are in turn related to Gopakumar-Vafa (GV) invariants entering the A-model topological string partition function Z_{top} on $\mathfrak{Y}$ [11], and are in principle computable by integrating the holomorphic anomaly equations satisfied by Z_{top}, a procedure sometimes called 'direct integration' [12, 13, 14]. These relations between D4-D2-D0 indices and topological strings are in the spirit of the OSV conjecture [15], and in fact imply a special case of the latter [16, 4, 8].

On the other hand, the fact that D4-D2-D0 bound states in type IIA string theory lift to M5-branes wrapped on a divisor $\mathcal{D} \subset \mathfrak{Y}$ times a circle strongly suggests that suitable generating series of rank 0 DT invariants should exhibit modular properties [17]. Specifically, in the simplest case of a single M5-brane wrapped on an ample divisor $\mathcal{D}$, the corresponding series of rank 0 DT invariants, which we call Abelian D4-D2-D0 indices, should transform as a vector-valued modular form, arising from the theta-series decomposition of the elliptic genus of the $(0, 4)$ superconformal field theory obtained by reducing the M5-brane along $\mathcal{D}$ [18, 19, 20, 21]. More generally, for a reducible divisor the generating series should transform as a vector-valued mock modular form of higher depth, with a fixed modular anomaly

[22, 23, 24] (see [25, 26, 27, 28] for related work). Since the space of such vector-valued (mock) modular forms is finite-dimensional, this opens up the possibility of computing infinite families of D4-D2-D0 indices, provided the singular terms in the generating series (also known as polar terms) can be determined independently.

This approach was applied long ago for a few CY threefolds $\mathfrak{Y}$ with $b_2(\mathfrak{Y}) = 1$ in [19, 20, 29, 30]. It was extended recently in [31] to the full list of 13 smooth complete intersections in weighted projected space (the so-called hypergeometric CY threefolds), see Table 1. Unfortunately, the analysis in [31] was based on an educated guess for the coefficients of the polar terms, which reproduced earlier results in [19, 20] and provided plausible answers for 5 additional models, but failed to produce a modular form for the last 3 models in Table 1. Although a strategy to compute non-Abelian D4-D2-D0 indices was spelled out, it was eventually inconclusive, again due to lack of control on the polar coefficients.

In this work, we revisit the analysis in [31] in light of the recent mathematical results in [8]. More specifically, we exploit a new and powerful explicit formula (4.12) relating PT and rank 0 DT invariants, which is proven by one of the authors in Appendix A of this paper, and depicted by the horizontal arrow at the bottom of Figure 1. Among other applications, this formula allows to prove rigorous Castelnuovo-type bounds for PT and GV invariants, and determines the GV invariants $\mathrm{GV}_Q^{(g)}$ for maximal genus $g = g_{\max}(Q)$, assuming some congruence condition on the degree Q. Along with various optimizations of the computer implementation, this allows us to push the direct integration method of [13] to high genus. By
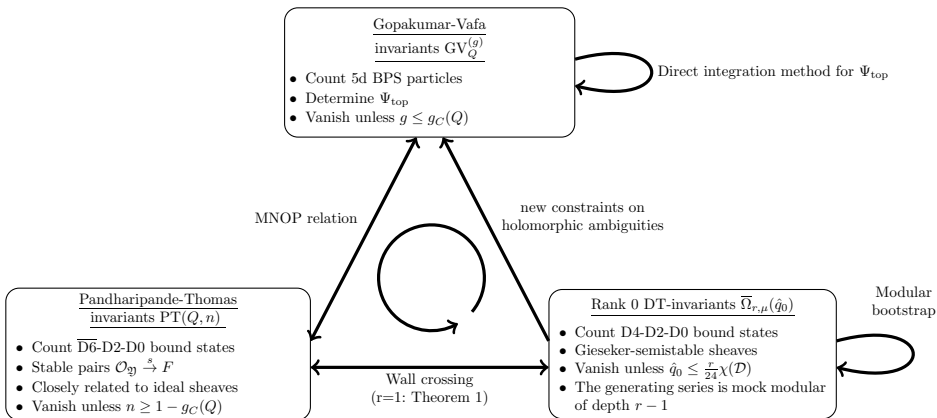

Figure 1: The triangle of relations between GV/PT/DT invariants.

Table 1: Relevant data for the 13 hypergeometric CY threefolds. The second to fifth columns indicate the Euler number of $\mathfrak{Y}$, the self-intersection $\kappa = H^3$ of the generator of $\mathrm{Pic}\,\mathfrak{Y}$, the second Chern class $c_2 = c_2(T\mathfrak{Y}).H$ and the holomorphic Euler characteristic $\chi_{\mathcal{D}} = \chi(\mathcal{O}_{\mathcal{D}})$ of the primitive divisor $\mathcal{D}$ dual to H (not to be confused with its topological Euler characteristic $\chi(\mathcal{D})$). The columns n_1^p and n_1^c indicate the number of polar terms and modular constraints on the generating series of Abelian D4-D2-D0 invariants, taken from [31]. The columns (a_i) and "type" indicate the local exponents in the Picard-Fuchs equation and the resulting degeneration type at $z = \infty$ in the notation of [33]. The column g_{integ} and g_{mod} indicate the maximal genus for which GV invariants $\mathrm{GV}_Q^{(g)}$ can be determined by the direct integration method, either using only the usual regularity conditions and the expression (3.30) for GV invariants saturating the bound $g \leq g_{\max}(Q)$ for $Q = 0$ mod κ, or also including GV invariants predicted by the modular series of Abelian D4-D2-D0 indices. The column g_{avail} indicates the genus up to which complete tables of GV invariants are currently known. For updates check [32]

$\mathfrak{Y}$	$\chi_{\mathfrak{Y}}$	κ	c_2	$\chi_{\mathcal{D}}$	n_1^p	n_1^c	(a_i)	type	g_{integ}	g_{mod}	g_{avail}
$X_5(1^5)$	-200	5	50	5	7	0	$(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$	F	53	69	64
$X_6(1^4, 2)$	-204	3	42	4	4	0	$(\frac{1}{6}, \frac{2}{6}, \frac{4}{6}, \frac{5}{6})$	F	48	66	48
$X_8(1^4, 4)$	-296	2	44	4	4	0	$(\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8})$	F	60	84	64
$X_{10}(1^3, 2, 5)$	-288	1	34	3	2	0	$(\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10})$	F	50	70	68
$X_{4,3}(1^5, 2)$	-156	6	48	5	9	0	$(\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4})$	F	20	24	24
$X_{6,4}(1^3, 2^2, 3)$	-156	2	32	3	3	0	$(\frac{1}{6}, \frac{1}{4}, \frac{3}{4}, \frac{5}{6})$	F	14	17	17
$X_{3,3}(1^6)$	-144	9	54	6	14	1	$(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3})$	K	29	33	33
$X_{4,4}(1^4, 2^2)$	-144	4	40	4	6	1	$(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4})$	K	26	34	34
$X_{6,6}(1^2, 2^2, 3^2)$	-120	1	22	2	1	0	$(\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6})$	K	18	21	21
$X_{6,2}(1^5, 3)$	-256	4	52	5	7	0	$(\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{5}{6})$	C	63	84	49
$X_{4,2}(1^6)$	-176	8	56	6	15	1	$(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4})$	C	50	64	50
$X_{3,2,2}(1^7)$	-144	12	60	7	21	1	$(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3})$	C	14	?	14
$X_{2,2,2,2}(1^8)$	-128	16	64	8	33	3	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	M	17	?	32

combining the formula (4.12) with these results for GV invariants, we are able to rigorously compute all polar terms and a large number of non-polar terms for most of the 13 hypergeometric CY threefolds, and find striking confirmations of the modularity of the corresponding generating series (as well as supporting evidence for the validity of the BMT inequality in those models where it is not yet known to hold). Expanding these generating series to arbitrary order, we predict an infinite set of Abelian D4-D2-D0 indices.

Turning the logic around and assuming that the generating series of Abelian D4-D2-D0 indices is indeed the one dictated by modularity, we predict infinite series of GV invariants $GV_Q^{(g)}$ lying at finite distance from the Castelnuovo bound $g = g_C(Q)$. This in turn provides additional boundary conditions for the direct integration method, which in principle allows us to push it beyond the maximal genus (indicated as g_{integ} in Table 1) at which the leading behaviour at special points in the moduli space and the Castelnuovo vanishing conditions no longer suffice to fix the holomorphic ambiguities. The maximal genus attainable using these additional boundary conditions is indicated in the column g_{mod} in Table 1. The updated data are available at [32].

More specifically, we find the generating series of D4-D2-D0 indices for 11 out of 13 models listed in Table 1. For 5 models, namely X_{10}, $X_{4,3}$, $X_{6,2}$, $X_{6,4}$ and $X_{4,2}$, our results imply that the polar terms differ from the naive Ansatz of [31] (in particular, the result for X_{10} disagrees with [20] but confirms the proposal in [30]). In all these cases, we find spectacular confirmation that the generating series is modular. For the last 2 models in this Table, namely $X_{3,2,2}$ and $X_{2,2,2,2}$, we are not yet able to uniquely fix the generating series due to our limited knowledge of GV invariants for these models.

The outline of this work is as follows. In §2, we give a rather extensive introduction to the main mathematical concepts which underlie this work, including the space of Bridgeland stability conditions on the derived category of coherent sheaves $\mathcal{C} = D^b \operatorname{Coh} \mathfrak{Y}$ and the associated generalized DT invariants. We also introduce the family of weak stability conditions $\nu_{b,w}$, which plays a central role in relating rank 0 DT invariants and PT invariants, and spell out the expected modular properties of generating series of Abelian D4-D2-D0 indices. In §3, we recall the relation between PT invariants and GV invariants, and explain how the latter can be computed using the direct integration method. We further give a heuristic computation of GV invariants for maximal genus $g = g_{\max}(Q)$ and submaximal genus $g = g_{\max}(Q) - 1$, which is confirmed in §4.2 as a consequence of Theorem 1 in Appendix §A. In §4, we explain the main results of Appendix A in more physical terms, starting in §4.1 with Theorem 4 which expresses D4-D2-D0 indices as contributions of D6-$\overline{\text{D6}}$-bound states, but whose applicability is limited to the most polar terms, and continuing in §4.2 with Theorem 1, which is less transparent physically but of much wider applicability. In §5 we use Theorem 1 to compute D4-D2-D0 indices and test modularity in three representative models, namely X_5, X_{10} and $X_{4,2}$, leaving the details of other models to Appendix B. Finally, in §6 we summarize our findings and discuss

avenues for future research. Extensive tables of GV, PT and DT invariants computed in the course of this project are available in Mathematica-readable form at the website [32].

Glossary of invariants

For the reader's convenience we summarize the notations for the various types of enumerative invariants that appear in this work. More details will be provided in the corresponding sections.

We generally denote by $\overline{\Omega}_\bullet(\gamma) \in \mathbb{Q}$ the rational Donaldson-Thomas invariants counting $\bullet$-semistable objects of class γ defined as in [3], where $\bullet$ denotes a (weak) stability condition or a limit thereof, and by $\Omega_\bullet(\gamma)$ the (conjecturally integral) generalized Donaldson-Thomas invariants obtained from $\overline{\Omega}_\bullet(\gamma)$ via the 'multicover formula' (2.16). This applies to the following invariants:

- $\overline{\Omega}_\sigma$, with σ a general (weak) stability condition on $\mathcal{C} = D^b \operatorname{Coh}(\mathfrak{Y})$, introduced in §2.2;
- $\overline{\Omega}_{b,w} = \overline{\Omega}_{\nu_{b,w}}$, with $\nu_{b,w}$ the slope function (2.33) on the heart $\mathcal{A}_b$;
- $\overline{\Omega}_\infty = \lim\limits_{w \to +\infty} \overline{\Omega}_{b,w}$, introduced above (2.37);
- $\overline{\Omega}_H$ counting Gieseker-semistable sheaves with respect to an ample class H, defined below (2.37);
- $\overline{\Omega}_{\mathrm{z}}^\Pi$, the DT invariant along the Π-stability slice, defined in §2.4.

We deviate from this notation for the D4-D2-D0 index $\overline{\Omega}_{r,\mu}(\hat{q}_0)$ introduced in §2.6, which determines the rank 0 DT invariant $\overline{\Omega}_{\mathrm{z}}^\Pi(0, r, q_1, q_0)$ in the large volume attractor chamber. In the special case of CY threefolds with Picard group $\operatorname{Pic}\mathfrak{Y} = H\mathbb{Z}$, it coincides with the index $\overline{\Omega}_H(\gamma)$, see (2.54). In §2.7, we also introduce lighter notations for rank ± 1 DT invariants at large volume,

- Donaldson-Thomas invariants $\mathrm{I}_{n,\beta} = \mathrm{DT}(\beta.H, n)$;
- Pandharipande-Thomas invariant $\mathrm{P}_{n,\beta} = \mathrm{PT}(\beta.H, n)$.

As explained in §3.1, these invariants are closely related to Gromov-Witten invariants $\mathrm{GW}_Q^{(g)} \in \mathbb{Q}$ and Gopakumar-Vafa invariants $\mathrm{GV}_Q^{(g)} \in \mathbb{Z}$.

2. Preliminaries

In this section, we recall the basic definitions of the mathematical structures which we use in this work, emphasizing their physical interpretation. In §2.1 we introduce the derived category of coherent sheaves $\mathcal{C} = D^b \operatorname{Coh} \mathfrak{Y}$,

which formalizes the notion of BPS states in type IIA string theory compactified on a Calabi-Yau threefold $\mathfrak{Y}$. In §2.2 we recall the definition of the space of Bridgeland stability conditions $\text{Stab}\,\mathcal{C}$ and the associated generalized Donaldson-Thomas invariants $\Omega_\sigma(\gamma)$, which are the mathematical counterpart of BPS indices. In §2.3 we review the mathematical construction of Bridgeland stability conditions in an open set around the large volume point. As an intermediate step, we introduce a two-parameter family of weak stability conditions defined by the central charge (2.28) which will play a central role in §4. In §2.4 we identify the physical slice of Π-stability conditions inside $\text{Stab}\,\mathcal{C}$. In §2.6, we introduce the rank 0 DT invariants counting D4-D2-D0 bound states, and state the modular properties of generating series of these invariants predicted by string theory arguments, restricting to the Abelian case (one unit of D4-brane charge). Finally, in §2.7 we introduce the rank 1 DT and PT invariants, $\text{DT}(Q, n)$ and $\text{PT}(Q, n)$, which count bound states with ± 1 unit of D6-brane charge at large volume. Their relation to Gopakumar-Vafa invariants is deferred to §3.

After reading §2.1 where notations for charge vectors are introduced, a reader uninterested in mathematical details may skip ahead to §2.5, where we briefly summarize the necessary mathematical constructions. In the last two subsections we introduce the main objects studied in this work, namely the D4-D2-D0 indices and the rank 1 DT and PT invariants.

2.1. BPS branes and derived category of coherent sheaves

As explained in [34, 35, 36], BPS states in type IIA string theory compactified on a Calabi-Yau (CY) threefold $\mathfrak{Y}$ are identified with B-branes in the A-twisted topological sigma model on $\mathfrak{Y}$. Mathematically, they are best understood as objects in the bounded derived category of coherent sheaves $\mathcal{C} = D^b \text{Coh}\,\mathfrak{Y}$. Such an object is a bounded complex

$$(2.1) \qquad E = \left(\dots \overset{d^{-2}}{\to} \mathcal{E}^{-1} \overset{d^{-1}}{\to} \mathcal{E}^0 \overset{d^0}{\to} \mathcal{E}^1 \overset{d^1}{\to} \dots \right),$$

where at each place $k \in \mathbb{Z}$, $\mathcal{E}^k$ is a coherent sheaf on $\mathfrak{Y}$ which vanishes for all but a finite set of indices k, and $d^k : \mathcal{E}^k \to \mathcal{E}^{k+1}$ a morphism such that $d^{k+1}d^k = 0$ for all $k \in \mathbb{Z}$. Up to quasi-isomorphisms (which preserve the cohomology of the complex and physically correspond to irrelevant boundary deformations), the coherent sheaf $\mathcal{E}^k$ can be assumed to be a vector bundle on $\mathfrak{Y}$, and is physically interpreted as a stack of wrapped D6-branes for k even, respectively anti-D6-branes for k odd. The morphism d^k is then interpreted as an open string tachyon field. More generally, the extension group

$\mathrm{Ext}^n(E, E') := \mathrm{Hom}(E, E'[n])$, where $[n]$ is the translation functor mapping $E = (\mathcal{E}^k, d^k)_{k \in \mathbb{Z}} \mapsto E[n] = (\mathcal{E}^{k-n}, d^{k-n})_{k \in \mathbb{Z}}$, is interpreted physically as the space of open strings of ghost number n.

Besides the grading by ghost number, the category $\mathcal{C}$ is also graded by the numerical Grothendieck group $K(\mathcal{C})$, which plays the role of the lattice of electromagnetic charges. Using the Chern character map $E \mapsto \mathrm{ch}(E) = \sum_k (-1)^k \mathrm{ch}(\mathcal{E}^k)$, $K(\mathcal{C})$ can be identified with the lattice $\Gamma \subset H^{\mathrm{even}}(\mathfrak{Y}, \mathbb{Q})$ spanned by vectors $\mathsf{v} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3)$ satisfying the quantization conditions [3, Theorem 4.19]

$$(2.2) \qquad \begin{aligned} &\mathrm{ch}_0 \in H^0(\mathfrak{Y}, \mathbb{Z}), \qquad \mathrm{ch}_1 \in H^2(\mathfrak{Y}, \mathbb{Z}), \\ &\mathrm{ch}_2 - \frac{1}{2}\mathrm{ch}_1^2 \in H^4(\mathfrak{Y}, \mathbb{Z}), \qquad \mathrm{ch}_3 + \frac{1}{2}c_2(T\mathfrak{Y})\,\mathrm{ch}_1 \in H^6(\mathfrak{Y}, \mathbb{Z}). \end{aligned}$$

The respective integer cohomology classes correspond physically to the D6, D4, D2 and D0 brane charges. The lattice Γ is endowed with the integer skew-symmetric pairing

$$(2.3) \qquad \langle \mathrm{ch}(E), \mathrm{ch}(E') \rangle := \int_{\mathfrak{Y}} (\mathrm{ch}\, E')^\vee \, \mathrm{ch}(E)\, \mathrm{Td}(T\mathfrak{Y}),$$

where $\vee$ acts as $(-1)^p$ on a form of degree $2p$ and $\mathrm{Td}(T\mathfrak{Y}) = 1 + \frac{1}{12}c_2(T\mathfrak{Y})$ is the Todd class of the tangent bundle. This pairing is skew-symmetric due to Serre duality $\mathrm{Ext}^n(E, E') = \mathrm{Ext}^{3-n}(E', E)$, and integer valued by the Grothendieck-Riemann-Roch (GRR) theorem, which identifies it with the alternating sum of the dimensions

$$(2.4) \qquad \chi(E', E) := \sum_n (-1)^n \dim \mathrm{Ext}^n(E', E) = \langle \mathrm{ch}(E), \mathrm{ch}(E') \rangle.$$

Physically, (2.3) is interpreted as the Dirac-Schwinger-Zwanziger pairing between electromagnetic charge vectors. It is useful to introduce the Mukai vector[1]

$$(2.5) \qquad \gamma(E) = \mathrm{ch}(E)\sqrt{\mathrm{Td}(T\mathfrak{Y})},$$

such that the pairing (2.3) takes the Darboux form $\int_{\mathfrak{Y}} \gamma(E')^\vee \gamma(E)$. We shall abuse notation and denote it by $\langle \mathsf{v}, \mathsf{v}' \rangle$ or $\langle \gamma, \gamma' \rangle$ interchangeably. We note that both $\mathrm{ch}(E)$ and $\gamma(E)$ change sign under the translation functor $E \mapsto$

---

[1]Note that a different convention $\gamma(E) = \mathrm{ch}(E)^\vee \sqrt{\mathrm{Td}(T\mathfrak{Y})}$ also appears in the literature.

$E[1]$, corresponding to CPT symmetry in physics, which maps D-branes to anti-D-branes. Instead, the transformation $\mathrm{ch}(E) \mapsto (\mathrm{ch}\, E)^\vee$ follows by taking the derived dual $E \mapsto E^\vee$, which is the physical counterpart of a parity transformation.

In this paper, unless mentioned otherwise, we always assume that $\mathfrak{Y}$ is a smooth projective CY threefold $\mathfrak{Y}$ with $b_2(\mathfrak{Y}) = 1$ and $H^2(\mathfrak{Y}, \mathbb{Z})_{\mathrm{tors}} = 0$. This last property holds for any general complete intersection in weighted projective spaces by a generalisation of Grothendieck-Lefschetz theorem proved in [37, Theorem 1], in particular for all models in Table 1. We denote by H the generator of $\Lambda := H^2(\mathfrak{Y}, \mathbb{Z}) = H\mathbb{Z}$. The lattice $\Lambda^* = H^4(\mathfrak{Y}, \mathbb{Z})$ is then generated by H^2/κ where $\kappa = \int_{\mathfrak{Y}} H^3$. Poincaré duality maps H to a primitive divisor class $[\mathcal{D}]$ in $H_4(\mathfrak{Y}, \mathbb{Z})$, where $\mathcal{D}$ is an ample divisor with cubic self-intersection $\kappa = [\mathcal{D}]^3$, and H^2/κ to a primitive curve class $[C] \in H_2(\mathfrak{Y}, \mathbb{Z})$.

We identify the Chern character $\mathrm{ch}(E)$ with the vector of rational numbers

(2.6)
$$[C_0, C_1, C_2, C_3](E) := \int_{\mathfrak{Y}} [H^3\, \mathrm{ch}_0(E),\, H^2.\, \mathrm{ch}_1(E),\, H.\, \mathrm{ch}_2(E),\, \mathrm{ch}_3(E)] \in \mathbb{Q}^4\,,$$

such that $\mathrm{ch} = (C_0 + C_1 H + C_2 H^2 + C_3 H^3)/\kappa$. Its components satisfy the quantization conditions

$$(2.7) \quad C_0 \in \kappa\mathbb{Z}, \qquad C_1 \in \kappa\mathbb{Z}, \qquad C_2 \in \mathbb{Z} + \frac{C_1^2}{2\kappa}, \qquad C_3 \in \mathbb{Z} - \frac{c_2}{12\kappa}\, C_1,$$

where we use the shorthand notation $c_2 := H.c_2(T\mathfrak{Y})$. We also define the charge vector $\gamma(E) = (p^0, p^1, q_1, q_0)$ obtained by expanding the Mukai vector (2.5) as in [38, (4.8)],

$$(2.8) \qquad\qquad \gamma(E) = p^0 + p^1 H - \frac{q_1}{\kappa}\, H^2 + \frac{q_0}{\kappa}\, H^3\,.$$

The Chern and Mukai vectors are related by

$$(2.9) \quad \begin{aligned} p^0 &= \mathrm{ch}_0, \quad p^1 = \frac{1}{\kappa}\, H^2.\, \mathrm{ch}_1, \\ q_1 &= -H.\, \mathrm{ch}_2 - \frac{c_2}{24\kappa}\, H^3\, \mathrm{ch}_0, \quad q_0 = \mathrm{ch}_3 + \frac{c_2}{24\kappa}\, H^2.\, \mathrm{ch}_1, \end{aligned}$$

such that

$$(2.10) \quad p^0 \in \mathbb{Z}, \qquad p^1 \in \mathbb{Z}, \qquad q_1 \in \mathbb{Z} + \frac{\kappa}{2}\, (p^1)^2 - \frac{c_2}{24}\, p^0, \qquad q_0 \in \mathbb{Z} - \frac{c_2}{24}\, p^1.$$

In this basis, the Dirac pairing (2.4) takes the Darboux form

$$(2.11) \qquad \langle \gamma, \gamma' \rangle = q_0 p'^0 + q_1 p'^1 - q_1' p^1 - q_0' p^0 \,.$$

Under the action of the auto-equivalence $E \mapsto E(k) := E \otimes \mathcal{O}_{\mathfrak{Y}}(kH)$ with $k \in \mathbb{Z}$, the Chern character transforms as $\mathrm{ch}(E) \mapsto e^{kH} \mathrm{ch}(E)$, while the components of the Mukai vector transform as

$$(2.12) \qquad p^0 \mapsto p^0, \qquad p^1 \mapsto p^1 + k p^0, \qquad q_1 \mapsto q_1 - \kappa k \, p^1 - \frac{\kappa k^2}{2} \, p^0 \,,$$

$$q_0 \mapsto q_0 - k q_1 + \frac{\kappa k^2}{2} \, p^1 + \frac{\kappa k^3}{6} \, p^0 \,.$$

We refer to this transformation as a spectral flow.

For later reference, we record the Mukai vectors for the primitive D6, D4, D2 and D0-branes, represented by the structure sheaves of the threefold $\mathfrak{Y}$, of the ample divisor $\mathcal{D}$, of the curve C and of a point $x \in \mathfrak{Y}$ Poincaré dual to H^3/κ,

$$(2.13) \qquad \begin{aligned} \gamma(\mathcal{O}_{\mathfrak{Y}}) &= \left(1, 0, -\frac{c_2}{24}, 0\right), & \gamma(\mathcal{O}_{\mathcal{D}}) &= \left(0, 1, \frac{\kappa}{2}, \frac{\kappa}{6} + \frac{c_2}{24}\right), \\ \gamma(\mathcal{O}_C) &= (0, 0, 1, -1), & \gamma(\mathcal{O}_x) &= (0, 0, 0, 1). \end{aligned}$$

It is immediate to check that the quantization conditions (2.10) are obeyed, using the fact that $\chi_{\mathcal{D}} := \chi(\mathcal{O}_{\mathcal{D}}) = \frac{\kappa}{6} + \frac{c_2}{12}$ is integer (and equal to the arithmetic genus plus one).

2.2. Bridgeland stability conditions and Donaldson-Thomas invariants

Physically, BPS states are elements in the point particle spectrum whose mass M saturate the Bogomolnyi-Prasad-Sommerfeld bound $M \geq |Z(\gamma)|$, where $Z(\gamma)$ is a central generator in the super-Poincaré algebra, which depends linearly on the electromagnetic charge vector γ and is otherwise a transcendental function of the complexified Kähler moduli $z \in \mathcal{M}_K(\mathfrak{Y})$. The BPS index $\Omega_z(\gamma)$ counts the number of BPS states with charge γ, weighted with a sign $(-1)^{2J_3}$ where J_3 is the projection of the angular momentum along a fixed axis, such that $\Omega_z(\gamma)$ becomes robust under complex deformations of $\mathfrak{Y}$. Mathematically, this is formalized by introducing the notion of Π-stability conditions, which are special cases of Bridgeland stability

conditions[2], and the associated generalized Donaldson-Thomas invariants.

A Bridgeland stability condition consists of a pair $\sigma = (Z, \mathcal{A})$ satisfying the following axioms [39]:

i) $Z : \Gamma \to \mathbb{C}$ is a linear map, known as the (holomorphic) central charge (we abuse notation and denote $Z(E) = Z(\gamma(E))$ for any $E \in \mathcal{C}$);

ii) $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{C}$ (i.e. $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ where $(\mathcal{D}^{\leq 0}, \mathcal{D}^{> 0})$ is a pair of orthogonal subcategories of $\mathcal{C}$ which are invariant under the left and right translation functors [1] and [−1], respectively), in particular $\mathcal{A}$ is an Abelian subcategory of $\mathcal{C}$;

iii) For any non-zero $E \in \mathcal{A}$, the central charge $Z(E)$ is contained in the Poincaré upper half-plane $\mathbb{H}_B = \mathbb{H} \cup (-\infty, 0)$, i.e. $Z(E) = \rho(E)e^{i\pi\phi(E)}$ where $\rho(E) > 0$ and $0 < \phi(E) \leq 1$;

iv) *(Harder-Narasimhan property)* Every non-zero $E \in \mathcal{A}$ admits a finite filtration $0 \subset E_0 \subset E_1 \cdots \subset E_n = E$ by objects E_i in $\mathcal{A}$, such that each factor $F_i := E_i/E_{i-1}$ is σ-semistable (as defined below) and $\phi(F_1) > \phi(F_2) \cdots > \phi(F_n)$;

v) *(Support property)* There exists a constant $C > 0$ such that, for all σ-semistable objects $E \in \mathcal{A}$, $\|\gamma(E)\| \leq C |Z(E)|$ where $\| \cdot \|$ is any fixed Euclidean norm on $\Gamma \otimes \mathbb{R}$.

In the last two items above, an object $F \in \mathcal{A}$ is called σ-semistable if $\phi(F') \leq \phi(F)$ for every non-zero subobject F' of F. More generally, an object $F \in \mathcal{C}$ is called σ-semistable if there exists $n \in \mathbb{Z}$ such that $F[n] \in \mathcal{A}$ and $F[n]$ is σ-semistable in the previous sense. For most purposes in this paper, we shall only need the notion of *weak stability condition* (as defined in [10, Appendix B]), which essentially amounts to relaxing the axiom iii) and allowing $\mathcal{A}$ to contain objects with vanishing central charge.

For any weak stability condition σ (subject to certain technical conditions spelled out in [3]) and any charge vector $\gamma \in \Gamma$, one defines the generalized Donaldson-Thomas invariant $\Omega_\sigma(\gamma)$ as follows. Let $\mathcal{M}_\sigma(\gamma)$ be the moduli stack of σ-semistable objects in $\mathcal{A}$ with $\gamma(E) = \pm\gamma$, where the sign is chosen such that $\pm Z(\gamma) \in \mathbb{H}_B$. If γ is primitive and σ generic, $\Omega_\sigma(\gamma)$ can be defined as the weighted Euler number

$$(2.14) \qquad \Omega_\sigma(\gamma) = \chi(\mathcal{M}_\sigma(\gamma), \nu) := \sum_{m \in \mathbb{Z}} m\chi(\nu^{-1}(m)),$$

---

[2]Stability conditions are defined on triangulated categories, which include the data of a translation functor $E \mapsto E[1]$ and a collection of distinguished triangles $A \to B \to C \to A[1]$ satisfying various axioms. The derived category of coherent sheaves is automatically endowed with a triangulated structure. For simplicity, we conflate distinguished triangles with short exact sequences $0 \to A \to B \to C \to 0$.

where $\nu : \mathcal{M}_\sigma(\gamma) \to \mathbb{Z}$ is Behrend's constructible function [40].[3] In the simplest case when $\mathcal{M}_\sigma(\gamma)$ is a smooth projective variety (up to the trivial $\mathbb{C}^\times$ action), $\Omega_\sigma(\gamma)$ is equal to the topological Euler characteristic up to a sign,

$$(2.15) \qquad \Omega_\sigma(\gamma) = (-1)^{\dim_\mathbb{C} \mathcal{M}_\sigma(\gamma)} \chi(\mathcal{M}_\sigma(\gamma)).$$

For non-primitive charge vectors, one first defines a rational invariant $\overline{\Omega}_\sigma(\gamma) \in \mathbb{Q}$ following [3], and then sets

$$(2.16) \qquad \Omega_\sigma(\gamma) = \sum_{k|\gamma} \frac{\mu(k)}{k^2} \overline{\Omega}_\sigma(\gamma/k),$$

where $\mu(k)$ is the Möbius function.[4] While $\Omega_\sigma(\gamma)$ is manifestly integral when γ is primitive, its integrality for general charge γ and σ generic remains conjectural. We shall often abuse notation and denote $\Omega_\sigma(\mathsf{v}) = \Omega_\sigma(\gamma)$ where $\mathsf{v} = \gamma/\sqrt{\mathrm{Td}(T\mathfrak{Y})}$ is the Chern character associated to the Mukai vector γ.

For a compact CY threefold, the space of Bridgeland stability conditions $\mathrm{Stab}(\mathcal{C})$ is hard to construct and poorly understood in general. Assuming that it is non-empty (as physics strongly suggests), one can show [39] that it is a complex manifold of dimension $\mathrm{rk}\,\Gamma$, such that the forgetful map $\mathrm{Stab}(\mathcal{C}) \to \mathrm{Hom}(\Gamma, \mathbb{C})$ which sends $\sigma = (Z, \mathcal{A}) \mapsto Z$ is a local homeomorphism. In other words, the heart $\mathcal{A}$ is locally determined by the central charge function Z. In particular, the complex dimension $\mathrm{rk}\,\Gamma = b_{\mathrm{even}}(\mathfrak{Y}) = 2b_2(\mathfrak{Y}) + 2$ is larger than the dimension $b_2(\mathfrak{Y})$ of Kähler moduli space $\mathcal{M}_K(\mathfrak{Y})$, which is conjecturally embedded as a co-dimension $b_2(\mathfrak{Y}) + 2$ submanifold $\Pi \subset \mathrm{Stab}(\mathcal{C})$, as we discuss in §2.4.

Moreover, $\mathrm{Stab}(\mathcal{C})$ admits an action of $\widetilde{GL^+}(2, \mathbb{R}) \times \mathrm{Aut}\,\mathcal{C}$ [39, Lemma 8.2], where $\widetilde{GL^+}(2, \mathbb{R})$ is the universal cover of the group of 2×2 real matrices with positive determinant and $\mathrm{Aut}\,\mathcal{C}$ is the group of autoequivalences of $\mathcal{C}$. The group $GL^+(2, \mathbb{R})$ acts on the central charge Z via

$$(2.17) \qquad \begin{pmatrix} \mathrm{Re}\,Z \\ \mathrm{Im}\,Z \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathrm{Re}\,Z \\ \mathrm{Im}\,Z \end{pmatrix}, \qquad ad - bc > 0 \,,$$

---

[3]As explained e.g. in [41, §2.3], the weight $\nu(p)$ can be interpreted physically as the dimension of the chiral ring of the superpotential whose critical locus determines the moduli space $\mathcal{M}_\sigma(\gamma)$.

[4]Recall that $\mu(k) = 0$ if k has repeated prime factors, otherwise $\mu(k) = (-1)^n$ with n the number of prime factors.

preserving the orientation on $\mathbb{R}^2$, hence the phase ordering of the central charges and hence stability of objects. Its universal cover acts on the stability condition $(Z, \mathcal{A})$ by suitably tilting the heart $\mathcal{A}$. By construction, the Donaldson-Thomas invariant $\Omega_\sigma(\gamma)$ is invariant under the action of $\widetilde{GL^+}(2, \mathbb{R})$ on σ, and under the combined action of $\mathrm{Aut}(\mathcal{C})$ on (γ, σ).

Importantly, being integer valued, the generalized DT invariants $\Omega_\sigma(\gamma) \in \mathbb{Z}$ are locally constant on $\mathrm{Stab}\,\mathcal{C}$, but they may jump when some object $E \in \mathcal{A}$ of charge γ goes from being stable to unstable. This may happen when the central charge $Z(\gamma')$ of a subobject $E' \subset E$ of charge γ' becomes aligned with $Z(\gamma)$, therefore along the real-codimension one *wall of instability* (or marginal stability)

$$(2.18) \qquad \mathcal{W}(\gamma, \gamma') \coloneqq \{\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}\,\mathcal{C} : \mathrm{Im}\,(Z(\gamma')\overline{Z(\gamma)}) = 0\}.$$

The discontinuity across $\mathcal{W}(\gamma, \gamma')$ is determined from the invariants on either side of the wall by the wall-crossing formulae of [2, 3]. Physically, the jump in the BPS index is due to the appearance or disappearance of multi-centered black hole bound states [16]. Of course, this physical interpretation only holds along the physical slice of Π-stability conditions.

2.3. Stability conditions for one-modulus CY threefolds

We now restrict again to compact CY threefolds with $b_2(\mathfrak{Y}) = 1$, and explain a general construction of an open set of Bridgeland stability conditions around the large volume limit following [9, 10]. While the full construction is not needed for the rest of the paper, it allows us to introduce, as an intermediate step, a family of weak stability conditions (2.28) (called tilt-stability in [9, 10]) and a conjectural inequality (2.31), which will play an essential role in relating rank 1 and rank 0 DT invariants in §4.

Parametrizing central charge functions modulo $\widetilde{GL^+}(2, \mathbb{R})$ action
As explained in the previous subsection, the space of Bridgeland stability conditions is parametrized locally by the central charges of the objects (2.13), or equivalently by the components $(X^0, X^1, F_1, F_0) \in \mathbb{C}^4$ of the holomorphic central charge in the Mukai basis,

$$(2.19) \qquad\qquad Z(\gamma) = q_0 X^0 + q_1 X^1 - p^1 F_1 - p^0 F_0.$$

Using the $\widetilde{GL^+}(2, \mathbb{R})$ action, we may restrict to the real four-dimensional slice with central charge [10, §8][5] parametrized by $(a, b, \alpha, \beta) \in \mathbb{R}^4$,

$$(2.20) \qquad Z_{a,b,\alpha,\beta}(\gamma) = \left(-\operatorname{ch}_3^b + \beta \operatorname{ch}_2^b + \alpha \operatorname{ch}_1^b \right) + \mathrm{i} \left(a \operatorname{ch}_2^b - \frac{1}{2} a^3 \operatorname{ch}_0^b \right),$$

where $\operatorname{ch}_k^b(E) = \int_{\mathfrak{Y}} e^{-bH} . H^{3-k} . \operatorname{ch}(E)$, or more explicitly

$$(2.21) \qquad \operatorname{ch}_0^b = C_0, \qquad \operatorname{ch}_1^b = C_1 - bC_0, \qquad \operatorname{ch}_2^b = C_2 - bC_1 + \frac{1}{2} b^2 C_0,$$
$$\operatorname{ch}_3^b = C_3 - bC_2 + \frac{1}{2} b^2 C_1 - \frac{1}{6} b^3 C_0.$$

This slice is invariant under the spectral flow transformation (2.12) provided it is accompanied by a translation $b \mapsto b + k$. We note that under derived duality $\gamma \mapsto \gamma^\vee$ (see below (2.5)) accompanied by a sign flip of (b, β), the central charge (2.20) transforms into its complex conjugate,

$$(2.22) \qquad Z_{a,b,\alpha,\beta}(\gamma^\vee) = -\overline{Z_{a,-b,\alpha,-\beta}(\gamma)}.$$

Upon setting

$$(2.23) \qquad a = \sqrt{\frac{1}{3} t^2 - \frac{c_2}{12\kappa}}, \qquad \alpha = \frac{1}{2} t^2 - \frac{c_2}{24\kappa}, \qquad \beta = 0,$$

the function (2.20) coincides with the large volume central charge[6]

$$(2.24) \qquad Z_{b,t}^{\mathrm{LV}}(E) = -\int_{\mathfrak{Y}} e^{-(b+\mathrm{i}t)H} \gamma(E),$$

up to rescaling of its imaginary part by t/a using the $\widetilde{GL^+}(2, \mathbb{R})$ action.

In [9, 10], a method to construct a heart $\mathcal{A}_{a,b}$ (depending only on a and b) is introduced so that the pair $(Z_{a,b,\alpha,\beta}, \mathcal{A}_{a,b})$ is a Bridgeland stability condition on $\mathcal{D}^b(\mathfrak{Y})$ whenever the inequalities

$$(2.25) \qquad a > 0, \qquad \alpha > \frac{1}{6} a^2 + \frac{1}{2} a|\beta|$$

---

[5]We swap (a, b) and (α, β) compared to [10], and rescale the imaginary part by the positive factor a.

[6]As discussed below (2.49), this formula agrees with the physical central charge in the large volume $t \to \infty$, up to an $\mathcal{O}(t^0)$ correction proportional to $\zeta(3)\chi_{\mathfrak{Y}}$. Agreement up to $\mathcal{O}(e^{-t})$ can be achieved by replacing $\sqrt{\operatorname{Td}(T\mathfrak{Y})}$ in $\gamma(E)$ (2.5) by the Γ-class [42].

are satisfied. The second condition ensures that the central charge (2.20) never vanishes on objects $\mathcal{O}_{\mathfrak{Y}}(mH)$ with $m \in \mathbb{Z}$. In particular, the region (2.25) includes the large volume slice (2.23) for $t^2 > \frac{c_2}{4\kappa}$. As we review in the remainder of this subsection, the construction of [9, 10] proceeds in two steps,

$$\text{slope stability} \qquad N_{b,a}\text{-stability} \qquad \text{Bridgeland stability}$$
$$\left(\operatorname{Coh}\mathfrak{Y},\, -\operatorname{ch}_1^b + i\operatorname{ch}_0^b\right) \xrightarrow{\text{tilt}} (\mathcal{A}_b,\, Z_{b,a}) \xrightarrow{\text{tilt}} (\mathcal{A}_{b,a},\, Z_{a,b,\alpha,\beta})$$

Independently of its use for constructing Bridgeland stability conditions, the family of weak stability conditions $N_{b,a}$ appearing in the intermediate step plays an essential role in relating rank 0 DT invariants to rank 1 DT invariants.

Step 1 We first start with the Abelian category of coherent sheaves $\operatorname{Coh}\mathfrak{Y}$ where for any $b \in \mathbb{R}$ we define the slope function

$$(2.26) \qquad \mu_b(\mathcal{E}) = \frac{\operatorname{ch}_1^b(\mathcal{E})}{\operatorname{ch}_0^b(\mathcal{E})}$$

for $\operatorname{ch}_0(\mathcal{E}) \neq 0$, and $\mu_b(\mathcal{E}) = +\infty$ otherwise. We say a coherent sheaf $\mathcal{E}$ is slope semi-stable if $\mu_b(\mathcal{E}') \leq \mu_b(\mathcal{E})$ for any subsheaf $\mathcal{E}' \subset \mathcal{E}$. We know that any slope semistable sheaf satisfies the classical Bogomolov-Gieseker inequality [10, Theorem 3.2]:

$$(2.27) \qquad \Delta_H(E) := (\operatorname{ch}_1^b(E))^2 - 2\operatorname{ch}_0^b(E)\operatorname{ch}_2^b(E) = C_1^2 - 2C_0 C_2 \geq 0.$$

Following [9], one defines

- $\mathcal{T}_b \subset \operatorname{Coh}\mathfrak{Y}$ as the subcategory generated by slope-semi-stable sheaves $\mathcal{E}$ with $\mu_b(\mathcal{E}) > 0$,
- $\mathcal{F}_b \subset \operatorname{Coh}\mathfrak{Y}$ as the subcategory generated by slope-semi-stable sheaves $\mathcal{E}$ with $\mu_b(\mathcal{E}) \leq 0$.

Then $\mathcal{A}_b := \langle \mathcal{F}_b[1], \mathcal{T}_b \rangle$ is the heart of a bounded t-structure on $\mathcal{D}^b(\mathfrak{Y})$ generated by length two complexes of the form $E = (F \xrightarrow{d} T)$ with $\ker d \in \mathcal{T}_b$ and $\operatorname{cok} d \in \mathcal{F}_b$. For objects in the heart $\mathcal{A}_b$, we consider the central charge function

$$(2.28) \qquad Z_{b,a}(\gamma) = -a\operatorname{ch}_2^b + \frac{1}{2}a^3\operatorname{ch}_0^b + i\,a^2\operatorname{ch}_1^b.$$

Note that up to $\widetilde{GL}(2,\mathbb{R})$-action, it can be obtained by setting $\beta = 0$ and $\alpha = \infty$ in (2.20), effectively getting rid of the dependence on ch_3. The resulting pair $(Z_{b,a}, \mathcal{A}_b)$ satisfies the axioms (i,ii,iv,v) in the previous subsection, but not iii), since the central charge of skyscraper sheaves vanishes. Nonetheless, it defines a family of weak stability conditions in the sense of [10, Appendix B].

For an object $E \in \mathcal{A}_b$, we define[7]

$$(2.29) \qquad N_{b,a}(E) := -\frac{\text{Re}\,[Z_{b,a}(E)]}{\text{Im}\,[Z_{b,a}(E)]} = \frac{\text{ch}_2^b(E) - \frac{1}{2}\,a^2\,\text{ch}_0^b(E)}{a\,\text{ch}_1^b(E)}\,,$$

with $N_{b,a}(E) = +\infty$ if $\text{ch}_1^b(E) = 0$. Then by definition, $E \in \mathcal{A}_b$ is semistable with respect to the pair $(Z_{b,a}, \mathcal{A}_b)$ if and only if for any non-trivial subobject $F \subset E$ in $\mathcal{A}_b$, we have $N_{b,a}(F) \leq N_{b,a}(E)$. By [10, Theorem 3.5], any such semistable object $E \in \mathcal{A}_b$ satisfies the classical Bogomolov inequality (2.27). Moreover, it is conjectured in [10, Conjecture 4.1] that it satisfies the following inequality involving the third Chern class $\text{ch}_3(E)$:

$$(2.30) \qquad a^2\left[(\text{ch}_1^b)^2 - 2\,\text{ch}_0^b\,\text{ch}_2^b\right] + 4(\text{ch}_2^b)^2 - 6\,\text{ch}_1^b\,\text{ch}_3^b \geq 0\,,$$

which we refer to as the BMT inequality. Moreover [10, Theorem 4.2] shows that the inequality (2.30) is equivalent to the original Conjecture 1.3.1 in [9], which says that for any object $E \in \mathcal{A}_b$ which is semistable with respect to the stability function $Z_{b,a}$ and satisfies $N_{b,a}(E) = 0$, i.e. $\text{ch}_2^b = \frac{1}{2}\,a^2\,\text{ch}_0^b$, one has

$$(2.31) \qquad \text{ch}_3^b \leq \frac{a^2}{6}\,\text{ch}_1^b\,.$$

Step 2 Similar to the construction of $\mathcal{A}_b$ in the first step, one defines

- $\mathcal{T}_{b,a} \subset \mathcal{A}_b$ as the subcategory generated by semi-stable objects in $\mathcal{A}_b$ with $N_{b,a}(E) > 0$,
- $\mathcal{F}_{b,a} \subset \mathcal{A}_b$ as the subcategory generated by semi-stable objects in $\mathcal{A}_b$ with $N_{b,a}(E) \leq 0$.

Then we define $\mathcal{A}_{b,a} = \langle \mathcal{F}_{b,a}[1], \mathcal{T}_{b,a} \rangle$. By construction, $\text{Im}\,Z_{a,b,\alpha,\beta}(E) \geq 0$ for any object $E \in \mathcal{A}_{b,a}$. The conjectural inequality (2.31) further guarantees that $\text{Re}\,Z_{a,b,\alpha,\beta}(E) < 0$ whenever $\text{Im}\,Z_{a,b,\alpha,\beta}(E) = 0$ [10], which shows that the axioms of §2.2 are indeed satisfied. This was in fact the original motivation for the conjectural BMT inequality.

---

[7]The ratio (2.29) agrees with $\sqrt{3}N_{bH,tH}(E)$ in [9, 10], upon setting $t = a\sqrt{3}$.

Wall-crossing in the space of weak stability condition To obtain the formula relating rank zero DT invariants to rank one DT invariants in Appendix A, we shall apply the wall-crossing formula in the space of weak stability conditions $(Z_{b,a}, \mathcal{A}_b)$, rather than in the space of Bridgeland stability conditions $\mathrm{Stab}(\mathcal{C})$, as walls are much easier to control.

It will be convenient to rescale and shift the slope function $N_{b,a}$ (2.29) into

$$(2.32) \qquad \nu_{b,w} := a N_{b,a} + b, \quad \text{where } w := \frac{1}{2}(a^2 + b^2),$$

for $w > b^2/2$. This is because the new slope

$$(2.33) \qquad \nu_{b,w}(E) = \begin{cases} \dfrac{C_2(E) - w C_0(E)}{C_1(E) - b C_0(E)} & \text{if } \mathrm{ch}_1^b(E) \neq 0, \\ +\infty & \text{if } \mathrm{ch}_1^b(E) = 0 \end{cases}$$

has a denominator that is linear in b and numerator linear in w, so the walls of $\nu_{b,w}$-instability (which is by construction equivalent to $N_{b,a}$-instability) are line segments in the region

$$(2.34) \qquad U := \left\{ (b, w) \in \mathbb{R}^2 \colon w > \tfrac{1}{2} b^2 \right\}$$

of the (b, w) plane (see the green line in Fig. 2). We shall abuse notation and denote by $\overline{\Omega}_{b,w}(\gamma)$ the rational DT invariant $\overline{\Omega}_{\nu_{b,w}}(\gamma)$ counting $\nu_{b,w}$-stable objects of class γ.

More precisely, the slope (2.33) coincides for two objects E and E' of Chern character C_i and C_i' along the line

$$(2.35) \qquad (C_1 C_0' - C_1' C_0) w + b(C_0 C_2' - C_0' C_2) + (C_2 C_1' - C_1 C_2') = 0,$$

passing through the points $\varpi(\gamma)$ and $\varpi(\gamma')$ defined by

$$(2.36) \qquad \varpi(\gamma) = \left(\frac{C_1}{C_0}, \frac{C_2}{C_0} \right).$$

Note that the points $\varpi(\gamma)$ lie outside the region U when E and E' are $\nu_{b,w}$-semistable objects, due to the Bogomolov-Gieseker inequality (2.27). In the original coordinates (b, a), walls of $N_{b,a}$-instability are half-circles centered at $b = \frac{C_0 C_2' - C_0' C_2}{C_0 C_1' - C_0' C_1}$ along the axis $a = 0$, or vertical lines going through $b = \frac{C_1 C_2' - C_1' C_2}{C_0 C_2' - C_0' C_2}$ when $C_1 C_0' - C_1' C_0 = 0$.
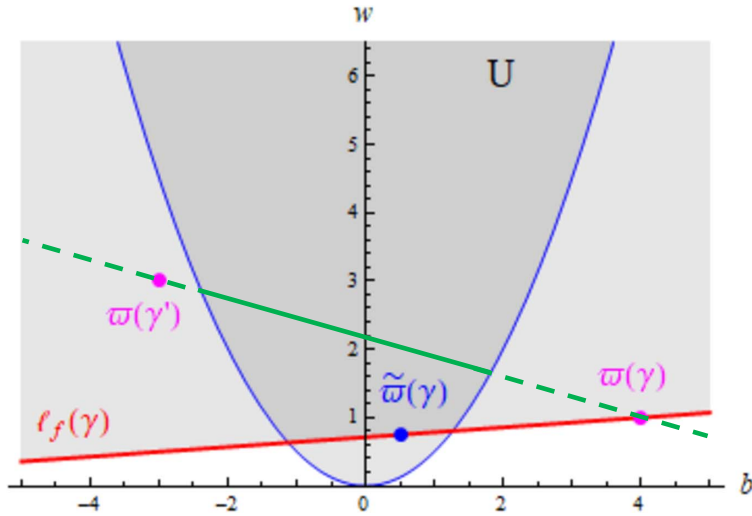
Figure 2: In the (b, w)-plane, walls of $\nu_{b,w}$-instability are straight lines between $\varpi(\gamma)$ and $\varpi(\gamma')$, where γ and γ' are the charges of the destabilizing objects. The BMT inequality (2.38) is saturated along the red line going through $\varpi(\gamma)$ and $\widetilde{\varpi}(\gamma)$. The parabola $w = \frac{1}{2}b^2$ is shown in blue.

Wall and chamber structure For any fixed class γ with $C_0 \neq 0$, or $C_0 = 0$ and $C_1 \neq 0$, there exists a set of lines $\{\ell_i\}_{i \in I}$ in $\mathbb{R}^2$ [43, Proposition 4.1] such that the segments $\ell_i \cap U$ (called 'walls') are locally finite and satisfy

1. If $C_0 \neq 0$, then all lines ℓ_i pass through $\varpi(\gamma)$, and if $C_0 = 0$ then all lines ℓ_i are parallel of slope $\frac{C_2}{C_1}$.
2. The $\nu_{b,w}$-semistability of any object $E \in \mathcal{C}$ of class γ is unchanged as (b, w) varies within any connected component (called a "*chamber*") of $U \setminus \bigcup_{i \in I} \ell_i$.
3. For any wall $\ell_i \cap U$, there is an object $E \in \mathcal{C}$ of class γ which is strictly $\nu_{b,w}$-semistable for all $(b, w) \in \ell_i \cap U$.

The DT invariant $\overline{\Omega}_{b,w^+}(\gamma)$ at a point just above ℓ_i is determined from the invariant $\overline{\Omega}_{b,w^-}(\gamma)$ at a point just below ℓ_i by the wall-crossing formula of [3]. Note that with this definition, the DT invariant $\overline{\Omega}_{b,w}(\gamma)$ is not necessarily discontinuous across the wall.

Tilt-stability and Gieseker stability Since the number of walls for fixed charge γ which are crossed as $w \to +\infty$ is finite [7, Proposition 1.4], the index $\overline{\Omega}_{b,w}(\gamma)$ reaches a fixed value as $w \to +\infty$. For $p^0 = 0$, there is no

vertical wall, so this value is independent of b, and we denote it by $\overline{\Omega}_\infty(\gamma)$. For $p^0 \neq 0$, the index may jump across the vertical wall at $b = \frac{C_1}{C_0}$ given by the vanishing of the slope (2.26). We denote by $\overline{\Omega}_\infty(\gamma)$ the limit of the index $\overline{\Omega}_{b,w}(\gamma)$ as $w \to +\infty$ on the side $b < \frac{C_1}{C_0}$ for positive rank $p^0 > 0$, or on the side $b > \frac{C_1}{C_0}$ for negative rank $p^0 < 0$.

For non-negative rank $p^0 \geq 0$ and γ primitive, it turns out that $\overline{\Omega}_\infty(\gamma)$ agrees with the weighted Euler number $\chi(\mathcal{M}_{\text{tilt}}(\gamma), \nu)$ of the moduli space $\mathcal{M}_{\text{tilt}}(\gamma)$ of tilt-semi-stable sheaves of charge γ [8, Lemma 2.4]. Here, tilt-stability is a variant of Gieseker semi-stability defined as follows: let $P_E(k)$ be the Hilbert polynomial

$$
\begin{aligned}
(2.37) \quad P_E(k) &:= \chi(\mathcal{O}(-kH), E) = \int_{\mathfrak{Y}} e^{kH} \operatorname{ch} E \operatorname{Td}(T\mathfrak{Y}) \\
&= \frac{\kappa p^0}{6} k^3 + \frac{\kappa p^1}{2} k^2 - \left(q_1 + \frac{c_2}{24} p^0 \right) k + \left(q_0 - \frac{c_2}{24} p^1 \right),
\end{aligned}
$$

and $p_E(k) = P_E(k)/a_E$ the associated monic Hilbert polynomial, with a_E the coefficient of the highest degree term in k. Gieseker-(semi)stability for a coherent sheaf E is the requirement that for all exact sequences $0 \to E' \to E \to E'' \to 0$ of coherent sheaves, we have $\deg p_{E'} > \deg p_{E''}$, or $\deg p_{E'} = \deg p_{E''}$ and $p_{E'}(k) < (\leq) p_{E''}(k)$ for $k \gg 1$. We denote by $\overline{\Omega}_H(\gamma)$ the rational index counting Gieseker-semistable sheaves with class γ, defined as in [3]. Tilt-stability is defined in the same way, but discarding the constant term of the Hilbert polynomial before dividing by its top coefficient as before. However, for threefolds with $\operatorname{Pic}\mathfrak{Y} = H\mathbb{Z}$ and two-dimensional class (i.e. $p^0 = 0$, $p^1 \neq 0$), the index $\overline{\Omega}_\infty(\gamma)$ counting tilt-semistable objects coincides with the index $\overline{\Omega}_H(\gamma)$ counting Gieseker-semistable sheaves [8, Lemma 5.2]. In §4, we shall present explicit formulae relating $\overline{\Omega}_H(\gamma)$ for rank 0 charges (counting D4-D2-D0 bound states) and rank ± 1 charges (counting D6-D2-D0 bound states), which follow by a sequence of wall-crossings from an empty chamber provided by the conjectural BMT inequality (2.30).

Conjectural BMT inequality In the plane parametrized by $(b, w = \frac{1}{2}(a^2 + b^2))$, the BMT inequality (2.30) implies the linear inequality

$$
(2.38) \quad L_{b,w}(\gamma) := (C_1^2 - 2C_0 C_2) w + (3C_0 C_3 - C_1 C_2) b + (2C_2^2 - 3C_1 C_3) \geq 0,
$$

whenever there exists a $\nu_{b,w}$-semistable object $E \in \mathcal{D}^b(\mathfrak{Y})$ of class γ. From (2.27), the coefficient of w in the above equation is $\Delta_H(E) \geq 0$. If $\Delta_H(E) > 0$, the inequality (2.38) says that E can be $\nu_{b,w}$-semistable only for points

$(b, w) \in U$ above the line $\ell_f(\gamma)$ defined by the equation $L_{b,w}(\gamma) = 0$ (see the red line in Fig. 2). This line passes through the points $\varpi(\gamma)$ defined in (2.36) and

$$(2.39) \qquad \widetilde{\varpi}(\gamma) = \left(\frac{2C_2}{C_1}, \frac{3C_3}{C_1} \right).$$

The conjectural BMT inequality (2.38) has now been proved for the quintic threefold X_5 and for a degree $(4, 2)$ complete intersection $X_{4,2}$ in $\mathbb{P}^5$ when (b, w) satisfy [44, 45]

$$(2.40) \qquad w - \frac{1}{2} b^2 > \frac{1}{2} [b](1 - [b]), \qquad [b] := b - \lfloor b \rfloor.$$

Moreover, a slightly weaker version of (2.38) is proved for the sextic and octic CY threefolds, X_6 and X_8, in the same restricted region (2.40) [46]. The proofs of the BMT inequality for these models rely on a strengthening of the classical Bogomolov-Gieseker inequality (2.27), i.e. the existence of a function $G \colon \mathbb{R} \to \mathbb{R}$ such that any slope-semistable sheaf E satisfies $\frac{\mathrm{ch}_2(E).H}{\mathrm{ch}_0(E)H^3} \leq G\left(\frac{\mathrm{ch}_1(E).H^2}{\mathrm{ch}_0(E)H^3} \right)$ and $G(b) \leq \frac{b^2}{2}$ for all $b \in \mathbb{R}$. When such a function is available, one can enlarge the space of weak stability conditions U to $U_G := \{(b, w) \in \mathbb{R}^2 \colon w > G(b)\}$, see Figure 3 for the quintic threefold. The existence of such a function and the status of the BMT inequality for the other hypergeometric models in Table 1 remains open at the time of writing.

2.4. Kähler moduli and Π-stability

While the DT invariants $\Omega_\sigma(\gamma)$ are mathematically well-defined throughout the space of Bridgeland stability conditions $\mathrm{Stab}\,\mathcal{C}$ (away from walls of marginal stability), they only acquire physical meaning along a particular complex one-dimensional slice $\Pi \subset \mathrm{Stab}\,\mathcal{C}$ where the central charge $Z(\gamma)$ coincides with the physical central charge $Z_z(\gamma)$ determined by the complexified Kähler structure on $\mathfrak{Y}$, or equivalently by the complex structure parametrized by z of the mirror family $\hat{\mathfrak{Y}}$. On the mirror side, the central charge is given by the period integral

$$(2.41) \qquad Z_z(\gamma) = \Pi_z(\hat{\gamma}) = \int_{\hat{\gamma}} \Omega_{3,0},$$

of the holomorphic 3-form on the cycle $\hat{\gamma} \in H_3(\hat{\mathfrak{Y}}, \mathbb{Z})$ dual to γ.
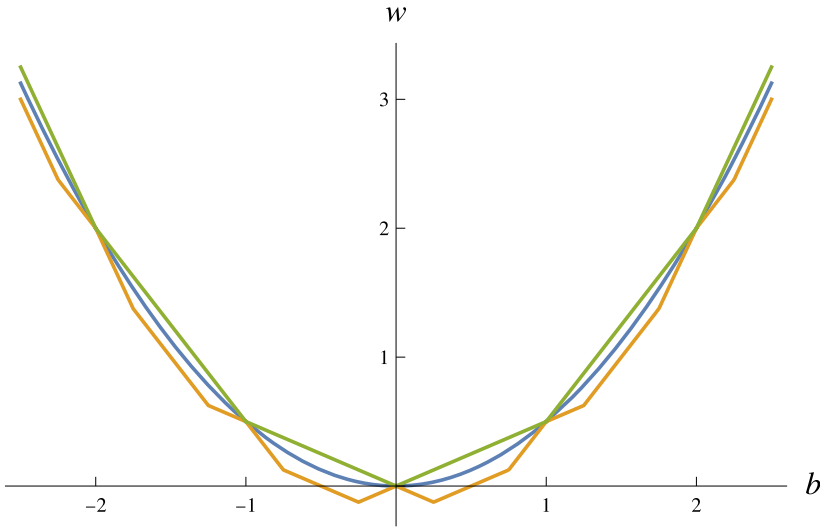
Figure 3: For the quintic threefold X_5, the stronger Bogomolov-Gieseker inequality established in [44, Theorem 1.1] implies that for any $\nu_{b,w}$-semistable object E, the point $\varpi(E)$ lies below the orange curve $w = G(b)$, where $G(b) = -\frac{1}{2}|b|$ for $|b| < \frac{1}{4}$, $G(b) = \frac{1}{2}|b| - \frac{1}{4}$ for $\frac{1}{4} < |b| < \frac{3}{4}$, $G(b) = \frac{3}{2}|b| - 1$ for $\frac{3}{4} < |b| < 1$ and $G(b) = G(b - \lfloor b \rfloor) + \lfloor b \rfloor b - \frac{1}{2}\lfloor b \rfloor^2$ when b lies outside the interval $(-1, 1)$. Moreover, the BMT inequality is known to hold in the region above the green curve given in (2.40) [44, Theorem 1.2]. These two curves intersect the blue curve $w = \frac{1}{2}b^2$ for integer values of b.

We shall restrict to CY threefolds obtained as a smooth complete intersection of degree $(d_1, \ldots, d_n)$ in weighted projective space $\mathbb{P}^{n+3}_{w_1,\ldots,w_{n+4}}$. There are 13 such threefolds $\mathfrak{Y}$, whose basic topological data are tabulated in Table 1. In particular, we note that $\sum_j d_j = \sum_i w_i$ by the CY condition, and $\kappa = H^3 = \prod_i d_i / \prod_j w_j$. The mirror threefold $\hat{\mathfrak{Y}}$ can be obtained, for example, by applying the general construction of [47]. For all these models, the periods satisfy a Picard-Fuchs equation of hypergeometric type,

$$(2.42) \qquad \mathcal{L}\,\Pi_z(\hat{\gamma}) = \left[(z\partial_z)^4 - \mu^{-1}z \prod_{k=1}^4 (z\partial_z + a_k) \right] \Pi_z(\hat{\gamma}) = 0\,,$$

where $\mu = \prod w_i^{w_i} / \prod d_j^{d_j}$, the 'local exponents' a_k satisfy $\sum_k a_k = 2$ and are ordered in increasing order for definiteness. The equation has singularities at $z = 0$, $z = \mu$ and at $z = \infty$, such that the Kähler moduli space

of $\mathfrak{Y}$ (or complex structure moduli space of $\hat{\mathfrak{Y}}$) consists of the punctured sphere $\mathcal{M}_K(\mathfrak{Y}) = \mathbb{P}\backslash\{0, \mu, \infty\}$. The two singularities at $z = 0$ and $z = \mu$ are universal, and correspond to the large volume limit and conifold point, respectively. Following [33], we denote these two types of degeneration by M (for maximal unipotent monodromy) and C (for conifold). The type of degeneration at $z = \infty$ depends on the local exponents a_k, and may be of type F (when all a_k are distinct, corresponding to a monodromy of finite order), C (when $a_2 = a_3$), K (when $a_1 = a_2$ and $a_3 = a_4$), or M (when all a_k's coincide). Degenerations of type K and M occur at infinite distance with respect to the special Kähler metric on $\mathcal{M}_K$, while degenerations of type F and C occur at finite distance. Under a type C degeneration, the conformal field theory on $\mathfrak{Y}$ becomes singular, due to a brane becoming massless, while a type F degeneration leads to a regular CFT, often with a Gepner-model type description. The regulator ρ, that will be relevant for the direct integration of the holomorphic anomaly equations discussed in Section 3.2, is defined to be the smallest denominator among the local exponents at $z = \infty$. The exponents and the type of the singularity at $z = \infty$ are indicated in Table 1.

To construct a basis of solutions adapted to the maximal unipotent monodromy at $z = 0$ (corresponding to the large volume limit on the mirror), we apply the Frobenius method. For $\epsilon \in \mathbb{C}$ let

$$(2.43) \qquad \Pi(z, \epsilon) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{n} \Gamma(d_j(k + \epsilon) + 1)}{\prod_{i=1}^{n+4} \Gamma(w_i(k + \epsilon) + 1)} z^{k+\epsilon}.$$

Using the identity

$$(2.44) \quad \frac{\prod_{j=1}^{n} \left(d_j \prod_{\ell=1}^{d_j-1} (d_j k + \ell) \right)}{\prod_{i=1}^{n+4} \left(w_i \prod_{\ell=1}^{w_i-1} (w_i k + \ell) \right)} = \mu^{-1}(k + a_1)(k + a_2)(k + a_3)(k + a_4),$$

one easily checks that

$$(2.45) \qquad \mathcal{L}\Pi(z, \epsilon) = \epsilon^4 z^\epsilon \frac{\prod_j \Gamma(d_j \epsilon + 1)}{\prod_i \Gamma(w_i \epsilon + 1)}.$$

Thus, the first three terms $\Pi_{0 \leq p \leq 3}(z)$ in the Taylor expansion around $\epsilon = 0$

$$(2.46) \qquad \Pi(z, \epsilon) = \sum_{p=0}^{\infty} \Pi_p(z)(2\pi i \epsilon)^p,$$

are annihilated by $\mathcal{L}$. In the Mukai basis (2.19), the coefficients (X^0, X^1, F_1, F_0) are given by

$$
(2.47) \quad \begin{aligned}
X^0 &= \Pi_0(z), & F_0 &= \kappa \Pi_3(z) + \frac{c_2}{24} \Pi_1(z), \\
X^1 &= \Pi_1(z), & F_1 &= -\kappa \Pi_2(z) - \frac{c_2}{24} \Pi_0(z).
\end{aligned}
$$

We define the flat coordinate $z = b + it = X^1/X^0$, such that $z \mapsto z+1$ under monodromy $z \mapsto e^{2\pi i}z$ around $z = 0$. The components can be integrated to a prepotential $F(z)$ such that

$$
(2.48) \quad \begin{aligned}
F_1/X^0 &= \partial_z F(z), \\
F_0/X^0 &= 2F(z) - z\partial_z F(z).
\end{aligned}
$$

In the large volume limit $t \to \infty$, the prepotential has an asymptotic expansion

$$
(2.49) \quad F(z) = -\frac{\kappa}{6} z^3 + \frac{\zeta(3)\chi_{\mathfrak{Y}}}{2(2\pi i)^3} - \frac{1}{(2\pi i)^3} \sum_{Q=1}^{\infty} \mathrm{GV}_Q^{(0)} \mathrm{Li}_3\left(e^{2\pi i Q z}\right),
$$

where $\mathrm{GV}_Q^{(0)}$ are the genus-zero Gopakumar-Vafa invariants. Keeping only the leading cubic term in (2.49) and fixing the Kähler gauge $X^0 = -1$ in (2.19), one arrives at

$$
(2.50)
$$
$$
\begin{aligned}
Z_{b,t}^{LV}(\gamma) &= \frac{\kappa}{6} z^3 p^0 - \frac{\kappa}{2} p^1 z^2 - q_1 z - q_0 \\
&= \left(-\mathrm{ch}_3^b + \left(\frac{1}{2} t^2 - \frac{c_2}{24\kappa}\right) \mathrm{ch}_1^b\right) + it \left(\mathrm{ch}_2^b - \left(\frac{1}{6} t^2 - \frac{c_2}{24\kappa}\right) \mathrm{ch}_0^b\right).
\end{aligned}
$$

which reproduces (2.24) and coincides with the standard slice (2.20) upon making the identifications in (2.23).

Taking into account subleading corrections, it is necessary to apply a $\widetilde{GL^+}(2, \mathbb{R})$ transformation in order to reach the form (2.20). The resulting values of a, b, α, β can be computed by equating the products $x_{ij} = \mathrm{Im}\left(Z_i \overline{Z_j}\right)$, $0 \le i < j \le 3$ where Z_i is the central charge for the Chern vector defined by $C_k = \delta_i^k$. Indeed, these quantities are invariant up to scale under $\widetilde{GL(2, \mathbb{R})}^+$ and satisfy the quadratic constraint $x_{01}x_{23} + x_{02}x_{31} + x_{03}x_{12} = 0$, so give the desired 4 local real coordinates. In this way, one finds

(2.51)

$$a(z) = \frac{\sqrt{(\operatorname{Im}\tilde{F}_1)^2 - 2\kappa\operatorname{Im}z\operatorname{Im}\tilde{F}_0}}{\kappa\sqrt{\operatorname{Im}z}}\,, \qquad b(z) = -\frac{\operatorname{Im}\tilde{F}_1}{\kappa\operatorname{Im}z}\,,$$

$$\alpha(z) = -\frac{\operatorname{Im}(\bar{z}\tilde{F}_1)}{\kappa\operatorname{Im}z} - \frac{(\operatorname{Im}\tilde{F}_1)^2}{2\kappa^2(\operatorname{Im}z)^2}\,,$$

$$\beta(z) = \frac{6\kappa^2(\operatorname{Im}z)^2\operatorname{Im}(\bar{z}\tilde{F}_0) + 6\kappa\operatorname{Im}z\operatorname{Im}\tilde{F}_1\operatorname{Im}(\tilde{F}_0 - \bar{z}\tilde{F}_1) - 4(\operatorname{Im}\tilde{F}_1)^3}{3\kappa\operatorname{Im}z\left(2\kappa\operatorname{Im}z\operatorname{Im}\tilde{F}_0 - (\operatorname{Im}\tilde{F}_1)^2\right)}\,,$$

where we denoted[8] $\tilde{F}_0 = 2F - z\partial_z F + \frac{c_2}{24}z$, $\tilde{F}_1 = \partial_z F - \frac{c_2}{24}$. In the region where $\alpha > \frac{1}{6}a^2 + \frac{1}{2}a|\beta|$, the heart $\mathcal{A}(z) = \mathcal{A}_{b(z),a(z)}$ is given by the double-tilt construction explained in the previous subsection. The construction of the heart on the full physical slice Π, including the vicinity of the singularities at $z = \mu$ and $z = \infty$, remains a challenging open problem. Assuming that this problem has been solved, we denote by $\overline{\Omega}_z^\Pi(\gamma)$ the generalized DT invariant along the physical slice. Fortunately, the relation between rank 0 and higher rank DT invariants at large volume can be derived using only the family of weak stability conditions $\nu_{b,w}$, assuming that the BMT inequality holds.

2.5. Interlude – summary

Let us briefly summarize the previous subsections. First, we introduced the space of Bridgeland stability conditions $\operatorname{Stab}\mathcal{C}$ on the derived category $\mathcal{C}$ of coherent sheaves, which is the appropriate mathematical framework for BPS branes in type IIA string theory. A stability condition is a pair $(Z, \mathcal{A})$ of a central charge function Z and a heart $\mathcal{A} \subset \mathcal{C}$ which determines which constituents may bind into stable objects. For one-modulus CY threefolds, after dividing out by the $\widetilde{GL^+}(2, \mathbb{R})$ action (which preserves the phase ordering of central charges), the space of stability conditions effectively has real dimension 4. Assuming the BMT conjecture (2.31), we outlined the construction of an open set $\mathcal{U}$ in $\operatorname{Stab}\mathcal{C}$ parametrized by $(a, \alpha, b, \beta) \in \mathbb{R}^4$ subject to the inequalities (2.25). The physical subspace of Π-stability conditions provides a real-codimension two slice Π in this open set, determined by the prepotential $F(z)$ (see Fig. 4). For $z \to i\infty$, this slice asymptotically coincides with the large volume slice (2.50) or equivalently (2.24). On the boundary of $\mathcal{U}$, there is also an important family of weak stability conditions with central charge (2.28) parametrized by $(b, a) \in \mathbb{R} \times \mathbb{R}^+$, which

---

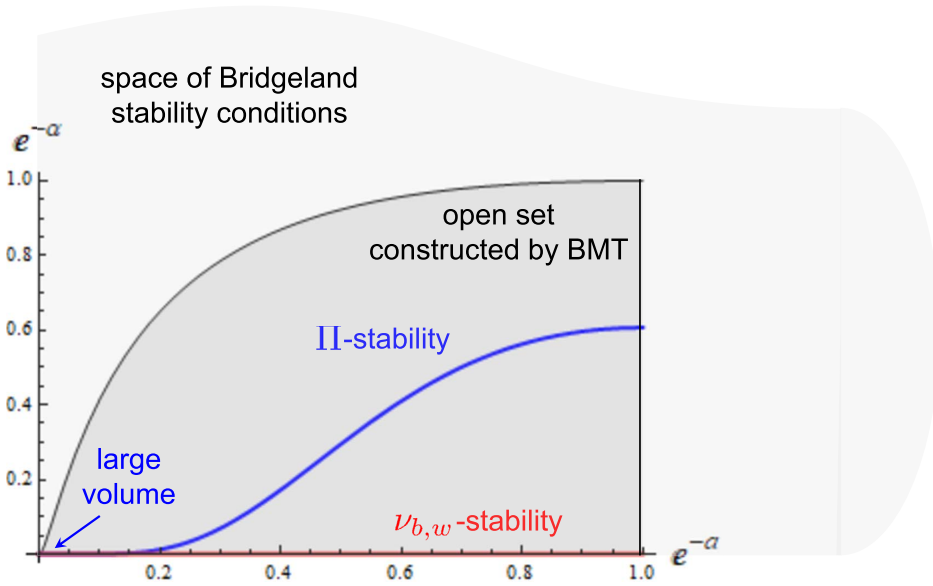[8]Note that the signs are such that shift by $\frac{c_2}{24}$ cannot be absorbed into a linear shift of F!

Figure 4: Section of the space of Bridgeland stability conditions by the plane $\beta = 0$, $b = \text{const}$, drawn in coordinates $(e^{-a}, e^{-\alpha})$. The boundary of the set constructed in [9, 10] corresponds to the inequalities (2.25). The red line is the slice of weak $\nu_{b,w}$-stability conditions with the central charge (2.28) and the blue line represents the slice of Π-stability conditions parametrized by the complexified Kähler structure of $\mathfrak{Y}$. The large volume limit corresponds to the region near the origin where the two slices approach each other.

is obtained from (2.50) by omitting the contributions proportional to the D0-brane charge ch_3^b and to the second Chern class $c_2(T\mathfrak{Y})$, and setting $t = a\sqrt{3}$. This family, interchangeably called $N_{b,a}$-stability or $\nu_{b,w}$-stability with $w = \frac{1}{2}(a^2 + b^2)$, serves as a key intermediate step for the construction of the heart $\mathcal{A}$ throughout the open set $\mathcal{U}$, and is the subject of the BMT conjecture (2.31), which constrains the existence of $\nu_{b,w}$-semi-stable objects for small w. We denote by $\overline{\Omega}_{b,w}(\gamma)$ the rational Donaldson-Thomas invariant counting $\nu_{b,w}$-semistable objects of class γ in the heart $\mathcal{A}_b$, and by $\overline{\Omega}_H(\gamma)$ the rational Donaldson-Thomas invariant counting H-Gieseker-semi-stable sheaves of class γ defined following [3]. In the next two subsections, we spell out the modularity properties predicted by string theory for DT invariants $\overline{\Omega}_H(\gamma)$ in the rank 0 case, and the relation with ordinary DT invariants and PT invariants in the rank ± 1 case.

2.6. D4-D2-D0 indices and modularity conjecture

Here we consider the case $p^0 = 0$, $p^1 := r > 0$, corresponding to D4-D2-D0 bound states. As explained below (2.37), for a fixed charge γ, the index $\overline{\Omega}_{b,w}(\gamma)$ reaches a finite value $\overline{\Omega}_\infty(\gamma)$ as $w \to +\infty$, which turns out to coincide with the index $\overline{\Omega}_H(\gamma)$ counting Gieseker-semi-stable sheaves. For CY threefolds with Picard rank one, this index also agrees with the 'large volume attractor index' (also called MSW index in [48, 22, 23, 24])[9]

$$(2.52) \qquad \overline{\Omega}_\infty(0, r, q_1, q_0) = \lim_{\lambda \to +\infty} \overline{\Omega}^\Pi_{-\frac{q_1}{\kappa r} + \mathrm{i}\lambda r}(0, r, q_1, q_0) \,,$$

where $\overline{\Omega}^\Pi_{z=b+\mathrm{i}t}(\gamma)$ denotes the DT invariant along the Π-stability slice. The index $\Omega_\infty(0, r, q_1, q_0)$ is preserved under spectral flow (2.12) with $k \in \mathbb{Z}$, which leaves the D4-brane charge r invariant, as well as the reduced D0-brane charge

$$(2.53) \qquad \hat{q}_0 := q_0 - \frac{q_1^2}{2\kappa r} \,,$$

and the class of $\mu := q_1 - \frac{1}{2}\kappa r^2$ in $\Lambda^*/\Lambda = \mathbb{Z}/(\kappa r\mathbb{Z})$. Accordingly, we denote

$$(2.54) \qquad \overline{\Omega}_{r,\mu}(\hat{q}_0) = \overline{\Omega}_\infty(\gamma) = \overline{\Omega}_H(\gamma) \,.$$

Note that for fixed $r \geq 1$, $\mu \in \mathbb{Z}$, the argument $\hat{q}_0$ is such that the combination

$$(2.55) \qquad n = \frac{\chi(\mathcal{D}_r)}{24} - \frac{\mu^2}{2\kappa r} - \frac{r\mu}{2} - \hat{q}_0 \,,$$

is an integer. Here $\chi(\mathcal{D}_r)$ is the topological Euler characteristic of the divisor $\mathcal{D}_r$ Poincaré dual to rH, [17, (3.8)][10]

$$(2.56) \qquad \chi(\mathcal{D}_r) = \kappa r^3 + c_2 r \,.$$

It follows from derived duality $E \mapsto E^\vee$, which acts on the Chern vector by $\mathrm{ch}(E) \mapsto \mathrm{ch}(E)^\vee$ as defined below (2.3), that the index (2.54) is invariant

---

[9]In general, the MSW index is defined as the value of $\overline{\Omega}^\Pi_z(\gamma)$ in the asymptotic direction $z^a = -\kappa^{ab}q_b + \mathrm{i}\lambda p^a$ with $\lambda \gg 1$, where κ^{ab} is the inverse of the matrix $\kappa_{ab} = \kappa_{abc}p^c$. When $b_2(\mathfrak{Y}) = 1$, the distinction between Gieseker index and MSW index becomes irrelevant.

[10]We urge the reader not to confuse $\chi(\mathcal{D}_r)$ with the holomorphic Euler characteristic $\chi_{\mathcal{D}_r}$ defined in (2.62).

under $\mu \mapsto -\mu$, on top of the periodicity $\mu \mapsto \mu + \kappa r$ (note however that the integer (2.55) is not invariant under these symmetries). Furthermore, $\Omega_{r,\mu}(\hat{q}_0)$ vanishes unless the reduced charge $\hat{q}_0$ is bounded from above by [4, Corollary 3.3]

$$(2.57) \qquad \hat{q}_0 \leq \hat{q}_0^{\max} = \frac{1}{24} \chi(\mathcal{D}_r).$$

Upon identifying $\hat{q}_0 = \frac{c_L}{24} - L_0$ with $c_L = \chi(\mathcal{D}_r)$, this coincides with the unitarity bound $L_0 \geq 0$ in the two-dimensional $(0,4)$ superconformal field theory obtained by reducing the worldvolume theory of an M5-brane wrapped on $\mathcal{D}_r$ [17].

Since the reduced D0-brane charge is bounded from above for fixed D4-brane charge $r > 0$ and D2-brane charge $q_1 = \mu + \frac{1}{2}\kappa r^2$, one can define the generating series of rational invariants

$$(2.58) \qquad h_{r,\mu}(\tau) = \sum_{\hat{q}_0 \leq \hat{q}_0^{\max}} \overline{\Omega}_{r,\mu}(\hat{q}_0)\, \mathrm{q}^{-\hat{q}_0}, \qquad \mathrm{q} = e^{2\pi i \tau},$$

Since μ takes values in $\mathbb{Z}/(\kappa r \mathbb{Z})$, (2.58) defines a vector with κr entries (half of which being redundant due to the symmetry under $\mu \mapsto -\mu$). For $r = 1$, the case of interest in this paper, the charge vector is primitive and therefore the rational DT invariant $\overline{\Omega}_{1,\mu}(\hat{q}_0)$ coincides with the integer DT invariant $\Omega_{1,\mu}(\hat{q}_0)$, defined by replacing $\overline{\Omega}_\infty(\gamma)$ by $\Omega_\infty(\gamma)$ in (2.54).

By exploiting the constraints of S-duality in string theory, it has been argued that the generating series $h_{r,\mu}$ must possess specific modular properties under $SL(2,\mathbb{Z})$ transformations of the parameter τ [48, 22, 24, 49]. More precisely, $h_{r,\mu}$ should transform as a weakly holomorphic vector valued mock modular form of depth $r - 1$, with a specific modular anomaly. In this paper, we restrict ourselves to the simplest Abelian case $r = 1$, and refer to the invariants $\Omega_{1,\mu}(\hat{q}_0)$ as Abelian D4-D2-D0 indices. In this situation, the modular anomaly disappears and $h_{1,\mu}$ must transform as a vector-valued modular form of weight $-3/2$ in the Weil representation attached to the lattice $\mathbb{Z}[\kappa]$ with quadratic form $m \mapsto \kappa m^2$. Equivalently, it must transform with the following matrices under $T : \tau \mapsto \tau + 1$ and $S : \tau \mapsto -1/\tau$ [49, Eq.(2.10)] (see also [19, 18, 16, 21])

$$(2.59) \qquad \begin{aligned} M_{\mu\nu}(T) &= e^{\frac{\pi i}{\kappa}(\mu + \frac{\kappa}{2})^2 + \frac{\pi i}{12} c_2}\, \delta_{\mu\nu}, \\ M_{\mu\nu}(S) &= \frac{(-1)^{\chi_{\mathcal{D}}}}{\sqrt{i\kappa}}\, e^{-\frac{2\pi i}{\kappa} \mu\nu}, \end{aligned}$$

where $\chi_{\mathcal{D}} = \frac{\kappa}{6} + \frac{c_2}{12}$ (see below (2.13)). We denote by $\mathscr{M}_1(\mathfrak{Y})$ the space of weakly holomorphic vector-valued modular forms with these transformation properties under $SL(2, \mathbb{Z})$.

It is well known that any weakly holomorphic vector-valued modular form of negative weight $w < 0$ is completely determined only by its 'polar coefficients', i.e. the terms in its Fourier expansion that become singular in the limit $\tau \to i\infty$. Such terms correspond to the terms with $\hat{q}_0 > 0$ in (2.58). Once the polar terms are known, the full modular form can then be determined, for example by constructing the Poincaré-Rademacher sum (see e.g. [50]). It is important however, that the dimension of the space of modular forms can be strictly smaller than the number of polar terms, which means that the polar coefficients must satisfy certain linear constraints, which are related to the existence of cusp forms in dual weight $2 - w$ [51, 21, 52]. In Table 1, we list the number of polar terms (denoted by n_1^p) and constraints (denoted by n_1^c) for the 13 smooth hypergeometric threefolds computed in [31], such that the dimension of $\mathscr{M}_1(\mathfrak{Y})$ is given by $n_1^p - n_1^c$.

2.7. Rank 1 DT invariants and stable pair invariants

We now turn to the case $p^0 = \pm 1$, as the corresponding invariants will turn out to provide the information needed to compute the polar coefficients in the generating series of D4-D2-D0 Abelian indices.

For $p^0 = 1$ and $p^1 = 0$, the index $\Omega_\infty(1, 0, q_1, q_0)$ reduces to the invariant originally defined in [53], counting ideal sheaves E with $\mathrm{ch}(E) = 1 - \beta - nH^3$, with $\beta = (q_1 + \frac{c_2}{24})H^2/\kappa$ (identified by Poincaré duality with a class in $H_2(\mathfrak{Y}, \mathbb{Z})$) and $n = -q_0 \in \mathbb{Z}$. Equivalently, it counts dimension-one subschemes $C \subset \mathfrak{Y}$ with class $[C] = \beta$ and holomorphic Euler number $\chi(\mathcal{O}_C) = n$. The moduli space of such subschemes is projective and admits a perfect symmetric obstruction theory (see e.g. [54] and references therein). We denote the corresponding DT invariant by

$$(2.60) \qquad \mathrm{I}_{n,\beta} = \mathrm{DT}(\beta.H, n) = \Omega_\infty\left(1, 0, -\beta.H - \frac{c_2}{24}, -n\right),$$

where the first notation is standard in the mathematics literature and the second was used in [31]. The case $p^0 = 1, p^1 \neq 0$ can be reached by tensoring by a line bundle $\mathcal{L}$ on $\mathfrak{Y}$, or equivalently using the spectral flow (2.12). As a result the index $\Omega_\infty(\gamma) = \mathrm{DT}(Q_+, n_+)$ depends only on the invariant combinations

$$(2.61) \qquad Q_+ = q_1 + \frac{1}{2}\kappa(p^1)^2 + \frac{c_2}{24}, \qquad n_+ = -q_0 - p^1 q_1 - \frac{1}{3}\kappa(p^1)^3,$$

which both take integral values as a consequence of the quantization conditions (2.10) and the integrality of the arithmetic genus of the divisor class $\mathcal{D}_r$ with $r = p^1$,

$$(2.62) \qquad \chi_{\mathcal{D}_r} := \chi(\mathcal{O}_{\mathcal{D}_r}) = \frac{1}{6} \kappa r^3 + \frac{1}{12} c_2 r \,.$$

For $p^0 = -1$ and $p^1 = 0$, the index $\Omega_\infty(-1, 0, q_1, q_0)$ instead counts stable pairs [4, §3] (more precisely, derived dual of stable pairs) $E = (\mathcal{O}_{\mathfrak{Y}} \xrightarrow{s} F)^\vee$, where F is a pure one-dimensional sheaf with $[F] = \beta = -q_1 H^2/\kappa$ and $\chi(F) = n = -q_0$, and s is a section with zero-dimensional cokernel. The Chern character for this object is $\mathrm{ch}(E) = -1 + \beta - nH^3$. As shown by Pandharipande and Thomas [55], the moduli space of stable pairs is also projective and admits a perfect symmetric obstruction theory. We denote the corresponding PT invariant by

$$(2.63) \qquad \mathrm{P}_{n,\beta} = \mathrm{PT}(\beta.H, n) = \Omega_\infty\left(-1, 0, \beta.H + \frac{c_2}{24}, -n\right) \,,$$

where the first notation is standard in the mathematics literature and the second is similar to the one used for DT invariants. The case $p^0 = -1, p^1 \neq 0$ can again be reached by tensoring by a line bundle $\mathcal{L}$ on $\mathfrak{Y}$, so that the index $\Omega_\infty(\gamma) = \mathrm{PT}(Q_-, n_-)$ depends only on the invariant combinations

$$(2.64) \qquad \begin{aligned} Q_- &= -q_1 + \frac{1}{2} \kappa(p^1)^2 + \frac{c_2}{24} \,, \\ n_- &= -q_0 + p^1 q_1 - \frac{1}{3} \kappa(p^1)^3 \,. \end{aligned}$$

As shown in Appendix A, Theorem 2, the invariants $\mathrm{DT}(Q, n)$ and $\mathrm{PT}(Q, n)$ vanish unless

$$(2.65) \qquad Q \geq 0 \quad \text{and} \quad n \geq -\left\lfloor \frac{Q^2}{2\kappa} + \frac{Q}{2} \right\rfloor \,.$$

where the first condition follows from the Bogomolov-Gieseker inequality (2.27). In fact, in the range $0 \leq Q < \kappa$, the BMT inequality (2.30) implies the slightly stronger bound

$$(2.66) \qquad n \geq -\left\lfloor \frac{2Q^2}{3\kappa} + \frac{Q}{3} \right\rfloor \,.$$

Given these lower bounds on Q and n we can define the generating series

$$Z_{DT}(y, q) = \sum_{Q,n} \mathrm{DT}(Q, n)\, y^Q\, q^n,$$

(2.67)

$$Z_{PT}(y, q) = \sum_{Q,n} \mathrm{PT}(Q, n)\, y^Q\, q^n.$$

In terms of these formal series, the DT/PT relation conjectured in [55] and proven in [56, 57] takes the simple form

(2.68) $$Z_{DT}(y, q) = M(-q)^{\chi_{\mathfrak{Y}}}\, Z_{PT}(y, q),$$

where $M(q) = \prod_{k>0}(1 - q^k)^{-k}$ is the Mac-Mahon function. In §3.2, we shall explain how PT invariants, hence also DT invariants, can be computed from the knowledge of the topological string partition function.

3. Gopakumar-Vafa invariants and direct integration

In this section, we recall how Gopakumar-Vafa (GV) invariants can be determined by integrating the holomorphic anomaly equations satisfied by the topological string partition function. Physically, GV invariants were introduced as multiplicities of five-dimensional BPS states that arise from M2-branes wrapping curves in a CY threefold [58, 59]. We shall not go into the details of the mathematical definition of GV invariants but instead refer to [60] and for an introduction to [61].

3.1. PT/GV relation

As explained in [62], the A-twisted topological string associates to any CY threefold $\mathfrak{Y}$ an infinite family of genus g topological string free energies $\mathcal{F}^{(g)}(z, \bar{z})$, which depend on the Kähler moduli z in a non-holomorphic fashion. In the 'holomorphic limit' $\bar{z} \to -i\infty$, and in an appropriate Kähler gauge, $\mathcal{F}^{(g)}(z, \bar{z})$ reduces to the generating series of Gromov-Witten invariants,

(3.1) $$F^{(g)}(z) \equiv \lim_{\bar{z} \to -i\infty} (X^0)^{2g-2} \mathcal{F}^{(g)}(z, \bar{z}) = \sum_{Q=1}^{\infty} \mathrm{GW}_Q^{(g)} e^{2\pi i Q z},$$

where $\mathrm{GW}_Q^{(g)} \in \mathbb{Q}$ depends only on the symplectic structure of $\mathfrak{Y}$. For $g = 0, 1$, there are additional polynomial terms in z which we have dropped for brevity. For $g = 0$, $F^{(0)}$ coincides (up to an overall factor $-1/(2\pi i)^3$) with the worldsheet instanton contribution to the tree-level prepotential (2.49). The instanton part of the topological string partition function is then given by

$$(3.2) \qquad \Psi_{\text{top}}(z, \lambda) = \exp\left(\sum_{g \geq 0} \lambda^{2g-2} F^{(g)}(z)\right),$$

According to [58, 59], the Gromov-Witten invariants $\text{GW}_Q^{(g)}$ can be traded for new invariants $\text{GV}_Q^{(g)}$ defined by equating

$$(3.3) \qquad \log \Psi_{\text{top}}(z, \lambda) = \sum_{g=0}^{\infty} \sum_{k=1}^{\infty} \sum_{Q=1}^{\infty} \frac{\text{GV}_Q^{(g)}}{k}\left(2\sin\frac{k\lambda}{2}\right)^{2g-2} e^{2\pi i k Q z}.$$

The integrality of the Gopakumar-Vafa invariants $\text{GV}_Q^{(g)}$ defined by (3.3) was shown in [63][11]. More recently, it was shown in [64] that for fixed degree Q, there is only a finite number of non-vanishing invariants $\text{GV}_Q^{(g)}$. We shall denote by $g_{\max}(Q)$ the maximal genus g such that $\text{GV}_Q^{(g)} \neq 0$.

It was conjectured in [11, 65] that the topological string partition function is related to the generating series of PT invariants defined in (2.67) via

$$(3.4) \qquad \Psi_{\text{top}}(z, \lambda) = M(-e^{i\lambda})^{\frac{1}{2}\chi\mathfrak{Y}} Z_{PT}\left(e^{2\pi i z/\lambda}, e^{i\lambda}\right).$$

The corresponding relation to the partition function Z_{DT}, which follows by using (2.68), was motivated physically in [66] and a derivation in M-theory was given in [67]. The MNOP conjecture (3.4) is known to hold for non-compact toric CY threefolds [11, 65], and for complete intersections in products of projective spaces [68]. We shall assume that it continues to hold for complete intersections in weighted projective spaces.

The MNOP relation (3.4) can be expressed in a product form such that the PT invariants are related to the GV invariants by the following PT/GV relation [11, 65]

$$(3.5)$$
$$Z_{PT}(y, \mathsf{q}) = \prod_{Q>0} \prod_{k>0} \left(1 - (-\mathsf{q})^k y^Q\right)^{k\text{GV}_Q^{(0)}}$$
$$\times \prod_{Q>0} \prod_{g=1}^{g_{\max}(Q)} \prod_{\ell=0}^{2g-2} \left(1 - (-\mathsf{q})^{g-\ell-1} y^Q\right)^{(-1)^{g+\ell}\binom{2g-2}{\ell}\text{GV}_Q^{(g)}}.$$

---

[11]As discussed in §3.3, an independent definition of GV invariants which makes integrality manifest was proposed in [60], but its compatibility with (3.3) remains conjectural.

After some elementary algebra, one can rewrite (3.5) as a plethystic exponential [54]
(3.6)

$$Z_{PT}(y, \mathrm{q}) = \mathrm{PE} \left[\sum_{Q>0} \sum_{g=0}^{g_{\max}(Q)} (-1)^{g+1} \mathrm{GV}_Q^{(g)} (1-x)^{2g-2} x^{(1-g)} y^Q \right] (-\mathrm{q}, y),$$

where

$$(3.7) \qquad \mathrm{PE}[f](x, y) = \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} f(x^k, y^k) \right).$$

Conversely, GV invariants may be expressed in terms of PT invariants by taking the plethystic logarithm (see e.g. [69]),

$$(3.8)$$
$$\sum_{Q>0} \sum_{g=0}^{g_{\max}(Q)} \mathrm{GV}_Q^{(g)} (1+\mathrm{q})^{2g-2} \mathrm{q}^{1-g} y^Q$$
$$= \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log \left[1 + \sum_{Q>0,m} (-1)^m (-\mathrm{q})^{km} \mathrm{PT}(Q, m) y^{kQ} \right],$$

where $\mu(k)$ is the Möbius function (see Footnote 4), and expanding in powers of q and y on either side. The plethystic representation of the MNOP relation turns out to be computationally much more efficient than the original formula (3.5).

It easily follows from this relation and the bound (2.65) that for any $Q > 0$, the maximal genus $g_{\max}(Q)$ is bounded from above by

$$(3.9) \qquad g_{\max}(Q) \le g_C(Q) := \left\lfloor \frac{Q^2}{2\kappa} + \frac{Q}{2} \right\rfloor + 1.$$

As in (2.66), the bound is strengthened to $g_{\max}(Q) \le \left\lfloor \frac{2Q^2}{3\kappa} + \frac{Q}{3} \right\rfloor + 1$ when $0 < Q < \kappa$. We refer to (3.9) as the Castelnuovo bound, in reference to Castelnuovo's work on the maximal arithmetic genus of irreducible curves in projective space [70] (see e.g. [71] for a more recent account). In this work, we have obtained (3.9) using rather different methods pioneered in [72] (see Appendix A.3). We note that the bound (3.9) was established recently for the quintic threefold in [73].

It is worth noting that for m and g sufficiently close to the Castelnuovo bound, the relation between PT and GV invariants becomes linear,

$$(3.10) \qquad \mathrm{PT}(Q, m) = \sum_{g=1}^{g_{\max}(Q)} \binom{2g-2}{g-1-m} \mathrm{GV}_Q^{(g)}.$$

The exact range of validity of this relation depends on $\mathfrak{Y}$, but it is easy to check that it holds true for $m = m_{\min}(Q) + \delta$ with $m_{\min}(Q) := 1 - g_{\max}(Q)$ and $\delta = 0, 1$:

$$(3.11) \qquad \begin{aligned} \mathrm{PT}(Q, m_{\min}(Q)) &= \mathrm{GV}_Q^{(g_{\max}(Q))}, \\ \mathrm{PT}(Q, m_{\min}(Q) + 1) &= \mathrm{GV}_Q^{(g_{\max}(Q)-1)} \\ &\quad + (2g_{\max}(Q) - 2)\mathrm{GV}_Q^{(g_{\max}(Q))}. \end{aligned}$$

In particular, $m_{\min}(Q)$ is the minimal value of m such that $\mathrm{PT}(Q, m) \neq 0$, and satisfies

$$(3.12) \qquad m_{\min}(Q) \geq m_C(Q) := - \left\lfloor \frac{Q^2}{2\kappa} + \frac{Q}{2} \right\rfloor.$$

3.2. Direct integration method for computing GV invariants

As shown in [12, 62], the topological string free energies $\mathcal{F}^{(g)}(\mathrm{z}, \bar{\mathrm{z}})$ satisfy the holomorphic anomaly equations

$$(3.13) \qquad \frac{\partial}{\partial \bar{\mathrm{z}}} \frac{\partial}{\partial \mathrm{z}} \mathcal{F}^{(1)} = \frac{1}{2} C_{\bar{\mathrm{z}}}^{\mathrm{zz}} C_{\mathrm{zzz}} + \left(\frac{\chi_{\mathfrak{Y}}}{24} - 1 \right) G_{\bar{\mathrm{z}}\mathrm{z}},$$

$$(3.14) \quad \frac{\partial}{\partial \bar{\mathrm{z}}} \mathcal{F}^{(g)} = \frac{1}{2} \bar{C}_{\bar{\mathrm{z}}}^{\mathrm{zz}} \left(D^2 \mathcal{F}^{(g-1)} + \sum_{h=1}^{g-1} D\mathcal{F}^{(g-h)} D\mathcal{F}^{(h)} \right), \quad \text{for } g \geq 2,$$

where $C_{\mathrm{zzz}} = \partial_{\mathrm{z}}^3 F^{(0)}(\mathrm{z})$ is the so-called Yukawa coupling, $\bar{C}_{\bar{\mathrm{z}}\bar{\mathrm{z}}\bar{\mathrm{z}}}$ is its complex conjugate, and indices are raised using the Kähler metric $G_{\mathrm{z}\bar{\mathrm{z}}} = \partial_{\mathrm{z}}\partial_{\bar{\mathrm{z}}}K$ with $K = -\log(\bar{\mathrm{z}}\partial_{\mathrm{z}}F - \mathrm{z}\partial_{\bar{\mathrm{z}}}\bar{F})$. In the flat coordinates $\mathrm{z}, \bar{\mathrm{z}}$, the Christoffel symbols $\Gamma_{\mathrm{zz}}^{\mathrm{z}} = G^{\bar{\mathrm{z}}\mathrm{z}}\partial_{\mathrm{z}}G_{\bar{\mathrm{z}}\mathrm{z}}$ vanish. Denoting the Hodge line bundle with connection $K_{\mathrm{z}} = \partial_{\mathrm{z}}K$ on the moduli space by $\mathcal{L}$, the free energies $\mathcal{F}^{(g)}$ are sections of $\mathcal{L}^{2-2g}$ and the covariant derivative acting on a section of $\mathcal{L}^n$ takes the form $D = \partial_{\mathrm{z}} + nK_{\mathrm{z}}$. Given the amplitudes $\mathcal{F}^{(h)}(\mathrm{z}, \bar{\mathrm{z}})$ for $h < g$, the equations (3.14) determine $\mathcal{F}^{(g)}(\mathrm{z}, \bar{\mathrm{z}})$ up to a holomorphic ambiguity $f^{(g)}(\mathrm{z})$.

The non-holomorphic dependence of the free energies can be absorbed in a set of 'propagators' S^{zz}, S^{z}, S, satisfying [62]

$$(3.15) \qquad \partial_{\bar{z}} S^{zz} = \bar{C}_{\bar{z}}^{\ zz}, \qquad \partial_{\bar{z}} S^{z} = G_{z\bar{z}} S^{zz}, \qquad \partial_{\bar{z}} S = G_{z\bar{z}} S^{z}.$$

More precisely, $\mathcal{F}^{(g)}(z, \bar{z})$ is an inhomogeneous polynomial of degree $3g - 3$ in K_z, S^{zz}, S^{z}, S (of respective degrees $1, 1, 2, 3$) with holomorphic coefficients [74, 75, 76]. It turns out that the dependence on the connection K_z can also be absorbed by introducing the shifted propagators [75, 76]

$$(3.16)$$

$$\tilde{S}^{zz} = S^{zz}, \qquad \tilde{S}^{z} = S^{z} - S^{zz} K_z, \qquad \tilde{S} = S - S^{z} K_z + \frac{1}{2} S^{zz} K_z K_z.$$

Up to a holomorphic ambiguity $f^{(1)}(z)$, the equation (3.13) can then be integrated to obtain

$$(3.17) \qquad \partial_z \mathcal{F}^{(1)} = \frac{1}{2} C_{zzz} \tilde{S}^{zz} - \left(\frac{\chi_{\mathfrak{Y}}}{24} - 1 \right) K_z + f^{(1)}(z),$$

and the holomorphic anomaly equations (3.14) for $g \geq 2$ can be rewritten as

$$(3.18)$$

$$\frac{\partial \mathcal{F}^{(g)}}{\partial \tilde{S}^{zz}} - K_z \frac{\partial \mathcal{F}^{(g)}}{\partial \tilde{S}^{z}} + \frac{1}{2} K_z K_z \frac{\partial \mathcal{F}^{(g)}}{\partial \tilde{S}} = \frac{1}{2} D^2 \mathcal{F}^{(g-1)} + \frac{1}{2} \sum_{h=1}^{g-1} D \mathcal{F}^{(g-h)} D \mathcal{F}^{(h)}.$$

The holomorphic limit is obtained by replacing $K_z, \tilde{S}^{zz}, \tilde{S}^{z}, \tilde{S}$ with the corresponding holomorphic limits K_z, S^{zz}, S^{z}, S and $\mathcal{F}^{(g)}(z, \bar{z})$ by $F^{(g)}(z)$. Since the dependence of the free energies on K_z is absorbed in the shifted propagators, the equations (3.18) can be integrated by collecting the powers of K_z on the right-hand side, and identifying them with the corresponding derivatives on the left-hand side.

In terms of the algebraic coordinate z, special geometry implies

$$(3.19) \qquad \Gamma_{zz}^{z} = 2K_z - C_{zzz} \tilde{S}^{zz} + s_{zz}^{z},$$

and the propagators are partially determined by the BCOV ring [75]

$$(3.20) \qquad \begin{aligned} \partial_z \tilde{S}^{zz} &= C_{zzz} \tilde{S}^{zz} \tilde{S}^{zz} + 2\tilde{S}^{z} - 2s_{zz}^{z} \tilde{S}^{zz} + h_z^{zz}, \\ \partial_z \tilde{S}^{z} &= C_{zzz} \tilde{S}^{zz} \tilde{S}^{z} + 2\tilde{S} - s_{zz}^{z} \tilde{S}^{z} - h_{zz} \tilde{S}^{zz} + h_z^{z}, \end{aligned}$$

$$\partial_z \tilde{S} = \frac{1}{2} C_{zzz} \tilde{S}^z \tilde{S}^z - h_{zz} \tilde{S}^z + h_z \,,$$
$$\partial_z K_z = K_z K_z - C_{zzz} \tilde{S}^{zz} + s_{zz}^z K_z - C_{zzz} \tilde{S}^z + h_{zz} \,,$$

up to another set of holomorphic (propagator) ambiguities s_{zz}^z, h_z^{zz}, h_z^z, h_z, h_{zz}. There is no canonical way to fix these ambiguities and different choices lead to a different functional dependence of the free energies on the propagators [76, 77]. For the 13 hypergeometric families, it turns out that there are always solutions of the form

(3.21)
$$s_{zz}^z = \frac{1}{z} \tilde{s}_{zz}^z \,, \qquad h_z^{zz} = z \tilde{h}_z^{zz} \,, \qquad h_z^z = 0 \,, \qquad h_z = \frac{1}{z} \tilde{h}_z \,, \qquad h_{zz} = \frac{1}{z^2} \tilde{h}_{zz} \,,$$

with $\tilde{s}_{zz}^z, \tilde{h}_z^{zz}, \tilde{h}_z, \tilde{h}_{zz} \in \mathbb{Q}$. Such solutions are determined by a polynomial equation in s_{zz}^z, and we pick the root such that $\tilde{h}_z$ has the maximal possible value.

The free energies in terms of the propagators can be obtained by integrating (3.18). If necessary, the full non-holomorphic dependence can then be restored by inserting the corresponding expressions for the propagators. However, to obtain the enumerative invariants that are encoded in the free energies we only need to consider the holomorphic limit. Using $K_z = -\partial_z \log(X^0)$ and the Ansatz (3.21), the BCOV ring (3.20) can be used to calculate the holomorphic limits of the propagators. Before carrying out the direct integration procedure, it remains to discuss how the holomorphic ambiguities $f^{(g)}(z)$ that arise at each genus from the integration of (3.18) can be fixed.

Let us first discuss the solution at genus $g = 1$ for the hypergeometric families. Combining (3.13) with (3.19) and using the behaviour in the large volume limit [12] and at the conifold point [78], the holomorphic anomaly equation for the genus one free energy can be integrated to obtain

(3.22)
$$\mathcal{F}^{(1)} = -\frac{1}{2} \left(4 - \frac{\chi \mathfrak{Y}}{12} \right) K - \frac{1}{2} \log \det G_{\bar{z}z} - \frac{1}{24} (12 + c_2) \log z - \frac{1}{12} \log \Delta \,,$$

where Δ is the discriminant polynomial and c_2 is the numerical second Chern class defined below (2.7).

At genus $g \geq 2$, the ambiguity can be written as a rational function in terms of the algebraic coordinates. It follows from the analysis in [74, 13],

see [79] for a review of B-model techniques, that

$$(3.23) \quad f^{(g)}(z) = \frac{1}{\Delta^{2g-2}} \sum_{k=0}^{2g-2} f_k z^k + \sum_{k=1}^{N(g)} f_k' z^k \,, \qquad N(g) = \left\lfloor \frac{2(g-1)}{\rho} \right\rfloor ,$$

where the 'regulator' ρ is the smallest denominator among the local exponents a_i, and f_k, f_k' are rational coefficients. To fix the coefficients, one can use known enumerative invariants — for example due to Castelnuovo like vanishing — together with the behaviour of the free energies around special points in the moduli space.

The generic constant map contribution of the free energies in the large volume limit $z = 0$ and the so-called gap condition at the conifold point $z = \mu$ can be used to fix all of the f_k, $k = 0, \ldots, 2g - 2$. On the other hand, for most of the hypergeometric families the current knowledge about the behaviour at $z = \infty$ is limited to the degree of regularity of the free energies at this point and, as discussed in [13], determines the upper limit $N(g)$ of the second sum in (3.23). Additional constraints are currently only understood for $X_{6,2}$, $X_{4,2}$ and most recently also for $X_{2,2,2,2}$ [80]. For $X_{6,2}$, $X_{4,2}$ the point at infinity is of conifold type and the expansion of the free energies around this point satisfy an additional 'small gap'. This imposes an additional $\lfloor 2g/\rho' \rfloor$ constraints, with ρ' for the two geometries respectively given by 3 and 4. On the other hand, the point at infinity in the moduli space of $X_{2,2,2,2}$ corresponds to a 'non-commutative resolution' of a singular degeneration of X_8. The corresponding free energies encode certain $\mathbb{Z}_2$-refined GV-invariants that also exhibit a Castelnuovo-like vanishing [80].

Using the Castelnuovo bound (3.9), together with the closed expression (3.30) for the invariants that saturate the bound and, in the case of $X_{6,2}$, $X_{4,2}$, additional conditions at infinity, the coefficients of the holomorphic ambiguity can be completely fixed as long as

$$(3.24) \qquad N(g) \leq \left\lfloor -\frac{\kappa}{2} + \frac{1}{2}\sqrt{\kappa\big(8(g-1) + \kappa\big)} \right\rfloor + \begin{cases} \lfloor \frac{2g}{3} \rfloor & \text{for } X_{6,2} \\ \lfloor \frac{g}{2} \rfloor & \text{for } X_{4,2} \\ 0 & \text{else} \end{cases} .$$

Moreover, as discussed in [80], for $X_{2,2,2,2}$ taking into account the additional Castelnuovo bound for the $\mathbb{Z}_2$-refined GV invariants at the point at infinity determines the holomorphic ambiguity up to genus 32.

We denote by g_{integ} the maximal value of g for which the previously discussed boundary conditions are sufficient to fix the holomorphic ambi-

guity[12], and tabulate its values for the various hypergeometric models in Table 1. Due to computational limitations, we have not yet reached this genus for all models. In §5, we shall see that the knowledge of D4-D2-D0 invariants can be used to push the direct integration method to even higher genus, denoted by g_{mod} in Table 1.

3.3. GV invariants at maximal and submaximal genus

Although the definition of GV invariants via Gromov-Witten invariants presented in §3.1 makes it clear that they are robust under complex structure deformations of $\mathfrak{Y}$, its main drawback is that integrality of the resulting invariants is not manifest. In [60] an alternative definition using moduli of stable sheaves was proposed, inspired by the geometric picture developed in [81] (and earlier attempts in [82, 83]). While the mathematical definition in [60] is quite involved (see [61] for a review aimed at physicists), the approach of [81] can be used to calculate GV invariants near the Castelnuovo bound, at least heuristically. The results (3.30) and (3.32) will be justified rigorously in §4.2 by combining Theorem 2 with the MNOP conjecture.

Motivated by the interpretation in terms of bound states of D2-D0 branes in Type IIA string theory, one considers one dimensional (semi-)stable sheaves supported on a curve of class $\beta \in H_2(\mathfrak{Y}, \mathbb{Z})$. The invariants are conjecturally independent of the D0-brane charge [84, 60], which can therefore be taken to be 1, such that semi-stability implies stability. For a fixed curve class β, the corresponding moduli space of stable sheaves $\widehat{\mathcal{M}}_\beta$ is fibered over the Chow variety $\mathcal{M}_\beta = \mathrm{Chow}(\beta)$, which parametrizes effective curves $\mathcal{C} \subset \mathfrak{Y}$ with $[\mathcal{C}] = \beta$. If a point $p \in \widehat{\mathcal{M}}_\beta$ projects to a smooth curve of genus[13] $g_{\max}(\beta)$, the corresponding fiber is the Jacobian torus $T^{2g_{\max}(\beta)}$ of $\mathcal{C}$.

It was argued in [81] that the little group $Spin(4) = SU(2)_L \times SU(2)_R$ in the five-dimensional effective theory arising from M-theory compactified on $\mathfrak{Y}$, should be identified with the product of the Lefschetz actions on the cohomology of the fiber and base of $\widehat{\mathcal{M}}_\beta$. As a result, in cases where $\widehat{\mathcal{M}}_\beta$ is smooth, the genus zero GV invariants can be defined as $\mathrm{GV}_\beta^{(0)} = (-1)^{\dim_{\mathbb{C}} \widehat{\mathcal{M}}_\beta} \chi(\widehat{\mathcal{M}}_\beta)$. More generally, genus zero GV invariants are related to

---

[12]Ignoring the floor functions in (3.23) and (3.24), and absorbing the correction term in (3.24) into an effective regulator $\rho = 6$ for $X_{6,2}$ or $\rho = 4$ for $X_{4,2}$, one finds a rule-of-thumb estimate $g_{\mathrm{integ}} \simeq \frac{1}{2}\kappa\rho(\rho - 1) + 1$.

[13]In this section we abuse notation and denote $\mathrm{GV}_\beta^{(g)} = \mathrm{GV}_{\beta.H}^{(g)}$ and $g_{\max}(\beta) = g_{\max}(\beta.H)$.

generalized Donaldson-Thomas invariants via $\mathrm{GV}_\beta^{(0)} = \Omega_H(0,0,\beta,1)$, where H is any ample divisor on $\mathfrak{Y}$ [85, 3].

On the other hand, if the Chow variety $\mathcal{M}_\beta$ itself is smooth, one finds that for maximal genus $g = g_{\max}(\beta) := g_{\max}(\beta.H)$,

$$(3.25) \qquad \mathrm{GV}_\beta^{(g_{\max}(\beta))} = (-1)^{\dim_{\mathbb{C}} \mathcal{M}_\beta} \chi(\mathcal{M}_\beta)\,.$$

Invoking a localization argument motivated by [86], the authors of [81] propose further geometric expressions for the invariants at genera close to $g_{\max}(\beta)$. In particular, in favorable cases

$$(3.26) \quad \mathrm{GV}_\beta^{(g_{\max}(\beta)-1)} = (-1)^{\dim_{\mathbb{C}} \mathcal{M}_\beta + 1} \Big[\chi(\mathcal{C}_\beta) + (2g_{\max}(\beta) - 2)\chi(\mathcal{M}_\beta) \Big],$$

where $\mathcal{C}_\beta \subset \mathfrak{Y} \times \mathcal{M}_\beta$ is the the universal curve. We observe that these relations agree with (3.11), (3.12), after identifying [87, 88]

$$(3.27) \qquad \begin{aligned} \mathrm{PT}(\beta, m_{\min}(\beta)) &= (-1)^{\dim_{\mathbb{C}} \mathcal{M}_\beta} \chi(\mathcal{M}_\beta)\,, \\ \mathrm{PT}(\beta, m_{\min}(\beta) + 1) &= (-1)^{\dim_{\mathbb{C}} \mathcal{M}_\beta + 1} \chi(\mathcal{C}_\beta)\,. \end{aligned}$$

We shall now apply these relations to determine the GV invariants $\mathrm{GV}_Q^{(g_{\max}(Q))}$ and $\mathrm{GV}_Q^{(g_{\max}(Q)-1)}$ for degree $Q = \kappa d$ with $d \in \mathbb{N}$ for the 13 hypergeometric CY threefolds.

Let us consider a CY threefold $\mathfrak{Y}$ obtained as a complete intersection of n generic hypersurfaces of respective degrees $(d_1, \ldots, d_n)$ in weighted projective space $W := \mathbb{P}^{n+3}_{w_1,\ldots,w_{n+4}}$. Curves of degree $Q = \kappa d$ on $\mathfrak{Y}$ are obtained by intersecting $\mathfrak{Y}$ with two additional hypersurfaces of respective degrees 1 and d. Using the adjunction formula, one can check that a generic curve of this type has the maximal possible genus $g = g_C(\kappa d)$.

We can identify the restriction of the linear subspace to $\mathfrak{Y}$ with the ample divisor $\mathcal{D}$. By Bertini's theorem, $\mathcal{D}$ is smooth if the restriction of $\mathcal{O}_W(1)$ to $\mathfrak{Y}$ is base-point free.[14] This is generically the case if the number

$$(3.28) \qquad \chi_{\mathcal{D}} = \#\{\, i \,|\, w_i = 1\,,\ i = 1, \ldots, n + 4 \,\}\,,$$

of weights that are equal to one is strictly greater than three. The equality with $\chi(\mathcal{O}_D)$ can be verified using the Hirzebruch-Riemann-Roch (HRR) theorem. Then a generic $(1, d)$-curve is smooth if the restriction of $\mathcal{O}_W(d)$ to

---

[14]Recall that the base locus of a bundle consists of the points where all sections vanish simultaneously.

$\mathcal{D} \subset \mathfrak{Y}$ is basepoint free as well, which is automatically implied. Comparing with Table 1, we see that $\mathcal{D}$ is singular for X_{10}, $X_{6,4}$ and $X_{6,6}$.

For $d \geq 2$, the moduli space $\mathcal{M}(1, d)$ of complete intersection curves of degree $(1, d)$ is a projective bundle with fibers $\mathbb{P}\left[h^0\left(\mathcal{D}, \mathcal{O}_\mathcal{D}(d)\right) - 1\right]$ over $\mathbb{P}\left[\chi_\mathcal{D} - 1\right]$, using $\mathbb{P}[k] := \mathbb{P}^k$. Using again HRR or generating functions, we further calculate that

$$(3.29) \qquad h^0\left(\mathcal{D}, \mathcal{O}_\mathcal{D}(d)\right) = \begin{cases} \chi_\mathcal{D} - 1 & d = 1, \\ \frac{1}{2}\kappa d(d-1) + \chi_\mathcal{D} & d \geq 2. \end{cases}$$

Assuming that every smooth curve of degree $Q = \kappa d$ is a complete intersection, we conclude that

$$(3.30) \qquad \mathrm{GV}_{\kappa d}^{(g_C(\kappa d))} = \begin{cases} \frac{1}{2}\chi_\mathcal{D}(\chi_\mathcal{D} - 1) & d = 1, \\ (-1)^{\frac{1}{2}\kappa d(d-1)}\chi_\mathcal{D}\, h^0\left(\mathcal{D}, \mathcal{O}_\mathcal{D}(d)\right) & d \geq 2, \end{cases}$$

where for $d = 1$ we took into account the fact that the two linear sections play a symmetric role.

To calculate $\mathrm{GV}_{\kappa d}^{(g_C(\kappa d) - 1)}$, we first note that the universal curve $\mathcal{C}_{\kappa d}$ is fibered over $\mathfrak{Y}$, with the fiber over a point p being the subset $\mathcal{M}_p \subset \mathcal{M}$ of curves that intersect p (to avoid cluttering, we now suppress the subscript denoting the curve class). If the point p is sufficiently generic, which is always true if $\chi_\mathcal{D} > 3$, we obtain one condition on the linear section as well as on the degree d section, such that

$$(3.31) \qquad \chi(\mathcal{M}_p) = \begin{cases} \frac{1}{2}(\chi_\mathcal{D} - 1)(\chi_\mathcal{D} - 2) & d = 1, \\ (\chi_\mathcal{D} - 1)\left(\chi_\mathcal{D} - 1 + \frac{1}{2}\kappa d(d-1)\right) & d \geq 2. \end{cases}$$

For the nine hypergeometric cases with a smooth divisor $\mathcal{D}$, using (3.26) together with $\chi(\mathcal{C}) = \chi(\mathcal{M}_p) \times \chi(\mathfrak{Y})$ one then arrives at

$$(3.32)$$
$$\mathrm{GV}_{\kappa d}^{(g_C(\kappa d) - 1)} = J_1\left(h^0\left(\mathcal{D}, \mathcal{O}_\mathcal{D}(d)\right) - 1\right) + (2g_C(\kappa d) - 2)\chi_\mathcal{D}\, h^0\left(\mathcal{D}, \mathcal{O}_\mathcal{D}(d)\right)$$

for $d \geq 2$, and

$$(3.33) \quad \mathrm{GV}_{\kappa}^{(g_c(\kappa) - 1)} = \frac{1}{2}J_1\left(h^0\left(\mathcal{D}, \mathcal{O}_\mathcal{D}(1)\right) - 1\right) + (g_C(\kappa) - 1)(\chi_\mathcal{D} - 1)\chi_\mathcal{D}$$

for $d = 1$, with $J_1 = \chi_\mathfrak{Y}(\chi_\mathcal{D} - 1)$.

In Section 4.2 we shall derive the expressions (3.30) and (3.32) using the relation between PT invariants and rank 0 DT invariants. In particular,

we shall find that (3.30) holds also for X_{10}, $X_{6,4}$, $X_{6,6}$, and (3.32) holds for those geometries if $\kappa d \geq 4$ and J_1 is defined in terms of a particular rank 0 DT invariant counting D4-D0 bound states (see below (4.30)).

4. D4-D2-D0 indices from GV invariants

In this section, we explain how to compute the Abelian D4-D2-D0 indices $\Omega_{1,\mu}(\hat{q}_0)$ introduced in §2.6 in terms of the Gopakumar-Vafa invariants $\mathrm{GV}_Q^{(g)}$ determined by the topological string partition function. The strategy is to combine the relation between rank 0 DT invariants and PT invariants, investigated in the series of mathematical papers [4, 5, 6, 7, 8], with the PT/GV relation explained in §3.1. Unfortunately, the explicit formulae stated in Thm 1.1 and Thm 1.2 of [8] are not yet sufficient for our purposes. In Appendix A, one of the authors proves a generalization of both theorems which we present in the following two subsections using more physics-friendly notations. The first theorem has a close relationship to the physical picture based on D6-$\overline{\mathrm{D6}}$ bound states advocated in earlier works on D4-D2-D0 indices [19, 20, 16, 29, 30, 89, 31], but turns out to be much less powerful than the second theorem which is fully explicit and allows to compute a large number of Abelian D4-D2-D0 indices. It also implies the Castelnuovo bounds on PT and GV invariants, as we explain in §4.2.

4.1. Wall-crossing for rank 0 class

In Theorem 4 from Appendix A, a slightly stronger version of [8, Thm 1.1] is established by studying the walls of $\nu_{b,w}$-instability for rank 0 classes. In this subsection we reformulate this result by restricting to CY threefolds with $b_2(\mathfrak{Y}) = 1$ and vanishing torsion $H^2(\mathfrak{Y}, \mathbb{Z})_{\mathrm{tors}} = 0$, and translating to the notations of §2.6 and §2.7. To this end, we identify in Eqs. (A.20)-(A.24)

(4.1)
$$
\begin{aligned}
D &= rH, & \beta.H &= -\mu - \frac{1}{2}\kappa r^2, & m &= n - \frac{1}{6}\kappa r^3 = -q_0 + \frac{rc_2}{24}, \\
D_i &= (-1)^i r_i H, & \beta_i.H &= Q_i, & n_i &= (-1)^i m_i, & i = 1, 2.
\end{aligned}
$$

Under these identifications, we obtain that, provided the reduced D0-brane charge $\hat{q}_0$ (2.53) lies in the range

(4.2)
$$
0 \leq \frac{\chi(\mathcal{D}_r)}{24} - \hat{q}_0 < \frac{\kappa r}{12} \min\left(\frac{r^2}{2} - \frac{1}{8}, \, r - \frac{1}{2}\right),
$$

the rank 0 DT invariant (2.54) can be expressed as

$$(4.3) \qquad \overline{\Omega}_{r,\mu}(\hat{q}_0) = \sum_{r_i, Q_i, n_i} (-1)^{\gamma_{12}} \, \gamma_{12} \, \mathrm{PT}(Q_1, n_1) \, \mathrm{DT}(Q_2, n_2) \,,$$

where (using the notation (2.62))

$$(4.4) \qquad \gamma_{12} = r(Q_1 + Q_2) + n_1 + n_2 - \chi_{\mathcal{D}_r} \,,$$

and the sum runs over integers r_i, Q_i and n_i restricted to satisfy

$$(4.5) \qquad \begin{aligned} r_1 + r_2 &= r \,, \\ Q_2 - Q_1 &= \mu + \kappa r r_2 \,, \\ n_1 + n_2 &= n - r_1 Q_1 - r_2 Q_2 - \frac{\kappa}{2} r r_1 r_2 \,, \end{aligned}$$

and

$$(4.6) \quad \begin{aligned} & \left| r_i - \sqrt{\frac{r^2}{4} - \frac{6}{r\kappa} \left(\frac{\chi(\mathcal{D}_r)}{24} - \hat{q}_0 \right)} + (-1)^i \left(\frac{r}{2} + \frac{\mu}{r\kappa} \right) \right| < 1 \,, \\ & 0 \le Q_i \le \frac{3}{r} \left(\frac{\chi(\mathcal{D}_r)}{24} - \hat{q}_0 \right) + \frac{1}{2\kappa} \left(\frac{\mu}{r} + \frac{\kappa}{2} \left(r + 2(-1)^i r_i \right) \right)^2 - \frac{\kappa r}{8} \,, \\ & n_i \ge -\frac{2}{3} Q_i \left(\frac{Q_i}{\kappa} + \frac{1}{2} \right) \,. \end{aligned}$$

Physically, the r.h.s. of (4.3) can be interpreted as contributions of two-centered bound states of an anti-D6-brane bound to (Q_1, n_1) D2-D0 branes, carrying index $\mathrm{PT}(Q_1, n_1)$, and a D6-brane bound to (Q_2, n_2) D2-D0 branes, carrying index $\mathrm{DT}(Q_2, n_2)$, with the D4-brane charge arising from the fluxes r_i on either side.

Unfortunately, the condition (4.2) is so restrictive that the theorem can only apply, at best, to the most polar term in each component of the modular vector $h_{1,\mu}$. In particular, for $\mu = 0$ it is valid only for $\hat{q}_0 = \frac{\chi(\mathcal{D}_r)}{24}$ where only $Q_i = n_i = 0$ contribute, leading to

$$(4.7) \qquad \overline{\Omega}_{r,0} \left(\frac{\chi(\mathcal{D}_r)}{24} \right) = (-1)^{1+\chi_{\mathcal{D}_r}} \chi_{\mathcal{D}_r} \,,$$

where $\chi_{\mathcal{D}_r}$ was defined in (2.62). In practice however, it was observed in [31, §D] that the formula (4.3) predicts the correct polar terms in many

examples with $r = 1$, provided one restricts the sum only to $Q_1 = n_1 = 0$. Using $\mathrm{PT}(0,0) = 1$, one arrives at the naive Ansatz for polar coefficients in [31, (5.20)],

$$(4.8) \qquad \overline{\Omega}_{r,\mu}(\hat{q}_0) = (-1)^{r\mu+n+\chi_{\mathcal{D}_r}}(r\mu + n - \chi_{\mathcal{D}_r})\,\mathrm{DT}(\mu, n)\,,$$

where n is the integer defined in (2.55). The physical intuition for this Ansatz was that D4-D2-D0 branes at large volume arise as bound states of a D6-brane with D2-brane charge μ and D0-brane charge n, and an anti-D6-brane carrying $-r$ units of D4-brane flux. Unfortunately, it appears difficult to relax the condition (4.2), and to justify physically or mathematically the truncation to terms with $Q_1 = n_1 = 0$, which appears to work in many cases.

4.2. Wall-crossing for rank -1 class

In [8, Thm 1.2], one of the authors of the present work obtained a different formula relating rank 0 and rank 1 DT invariants, which is valid only for CY threefolds with $\mathrm{Pic}\,\mathfrak{Y} = \mathbb{Z}$ (hence $b_2(\mathfrak{Y}) = 1$ and $H^2(\mathfrak{Y}, \mathbb{Z})_{\mathrm{tors}} = 0$). The formula follows by studying the possible walls for objects of rank -1 class

$$(4.9) \qquad \mathsf{v}_k = \mathsf{v} - e^{-kH} = \left[-1, D + kH, \beta - \tfrac{1}{2}\,k^2 H^2, -m + \tfrac{1}{6}\,k^3 H^3\right]\,,$$

in the space of weak stability conditions for $k \gg 1$, and applies for arbitrary $D \in H^2(\mathfrak{Y}, \mathbb{Z})$, Poincaré dual to an arbitrary divisor class. Unfortunately, an explicit lower bound on k was not provided. In Appendix A, restricting to the case of primitive divisor, which is sufficient for computing Abelian D4-D2-D0 invariants, a more general formula is derived that does not require taking k large. Below, we rephrase Theorem 1 from Appendix A using the same notations as in the previous subsection, and explain how to use it to compute Abelian D4-D2-D0 invariants from the knowledge of GV invariants.

Main result   Let us fix $(Q, m) \in \mathbb{Z}_+ \times \mathbb{Z}$, and define the function $f :$ $\mathbb{R}^+ \to \mathbb{R}$ by

$$(4.10) \qquad f(x) := \begin{cases} x + \frac{1}{2} & \text{if } 0 < x < 1, \\ \sqrt{2x + \frac{1}{4}} & \text{if } 1 \le x < \frac{15}{8}, \\ \frac{2}{3}x + \frac{3}{4} & \text{if } \frac{15}{8} \le x < \frac{9}{4}, \\ \frac{1}{3}x + \frac{3}{2} & \text{if } \frac{9}{4} \le x < 3, \\ \frac{1}{2}x + 1 & \text{if } 3 \le x. \end{cases}$$
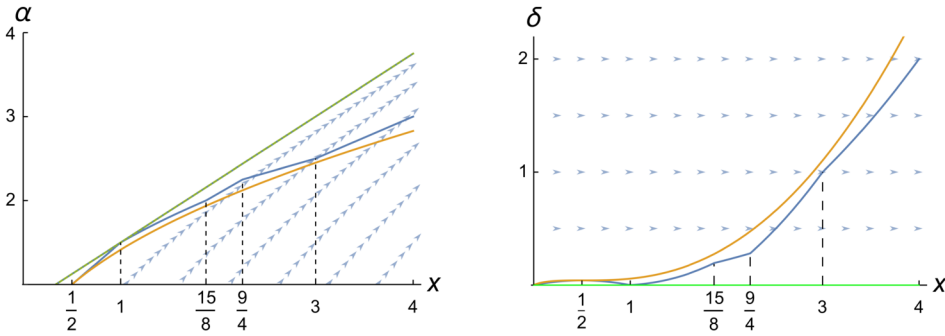
Figure 5: Left: The blue line represents the curve $f(x)$, above which Theorem 1 applies. The green line is the Castelnuovo line $\alpha = \frac{3}{4}(x+1)$, above which PT and DT invariants vanish. Below the orange line $\alpha = \sqrt{2x}$, the BMT line (2.38) does not intersect the parabola $w = \frac{1}{2}b^2$ in the (b, w) plane, so the argument in §A.7 fails. The dotted lines, oriented to the right, indicate the trajectories induced by spectral flow $(Q, m) \mapsto (Q+\kappa k, m-Qk-\frac{1}{2}\kappa k(k+1))$. Right: Same diagram in the (x, δ) plane, where $\delta = \frac{m}{\kappa} + \frac{1}{2}x(x+1)$. In these coordinates, Theorem 1 applies when (x, δ) lies below the blue line $\delta = -\frac{2}{3}xf(x) + \frac{1}{2}x(x+1)$. The Castelnuovo line, below which PT and DT invariants vanish, is the horizontal axis while the trajectories induced by spectral flow are now horizontal lines, oriented to the right.

Note that this function is uniformly bounded by $\frac{1}{2}(x+1) \le f(x) \le \frac{3}{4}(x+1)$ (see Figure 5, left). Theorem 1 then shows that, whenever $x > 0$ and $f(x) < \alpha$, with x, α defined by[15]

(4.11)
$$x = \frac{Q}{\kappa}, \qquad \alpha = -\frac{3m}{2Q},$$

the stable pair invariant $\text{PT}(Q, m)$ can be expressed in terms of invariants $\text{PT}(Q', m')$ with $Q' < Q$ *and* Abelian D4-D2-D0 invariants $\Omega_{1,\mu}(\hat{q}_0)$. More precisely,[16]

---

[15]The ratio α is unrelated to the parameter in (2.20). Instead, the variables (x, α) are the coefficients of the line $\ell_f(\mathsf{v})$ defined by $L_{b,w}(\mathsf{v}) = 2\kappa Q(w - \alpha b + x) = 0$ in (2.38) for the class $\mathsf{v} = [-1, 0, QH^2/\kappa, -m]$.

[16]Translating the formula (A.4) to the notations of this section, one finds that the index of the Abelian invariant should be $\mu = Q' - Q - \kappa$. Then we used the invariance of $\Omega_{1,\mu}(\hat{q}_0)$ under shifts of μ by κ and the flip of the sign to get (4.12).

$$(4.12) \quad \mathrm{PT}(Q,m) = \sum_{(Q',m')} (-1)^{\chi(Q',m')} \chi(Q',m') \, \mathrm{PT}(Q',m') \, \Omega_{1,Q-Q'}(\hat{q}'_0) \,,$$

where on the right-hand side

$$(4.13) \qquad \begin{aligned} \chi(Q',m') &= m - m' + Q + Q' - \chi_{\mathcal{D}} \,, \\ \hat{q}'_0 &= m' - m - \frac{1}{2\kappa}\left(Q' - Q\right)^2 - \frac{1}{2}\left(Q + Q'\right) + \frac{\chi(\mathcal{D})}{24} \,, \end{aligned}$$

with $\chi_{\mathcal{D}}$ and $\chi(\mathcal{D})$ defined in (2.62) and (2.56), respectively. The sum runs over pairs of integers (Q', m') such that

$$(4.14) \qquad\qquad 0 \leq \quad Q' \quad \leq Q + \kappa\left(\frac{1}{2} - \alpha\right),$$

$$(4.15) \qquad -\frac{Q'^2}{2\kappa} - \frac{Q'}{2} \leq \quad m' \quad \leq m + \frac{1}{2\kappa}(Q - Q')^2 + \frac{1}{2}(Q + Q').$$

Note that the lower bound on Q' simply follows from vanishing of PT invariants for negative degrees, while the upper bound on m' similarly corresponds to vanishing of Abelian invariants for charges spoiling (2.57). On the other hand, the upper bound on Q' implies that $Q' < Q$, since $\alpha > f(x) > 1/2$ which shows that (4.12) has a recursive nature.

Mathematically, the equality (4.12) follows by collecting the contributions from all walls for the Chern vector $\mathsf{v}_0 = (-1, 0, \beta, -m)$, between an empty chamber provided by the BMT inequality and the large volume limit $w \to \infty$ where the index $\Omega_{b,w}(\mathsf{v})$ coincides with $\mathrm{PT}(Q,m)$. Schematically, the formula (4.12) says that anti-D6-brane bound to (Q, m) D2-D0-branes arises from bound states of anti-D6-branes bound to (Q', m') D2-D0-branes and carrying -1 unit of D4-brane flux, and D4-brane bound to $(Q - Q', m - m')$ D2-D0-branes. The relation (4.12) in principle gives a recursive way of computing the PT invariants if Abelian D4-D2-D0 invariants are known, with the caveat that the terms (Q', m') contributing to the sum may not satisfy the condition $f(x') < \alpha'$.

A crucial observation is that the term $(Q', m') = (0, 0)$ with $\mathrm{PT}(0,0) = 1$ always contributes to the sum (4.12), so one may invert this relation to extract the Abelian D4-D2-D0 invariant $\Omega_{1,Q}(\hat{q}_0)$, where m should now be seen as a function of Q and $\hat{q}_0$ obtained by setting $Q' = 0, m' = 0$ in (4.13),

$$(4.16) \qquad\qquad m(Q, \hat{q}_0) = \frac{\chi(\mathcal{D})}{24} - \hat{q}_0 - \frac{Q^2}{2\kappa} - \frac{Q}{2} \,.$$

As above, the resulting formula may be used to recursively compute Abelian D4-D2-D0 invariants in terms of PT invariants, with the same caveat.

In practice, however, the condition $f(x) < \alpha$ is typically not satisfied for the charges of interest. Indeed, to compute the generating functions (2.58), we are interested in $Q \in [0, \kappa/2]$ and it is easy to see that for such small Q, in the best case, the condition is satisfied only for D0-brane charges very close to the bound (2.57). Fortunately, we can always use the spectral flow invariance to make Q large enough so that the condition becomes satisfied. Indeed, for $Q \geq 3\kappa$ the condition $f(x) < \alpha$ can be rewritten as

$$(4.17) \qquad \frac{\chi(\mathcal{D})}{24} - \hat{q}_0 < \frac{Q^2}{6\kappa} - \frac{Q}{6},$$

and is clearly satisfied if Q is sufficiently large.

Thus, we arrive at the following recipe. To compute $\Omega_{1,\mu}(\hat{q}_0)$, let us choose $k \in \mathbb{Z}_+$ such that

$$(4.18) \qquad f(Q_k) < -\frac{3m_k}{2Q_k}, \qquad \text{where} \quad m_k = m(Q_k, \hat{q}_0), \quad Q_k = \mu + \kappa k.$$

Then the Abelian index is given by the following formula

(4.19)

$$\Omega_{1,\mu}(\hat{q}_0) = \frac{(-1)^{m_k + Q_k - \chi_{\mathcal{D}}}}{m_k + Q_k - \chi_{\mathcal{D}}} \Big[\mathrm{PT}(Q_k, m_k) \\ - \sum_{(Q', m') \neq (0,0)} (-1)^{\chi(Q', m')} \chi(Q', m') \, \mathrm{PT}(Q', m') \, \Omega_{1, Q_k - Q'}(\hat{q}'_0) \Big],$$

where one should apply (4.13)-(4.15) with (Q, m) replaced by (Q_k, m_k). For practical computations, it is of course convenient to choose the minimal possible value of k satisfying (4.18), because PT invariants are usually known for small degrees Q only.

Before we proceed in the next section to apply this result to the CY threefolds listed in Table 1, we spell out two important consequences of the formula (4.12), which are also proven in Appendix A (see §A.3 and §A.2).

Castelnuovo bound   As a consequence of the wall structure established for the proof of Theorem 1, and using induction on Q, one obtains a Castelnuovo-type inequality for PT invariants: namely, for any $(Q, m) \in \mathbb{Z}_+ \times \mathbb{Z}$,

$\mathrm{PT}(Q, m) = 0$ unless

$$(4.20) \qquad m \geq -\mathcal{C}(Q), \qquad \mathcal{C}(Q) := \begin{cases} \left\lfloor \frac{Q^2}{2\kappa} + \frac{Q}{2} \right\rfloor & Q \geq \kappa, \\[2mm] \left\lfloor \frac{2Q^2}{3\kappa} + \frac{Q}{3} \right\rfloor & 0 < Q < \kappa. \end{cases}$$

As a result, we can replace the lower bound in (4.15) by $-\mathcal{C}(Q') \leq m'$, as stated in Appendix A. By the DT/PT relation, (4.20) implies the same statement for the DT invariant $\mathrm{DT}(Q, m)$, while the PT/GV relation implies the Castelnuovo bound for GV invariants in (3.9). Note that in terms of (x, α) defined in (4.11), the bounds in (4.20) take a universal form independent of κ:

$$(4.21) \qquad \alpha \leq \begin{cases} \frac{3}{4}(x+1) & x \geq 1, \\[1mm] x + \frac{1}{2} & x \leq 1. \end{cases}$$

Since (4.12) provides a way to compute PT invariants in the range $\frac{1}{2}x + 1 < \alpha \leq \frac{3}{4}(x+1)$ (assuming that $Q = \kappa x$ is large, for the sake of argument) in terms of PT invariants of lower degree, it follows that for fixed degree Q, the number of unknown GV invariants is effectively reduced from $Q^2/(2\kappa)$ to $Q^2/(3\kappa)$. Fixing instead the genus g, the number of constraints on holomorphic ambiguities from known GV invariants now grows as $\sqrt{3\kappa g}$, rather than $\sqrt{2\kappa g}$, therefore allowing to fix them up to genus $g \lesssim \frac{3}{4}\kappa\rho^2$ rather than $g \lesssim \frac{1}{2}\kappa\rho^2$ (see footnote 12). Thus, we expect that the additional constraints from (4.12) will allow to push the direct integration method to genus $g_{\mathrm{mod}} \simeq \frac{3}{2} g_{\mathrm{integ}}$, i.e. a factor $3/2$ higher than the maximal genus predicted by (3.24). Unfortunately, this reasoning overlooks the complicated relation between PT and GV invariants, and in practice the gain in genus will be slighter smaller (see the last column in Table 1).

Returning to the prescription (4.18), we note that the distance away from the Castelnuovo bound (4.20) is independent of k,

$$(4.22) \qquad m(Q_k) + \frac{Q_k^2}{2\kappa} + \frac{Q_k}{2} = \hat{q}_0 - \frac{\chi(\mathcal{D})}{24}.$$

In Figure 5 (right), we represent the region of validity of Thm 1 in the plane (x, δ) where $\delta = \frac{m}{\kappa} + \frac{Q^2}{2\kappa^2} + \frac{Q}{\kappa}$, where spectral flow acts by horizontal translations $x \mapsto x - k$, keeping δ fixed. This makes it clear that Thm 1 is always valid for $k \geq k_0$ large enough. Experimentally, we shall see in §5 and §B that the formula (4.19) often gives the correct result for $k = k_0 - 1$ or
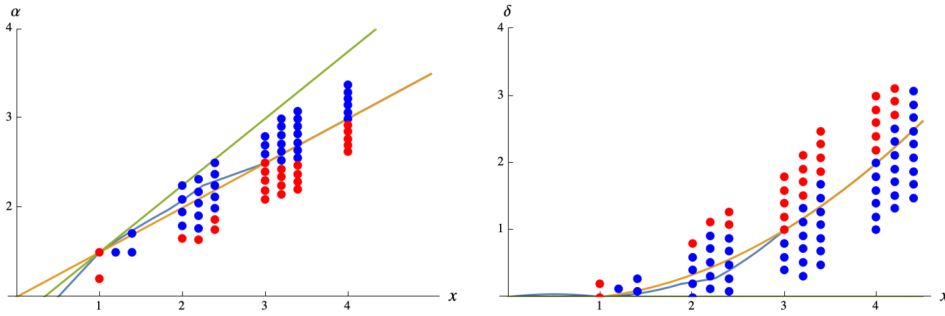
Figure 6: Left: Blue dots indicate values of (x, α) for which the formula (4.19) turns out to give the correct D4-D2-D0 indices for X_5. Red dots instead indicate values of (x, α) for which (4.19) fails to give the correct result. The values of (x, α) correspond to (Q_k, m_k) with $k \in \{k_0, k_0 - 1, k_0 - 2\}$ where k_0 is the minimal value of k such that (4.18) holds. All red dots lie below the line $\alpha = f(x)$ shown in blue, and in fact they all lie below the line $\alpha = \frac{x}{2} + 1$ shown in orange (see Remark 4). Interestingly, there are also some blue dots lying below this line, which indicates that the condition $f(x) < \alpha$ for the validity of (4.19) can probably be weakened. Right: same diagram in the (x, δ) plane.

(less often) $k = k_0 - 2$, even though the assumptions of Thm 1 are no longer satisfied, see in particular Figure 6 for the quintic threefold.

Optimal case   The formula (4.12) becomes particularly simple in cases where the sum over $(Q', m') \neq (0, 0)$ becomes empty. This occurs provided $(Q', m') = (0, 0)$ is the only solution to (4.14), (4.15) — in this case we call the pair (Q, m) optimal. A sufficient set of conditions is that

$$(4.23) \qquad \alpha > f(x), \qquad \Psi\left(x, 1/\kappa, \alpha\right) < 0, \qquad \Psi\left(x, x - \alpha + \tfrac{1}{2}, \alpha\right) < 0,$$

where

$$(4.24) \qquad \Psi(x, x', \alpha) := \frac{1}{2}(x - x')^2 + \frac{1}{2}(x + x') - \frac{2}{3}\alpha x + \frac{1}{2}x'^2 + \frac{1}{2}x'$$

is the difference between the upper and lower bounds in (4.15), after expressing the result in terms of $x = Q/\kappa$ and $x' = Q'/\kappa$ and rescaling by κ. The values $x' = 1/\kappa$ and $x' = x - \alpha + \frac{1}{2}$ correspond to the minimal and maximal values $Q' = 1$ and $Q' = \kappa(\frac{1}{2} - \alpha)$ to be ruled out. The condition

$\Psi\left(x, 1/\kappa, \alpha\right) < 0$ can be equivalently written as

$$(4.25) \qquad \frac{3}{4}(x+1) - \frac{3(x-1)}{2x\kappa} + \frac{3}{2\kappa^2 x} < \alpha.$$

This shows that the condition (4.25) can only be satisfied close to the Castelnuovo bound. The last condition in (4.23) turns out to be implied by the condition $\alpha > f(x)$ when $x > \frac{1}{4}(5 + \sqrt{7}) \simeq 1.91$. When the conditions in (4.23) are obeyed (or more generally when (Q, m) is optimal), (4.12) simply reduces to

$$(4.26) \qquad \mathrm{PT}(Q, m) = (-1)^{m+Q-\chi_{\mathcal{D}}} \left(m + Q - \chi_{\mathcal{D}}\right) \Omega_{1,Q}(\hat{q}_0).$$

In fact, using the invariance of $\Omega_{1,\mu}(\hat{q}_0)$ under spectral flow as above, one can always choose the spectral flow parameter k large enough such that (Q_k, m_k) is optimal. In particular, this implies that the ratio

$$(4.27) \qquad \Omega_{1,Q}(\hat{q}_0) = (-1)^{m_k + Q_k - \chi_{\mathcal{D}}} \frac{\mathrm{PT}(Q_k, m_k)}{m_k + Q_k - \chi_{\mathcal{D}}},$$

must stabilize to a constant value for k larger than a suitable $k_1 \geq k_0$.

It is also possible to use these relations to derive general formulae for GV invariants near the Castelnuovo bound. Let us choose, for example, $Q = m = 0$. Then using (4.26) and (4.7) with $r = 1$, we find that the optimality condition is satisfied for any $k \geq 2$, leading to[17]

$$(4.28) \qquad \mathrm{PT}\left(\kappa k, -\tfrac{1}{2}\kappa k(k+1)\right) = (-1)^{\frac{1}{2}\kappa k(k-1)} \left(\chi_{\mathcal{D}} + \tfrac{1}{2}\kappa k(k-1)\right) \chi_{\mathcal{D}}.$$

Since the second argument on the left-hand side is equal to $1 - g_C(\kappa k)$, one can use the relation (3.11) to obtain the GV invariant for $Q = \kappa k$ and maximal genus,

$$(4.29) \qquad \mathrm{GV}_{\kappa k}^{(g_{\max}(\kappa k))} = (-1)^{\frac{1}{2}\kappa k(k-1)} \left(\chi_{\mathcal{D}} + \tfrac{1}{2}\kappa k(k-1)\right) \chi_{\mathcal{D}}.$$

This reproduces the result which was obtained by heuristic arguments in (3.30) for $d = k \geq 2$.

---

[17]This result reduces to the first part of Theorem 3 for the quintic upon setting $\mu = \kappa k, \kappa = \chi_{\mathcal{D}} = 5$.

Similarly, choosing $Q = 0$ and $m = 1$, we find[18]

$$
\text{(4.30)} \quad
\begin{aligned}
&\text{PT}\left(\kappa k, -\tfrac{1}{2}\kappa k(k+1)+1\right) \\
&\quad = (-1)^{\frac{1}{2}\kappa k(k-1)+1}\left(\chi_{\mathcal{D}} + \tfrac{1}{2}\kappa k(k-1) - 1\right)J_1\,,
\end{aligned}
$$

where $J_1 := (-1)^{\chi_{\mathcal{D}}+1}\,\overline{\Omega}_{1,0}\left(\frac{\chi(\mathcal{D})}{24} - 1\right)$. Using (3.12), we conclude that the GV invariant for $Q = \kappa k$ and submaximal genus is given by

$$
\text{(4.31)}
$$
$$
\begin{aligned}
\text{GV}_{\kappa k}^{(g_{\max}(\kappa k)-1)} = (-1)^{\frac{1}{2}\kappa k(k-1)}&\left[-\frac{1}{2}\,\kappa^2 k^4 \chi_{\mathcal{D}} - \frac{1}{2}\,\kappa k^2\left(2\chi_{\mathcal{D}}^2 - \kappa\chi_{\mathcal{D}} + J_1\right)\right.\\
&\left.+\kappa k\left(\frac{J_1}{2} - \chi_{\mathcal{D}}^2\right) + J_1(1 - \chi_{\mathcal{D}})\right].
\end{aligned}
$$

This reproduces the result obtained by heuristic arguments in (3.32), although the constant J_1 is not determined by the present computation. In the examples in §5 and §B, we shall see that $J_1 = \chi_{\mathfrak{Y}}(\chi_{\mathcal{D}} - 1)$ when the divisor $\mathcal{D}$ is smooth, which is the case when $\chi_{\mathcal{D}} \geq 4$, but that it may otherwise differ from this value.

5. Testing the modularity of rank 0 DT invariants

In this section, we apply the results explained in §4.2 to determine the generating series of Abelian D4-D2-D0 indices for several examples of one-parameter threefolds, including X_5 (the quintic in $\mathbb{P}^4$), X_{10} (the decantic in weighted projective space $\mathbb{P}^4_{5,2,1,1,1}$ and $X_{4,2}$ (a complete intersection of degree $(4,2)$ in $\mathbb{P}^5$). In particular, we rigorously compute the polar coefficients and a large number of non-polar coefficients, and confirm the modularity property predicted by string theory. For X_5 our results coincide with those in [19], for X_{10} we confirm the proposal in [30] (which deviates from the original computation in [20]), while for $X_{4,2}$ we determine the generating series that was previously unknown. In Appendix B, we give similar results for all other hypergeometric models, except for $X_{3,2,2}$ and $X_{2,2,2,2}$ for which our current knowledge of GV invariants is still insufficient to determine (or just guess) the polar terms.

---

[18]By a case-by-case analysis, one checks that the optimality conditions are verified when $k \geq 2$ for $X_{4,3}, X_{3,3}, X_{4,2}, X_{3,2,2}, X_{2,2,2,2}$, when $k \geq 3$ for $X_8, X_{4,4}, X_{6,2}$, and when $k \geq 4$ for X_{10} and $X_{6,6}$.

5.1. Basis of vector-valued modular forms

As explained in §2.6, string theory predicts that the generating series (2.58) of Abelian D4-D2-D0 indices (for brevity we drop the rank index 1)

$$(5.1) \qquad h_\mu(\tau) = \sum_{\hat{q}_0 \leq \frac{\chi(\mathcal{D})}{24}} \Omega_{1,\mu}(\hat{q}_0) \, \mathrm{q}^{-\hat{q}_0}$$

should behave under $SL(2,\mathbb{Z})$ transformations as a vector-valued modular form of weight $-3/2$, transforming in the Weil representation of the lattice $\mathbb{Z}[\kappa]$. The space $\mathcal{M}_1(\mathfrak{Y})$ of such functions has dimension $n_1^p - n_1^c$, where n_1^p is the number of polar coefficients, corresponding to terms with negative power $\hat{q}_0 > 0$ in (5.1), and n_1^c is the number of linear relations which these coefficients must satisfy, in order for a modular form to exist (the numbers n_1^p and n_1^c are listed in Table 1).

An overcomplete basis of $\mathcal{M}_1(\mathfrak{Y})$ can be constructed as follows [31]. We define the theta series

$$(5.2) \qquad \vartheta_\mu^{(\kappa)}(\tau) = \begin{cases} (-1)^{\mu+\frac{\kappa}{2}} \sum_{k \in \mathbb{Z}+\frac{\mu}{\kappa}+\frac{\kappa}{2}} \mathrm{q}^{\frac{\kappa}{2} k^2} & \kappa \text{ even,} \\ -\mathrm{i}\kappa \sum_{k \in \mathbb{Z}+\frac{\mu}{\kappa}+\frac{\kappa}{2}} (-1)^{\kappa k} \, k \, \mathrm{q}^{\frac{\kappa}{2} k^2} & \kappa \text{ odd.} \end{cases}$$

They satisfy

$$(5.3) \qquad \vartheta_\mu^{(\kappa)}(\tau) = \vartheta_{\mu+\kappa}^{(\kappa)}(\tau) = \vartheta_{-\mu}^{(\kappa)}(\tau)$$

and transform under $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ as vector-valued modular forms of weight $1/2$ and $3/2$, respectively. For $\kappa = 1$, we note that $\vartheta_\mu^{(1)}(\tau) = \eta^3$ where $\eta(\tau)$ is the Dedekind theta function. More generally, for $\mu = 0$ one has

$$(5.4) \qquad \vartheta_0^{(\kappa)}(\tau) = \begin{cases} 2(-1)^{\frac{\kappa}{2}} \dfrac{\eta(2\kappa\tau)^2}{\eta(\kappa\tau)} & \kappa \text{ even,} \\ \kappa(-1)^{\frac{\kappa-1}{2}} \eta(\kappa\tau)^3 & \kappa \text{ odd.} \end{cases}$$

We claim that any element of $\mathcal{M}_1(\mathfrak{Y})$ is a linear combination of the form

$$(5.5) \qquad h_\mu(\tau) = \sum_{\ell=0}^{\ell_0} \sum_{k=0}^{k_\ell} a_{\ell,k} \, E_4^{\lfloor w_\ell/4 \rfloor - \epsilon_\ell - 3k}(\tau) \, E_6^{2k+\epsilon_\ell}(\tau) \, \frac{D^\ell \theta_\mu^{(\kappa)}(\tau)}{\eta^{4\kappa+c_2}(\tau)} \,,$$

where $E_4(\tau)$ and $E_6(\tau)$ are the standard Eisenstein series, and D^ℓ is the iterated Serre derivative[19], acting on holomorphic modular forms of weight w through $D_w = q\partial_q - \frac{w}{12}E_2$, where E_2 is the normalized quasi-modular Eisenstein series. Finally, the integers $k_\ell, \epsilon_\ell, w_\ell$ in (5.5) are given by

$$
\begin{aligned}
(5.6) \qquad & k_\ell = \lfloor w_\ell/12 \rfloor - \delta^{(12)}_{w_\ell-2}, \qquad \epsilon_\ell = \delta^{(2)}_{w_\ell/2-1}, \\
& w_\ell = 2\kappa + \frac{1}{2}c_2 - 3 - 2\ell + \delta^{(2)}_\kappa,
\end{aligned}
$$

where $\delta^{(n)}_x$ is equal to 1 if $x = 0 \mod n$ and 0 otherwise, and ℓ_0 should be chosen sufficiently large so that $\sum_{\ell=0}^{\ell_0}(k_\ell + 1)$ is not smaller than the dimension of the space $\mathscr{M}_1(\mathfrak{Y})$. The coefficients $a_{\ell,k}$ are not unique in general (since the basis is overcomplete), but the modular form $h_\mu(\tau)$ is uniquely fixed by providing $n_1^p - n_1^c$ of its Fourier coefficients (for example the polar coefficients). Having determined a suitable set of coefficients $a_{\ell,k}$, it is then straightforward to expand $h_\mu(\tau)$ to arbitrary order, and obtain a prediction for an infinite number of Abelian D4-D2-D0 invariants.

5.2. X_5

Abelian D4-D2-D0 invariants for the quintic threefold were first studied in [19], using a different basis of modular forms and an ingenuous but non-rigorous method for computing the polar terms. In this case, $\kappa = 5$, $n_1^p = 7$ and $n_1^c = 0$ so the vector-valued modular form is uniquely determined by computing 7 of its coefficients. Using the overcomplete basis of the previous subsection, the result of [19] can be written as

$$
\begin{aligned}
(5.7) \qquad h_\mu = \frac{1}{\eta^{70}} \Bigg[& -\frac{222887 E_4^8 + 1093010 E_4^5 E_6^2 + 177095 E_4^2 E_6^4}{35831808} \\
& + \frac{25\left(458287 E_4^6 E_6 + 967810 E_4^3 E_6^3 + 66895 E_6^5\right)}{53747712} D \\
& + \frac{25\left(155587 E_4^7 + 1054810 E_4^4 E_6^2 + 282595 E_4 E_6^4\right)}{8957952} D^2 \Bigg] \vartheta_\mu^{(5)},
\end{aligned}
$$

---

[19]Rather than the standard iterated Serre derivative, one can just as well use its improved version introduced in [90, Eq (35)] or Rankin-Cohen brackets. Unfortunately this does not lead to smaller denominators in the resulting coefficients $a_{\ell,k}$.

In view of the symmetry properties (5.3), there are only three distinct components, with the following expansion:
(5.8)

$$
h_0 = q^{-\frac{55}{24}} \Big(\underline{5 - 800q + 58500q^2} + 5817125q^3 + 75474060100q^4
$$
$$
+ 28096675153255q^5 + 3756542229485475q^6 + 277591744202815875q^7
$$
$$
+ 13610985014709888750q^8 + 490353109065219393125q^9 + \dots \Big),
$$
$$
h_1 = q^{-\frac{55}{24}+\frac{3}{5}} \Big(\underline{0 + 8625q} - \underline{\dotuline{1138500q^2}} + 3777474000q^3 + 3102750380125q^4
$$
$$
+ 577727215123000q^5 + 52559194851824125q^6
$$
$$
+ 2990604504777589125q^7 + \dots \Big),
$$
$$
h_2 = q^{-\frac{55}{24}+\frac{2}{5}} \Big(\underline{0 + 0q} - \underline{\dotuline{1218500q^2}} + 441969250q^3 + 953712511250q^4
$$
$$
+ 217571250023750q^5 + 22258695264509625q^6
$$
$$
+ 1374043315791020500q^7 + \dots \Big).
$$

Here and elsewhere, the polar coefficients are underlined. Using Eq. (4.19) and GV invariants up to genus 53, we have reproduced all terms up to (and including) orders q^9, q^5 and q^6 in these expansions, respectively.[20] In many cases, we find that (4.19) holds even though the assumption $f(x) < \alpha$ is not obeyed (see Figure 6), in particular we can also reproduce the coefficients of q^6 in h_1 and q^7 in h_2 using (4.19) with $k = k_0 - 1$, where k_0 is the minimal value of k for which (4.18) is satisfied.

As already noted in [31], the naive Ansatz (4.8) with $r = 1$ gives the correct polar terms in this case. In addition, it also correctly predicts the $\mathcal{O}(q^2)$ terms in h_1 and h_2, as indicated with dotted underline. The coefficient of the order $\mathcal{O}(q^3)$ term in h_0 can be understood as

$$
(5.9) \qquad 5817125 = -2\,\mathrm{DT}(0,3) + \mathrm{DT}(0,2) + \mathrm{DT}(1,1)^2 \,,
$$

where the first term is the naive ansatz prediction, the second is a correction from the locus where the 3 D0-branes are aligned, and the last term

---

[20]For the coefficients up to q^5 in h_0, q^2 in h_1 and q^3 in h_2, the relevant value of (Q, m) is optimal and the formula (4.12) has only one non-vanishing contribution (or none when the coefficient is zero). For the terms of order q^6, q^7, q^8, q^9 in in h_0, there are contributions from 2,3,4,5 walls, respectively. For the order q^3, q^4, q^5, q^6, q^7 in h_1, there are contributions from 2,3,4,4, 5 walls, respectively. For the terms of order q^4, q^5, q^6, q^7 in in h_2, there are contributions from 2,4,3,3,5 walls, respectively.

Table 2: GV invariants $\mathrm{GV}_Q^{(g_C(Q)-\delta)}$ for X_5, assuming modularity

Q	g_C	$\delta = 0$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 4$	$\delta = 5$
1	1	0	2875				
2	2	0	0	609250			
3	3	0	0	609250	317206375		
4	4	0	8625	534750	3721431625	242467530000	
5	6	10	1100	49250	−15663750	75478987900	12129909700200
6	7	0	−34500	−3079125	−7529331750	3156446162875	871708139638250
7	9	0	0	4874000	1300955250	−1917984531500	245477430615250
8	11	0	0	−6092500	−1670397000	2876330661125	−471852100909500
9	13	0	60375	5502750	18763368375	−12735865055000	1937652290971125
10	16	−50	−5700	−286650	50530375	−454092663150	150444095741780
11	18	0	−86250	−7357125	−29938013250	22562306494375	−4041708780324500
12	21	0	0	−13403500	−3937166500	8725919269125	−2017472506595500
13	24	0	0	−15840500	−4638330000	10690009494250	−2578098061480250
14	27	0	−138000	−10177500	−52227066000	42752384997625	−8759526658670500
15	31	−100	−9200	−342400	136695125	−1214106563650	484402370601245
16	34	0	181125	11178000	70714095125	−60120995398500	13182681427726625
17	38	0	0	−28025500	−7761538500	20623428936750	−5693356905665000
18	42	0	0	31681000	8578113250	−23636174920000	6726357908107750
19	46	0	−258750	−10246500	−103897578000	92567501962875	−22247603793898250
20	51	175	9700	113650	−271460000	2362533313525	−1059131220525950
21	55	0	319125	7158750	129691149375	−118821918509250	30276261813046500
22	60	0	0	48740000	11680440750	−37863219131500	12130764520281750
23	65	0	0	53614000	12356541750	−41972283930000	13849264699781000
24	70	0	422625	−2829000	174040666500	−165847969399750	46048552308175750
25	76	275	1950	−261225	−437171250	3908290893900	−1955377337896550
26	81	0	−500250	14145000	−207540563250	202764143836375	−59568660504287750
27	87	0	0	75547000	14268228250	−60651049880500	22529431755767500
28	93	0	0	−81639500	−14474860500	65883050745250	−25213918522757500
29	99	0	629625	−40175250	264127092375	−267394402192000	85401556513695875
30	106	−400	20800	−111400	631692625	−5861793912900	3278134921975475

corresponds to a bound state of D6-D2 and $\overline{\mathrm{D6}}$-$\overline{\mathrm{D2}}$-branes [29]. It would be interesting to have a similar bound state interpretation for other non-polar coefficients.

Using modularity we can also predict GV and PT invariants of arbitrary degree, provided they are close enough to the Castelnuovo bound. In Table 2, we list the GV invariants with $\delta = g_C(Q) - g \leq 5$, and similarly in Table 3 we list the PT invariants with $\delta = m - m_C(Q) \leq 5$, where $g_C(Q)$ and $m_C(Q)$ were defined in (3.9) and (3.12), respectively. Using these GV invariants, we have in principle sufficiently many boundary conditions to fix the holomorphic ambiguity up to genus 69. Due to limited computer resources, we have currently pushed up the direct integration to genus 64, and confirmed the predictions of modularity up to that order.

Table 3: Stable pair invariants $\mathrm{PT}(Q, m_C(Q) + \delta)$ for X_5, assuming modularity

Q	m_C	$\delta = 0$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 4$	$\delta = 5$
1	0	0	2875	−5750	8625	−11500	14375
2	−1	0	0	609250	2912875	−14703500	38888250
3	−2	0	0	609250	317206375	1117181000	−2098275750
4	−3	0	8625	569250	3722552875	244219693000	609122565875
5	−5	10	1200	58500	−15336250	75441932225	12282361758020
6	−6	0	−34500	−3395375	−7552124750	3111341190625	884181641560000
7	−8	0	0	4874000	1359443250	−1904746390000	230184283873875
8	−10	0	0	−6092500	−1767877000	2852214003125	−437477532060500
9	−12	0	60375	6831000	18887370000	−12396985924250	1736738444379375
10	−15	−50	−7200	−468000	40719875	−452993138850	140467307991350
11	−17	0	−86250	−10117125	−30201650750	21720393561500	−3466439656488000
12	−20	0	0	−13403500	−4419692500	8583611403125	−1740547789348750
13	−23	0	0	−15840500	−5303631000	10490837623750	−2175677447038750
14	−26	0	−138000	−17077500	−52884636000	40335754941625	−6932684543525000
15	−30	−100	−15200	−1053000	98891125	−1207584961600	421450499252120
16	−33	0	181125	22770000	71772279125	−55849465988500	9821365434297875
17	−37	0	0	−28025500	−9723323500	20027962736250	−4351425496412500
18	−41	0	0	31681000	11049231250	−22889100270000	5004138750546250
19	−45	0	−258750	−33016500	−105769272000	83774260263375	−15020907593198000
20	−50	175	27200	1930500	−186148000	2339695863100	−842900254597650
21	−54	0	319125	40986000	132211590000	−105493513413000	19064998024136500
22	−59	0	0	48740000	17236800750	−36241075427500	8050136250878750
23	−64	0	0	53614000	19004677750	−40055925472500	8920421250973750
24	−69	0	422625	54648000	177541278000	−142726517485750	25997724680535000
25	−75	275	43200	3100500	−302490500	3849177065100	−1404833757662750
26	−80	0	−500250	−64894500	−211538544000	170651270906875	−31197269617418250
27	−86	0	0	75547000	26960124250	−57222750675000	12836703751401250
28	−92	0	0	−81639500	−29169970500	61991313231250	−13924560001520000
29	−98	0	629625	81972000	268200654000	−217192526608750	39863177843487000
30	−105	−400	−63200	−4563000	447918625	−5736028567600	2107250636494125

5.3. X_{10}

We now turn to the decantic in $\mathbb{P}^4_{5,2,1,1,1}$, which was first studied in [20] and revisited in [30]. In this case, $\kappa = 1$, $n_1^p = 2$ and $n_1^c = 0$ so the scalar modular form $h := h_0$ is uniquely fixed by 2 coefficients. In [20], it was suggested that

$$
h \stackrel{?}{=} \frac{541 E_4^4 + 1187 E_4 E_6^2}{576 \, \eta^{35}}
$$

(5.10)

$$
= q^{-\frac{35}{24}} \left(\underline{3 - 576 q} + 271704 q^2 + 206401533 q^3 + \dots \right).
$$

The same result was found in [31] using the naive Ansatz (4.8). Instead, assuming that the BMT inequality is satisfied, Eq. (4.19) predicts that the

coefficient of the subleading polar term should be -575, as suggested in [30]. In fact, using (4.19) and GV invariants up to genus 47, we can check all the terms up to order q^{11} in the resulting expansion,

(5.11)

$$
\begin{aligned}
h &= \frac{203E_4^4 + 445E_4E_6^2}{216\,\eta^{35}} \\
&= q^{-\frac{35}{24}}\Big(3 - \underline{\underline{575}}q + 271955q^2 + 206406410q^3 + 21593817025q^4 \\
&\quad + 1054724115956q^5 + 32284130488575q^6 + 712354737460415q^7 \\
&\quad + 12285858824682770q^8 + 174458903522212025q^9 \\
&\quad + 2114022561929255740q^{10} + 22434520426025264925q^{11} \\
&\quad + 212611407819858981640q^{12} + \cdots\Big).
\end{aligned}
$$

Here and in Appendix B, the double-underline underscores the fact that the polar coefficient deviates from the naive ansatz (4.8). Interestingly, applying Eq. (4.19) with $k = k_0 - 1$, where k_0 is the minimal value for which (4.18) is satisfied, one can reproduce the expansion (5.11) to even higher order q^{14}.

As discussed in [30], the deviation from the naive ansatz arises because the moduli space of D4-D0 bound states is in general not a bundle over the moduli space of the D0-brane, which is $\mathfrak{Y}$ itself. When the D0-brane is at a generic position, the requirement that it should belong to the divisor imposes one condition on the defining equation of the divisor. Since the divisor is the vanishing locus of a linear polynomial in the three homogeneous coordinates of weight one, the moduli space of divisors containing a given generic point on $\mathfrak{Y}$ is $\mathbb{P}^1$. However, when the D0-brane lies at the special point where all homogeneous coordinates of weight one vanish, it no longer imposes any condition on the divisor, whose moduli space is then enhanced to $\mathbb{P}^2$. Hence, the index for a D4-brane bound to a single D0-brane should be [30]

(5.12) $$\chi(\mathbb{P}^1) \times (\chi_{\mathfrak{Y}} - \chi(\mathrm{pt})) + \chi(\mathbb{P}^2) \times \chi(\mathrm{pt}) = -575\,,$$

in agreement with (5.11). Ignoring the effect of the special point, one would instead find $\chi(\mathbb{P}^1) \times \chi_{\mathfrak{Y}} = -576$, as predicted by the naive ansatz.

While the maximal genus attainable by the standard direct integration method is 50, using modularity, we can predict GV invariants close to the Castelnuovo bound to arbitrary genus (see Table 4), and provide sufficiently many boundary conditions to push the direct integration method, in principle, up to genus 70.

Table 4: GV invariants $\mathrm{GV}_Q^{(g_C(Q)-\delta)}$ for X_{10}, assuming modularity

Q	g_C	$\delta = 0$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 4$	$\delta = 5$
1	2	3	280	231200			
2	4	−12	−1656	−537976	207680960	12215785600	
3	7	−18	−2646	−1057570	630052679	−46669244594	1264588024791
4	11	27	4060	1825541	−1268283512	125509540304	−5611087226688
5	16	39	5730	2814100	−2139555052	244759232792	−13239429980228
6	22	−54	−7507	−4004506	3254742758	−416587966648	25859458639950
7	29	−72	−9193	−5375708	4629222449	−655954806090	45976776864607
8	37	93	10554	6910207	−6280307986	981118531775	−77100442475920
9	46	117	11320	8597590	−8227101620	1413894771755	−124031731398850
10	56	−144	−11185	−10438670	10490492480	−1979933144970	193210634123311
11	67	−174	−9807	−12450166	13093396333	−2709028151150	293124778727973
12	79	207	6808	14669923	−16061324744	3635467145440	−434786567257064
13	92	243	1774	17162672	−19423381916	4798433274180	−632285283576376
14	106	−282	5745	−20026330	23213797570	−6242490557180	903422424012068
15	121	−324	16235	−23398840	27474114305	−8018190890070	1270440806044980

5.4. $X_{4,2}$

Finally, we turn to the degree $(4, 2)$ complete intersection in $\mathbb{P}^5$. For this model $\kappa = 8$ and there are 15 polar coefficients with one modular constraint. In [31] it was found that the naive Ansatz (4.8) is incompatible with modularity. Using (4.19) and GV invariants up to genus 50, we find that the first terms of the generating function are given by

(5.13)
$$
\begin{aligned}
h_0^{(\mathrm{p})} =\,& \mathrm{q}^{-\frac{8}{3}}\Big(\underline{-6 + 880\mathrm{q} - 60192\mathrm{q}^2 - 780416\mathrm{q}^3 + 23205244196\mathrm{q}^4} \\
&+ 36880172393344\mathrm{q}^5 + 10924546660884800\mathrm{q}^6 \\
&+ 1454816640629235200\mathrm{q}^7 + \cdots\Big), \\
h_1^{(\mathrm{p})} =\,& \mathrm{q}^{-\frac{8}{3}+\frac{9}{16}}\Big(\underline{0 - 5120\mathrm{q} - \tfrac{1}{30}\,\mathrm{PT}(25, -49)\mathrm{q}^2} \\
&+ \tfrac{1}{29}\left(222720 + \mathrm{PT}(25, -48)\right)\mathrm{q}^3 + \cdots\Big), \\
h_2^{(\mathrm{p})} =\,& \mathrm{q}^{-\frac{8}{3}+\frac{1}{4}}\Big(\underline{0 + 0\mathrm{q} + \tfrac{1}{33}\,\mathrm{PT}(26, -53)\mathrm{q}^2} - \tfrac{1}{32}\,\mathrm{PT}(26, -52)\mathrm{q}^3 + \cdots\Big), \\
h_3^{(\mathrm{p})} =\,& \mathrm{q}^{-\frac{8}{3}+\frac{1}{16}}\Big(\underline{0 + 0\mathrm{q} + 7680\mathrm{q}^2} + \tfrac{1}{35}\,\mathrm{PT}(27, -56)\mathrm{q}^3 + \cdots\Big), \\
h_4^{(\mathrm{p})} =\,& \mathrm{q}^{-\frac{8}{3}}\Big(\underline{0 + 12\mathrm{q} - 2112\mathrm{q}^2} - 34689216\mathrm{q}^3 + \tfrac{1}{37}\,\mathrm{PT}(28, -59)\mathrm{q}^4 + \cdots\Big).
\end{aligned}
$$

Although not all polar terms are found in this way, the result (5.13) provides an overdetermined set of coefficients which are compatible with the modular constraint and sufficient to fix uniquely the corresponding modular form. It is found to be

(5.14)
$$
\begin{aligned}
h_\mu = \frac{1}{\eta^{88}} \Bigg[&-\frac{827243 E_6^7}{13060694016} \\
&- \frac{E_4(-71601885840 E_4^9 - 69248772786 E_4^6 E_6^2 + 131750318292 E_4^3 E_6^4 + 14988448525 E_6^6)}{2190387225600} D \\
&+ \frac{(-7850108795 E_4^8 E_6 - 3026319343 E_4^5 E_6^3 + 15844024271 E_4^2 E_6^5)}{30422044800} D^2 \\
&+ \frac{(41784458605 E_4^9 + 14762282727 E_4^6 E_6^2 - 68049440469 E_4^3 E_6^4 - 1016731100 E_6^6)}{19013778000} D^3 \\
&- \frac{4(173171 E_4^7 E_6 + 342266 E_4^4 E_6^3 + 44435 E_4 E_6^5)}{229635} D^4 \\
&+ \frac{16(-93844535 E_4^8 - 89437029 E_4^5 E_6^2 + 93510063 E_4^2 E_6^4)}{132040125} D^5 \Bigg] \vartheta_\mu^{(8)} ,
\end{aligned}
$$

and produces the following expansions

(5.15)
$$
\begin{aligned}
h_0 = q^{-\frac{8}{3}} \Big(&-6 + 880q - 60192q^2 - 780416q^3 + 23205244196q^4 \\
&+ 36880172393344q^5 + 10924546660884800q^6 \\
&+ 1454816640629235200q^7 + \dots \Big), \\
h_1 = q^{-\frac{8}{3}+\frac{9}{16}} \Big(&0 - 5120q + 668160q^2 + 112032256q^3 + 2015342615552q^4 \\
&+ 1027768507417600q^5 + 184583137843579904q^6 \\
&+ 17979440506308718592q^7 + \dots \Big), \\
h_2 = q^{-\frac{8}{3}+\frac{1}{4}} \Big(&0 + 0q + 276864q^2 - 32485376q^3 + 176489687424q^4 \\
&+ 168522803580928q^5 + 39373360484128256q^6 \\
&+ 4527688807584194560q^7 + \dots \Big), \\
h_3 = q^{-\frac{8}{3}+\frac{1}{16}} \Big(&0 + 0q + 7680q^2 - 32203776q^3 + 27746555904q^4 \\
&+ 53778203675136q^5 \\
&+ 15108125739695104q^6 + 1937976067726382592q^7 + \dots \Big),
\end{aligned}
$$

Table 5: GV invariants $\mathrm{GV}_Q^{(g_C(Q)-\delta)}$ for $X_{4,2}$, assuming modularity

Q	g_C	$\delta = 0$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 4$	$\delta = 5$
1	1	0	1280				
2	2	0	0	92288			
3	3	0	0	2560	15655168		
4	4	0	−8	−672	17407072	3883902528	
5	5	0	0	7680	16069888	24834612736	1190923282176
6	6	0	0	276864	12679552	174937485184	23689021709568
7	7	0	7680	591360	−285585152	2016330670592	494602061689344
8	9	15	1520	67208	−8285120	−46434384200	37334304102560
9	10	0	−25600	−2270720	370290688	−4031209095680	1103462757073920
10	12	0	0	1384320	117390080	528559731712	−344741538150784
11	14	0	0	−46080	−160005120	−109083434240	163217721434624
12	16	0	−96	−12096	208486080	49221875968	−145360041245120
13	18	0	0	−61440	−223475712	−160179161088	272915443716096
14	20	0	0	2491776	175162624	1228486889728	−1047846937829632
15	22	0	−56320	−4428800	1220514304	−16165844458240	7742999973263360
16	25	84	7408	286784	−30323216	−231113426452	341194684288608
17	27	0	71680	5002240	−1685727232	22238429571584	−11254527777976576
18	30	0	0	4152960	230535424	2276356656640	−2136509421094912
19	33	0	0	130560	507426816	386536492032	−775389100867584
20	36	0	−240	−23808	626523936	144150871104	−587797370270104
21	39	0	0	161280	632463360	482500187136	−1002354648247296
22	42	0	0	6367872	217984256	3670593912832	−3643185915136000
23	45	0	133120	5255680	−3616804864	46617513355264	−26329088088999936
24	49	180	8240	90016	−91088144	−597217698472	974876677046816
25	52	0	−158720	−4172800	4443311104	−56810684083200	33070947498452480
26	56	0	0	9136512	71061760	5412945197824	−5609947543679488
27	60	0	0	−276480	−1095613440	−818893387776	1871635810564608
28	64	0	−480	−22848	1324638144	242234826816	−1337626038427488
29	68	0	0	−322560	−1278422016	−943519552512	2226739820757504
30	72	0	0	12458880	−289968896	7510718536448	−8087071198417408
31	76	0	−250880	5058560	7256038912	−93671359907840	59478452149884928
32	81	324	−5200	−169696	−165887120	−1137829570120	1982002329031968
33	85	0	286720	−10846720	−8197510144	108071476324864	−70629413377719296
34	90	0	0	16334976	−958134016	9978853510144	−11133687621246976

$$h_4 = \mathrm{q}^{-\frac{8}{3}}\left(0 + \underline{12}\mathrm{q} - \underline{2112}\mathrm{q}^2 - 34689216\mathrm{q}^3 + 10834429824\mathrm{q}^4\right.$$
$$+ 36099879476640\mathrm{q}^5 + 10900431340916352\mathrm{q}^6$$
$$\left. + 1454331023779312896\mathrm{q}^7 + \dots \right).$$

Furthermore, coefficients of q^2, q^3 in h_0, q^2, q^3, q^4 in h_1, up to q^5 in h_2, q^3, q^4, q^5 in h_3, and q^4 in h_4 are also reproduced by (4.19) with $k = k_0 - 1$. Thus, there is overwhelming evidence that (5.14) is correct. While the maximal genus attainable by the standard direct integration method is 50, using modularity we can predict GV invariants close to the Castelnuovo bound to arbitrary genus (see Table 5), and provide sufficiently many boundary conditions to push the direct integration method up to genus 64.

6. Discussion

In this work, we have exploited a triangle of relations between Gopakumar-Vafa invariants $\mathrm{GV}_Q^{(g)}$, which determine the topological string partition function on a CY threefold $\mathfrak{Y}$, Pandharipande-Thomas invariants $\mathrm{PT}(Q, m)$ which count bound states of a single anti-D6-brane with Q D2 and m D0 branes, and D4-D2-D0 invariants $\overline{\Omega}_{r,\mu}(\hat{q}_0)$, which count BPS black holes with r units of D4-brane charge along an ample divisor $\mathcal{D}$, and D2-D0 brane charge determined by μ and $\hat{q}_0$. Mathematically, these invariants count embedded curves, stable pairs and Gieseker-stable coherent sheaves supported on $\mathcal{D}$, respectively. While the relation between GV and PT invariants is standard [11], and relations between GV invariants and D4-D2-D0 indices were first proposed in [15], we used a novel explicit formula (4.19) proven in Appendix A, which applies for one-parameter CY threefolds with $\operatorname{Pic} \mathfrak{Y} = \mathbb{Z}H$ (or more generally, CY threefolds satisfying Assumption (*) in Prop. 2) and for $r = 1$ unit of D4-brane charge. We applied this formula for the 13 CY threefolds of hypergeometric type, for which we have computed GV invariants (and therefore PT invariants) to relatively high genus using the direct integration method of [13].

For most models, we could rigorously compute the first few coefficients in the generating series of Abelian D4-D2-D0 invariants, including both polar and non-polar terms, and find a unique vector-valued modular form which reproduces all of them, providing a striking confirmation of the modularity properties predicted by string theory. These results also provide indirect support for the BMT inequality which is assumed in the derivation of (4.19), in cases where it is not yet known to hold. For $X_{3,3}$, $X_{4,3}$, $X_{3,2,2}$ and $X_{2,2,2,2}$, we could not compute sufficiently many terms rigorously to uniquely fix the vector-valued modular form, but in the first two cases we could determine a unique candidate which agrees with the formula (4.19) for many coefficients, albeit outside the known regime of validity for this formula. For $X_{3,2,2}$ and $X_{2,2,2,2}$, our current knowledge of GV invariants is not sufficient to identify the modular form with sufficient confidence. Conversely, in cases where the vector-valued modular form could be identified, we used these modular predictions to determine GV invariants close to the Castelnuovo bound for arbitrarily high genus. These results provide new boundary conditions for the direct integration method, which in principle allow to reach higher genus than hitherto possible (in practice, some computational limitations need to be overcome in order to reach such high genera). The case of $X_{4,3}$ is particularly noteworthy, as it requires combining information from direct integration, modularity and wall-crossing to go beyond the restrictions imposed by each of these methods separately.

These results raise several natural questions. First, it is intriguing that the naive Ansatz (4.8), which was proposed as an educated guess in [31], so often manages to produce the correct polar terms. As discussed in §4.1, a similar result (4.3) arises by studying the chamber structure of rank 0 DT-invariants in the space of weak stability conditions. Unfortunately, the walls can only be controlled under the very restrictive assumption (4.2) which in practice limits its use to the most polar terms. It would be very interesting to relax the condition (4.2), but this seems to require a stronger bound on ch_3 than provided by the standard BMT inequality. Physically, such a result would give a clear physical origin of the polar coefficients in terms of bound states of D6-$\overline{\mathrm{D6}}$-branes, as proposed in [16]. Instead, the formula (4.12) expresses the spectrum of anti-D6-branes as a sum of bound states of D4-D2-D0 branes and anti-D6-branes with lower D4-D2-D0 brane charge, and does not provide any insight on the micro-structure of D4-D2-D0 bound states by themselves.

A second question is, why on earth should the generating series of Abelian D4-D2-D0 invariants be modular. Of course, physics gives a clear reason, by identifying them with the elliptic genus of the $(0,4)$-superconformal field theory obtained by wrapping an M5-brane on the divisor $\mathcal{D}$. From the mathematics viewpoint however, modularity is still largely mysterious. For non-compact CY threefolds of the form $\mathfrak{Y} = K_S$ where S is a complex projective surface, the generating series of Abelian D4-D2-D0 invariants supported on the divisor S is given by Göttsche's formula for the Euler number of the Hilbert scheme of points on S [91], which is manifestly modular. For K3-fibered CY threefolds, the modularity of vertical D4-D2-D0 indices (counting D4-branes supported on a K3-fiber) can be shown to follow from Göttsche's formula for the Hilbert scheme of points on $K3$ and from the modularity of the generating series of Noether-Lefschetz numbers determined by the fibration [41]. In our generating series (5.5) of Abelian D4-D2-D0 invariants, it is tempting to identify the factor $1/\eta^{\kappa+c_2}$ as coming from the Hilbert scheme of points on the divisor $\mathcal{D}$, and the remainder as the generating series of some Noether-Lefschetz-type numbers taking into account the moduli of the divisor $\mathcal{D} \subset \mathfrak{Y}$ equipped with a line bundle [5]. Eventually, one would hope that modularity can be derived from the existence of an underlying vertex operator algebra acting on the cohomology of the moduli space of semi-stable sheaves, similar to the case of Hilbert scheme of points on surfaces [92].

Third, it would be very interesting to generalize this approach to the case of non-Abelian D4-D2-D0 indices, where the generating series are expected to be mock modular. While the relation between rank 1 and rank

0 DT invariants from [8, Thm 1.2] covers this case, it requires taking the spectral flow parameter to be large enough, with unspecified lower bound. Nonetheless, we expect that the approach in Appendix A can be generalized and used to compute polar coefficients for $r > 1$ as well. The strategy outlined in [31] can then be used to construct a suitable mock modular series (using the generating series of Hurwitz class numbers to cancel the modular anomaly in the $r = 2$ case). In a subsequent work [93], we apply this strategy for the models X_8 and X_{10} at rank $r = 2$ and verify the mock modular properties predicted in [22, 23, 24, 49]. It would also be desirable to generalize this approach to other classes of one-parameter CY threefolds (such as freely acting orbifolds of hypergeometric models, or complete intersections in Grassmannians or Pfaffians), and to the CY threefolds with 2 or more Kähler parameters. In particular, we expect an interesting interplay between the modularity of D4-D2-D0 invariants and the modularity associated to elliptic fibrations [94, 95, 96].

Finally, having found the generating series of Abelian D4-D2-D0 invariants, it is now straightforward to extract the asymptotic growth of the Fourier coefficients, and produce a Rademacher-type series which computes them explicitly [97, 98, 99, 21]. It would be very interesting to reproduce these microstate degeneracies from localization in supergravity, in analogy to cases with $\mathcal{N} = 8$ or $\mathcal{N} = 4$ supersymmetry [100, 101, 102, 103, 104] (see [105, 106, 107] for some progress in this direction). We hope to return to these issues in near future.

Appendix A. New explicit formulae, by S. Feyzbakhsh

Let $(\mathfrak{Y}, H)$ be a smooth polarised Calabi-Yau threefold (i.e. $K_{\mathfrak{Y}} \cong \mathcal{O}_{\mathfrak{Y}}$ and $H^1(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) = 0$) with $\mathrm{Pic}(\mathfrak{Y}) = \mathbb{Z}.H$ satisfying the BMT conjecture. In §A.1, we improve the result of [8, Theorem 1.2] for rank zero classes with minimal D4-brane charge $\mathrm{ch}_1 = H$ and obtain, under some assumptions, an explicit formula for the stable pair invariants $\mathrm{PT}_{m,\beta}$ in terms of rank zero DT invariants and stable pair invariants $\mathrm{PT}_{m',\beta'}$ for $\beta'.H < \beta.H$. In §A.2, we explain how this result can be inverted to determine rank 0 DT invariants with minimal D4-brane charge from stable pair invariants. In §A.3, we apply the wall-crossing argument for Theorem 1 to establish a Castelnuovo-type bound for PT invariants, and compute the PT invariants saturating this bound explicitly for the quintic threefold. In §A.4, we extend Theorem 1 to a special case where non-primitive wall-crossing occurs. In §A.5, we state a generalisation of Theorem 1 under the weaker assumption ($\star$) that H^3 divides $H'.H^2$ for all $H' \in \mathrm{Pic}(\mathfrak{Y})$. Finally, in §A.6, we strengthen [8, Theorem 1.1] under assumption ($\star$). The proofs of these results are collected in §A.7.

Notation: In this section, we label the charges by the Chern character (rather than the Mukai vector), so for instance $\overline{\Omega}_H(\mathsf{v})$ counts H-Gieseker semi-stable sheaves of Chern character v. Furthermore, for readability we decompose v into its components in $H^0(\mathfrak{Y}, \mathbb{Z})$, $H^2(\mathfrak{Y}, \mathbb{Z})$, $H^4(\mathfrak{Y}, \mathbb{Q})$ and $H^6(\mathfrak{Y}, \mathbb{Q})$. For example, $\mathsf{v} = (-1, 0, \beta, -m)$ stands for $\mathsf{v} = -1 + \beta - m$. We also use the notation $\mathrm{ch}_{\leq 2} E$ for the projection of $\mathrm{ch}\, E$ on

$$H^0(\mathfrak{Y}, \mathbb{Z}) \oplus H^2(\mathfrak{Y}, \mathbb{Z}) \oplus H^4(\mathfrak{Y}, \mathbb{Q}).$$

A.1. Stable pair invariants from rank 0 DT invariants with minimal D4-brane charge

For any $\beta \in H_2(\mathfrak{Y}, \mathbb{Z})$ (which can also be regarded as a class in $H^4(\mathfrak{Y}, \mathbb{Z})$ by Poincaré duality), we define the integer

$$(A.1) \qquad \mathcal{C}(\beta) \;:=\; \begin{cases} \left\lfloor \frac{2}{3H^3}(\beta.H)^2 + \frac{\beta.H}{3} \right\rfloor & \text{if } \beta.H < H^3, \\[2mm] \left\lfloor \frac{1}{2H^3}(\beta.H)^2 + \frac{\beta.H}{2} \right\rfloor & \text{if } H^3 \leq \beta.H, \end{cases}$$

which determines the Castelnuovo bound as we explain below. Consider the function

$$(A.2) \qquad f(x) \;:=\; \begin{cases} x + \frac{1}{2} & \text{if } 0 < x < 1, \\[1mm] \sqrt{2x + \frac{1}{4}} & \text{if } 1 < x < \frac{15}{8}, \\[1mm] \frac{2}{3}x + \frac{3}{4} & \text{if } \frac{15}{8} \leq x < \frac{9}{4}, \\[1mm] \frac{1}{3}x + \frac{3}{2} & \text{if } \frac{9}{4} \leq x < 3, \\[1mm] \frac{1}{2}x + 1 & \text{if } 3 \leq x. \end{cases}$$

Theorem 1. *Fix* $m \in \mathbb{Z}$ *and* $\beta \in H_2(\mathfrak{Y}, \mathbb{Z})$ *such that* $\beta.H > 0$ *and*

$$(A.3) \qquad f\left(\frac{\beta.H}{H^3} \right) < -\frac{3m}{2\beta.H}\,.$$

Then

$$(A.4) \qquad \begin{aligned} \mathrm{P}_{m,\beta} = \sum_{(m', \beta') \in M_{m,\beta}} & (-1)^{\chi_{m',\beta'}} \chi_{m',\beta'}\; \mathrm{P}_{m',\beta'} \\ & \times \overline{\Omega}_H\left(0,\; H,\; \frac{1}{2}H^2 - \beta' + \beta\,,\; \frac{1}{6}H^3 + m' - m - \beta'.H \right), \end{aligned}$$

where

$$(A.5) \qquad \chi_{m',\beta'} = \beta.H + \beta'.H + m - m' - \frac{H^3}{6} - \frac{1}{12}c_2(\mathfrak{Y}).H \,.$$

The set $M_{m,\beta}$ consists of pairs $(m', \beta') \in H_0(\mathfrak{Y}, \mathbb{Z}) \oplus H_2(\mathfrak{Y}, \mathbb{Z})$ such that

$$(A.6) \qquad 0 \le \beta'.H \le \frac{H^3}{2} + \frac{3mH^3}{2\beta.H} + \beta.H$$

and

$$(A.7) \qquad -\mathcal{C}(\beta') \le \ m' \ \le \frac{1}{2H^3}(\beta.H - \beta'.H)^2 + \frac{1}{2}(\beta.H + \beta'.H) + m.$$

Since $f(x) > \frac{1}{2}$, (A.3) implies that $\frac{1}{2} < -\frac{3m}{2\beta.H}$, thus $\beta'.H < \beta.H$ in (A.6).

Remark 1. *Here are three comments regarding Theorem 1: a) to prove Theorem 1, we only need a weaker version of BMT conjecture explained in Remark 3, b) one can strengthen Theorem 1 for specific CY threefolds (e.g. quintic threefolds) where a stronger version of Bogomolov-Gieseker inequality holds, see Remark 4 for more details, and c) a generalisation of Theorem 1, when $\mathfrak{Y}$ is not of Picard rank one but satisfies assumption $(\star)$, is proved in Proposition 2.*

For any $(m, \beta) \in H_0(\mathfrak{Y}, \mathbb{Z}) \oplus H_2(\mathfrak{Y}, \mathbb{Z})$, consider the function $\Psi(x, x', \alpha)$ defined in (4.24) for $x = \frac{\beta.H}{H^3}$ and $\alpha = -\frac{3m}{2\beta.H}$. We define the function $\Psi_{m,\beta} \colon \mathbb{R} \to \mathbb{R}$ as

$$\Psi_{m,\beta}(x') := \Psi\left(\frac{\beta.H}{H^3}, \ x', \ -\frac{3m}{2\beta.H}\right)$$

$$(A.8) \qquad = \frac{1}{2}\left(\frac{\beta.H}{H^3} - x'\right)^2 + \frac{1}{2}\left(\frac{\beta.H}{H^3} + x'\right) + \frac{m}{H^3} + \frac{1}{2}x'^2 + \frac{1}{2}x' \,.$$

Note that $\Psi_{m,\beta}(x')$ is the difference of the right and the left hand side of (A.7), up to a factor of H^3, for $x' = \beta'.H/H^3$.

Definition 1. *A value of $(m, \beta) \in H_0(\mathfrak{Y}, \mathbb{Z}) \oplus H_2(\mathfrak{Y}, \mathbb{Z})$ is called optimal if*

- *the inequality (A.3) is satisfied, and*
- *$\Psi_{m,\beta}(\frac{1}{H^3}) < 0$ and $\Psi_{m,\beta}\left(\frac{1}{2} + \frac{3m}{2\beta.H} + \frac{\beta.H}{H^3}\right) < 0$ (i.e. $\Psi_{m,\beta}$ is negative for all possible positive values of $\beta'.H$ in (A.6)).*

For an optimal value of (m, β), the only possible value of β' in the wall-crossing formula (A.4) is $\beta' = 0$. Since $\mathrm{P}_{m',0} = 1$ if $m' = 0$ and otherwise vanishes, we get the following:

Corollary 1. *For any optimal value* $(m, \beta) \in H_0(\mathfrak{Y}, \mathbb{Z}) \oplus H_2(\mathfrak{Y}, \mathbb{Z})$, *we have*

$$(A.9) \qquad \mathrm{P}_{m,\beta} = (-1)^{1+\chi(\mathcal{O}_{\mathfrak{Y}}(H), \mathsf{w})} \chi(\mathcal{O}_{\mathfrak{Y}}(H), \mathsf{w}) \, \overline{\Omega}_H(\mathsf{w}),$$

where $\mathsf{w} = \left(0, H, \beta + \frac{H^2}{2}, -m + \frac{H^3}{6}\right).$

A.2. Application I: An explicit formula for minimal rank zero DT invariants

Corollary 1 gives us a way to write arbitrary minimal rank zero DT invariants in terms of PT invariants. Pick $\beta_0 \in H^4(\mathfrak{Y}, \mathbb{Q})$ and $m_0 \in \mathbb{Z}$. After tensoring by the line bundle $\mathcal{O}_{\mathfrak{Y}}(kH)$ with $k \in \mathbb{Z}$, the rank 0 DT invariant is unchanged,

$$(A.10) \quad \overline{\Omega}_H(0, H, \beta_0, m_0) = \overline{\Omega}_H\left(0, H, \beta_0 + kH^2, m_0 + k\beta_0.H + \frac{k^2}{2}H^3\right).$$

Then as a consequence of Corollary 1, one gets the following.

Corollary 2. *There exists* $k(\beta_0, m_0) \in \mathbb{Z}_{>0}$ *such that for any integer* $k \geq k(\beta_0, m_0)$, *the class* (m, β) *for*

$$(A.11) \qquad \beta := \beta_0 + kH^2 - \frac{H^2}{2} \, , \qquad m := \frac{H^3}{6} - m_0 - k\beta_0.H - \frac{k^2}{2}H^3$$

is optimal (see Definition 1), thus

$$(A.12) \qquad \overline{\Omega}_H(\mathsf{w}) = (-1)^{1+\chi\left(\mathcal{O}_{\mathfrak{Y}}((1-k)H), \mathsf{w}\right)} \frac{\mathrm{P}_{m,\beta}}{\chi\left(\mathcal{O}_{\mathfrak{Y}}((1-k)H), \, \mathsf{w}\right)} \, ,$$

where $\mathsf{w} = (0, H, \beta_0, m_0).$

A.3. Application II: Castelnuovo bound for PT invariants

As a result of wall-crossing for rank -1 classes and induction on $\beta.H$, we can prove the following Castelnuovo-type bound for stable pairs.

Theorem 2. *Fix $\beta \in H_2(\mathfrak{Y}, \mathbb{Z})$ and $m \in \mathbb{Z}$. Let $\mathcal{O}_{\mathfrak{Y}} \xrightarrow{s} F$ be a stable pair such that $\mathrm{ch}_2(F) = \beta$ and $\mathrm{ch}_3(F) = m$, then*

$$(A.13) \qquad\qquad m \geq -\mathcal{C}(\beta).$$

In particular, the invariant $\mathrm{P}_{m,\beta}$ vanishes unless the inequality (A.13) is satisfied.

The nature of the proof of Theorem 2 is similar to [72, Proposition 1.3] where the same result is proved under the extra assumption that for any $E \in \mathrm{Coh}(\mathfrak{Y})$, we have $\mathrm{ch}_2(E) \in \frac{H^2}{2}\mathbb{Z}$ and $\mathrm{ch}_3(E) \in \frac{H^3}{6}\mathbb{Z}$. Theorem 2 for quintic threefolds, as well as the first part of Theorem 3 below, is also proved in the recent paper [73] via different arguments.

Remark 2. *Note that when m is close to the Castelnuovo bound $-\mathcal{C}(\beta)$, then one can apply Theorem 1 to find an explicit formula for $\mathrm{P}_{m,\beta}$ in terms of rank zero invariants $\overline{\Omega}_H(\mathsf{w})$ with $\mathrm{ch}_1(\mathsf{w}) = H$. Since $\overline{\Omega}_H(\mathsf{w}) = \overline{\Omega}_H(\mathsf{w} \otimes \mathcal{O}_{\mathfrak{Y}}(kH))$ for any $k \in \mathbb{Z}$, the knowledge of a few rank zero DT invariants determines PT invariants along the boundary of Castelnuovo bound.*

Let us spell out Remark 2 for the case of the quintic threefold X_5. Based on physical arguments in §5.2, the following vanishings are expected (here $\kappa = H^3 = 5$):

Conjecture 1. *(i) $\overline{\Omega}_H\left(0, H, \pm(\frac{1}{2} + \frac{1}{\kappa})H^2, \frac{H^3}{6}\right) = 0$,*

(ii) $\overline{\Omega}_H\left(0, H, \pm(\frac{1}{2} + \frac{2}{\kappa})H^2, (-\frac{m}{\kappa} + \frac{1}{6})H^3\right) = 0$ for $m = 0, -1$.

Theorem 3. *Take an integer $\mu \geq 13$ or $\mu = 10$. If $\mu \overset{\kappa}{\equiv} 0$ and $m = -\mathcal{C}(\frac{\mu}{\kappa}H^2)$, then*

$$(A.14) \qquad\qquad \mathrm{P}_{m, \frac{\mu}{\kappa}H^2} = (-1)^{m+\mu}(5 - m - \mu) \times 5.$$

If $\mu \overset{\kappa}{\equiv} \pm p$ where $p = 1$ (resp. 2) and Conjecture 1(i) (resp. Conjecture 1(ii)) holds, then $\mathrm{P}_{m, \frac{\mu}{\kappa}H^2}$ vanishes if $m < -\mathcal{C}(\frac{\mu}{\kappa}H^2) + p$; moreover, for $m = -\mathcal{C}(\frac{\mu}{\kappa}H^2) + p$, then

$$(A.15) \qquad\qquad \mathrm{P}_{m, \frac{\mu}{\kappa}H^2} = (-1)^{1+\chi(\mathcal{O}_{\mathfrak{Y}}(H),\mathsf{w})}\chi(\mathcal{O}_{\mathfrak{Y}}(H), \mathsf{w})\, \overline{\Omega}_H(\mathsf{w}),$$

where $\mathsf{w} = \left(0, H, (\frac{1}{2} + \frac{\mu}{\kappa})H^2, (-\frac{m}{\kappa} + \frac{1}{6})H^3\right)$.

A.4. An example with non-primitive wall-crossing

In this part, we generalize Theorem 1 for some examples of pairs (m, β) where the inequality (A.3) is saturated. Similar results can be obtained when the inequality is mildly violated.

Proposition 1. *If $\beta.H > 4H^3$ and $\alpha = -\frac{3m}{2\beta.H} = f(x)$, where (x, α) are defined in (4.11), then Theorem 1 holds true. However if $\beta.H = 4H^3$ and $m = -8H^3$, then there is an additional wall with non-primitive wall-crossing, leading to*

$$
\begin{aligned}
\mathrm{P}_{m,\beta} = &\sum_{(m', \beta') \in M_{m,\beta}} (-1)^{\chi_{m',\beta'}} \chi_{m',\beta'} \mathrm{P}_{m',\beta'} \\
&\times \overline{\Omega}_H \left(0, \ H, \ \frac{1}{2}H^2 - \beta' + \beta, \ \frac{1}{6}H^3 + m' - m - \beta'.H \right) \\
&+ \frac{1}{2} \left(\chi(\mathcal{O}_{\mathfrak{Y}}, \mathcal{O}_{\mathfrak{Y}}(2H)) \right)^2 - \frac{1}{2} \chi(\mathcal{O}_{\mathfrak{Y}}, \mathcal{O}_{\mathfrak{Y}}(2H)) ,
\end{aligned}
$$
(A.16)

where we recall that $\chi(\mathcal{O}_{\mathfrak{Y}}, \mathcal{O}_{\mathfrak{Y}}(2H)) = \chi_{2\mathcal{D}} = \frac{4}{3}\kappa + \frac{1}{6}c_2$.

A.5. Relaxing the Picard rank one assumption

In this subsection, we relax the Picard rank one assumption to

Assumption ($\star$) H^3 divides $H'.H^2$ for all $H' \in \mathrm{Pic}(\mathfrak{Y})$.

The following proposition generalises Theorem 1.

Proposition 2. *Let $(\mathfrak{Y}, H)$ be a smooth polarised Calabi-Yau threefold $\mathfrak{Y}$ satisfying the BMT conjecture and assumption ($\star$). Fix $m \in \mathbb{Z}$ and $\beta \in H_2(\mathfrak{Y}, \mathbb{Z})$ such that the condition (A.3) is satisfied. Then*

$$
\begin{aligned}
\mathrm{P}_{m,\beta} = &\sum_{(H', m', \beta') \in \widetilde{M}_{m,\beta}} (-1)^{\chi_{H',m',\beta'}} \chi_{H',m',\beta'} \ \mathrm{P}_{m',\beta'} \\
&\times \overline{\Omega}_H \left(0, \ H', \ \frac{1}{2}H'^2 - \beta' + \beta, \ \frac{1}{6}H'^3 + m' - m - \beta'.H' \right),
\end{aligned}
$$
(A.17)

where

$$
\chi_{H',m',\beta'} = \beta.H' + \beta'.H' + m - m' - \frac{H'^3}{6} - \frac{1}{12}c_2(\mathfrak{Y}).H' ,
$$
(A.18)

and the set $\widetilde{M}_{m,\beta}$ consists of triples $(H', m', \beta') \in H^2(X, \mathbb{Q}) \oplus H_0(\mathfrak{Y}, \mathbb{Z}) \oplus H_2(\mathfrak{Y}, \mathbb{Z})$ such that

(i) $H'.H^2 = H^3$,

(ii) $0 \leq \beta'.H \leq \frac{H'^2.H}{2} + \frac{3mH^3}{2\beta.H} + \beta.H$,

(iii) $m' \geq -\frac{2}{3}\beta'.H\left(\frac{\beta'.H}{H^3} + \frac{1}{2}\right)$, and

(iv) $m' \leq \frac{1}{2H^3}\left(\frac{1}{2}H'^2 H - \beta'.H + \beta.H\right)^2 + \frac{H^3}{24} - \frac{H'^3}{6} + m + \beta'.H'.$

A.6. Rank 0 DT invariants from rank one DT and PT invariants

Finally, we provide a strengthening of the wall-crossing formula [8, Theorem 1.1] for rank zero classes, which holds for CY threefolds satisfying the BMT inequality and assumption $(\star)$.

Given a polarization $H \in H^2(\mathfrak{Y}, \mathbb{Z})$ and a rank 0 Chern class[21]

$$(A.19) \qquad \mathsf{v} = [0, D, \beta, -m] \in H^2(\mathfrak{Y}, \mathbb{Q}) \oplus H^4(\mathfrak{Y}, \mathbb{Q}) \oplus H^6(\mathfrak{Y}, \mathbb{Q}),$$

with $D \neq 0$, let us define

$$(A.20) \qquad Q_H(\mathsf{v}) = \frac{1}{2}\left(\frac{D.H^2}{H^3}\right)^2 + 6\left(\frac{\beta.H}{D.H^2}\right)^2 + \frac{12m}{D.H^2}.$$

Theorem 4. *Let $(\mathfrak{Y}, H)$ be a smooth polarised CY threefold $\mathfrak{Y}$ satisfying the BMT conjecture and assumption $(\star)$. If a rank zero class $\mathsf{v} = [0, D, \beta, -m]$ satisfies $Q_H(\mathsf{v}) < \min\{\frac{r^2}{2} - \frac{1}{8}, r - \frac{1}{2}\}$ for $r = \frac{D.H^2}{H^3}$ then*

$$(A.21)$$
$$\overline{\Omega}_H(\mathsf{v})$$
$$= \left(\#H^2(X, \mathbb{Z})_{\text{tors}}\right)^2 \sum_{\substack{\mathsf{v}_1 = -e^{D_1}(1,0,-\beta_1,-m_1) \\ \mathsf{v}_2 = e^{D_2}(1,0,-\beta_2,-m_2) \\ \mathsf{v}_1 + \mathsf{v}_2 = \mathsf{v} \\ (D_i,\ \beta_i,\ m_i) \in M_i(\mathsf{v})}} (-1)^{\chi(\mathsf{v}_2,\mathsf{v}_1)-1}\, \chi(\mathsf{v}_2,\mathsf{v}_1)\, \mathrm{P}_{-m_1,\beta_1}\, \mathrm{I}_{m_2,\beta_2}.$$

Here $M_i(\mathsf{v})$ for $i = 1, 2$ is the set all classes $(D_i, \beta_i, m_i) \in H^2(X, \mathbb{Q}) \oplus H^4(X, \mathbb{Q}) \oplus H^6(X, \mathbb{Q})$ such that

$$(A.22) \quad \left|\frac{D_i.H}{H^3} - \theta_i\right| < 1 \qquad for \qquad \theta_i := \frac{\beta.H}{D.H^2} + (-1)^i\sqrt{\frac{r^2}{4} - \frac{1}{2}Q_H(\mathsf{v})},$$

---

[21]Note that the sign of m is flipped compared to [8, Theorem 1.1].

(A.23)
$$-\frac{D_i^2.H}{2H^3} + \frac{\beta_i.H}{H^3} + \frac{\beta.H}{D.H^2}\frac{D_i.H^2}{H^3} \leq -\frac{1}{8}\left(\frac{D.H^2}{H^3}\right) + \frac{1}{2}\left(\frac{\beta.H}{D.H^2}\right)^2 + \frac{1}{4}Q_H(\mathsf{v}),$$

and

(A.24)
$$(-1)^{i+1}m_i \leq \frac{2}{3}\beta_i.H\left(\frac{\beta_i.H}{H^3} + \frac{1}{2}\right).$$

Note that if $Q_H(\mathsf{v})$ is as small as required in [8, Theorem 1.1], then one can apply [8, Proposition 3.5 & 3.6] to show that the above sets $M_i(\mathsf{v})$ reduce to $M(\mathsf{v})$ so that the final wall-crossing formulae agree.

A.7. Proofs

In this subsection, we collect the proofs of the various claims up to now.

To prove Theorem 1, we use weak stability conditions $\nu_{b,w}$ for $(b,w) \in U$ where $U = \{(b,w) \in \mathbb{R}^2 : w > \frac{b^2}{2}\}$, see Fig. 2 and [6, §1] for all details. We study wall-crossing for the fixed class

(A.25)
$$\mathsf{v} = (-1, 0, \beta, -m) \in \bigoplus_{i=0}^3 H^{2i}(\mathfrak{Y}, \mathbb{Q}).$$

We start in the large volume limit $b > 0$ and $w \gg 1$ where an object $E \in \mathcal{A}_b$ of class v is $\nu_{b,w}$-semistable if and only if $E^\vee \otimes \det(E)^{-1}[1]$ is a stable pair, see [4, §3]. Then we move down and investigate all walls of instability for objects of class v.

By the conjectural BMT inequality (2.38), if there is a $\nu_{b,w}$-semistable object in $\mathcal{A}_b$ of class v, then $L_{b,w}(\mathsf{v}) = w(2H^3\beta.H) + 3b(-H^3)(-m) + 2(\beta.H)^2 \geq 0$, i.e.

(A.26)
$$w \geq -b\frac{3m}{2\beta.H} - \frac{\beta.H}{H^3}.$$

Hence any wall for class v lies above or on the line ℓ_f of equation $w = \alpha b - x$ where

(A.27)
$$\alpha = -\frac{3m}{2\beta.H} \qquad \text{and} \qquad x = \frac{\beta.H}{H^3}.$$

Let $b_1 < b_2$ be the values of b at the intersection points of the line ℓ_f with the boundary ∂U.

Assume there is a wall ℓ for class v. Then there is an object $E \in \mathcal{A}_b$ of class v which is strictly $\nu_{b,w}$-semistable for all $(b,w) \in \ell \cap U$. Let $E' \to E \to E''$ be a destabilising sequence along the wall ℓ.

Lemma 1. *Suppose*

(i) $0 < b_1 < 2$,

(ii) $1 < b_2 - b_1$, and

(iii) if $b_2 r \le c \le b_1(r+1)$ for some $(r,c) \in \mathbb{Z}_{\ge 0} \oplus \mathbb{Z}_{\ge 0}$, then either $(r,c) = (0,1)$ or $(0,0)$.

Then $b_1 \ge 1$ and there is an ordering E_0, E_1 of E', E'' such that

- *E_0 is a rank zero Gieseker-stable sheaf with $\mathrm{ch}_1(E_0) = H$.*
- *E_1 is a rank -1 object and $\nu_{b,w}$-stable for $b > -1$ and $w \gg 1$.*

Moreover, there is no $\nu_{b,w}$-semistable object of class v for $(b,w) \in U$ below ℓ_f.

Proof. Since $\mathrm{rk}(E) = -1$, one of the objects E', E'' has rank < 0; call it E_1. The other E_0 has rank ≥ 0. Let

$$(A.28) \qquad \mathrm{ch}_{\le 1}(E_0) = (r, cH) \qquad \text{and} \qquad \mathrm{ch}_{\le 1}(E_1) = (-1-r, -cH).$$

By [6, Remark 1.2] for any point $(b,w) \in \ell \cap U$, we have $\mathrm{ch}_1(E_i)H^2 - b\,\mathrm{ch}_0(E_i)H^3 \ge 0$ for $i = 1, 2$. Since ℓ lies above ℓ_f, this in particular holds for $b = b_1, b_2$ which implies

$$(A.29) \qquad\qquad b_2 r \le c \le b_1(1+r).$$

By assumption (iii), it follows that either $(r,c) = (0,0)$ or $(r,c) = (0,1)$. In the first case, E_0 is of $\nu_{b,w}$-slope $+\infty$ for all $(b,w) \in U$, thus it cannot have the same $\nu_{b,w}$-slope as E along the wall ℓ. Thus $(r,c) = (0,1)$ and $1 \le b_1$.

Hence E_1 is a rank -1 object with $\mathrm{ch}_1(E_1) = -H$. By [108, Lemma 3.5], there is no wall for E_1 crossing the vertical line $b = 2$, so if E_1 is $\nu_{b,w}$-semistable for some $b = 2$ and $w > 2$, then it is $\nu_{b,w}$-stable for $b = 2$ and any $w > 2$. By conditions (i) and (ii), the wall ℓ_f and so the wall ℓ intersects the vertical line $b = 2$ at a point inside U, thus $\nu_{b,w}$-semistability of E_1 along the wall implies that E_1 is $\nu_{b,w}$-stable for $b = 2$ and $w \gg 1$. Then the wall and chamber structure for the class $\mathrm{ch}(E_1)$ implies that E_1 is $\nu_{b,w}$-stable for any $b > -1$ and $w \gg 1$.

We claim that there is no wall for E_0 when we move up from the wall ℓ to the large volume limit. Suppose for a contradiction that there was a wall ℓ_0

with the destabilising sequence $F_1 \hookrightarrow E_0 \twoheadrightarrow F_2$. Set $r_i := \text{rk}\big(\mathcal{H}^{-1}(F_i)\big) \geq 0$. By definition of the heart $\mathcal{A}_b$, for $(b, w) \in \ell_0 \cap U$,

(A.30)
$$\text{ch}_1\big(\mathcal{H}^{-1}(F_i)\big).H^2 \leq b\, r_i H^3 \quad \text{and} \quad \text{ch}_1\big(\mathcal{H}^0(F_i)\big).H^2 \geq b\,(\text{ch}_0(F_i) + r_i)H^3.$$

Since ℓ_0 lies above or on ℓ_f, we may apply the left hand inequality for $b = b_1$ and the right hand for $b = b_2$. Then subtracting gives $\text{ch}_1(F_i).H^2 \geq b_2 \text{ch}_0(F_i)H^3 + (b_2 - b_1)r_i H^3$. Adding over $i = 0, 1$ gives

(A.31) $$H^3 = \text{ch}_1(E).H^2 \geq (b_2 - b_1)(r_0 + r_1)H^3.$$

Since $b_2 - b_1 > 1$ by assumption (ii), we get $r_1 = r_2 = 0$. Thus $\mathcal{H}^{-1}(F_i) = 0$ as they are torsion-free sheaves by definition of the heart $\mathcal{A}_b$. Thus F_i's are sheaves with $\text{ch}_0(F_1) = \text{ch}_0(F_2) = 0$. Hence they have the same $\nu_{b,w}$-slope as E_0 with respect to any $(b, w) \in U$, so they cannot induce a wall. Thus E_0 is $\nu_{b,w}$-semistable for $w \gg 1$, hence is a torsion sheaf by [10, Lemma 2.7(c)].

By definition of the heart $\mathcal{A}_b$, any torsion sheaf F lies in $\mathcal{A}_b$ and $\nu_{b,w}(F) = \frac{\text{ch}_2(F).H}{\text{ch}_1(F).H^2}$ if $\text{ch}_1(F) \neq 0$, otherwise $\nu_{b,w}(F) = +\infty$. In our case, since $\text{ch}_1(E_0).H^2$
$= H^3$ is minimal, the $\nu_{b,w \gg 1}$-semistability of E_0 immediately implies that E_0 is a Gieseker-stable sheaf. $\qquad\square$

Recall that the equation of ℓ_f is $w = \alpha b - x$ for the values of α, x in (A.27). The b-values of the intersection point of ℓ_f with ∂U, which is the parabola of equation $w = \frac{1}{2}b^2$, are

(A.32) $$b_1 = \alpha - \sqrt{\alpha^2 - 2x}, \quad b_2 = \alpha + \sqrt{\alpha^2 - 2x}\,.$$

The condition (i) that $0 < b_1 < 2$ is equivalent to

(A.33) $$\alpha - \sqrt{\alpha^2 - 2x} < 2 \quad \text{i.e.} \quad 0 < \alpha < 2 \quad \text{or} \quad 1 + \frac{x}{2} < \alpha\,.$$

Also the condition (ii) that $b_2 - b_1 > 1$ is equivalent to

(A.34) $$2\sqrt{\alpha^2 - 2x} > 1 \quad \text{i.e.} \quad \sqrt{2x + \frac{1}{4}} < \alpha\,.$$

Hence, a simple case by case analysis verifies the following:

Lemma 2. *Consider the function $f(x)$ defined in (A.2). If $x > 1$ and $\alpha > f(x)$, then the conditions (i), (ii) and (iii) in Lemma 1 hold.*

By Lemma 1, the destabilising objects are of Chern character
(A.35)
$$\mathsf{v}^0_{m',\beta'} := \mathrm{ch}(E_0) = \left(0, \ H, \ \frac{1}{2}H^2 - \beta' + \beta, \ \frac{1}{6}H^3 + m' - m - \beta'.H \right)$$

and

$$(A.36) \quad \mathsf{v}^1_{m',\beta'} := \mathrm{ch}(E_1) = \left(-1, \ -H, \ -\frac{H^2}{2} + \beta', \ -\frac{1}{6}H^3 - m' + \beta'.H \right).$$

We know that the point $\varpi(\mathrm{ch}(E_1)) = (1, \ -\frac{\beta'.H}{H^3} + \frac{1}{2})$ (defined in (2.36)) lies above or on ℓ_f, so

$$(A.37) \qquad \frac{\beta'.H}{H^3} \leq \frac{3m}{2\beta.H} + \frac{\beta.H}{H^3} + \frac{1}{2}.$$

Moreover, applying [6, Lemma B.3] for E_0 implies

$$(A.38) \quad \frac{1}{6}H^3 + m' - m - \beta'.H \leq \frac{1}{2H^3}\left(\frac{1}{2}H^3 - \beta'.H + \beta.H \right)^2 + \frac{H^3}{24}$$

which is equivalent to

$$(A.39) \qquad m' \leq \frac{1}{2H^3}(\beta.H - \beta'.H)^2 + \frac{1}{2}(\beta.H + \beta'.H) + m.$$

Proof of Theorem 2. Suppose $\mathrm{P}_{m,\beta} \neq 0$, i.e. there is a $\nu_{b,w}$-stable object of class $\mathsf{v} = (-1, 0, \beta, -m)$ for $b > 0$ and $w \gg 1$. By [108, Lemma 3.5], there is no wall for class v crossing the vertical line $b = 1$. Then the conjectural BMT inequality (2.38) at the boundary point $(b, w) = (1, \frac{1}{2})$ implies that

$$(A.40) \qquad -\frac{m}{H^3} \leq \frac{2}{3}\left(\frac{\beta.H}{H^3} \right)^2 + \frac{\beta.H}{3H^3}$$

which proves the claim (A.13) if $\frac{\beta.H}{H^3} \leq 1$. Hence we may assume $\beta.H^3 > H^3$. If (A.13) does not hold, then

$$(A.41) \qquad -\frac{3m}{2\beta.H} > \frac{3\beta.H}{4H^3} + \frac{3}{4} > f\left(\frac{\beta.H}{H^3} \right).$$

Then by combining Lemma 1 and Lemma 2, it follows that as we move down from the large volume limit, any large volume limit stable object

E of class v gets destabilised along a wall with the destabilising objects E_0 and E_1 as described in Lemma 1. By [10, Corollary 3.10], we know that the discriminants Δ_H of the destabilising factors are less than $\Delta_H(\mathsf{v})$, so $\beta'.H < \beta.H$. Thus by applying induction on $\beta.H$, we may assume the object E_1, which is large volume limit stable of rank -1, satisfies the claim. Combining it with (A.39) implies that

$$(A.42)$$
$$-\frac{1}{2H^3}(\beta'.H)^2 - \frac{\beta'.H}{2} \leq m' \leq \frac{1}{2H^3}(\beta.H - \beta'.H)^2 + \frac{1}{2}(\beta.H + \beta'.H) + m.$$

This in particular implies that

$$0 \leq \frac{1}{2H^3}(\beta.H - \beta'.H)^2 + \frac{1}{2}(\beta.H + \beta'.H) + \frac{1}{2H^3}(\beta'.H)^2 + \frac{\beta'.H}{2} + m.$$

If the claim (A.13) does not hold, then

$$0 < \frac{(\beta.H - \beta'.H)^2}{2H^3} + \frac{1}{2}(\beta.H + \beta'.H) + \frac{(\beta'.H)^2}{2H^3} + \frac{\beta'.H}{2} - \frac{(\beta.H)^2}{2H^3} - \frac{1}{2}\beta.H$$
$$= \beta'.H\left(\frac{\beta'.H}{H^3} - \frac{\beta.H}{H^3} + 1\right).$$

Since $\beta'.H \geq 0$, we get $\beta.H - H^3 < \beta'.H$, then (A.37) gives $\beta.H - H^3 < \frac{3mH^3}{2\beta.H} + \beta.H + \frac{H^3}{2}$, i.e. $-\frac{3m}{2\beta.H} < \frac{3}{2}$ which is not possible by (A.41) for $\beta.H > H^3$. $\qquad\square$

Proof of Theorem 1. By [4, §3], any $\nu_{b,w}$-semistable object $E \in \mathcal{A}_b$ of class v for $b > 0$ and $w \gg 1$ is derived-dual of a stable pair up to tensoring by a line bundle with torsion c_1, so

$$(A.43) \qquad \overline{\Omega}_\infty(\mathsf{v}) := \overline{\Omega}_{b>0,\ w\to+\infty}(\mathsf{v}) = \left(\#H^2(\mathfrak{Y}, \mathbb{Z})_{\mathrm{tors}}\right)\mathrm{P}_{m,\beta}.$$

On the other hand, we know that there is no $\nu_{b,w}$-semistable object of class v when $(b,w) \in U$ lies below ℓ_f, so $\overline{\Omega}_{b,w}(\mathsf{v}) = 0$. Between, this point and large volume limit, there are several walls ℓ.

If $0 < x \leq 1$ and $\alpha > f(x)$, then $b_1 < 1$, implying $\mathrm{PT}_{m,\beta} = 0$ and no non-trivial wall exists, so the claim follows. Therefore, we may assume $x > 1$. Subsequently, by applying Lemma 2, we can utilize Lemma 1, which describes the destabilizing factors along any wall ℓ. We know that the first factor E_0 is Gieseker-stable, and any Gieseker-stable sheaf of class $\mathrm{ch}(E_0)$ is $\nu_{b,w}$-stable for $(b,w) \in \ell$ as there is no wall for $\mathrm{ch}(E_0)$ between the large

volume limit and ℓ. Thus for points $(b, w^{\pm})$ above and below the wall ℓ, we have

$$(A.44) \qquad \overline{\Omega}_{b,w^{\pm}}(\mathsf{v}^0_{m',\beta'}) = \overline{\Omega}_H(\mathsf{v}^0_{m',\beta'}).$$

We also know that there is no wall for the other factor E_1 between ℓ and the large volume limit, thus

$$(A.45) \quad \overline{\Omega}_{b,w^{\pm}}(\mathsf{v}^1_{m',\beta'}) = \overline{\Omega}_{b>1,\ w \to +\infty}(\mathsf{v}^1_{m',\beta'}) = \left(\#H^2(\mathfrak{Y}, \mathbb{Z})_{\text{tors}}\right)\mathrm{P}_{m',\beta'}.$$

Combining (A.37), (A.39) and Theorem 2 implies that $(\beta', m') \in M_{m,\beta}$ as defined in Theorem 1.

Then summing up the wall crossing formulae [3, Equation (5.13)] over all walls for class v between the large volume limit and ℓ_f gives

$$(A.46)$$
$$0 = \overline{\Omega}_{b,\,w<\alpha b-x}(\mathsf{v}) = \overline{\Omega}_{\infty}(\mathsf{v}) + \left(\#H^2(\mathfrak{Y}, \mathbb{Z})_{\text{tors}}\right)$$
$$\times \sum_{(m',\beta') \in M_{m,\beta}} (-1)^{\chi(\mathsf{v}^1_{m',\beta'},\mathsf{v}^0_{m',\beta'})+1}\chi(\mathsf{v}^1_{m',\beta'}, \mathsf{v}^0_{m',\beta'})\, \mathrm{P}_{m',\beta'}\, \overline{\Omega}_H\left(\mathsf{v}^0_{m',\beta'}\right).$$

This implies

$$(A.47)$$
$$\mathrm{P}_{m,\beta} = \sum_{(m',\beta') \in M_{m,\beta}} (-1)^{\chi(\mathsf{v}^1_{m',\beta'},\mathsf{v}^0_{m',\beta'})}\chi(\mathsf{v}^1_{m',\beta'}, \mathsf{v}^0_{m',\beta'})\, \mathrm{P}_{m',\beta'}\, \overline{\Omega}_H\left(\mathsf{v}^0_{m',\beta'}\right),$$

where $\chi(\mathsf{v}^1_{m',\beta'}, \mathsf{v}^0_{m',\beta'}) = \beta.H + \beta'.H + m - m' - \frac{H^3}{6} - \frac{1}{12}c_2(\mathfrak{Y}).H = \chi_{m',\beta'}$ as claimed. $\qquad \square$

Proof of Theorem 3. Suppose $\mu = n\kappa + p$ where $n \in \mathbb{Z}_{\geq 0}$ and $p = 0, \pm 1, \pm 2$. One can easily check that if $\mu \geq 13$ or $\mu = 10$, the classes (i) $\left(\mathcal{C}(\frac{\mu}{\kappa}H^2), \frac{\mu}{\kappa}H^2\right)$ and (ii) $\left(\mathcal{C}(\frac{\mu}{\kappa}H^2) - 1, \frac{\mu}{\kappa}H^2\right)$ for $p = \pm 1, \pm 2$, and if $\mu \geq 18$ the class (iii) $\left(\mathcal{C}(\frac{\mu}{\kappa}H^2) - 2, \frac{\mu}{\kappa}H^2\right)$ for $p = \pm 2$ are optimal in the sense of Definition 1. Thus combining Corollary 1 and Conjecture 1 implies the claim. Note that when $\mu = n\kappa$, [8, Theorem 1.1] implies that

$$(A.48) \qquad \overline{\Omega}_H\left(0, H, \left(n + \tfrac{1}{2}\right)H^2, \left(\tfrac{1}{6} + \tfrac{n(n+1)}{2}\right)H^3\right)$$
$$= (-1)^{\chi(\mathcal{O}((n+1)H),\mathcal{O}(nH))+1}\chi(\mathcal{O}((n+1)H), \mathcal{O}(nH)[1]) = 5.$$

If $\mu = 13$ or 17, one can directly apply the wall-crossing formula A.4 in Theorem 1 to show again $\beta' = 0$ which implies the claim. $\qquad \square$

Remark 3. *To prove Theorem 1 and Theorem 2 (and so Corollary 1 and 2, and Theorem 3), we applied the conjectural BMT inequality only for (i) the rank zero classes $\mathsf{v}^0_{m',\beta'}$ to get (A.56) using [6, Lemma B.3], and (ii) rank -1 class $(-1, 0, \beta, -m)$ with $\beta.H \leq H^3$ to get (A.40). Thus we only need the conjectural BMT inequality for the following two cases:*

> *(i) Rank zero Gieseker-stable sheaves of class $(0, H, \beta_1, m_1)$ for the values $(b, w) \in U$ lying along the line ℓ_1 which is of slope $\frac{\beta_1.H}{H^3}$ and intersects ∂U at two points with b-values $b' < b''$ so that $b'' - b' = 1$.*
>
> *(ii) Rank one torsion-free sheaves of class $(1, 0, -\beta_2, m_2)$ for $\beta_2.H \leq H^3$ and the point $(b, w) = (-1, \frac{1}{2})$ along the boundary ∂U.*

This, in particular, shows that the weaker version of BMT conjecture proved in [44] for quintic X_5 and in [45] for $X_{4,2}$ is sufficient for our result.

Remark 4. *Suppose our CY threefold $\mathfrak{Y}$ satisfies a stronger version of classical Bogomolov-Gieseker inequality (2.27), i.e. there is a function $G \colon \mathbb{R} \to \mathbb{R}$ such that any slope-semistable sheaf E satisfies $\frac{\mathrm{ch}_2(E).H}{\mathrm{ch}_0(E)H^3} \leq G\left(\frac{\mathrm{ch}_1(E).H^2}{\mathrm{ch}_0(E)H^3}\right)$ and $G(b) \leq \frac{b^2}{2}$ for all $b \in \mathbb{R}$. Then one can enlarge the space of weak stability conditions U to $U_G := \{(b, w) \in \mathbb{R}^2 \colon w > G(b)\}$, and apply all the arguments in this section within the enlarged space U_G instead of U. This, in particular, shows that the intersection of the line ℓ_f with ∂U_G has b-values $b_1^G < b_2^G$ so that $b_2^G - b_1^G \geq b_2 - b_1$, thus we can improve the function f in Theorem 1. For instance, for a quintic threefold, one can apply Li's version of stronger Bogomolov-Gieseker inequality [44, Theorem 1.1] to show that equation of the function f can be improved to $\frac{x}{2} + 1$ for any $x > 0$ (see Figures 3 and 6).*

Proof of Proposition 1. Since $\alpha = \frac{x}{2} + 1$, the line ℓ_f intersects ∂U at two points with b-values $b_1 = 2 < b_2$ such that $b_2 - b_1 \geq b_1$, where the inequality is strict if $x > 4$. Then, using the same notations as in Lemma 1, it implies that $b_2 r \leq c \leq 2(r + 1)$. Thus if $x > 4$, we can have (i) $(r, c) = (0, 1)$, or (ii) $(r, c) = (0, 2)$. If $x = 4$, there is a third possibility (iii) $r = 1$ and $c = 4$.

First consider a wall ℓ of type (i). We know ℓ lies above or on ℓ_f, so there is $w^+ > 2$ such that the point $(2, w^+) \in U$ lies just above the wall ℓ. Since no wall for E_1 can cross the vertical line $b = 2$, we get $\overline{\Omega}_{b=2,w^+}(\mathrm{ch}(E_1)) = \overline{\Omega}_{b=2,\, w \to +\infty}(\mathrm{ch}(E_1))$. Thus the proof of Theorem 1 goes through.

In case (ii), we know that $\mathrm{ch}_{\leq 1}(E_1) = (-1, -2H)$. Thus $\varpi(E_1)$ lies on ∂U, so $\mathrm{ch}_2(E_1) = -2H^2$. Then [109, Proposition 4.20(ii)] implies that $E_1 = \mathcal{O}_{\mathfrak{Y}}(2H)[1]$, so $\frac{\mathrm{ch}_2(E_0).H^2}{H^3} = x + 2$ and $\mathrm{ch}_3(E_0) = \frac{2}{3}\alpha x H^3 + \frac{4}{3} H^3 =$

$\frac{H^3}{3}x^2 + \frac{2H^3}{3}x + \frac{4}{3}H^3$. Applying [6, Lemma B.3] for E_3 gives

$$\frac{\mathrm{ch}_3(E_0)}{H^3} = \frac{1}{3}x^2 + \frac{2}{3}x + \frac{4}{3} \leq \frac{(x+2)^2}{4} + \frac{1}{3}$$

which holds only if $x = 4$. If $\beta = 4H^2$ and $m = -8H^3$, then $\mathrm{ch}(E_0) = (0, 2H, 6H^2, \frac{28}{3}H^3)$. Applying the conjectural BMT inequality (2.38) implies that the final wall for E_0 coincides with the line ℓ_f and this is the only wall that can happen for E_0 by [8, Theorem 1.1] where the destabilising factors are $\mathcal{O}_{\mathfrak{Y}}(4H)$ and $\mathcal{O}_{\mathfrak{Y}}(2H)[1]$.

In case (iii) when $x = 4$, we know that $\varpi(E_0)$ lies along the wall, so E_1 is of class $\mathrm{ch}_{\leq 2}(E_0) = (1, 4H, 8H^2)$. Since $\Delta_H(E_0) = 0$, there is no wall for E_0 up to the large volume limit by [10, Corollary 3.10], so E_0 is a slope-stable sheaf and $\mathrm{ch}_3(E_0) \leq \frac{32}{3}H^3$. The other factor is of class $\mathrm{ch}_{\leq 2}(E_1) = (-2, -4H, -4H^2)$. Applying the BMT inequality (2.38) at the point $(b = 2 + \epsilon, w)$ on the wall, where $0 < \epsilon \ll 1$, implies that $\mathrm{ch}_3(E_1) \leq -\frac{8}{3}H^3$. Given that $\mathrm{ch}_3(E_0) + \mathrm{ch}_3(E_1) = 8H^3$, it follows that $\mathrm{ch}(E_0) = \mathrm{ch}(\mathcal{O}_{\mathfrak{Y}}(4H))$ and $\mathrm{ch}(E_1) = -\mathrm{ch}(\mathcal{O}_{\mathfrak{Y}}(2H)^{\oplus 2})$.

To summarise, case (i) only contributes to the walls which are of the same form as described in Lemma 1. If $x > 4$, this is the only case that we need to consider and so Theorem 1 is valid. But if $\beta = 4H^2$, cases (ii) and (iii) contribute to the last wall ℓ_f. Define $\mathsf{v}_1 := \mathrm{ch}(\mathcal{O}_{\mathfrak{Y}}(2H)[1])$ and $\mathsf{v}_2 := \mathrm{ch}(\mathcal{O}_{\mathfrak{Y}}(4H))$, and let $(b, w^{\pm})$ be points in U just above and below the final wall ℓ_f. Then applying the wall-crossing formula [8, Equation (16)] shows that we have three contributions along this wall:

1. $\{\mathsf{v}_1, \mathsf{v}_1 + \mathsf{v}_2\}$ contribution (corresponding to case (ii)) to $\overline{\Omega}_{b,w^-}(\mathsf{v})$ is

   $$(\mathrm{A.49}) \qquad (-1)^{\chi(\mathsf{v}_1, \mathsf{v}_1 + \mathsf{v}_2)+1} \chi(\mathsf{v}_1, \mathsf{v}_1 + \mathsf{v}_2) \overline{\Omega}_{b,w^+}(\mathsf{v}_1) \overline{\Omega}_{b,w^+}(\mathsf{v}_1 + \mathsf{v}_2).$$

   As explained in case (ii), we know that $\overline{\Omega}_{b,w^+}(\mathsf{v}_i) = \overline{\Omega}_H(\mathsf{v}_i) = 1$ for $i = 1, 2$ and

   $$\overline{\Omega}_{b,w^+}(\mathsf{v}_1 + \mathsf{v}_2) = (-1)^{\chi(\mathsf{v}_1, \mathsf{v}_2)} \chi(\mathsf{v}_1, \mathsf{v}_2) \overline{\Omega}_{b,w^+}(\mathsf{v}_1) \overline{\Omega}_{b,w^+}(\mathsf{v}_2)$$

   thus the contribution (A.49) is $-\left(\chi(\mathcal{O}_{\mathfrak{Y}}, \mathcal{O}_{\mathfrak{Y}}(2H))\right)^2$.
2. $\{2\mathsf{v}_1, \mathsf{v}_2\}$ contribution (corresponding to case (iii)) to $\overline{\Omega}_{b,w^-}(\mathsf{v})$ is

   $$2\chi(\mathcal{O}_{\mathfrak{Y}}(2H), \mathcal{O}_{\mathfrak{Y}}(4H)) \times \overline{\Omega}_H(2\mathsf{v}_1)\overline{\Omega}_H(\mathsf{v}_2),$$

   where $\overline{\Omega}_H(2\mathsf{v}_1) = \frac{1}{4}\overline{\Omega}_H(\mathsf{v}_1) = \frac{1}{4}$ by [3, Example 6.2];

3. $\{v_1, v_1, v_2\}$ contribution to $\overline{\Omega}_{b,w^-}(v)$ is

$$(A.50) \qquad \frac{1}{2}\big(\chi(v_1, v_2)\big)^2 \overline{\Omega}_H(v_1)^2 \overline{\Omega}_H(v_2) = \frac{1}{2}\big(\chi(\mathcal{O}_{\mathfrak{Y}}, \mathcal{O}_{\mathfrak{Y}}(2H))\big)^2.$$

Hence the overall contribution of the wall ℓ_f to $\overline{\Omega}_{b,w^-}(v)$ is

$$(A.51) \qquad -\frac{1}{2}\big(\chi(\mathcal{O}_{\mathfrak{Y}}, \mathcal{O}_{\mathfrak{Y}}(2H))\big)^2 + \frac{1}{2}\chi(\mathcal{O}_{\mathfrak{Y}}, \mathcal{O}_{\mathfrak{Y}}(2H)).$$

Combining this with the wall-crossing formula (A.46) implies the claim. $\qquad\square$

Proof of Proposition 2. As before, we do wall-crossing for the class $v = (-1, 0, \beta, -m)$. The same argument as in Lemma 1 implies that the destabilising factors have Chern class
$$(A.52)$$
$$v^0_{m',\beta',H'} = \mathrm{ch}(E_0) = \left(0, \ H', \ \frac{1}{2}H'^2 - \beta' + \beta, \ \frac{1}{6}H'^3 + m' - m - \beta'.H'\right)$$

and $v^1_{m',\beta',H'}$ where

$$(A.53) \qquad v^1_{m',\beta',H'} \otimes \mathcal{O}_{\mathfrak{Y}}(-H') = \mathrm{ch}(E_1(-H')) = \left(-1, \ 0, \ \beta', \ -m'\right).$$

where $\frac{H'.H^2}{H^3} = 1$. Moreover E_0 is a Gieseker-stable sheaf as $\mathrm{ch}_1(E_0).H^2$ is still minimal and $E_1(-H')$ is $\nu_{b,w}$-stable for $b > 0$ and $w \gg 1$, thus $\beta'.H \geq 0$. We know that the point $\varpi(\mathrm{ch}(E_1)) = (1, -\frac{\beta'.H}{H^3} + \frac{H.H'^2}{2H^3})$ (defined in (2.36)) lies above or on ℓ_f, so

$$(A.54) \qquad \frac{\beta'.H}{H^3} \leq \frac{3m}{2\beta.H} + \frac{\beta.H}{H^3} + \frac{H.H'^2}{2H^3}.$$

Moreover, we know that E_1 is $\nu_{b,w}$-stable for $b > -1$ and $w \gg 1$, so $(E_1 \otimes \det(E_1))^\vee[1]$ is a stable pair, thus [5, Proposition 2.6] implies that

$$(A.55) \qquad -m' \leq \frac{2}{3}\beta'.H\left(\frac{\beta'.H}{H^3} + \frac{1}{2}\right).$$

On the other hand, applying [6, Lemma B.3] for E_0 gives

$$(A.56) \quad \frac{1}{6}H'^3 + m' - m - \beta'.H' \leq \frac{1}{2H^3}\left(\frac{1}{2}H'^2 H - \beta'.H + \beta.H\right)^2 + \frac{H^3}{24}.$$

Then the claim follows by a similar argument as in the wall-crossing formula (A.46). $\qquad\square$

Proof of Theorem 4. The argument is similar to [8], we include it for completeness. Define $r = \frac{D.H^2}{H^3}$ and $s = \frac{\beta.H}{H^3}$, then by our assumption $r \in \mathbb{Z}$. The conjectural BMT inequality implies that any wall ℓ for class v lies above or on the line ℓ_f with equation

$$(A.57) \qquad w = \frac{s}{r}b + \frac{r^2}{8} - \frac{s^2}{2r^2} - \frac{1}{4}Q_H(\mathsf{v})$$

which intersects ∂U at two points with b-values

$$(A.58) \qquad b_1 = \frac{s}{r} - \sqrt{\frac{r^2}{4} - \frac{1}{2}Q_H(\mathsf{v})}, \quad b_2 = \frac{s}{r} + \sqrt{\frac{r^2}{4} - \frac{1}{2}Q_H(\mathsf{v})}.$$

Our assumption on $Q_H(\mathsf{v})$ implies $b_2 - b_1 > \max\{r-1, \frac{1}{2}\}$. Let $E_1 \to E \to E_2$ be a destabilising sequence along a wall ℓ for class v. By definition of the heart $\mathcal{A}_b$,

$$(A.59) \qquad \mu_H^+(\mathcal{H}^{-1}(E_i)) \leq b_1 \qquad \text{and} \qquad b_2 \leq \mu_H^-(\mathcal{H}^0(E_i)).$$

Summing up over E_1 and E_2 and using $\mathrm{rk}(E) = \mathrm{rk}(E_1) + \mathrm{rk}(E_2) = 0$ imply

$$(A.60) \qquad r \geq (b_2 - b_1)\big(\mathrm{ch}_0(\mathcal{H}^0(E_1)) + \mathrm{ch}_0(\mathcal{H}^0(E_2))\big).$$

Since $b_2 - b_1 > \frac{r}{2}$, we get

$$(A.61) \quad \mathrm{ch}_0(\mathcal{H}^{-1}(E_1)) + \mathrm{ch}_0(\mathcal{H}^{-1}(E_2)) = \mathrm{ch}_0(\mathcal{H}^0(E_1)) + \mathrm{ch}_0(\mathcal{H}^0(E_2)) \leq 1.$$

Therefore, one of the factors E_1 is of rank -1 with $\mathcal{H}^{-1}(E_1)$ of rank one and $\mathcal{H}^0(E_1)$ of rank zero; and the other factor E_2 is a sheaf of rank one. We claim

$$(A.62) \qquad \mu_H(E_2) - b_2 < 1.$$

Otherwise, (A.59) gives

$$(A.63) \qquad b_2 - b_1 \leq \mu_H(E_2) - 1 - \mu_H(\mathcal{H}^{-1}(E_1)).$$

Since $\mathcal{H}^0(E_1)$ is of rank zero, we have $\mu_H(E_1) \leq \mu_H(\mathcal{H}^{-1}(E_1))$, thus

$$(A.64) \qquad b_2 - b_1 \leq \mu_H(E_2) - 1 - \mu_H(E_1) = r - 1.$$

The last equality comes from $\mathrm{rk}(E_2) = -\mathrm{rk}(E_1) = 1$. But the above is not possible by our assumption on $b_2 - b_1$. Combining it with (A.59) gives

$$(A.65) \qquad b_2 \leq \mu_H(E_2) = \frac{\mathrm{ch}_1(E_2).H^2}{H^3} < b_2 + 1.$$

We know that there is no wall for E_2 crossing the vertical lines $b = \mu(E) - \frac{1}{2}$ and $b = \mu(E) - 1$ [108, Lemma 3.5]. Since $b_2 - b_1 > \frac{1}{2}$ at least one of these vertical lines intersects the wall ℓ at a point inside U. Thus $\nu_{b,w}$-semistability of E_2 along the wall implies that E_2 is $\nu_{b,w}$-stable for $b < \mu(E_2)$ and $w \gg 1$, so E_2 is a rank one torsion-free sheaf. A similar argument also shows that E_1 is stable in the large volume limit, so is the derived dual of a stable pair (up to tensoring by a line bundle).

Hence the destabilising factors are of classes $\mathsf{v}_i = (-1)^i e^{D_i}(1, 0, -\beta_i, -m_i)$ for $i = 1, 2$. We know that the point $\varpi(E_i) = \left(\frac{D_i.H^2}{H^3}, \frac{D_i^2.H}{2H^3} - \frac{\beta_i.H}{H^3} \right)$ lies above or on ℓ_f, i.e.

(A.66)
$$\frac{D_i^2.H}{2H^3} - \frac{\beta_i.H}{H^3} - \frac{\beta.H}{D.H^2}\frac{D_i.H^2}{H^3} \geq \frac{1}{8}\left(\frac{D.H^2}{H^3}\right) - \frac{1}{2}\left(\frac{\beta.H}{D.H^2}\right)^2 - \frac{1}{4}Q_H(\mathsf{v}).$$

Finally applying [5, Proposition 2.5 & 2.6] to $F_i \otimes D_i^{-1}$ imply

$$(A.67) \qquad (-1)^{i+1}m_i \leq \frac{2}{3}\beta_i.H\left(\frac{\beta_i.H}{H^3} + \frac{1}{2}\right).$$

Conversely, take two classes $\mathsf{v}_i = (-1)^i e^{D_i}(1, 0, -\beta_i, -m_i)$ for $i = 1, 2$ satisfying $\mathsf{v}_1 + \mathsf{v}_2 = \mathsf{v}$ and conditions (A.22) and (A.23). Then we have

$$(A.68) \qquad \left|\frac{D_1.H^2}{H^3} - b_1\right| < 1, \qquad \left|\frac{D_2.H^2}{H^3} - b_2\right| < 1$$

and $\frac{D_2.H^2}{H^3} - \frac{D_1.H^2}{H^3} = r$. The Hodge index theorem implies

$$(A.69) \qquad \frac{D_i^2.H}{2H^3} - \frac{\beta_i.H}{H^3} \leq \frac{1}{2}\left(\frac{D_i.H^2}{H^3}\right)^2,$$

thus $\varpi(E_i)$ lies outside U and above or on ℓ_f by (A.23). Since by our assumption on $Q_H(\mathsf{v})$, we have $b_2 - b_1 > \max\{r - 1, \frac{1}{2}\}$, we get

$$(A.70) \qquad b_1 - 1 < \frac{D_1.H^2}{H^3} < b_1 < b_2 < \frac{D_2.H^2}{H^3} < b_2 + 1.$$

Then the same argument as above shows that there is no wall for classes v_1 and v_2 above or on ℓ_f. Hence large volume limit stable objects of classes v_i for $i = 1, 2$ are $\nu_{b,w}$-stable of the same $\nu_{b,w}$-slope along the line ℓ passing through $\varpi(\mathsf{v}_i)$ for $i = 1, 2$, which lies above or on ℓ_f. Thus they make a wall for objects of class v. This completes the proof of the claim. $\square$

Appendix B. Other hypergeometric CY threefolds

In this section, we extend the analysis of §5 to the other hypergeometric CY threefolds, with the exception of $X_{3,2,2}$ and $X_{2,2,2,2}$ for which the current knowledge of GV invariants is not sufficient yet to uniquely determine (or even guess) the generating series of Abelian D4-D2-D0 indices. In all cases, we assume that the BMT inequality is satisfied.

B.1. X_6

Here we consider the sextic in $\mathbb{P}^4_{2,1,1,1,1}$, first studied in [20]. In this case, $\kappa = 3$, $n_1^p = 4$ and $n_1^c = 0$ so the modular form h_μ is uniquely fixed by 4 coefficients. Using the basis (5.5), the generating function proposed in [20] reads

$$(\text{B.1}) \qquad h_\mu = \frac{1}{\eta^{54}} \left[\frac{7E_4^6 + 58E_4^3 E_6^2 + 7E_6^4}{216} + \frac{5E_4^4 E_6 + 3E_4 E_6^3}{2} D \right] \vartheta_\mu^{(3)},$$

and has the following expansion:

(B.2)
$$h_0 = \mathsf{q}^{-\frac{15}{8}} \left(\underline{-4 + 612\mathsf{q}} - 40392\mathsf{q}^2 + 146464860\mathsf{q}^3 + 66864926808\mathsf{q}^4 \right.$$
$$+ 8105177463840\mathsf{q}^5 + 503852503057596\mathsf{q}^6$$
$$\left. + 20190917119833144\mathsf{q}^7 + \dots \right),$$
$$h_1 = \mathsf{q}^{-\frac{15}{8}+\frac{2}{3}} \left(\underline{0 - 15768\mathsf{q}} + 7621020\mathsf{q}^2 + 10739279916\mathsf{q}^3 + 1794352963536\mathsf{q}^4 \right.$$
$$+ 134622976939812\mathsf{q}^5 + 6141990299963544\mathsf{q}^6$$
$$\left. + 196926747589177416\mathsf{q}^7 + \dots \right).$$

Using (4.19), we can rigorously compute and confirm the terms up to (and including) order q^9 and q^6 in these expansions. The term of order q^6 in h_0 can be further verified using Prop. 1. Furthermore, the terms of order

Table 6: GV invariants $\mathrm{GV}_Q^{(g_C(Q)-\delta)}$ for X_6, assuming modularity

Q	$g_C(Q)$	$\delta=0$	$\delta=1$	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$
1	1	0	7884				
2	2	0	7884	6028452			
3	4	6	576	17496	145114704	11900417220	
4	5	0	−47304	−14966100	10801446444	1773044322885	34600752005688
5	7	0	63072	22232340	−21559102992	1985113680408	571861298748384
6	10	−28	−3168	−146988	−583398600	207237771936	−18316495265688
7	12	0	−110376	−43329384	54521267292	−8041642037676	513634614205788
8	15	0	−141912	−57278448	76595605884	−12434437188576	904511824896888
9	19	−52	−5472	−225504	−1453991342	645551751060	−82281995054250
10	22	0	220752	90243180	−132472407960	24441320028348	−2094555362224356
11	26	0	−268056	−109069632	166408768980	−32325403958928	2952049189946940
12	31	88	7572	212904	2755381840	−1352963727576	204189584421816
13	35	0	378432	150306948	−246695539464	52656199163280	−5391865451528568
14	40	0	441504	172213236	−293223343680	65474719151724	−7076432910134952
15	46	136	7956	47736	4489872516	−2384492136120	414897391102896
16	51	0	−583416	−217181952	399497240700	−97481656444968	11697806611060704
17	57	0	662256	239613660	−459419696640	117150837604344	−14795431515539352
18	64	−196	−4680	225396	−6665394192	3810518530344	−758652854479632
19	70	0	−835704	−282637296	593248436100	−165165188729184	23060985834155292
20	77	0	−930312	−302472360	667301101092	−194106551379768	28471201009767792
21	85	−268	4632	365112	−9289038760	5717547855792	−1296313683456384
22	92	0	1135296	336739140	−829978779600	263496783986604	−42580355264714232
23	100	0	−1245672	−350287848	918685187964	−304661265971256	51631322400126468

q^{10} and q^{11} in h_0 as well as q^7 and q^8 in h_1 are reproduced by (4.19) with $k = k_0 - 1$. Thus, there is overwhelming evidence that (B.1) is correct. While the maximal genus attainable by the standard direct integration method is 48, using modularity, we can predict GV invariants close to the Castelnuovo bound to arbitrary genus (see Table 6), and provide sufficiently many boundary conditions in principle to push the direct integration method up to genus 63.

B.2. X_8

We now consider the octic in $\mathbb{P}^4_{4,1,1,1,1}$, first studied in [20]. In this case, $\kappa = 2$, $n_1^p = 4$ and $n_1^c = 0$ so the modular form h_μ is uniquely fixed by 4 coefficients. Using the basis (5.5), the generating function proposed in [20] reads

(B.3)
$$h_\mu = \frac{1}{\eta^{52}}\left[\frac{103E_4^6 + 1472E_4^3E_6^2 + 153E_6^4}{5184} + \frac{503E_4^4E_6 + 361E_4E_6^3}{108}\,D\right]\vartheta_\mu^{(2)},$$

and has the following expansion:

(B.4)

$$h_0 = q^{-\frac{46}{24}} \Big(\underline{-4 + 888q} - 86140q^2 + 132940136q^3 + 86849300500q^4$$
$$+ 11756367847000q^5 + 787670811260144q^6$$
$$+ 33531427162546608q^7 + \dots \Big),$$

$$h_1 = q^{-\frac{46}{24}+\frac{3}{4}} \Big(\underline{0 - 59008q} + 8615168q^2 + 21430302976q^3 + 3736977423872q^4$$
$$+ 289181439668352q^5 + 13588569634434304q^6$$
$$+ 448400041603851008q^7 + \dots \Big).$$

Using (4.19), we can rigorously compute and confirm the terms up to (and including) order q^9 and q^7 in these expansions. The term of order q^4 in h_0 can be further verified using Prop. 1. The terms of order q^{10}, q^{11} in h_0 as well as q^8, q^9 and q^{10} in h_1 are reproduced by (4.19) with $k = k_0 - 1$. Thus, there is overwhelming evidence that (B.3) is correct. While the maximal genus attainable by the standard direct integration method is 60, using modularity, we can predict GV invariants close to the Castelnuovo bound to arbitrary genus (see Table 7), and provide sufficiently many boundary conditions in principle to push the direct integration method up to genus 80.

B.3. $X_{4,3}$

We now consider the complete intersection of degree $(4,3)$ in $\mathbb{P}^5_{2,1,1,1,1,1}$, In this case, $\kappa = 6$, $n_1^p = 9$ and $n_1^c = 0$ so the modular form h_μ is uniquely fixed by 9 coefficients. This model was first considered in [31], assuming the naive Ansatz (4.8) for the polar terms. Unfortunately, with the GV invariants being known only up to genus 20 using direct integration, Eq. (4.19) only allows to determine 3 polar coefficients:

(B.5)

$$h_0 = q^{-\frac{9}{4}} \Big(\underline{5 - 624q} + \tfrac{1}{21} \mathrm{PT}(18,-34)q^2 - \tfrac{1}{20} \mathrm{PT}(18,-33)q^3 + \dots \Big),$$

$$h_1 = q^{-\frac{9}{4}+\frac{7}{12}} \Big(\underline{-\tfrac{1}{12} \mathrm{PT}(13,-20)} - \tfrac{1}{24} \mathrm{PT}(19,-38)q + \tfrac{1}{23} \mathrm{PT}(19,-37)q^2 + \dots \Big),$$

$$h_2 = q^{-\frac{9}{4}+\frac{1}{3}} \Big(\underline{-\tfrac{1}{14} \mathrm{PT}(14,-23)} + \tfrac{1}{13} \mathrm{PT}(14,-22)q - \tfrac{1}{26} \mathrm{PT}(20,-41)q^2 + \dots \Big),$$

$$h_3 = q^{-\frac{9}{4}+\frac{1}{4}} \Big(\underline{0 + \tfrac{1}{15} \mathrm{PT}(15,-25)q} - \tfrac{1}{14} \mathrm{PT}(15,-24)q^2 + \dots \Big).$$

Table 7: GV invariants $\mathrm{GV}_Q^{(g_C(Q)-\delta)}$ for X_8, assuming modularity

Q	g_C	$\delta=0$	$\delta=1$	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$
1	1	0	29504				
2	3	6	864	41312	128834912		
3	4	0	−177024	−16551744	21464350592	1423720546880	
4	7	24	4152	301450	396215800	−174859503824	12499667277744
5	9	0	354048	37529088	−86307810432	12063928269056	−674562224718848
6	13	40	7032	523434	918424384	−537735889892	67237956960504
7	16	0	−649088	−67977216	194884427520	−34549033260480	2730733623512576
8	21	64	10760	747160	1693127408	−1100325268755	163574439433328
9	25	0	1062144	97599232	−348278532864	70573905748736	−6573094863849216
10	31	96	14664	874648	2715237856	−1885455097488	317498157747448
11	36	0	−1593216	−115655680	547020195328	−124368823627264	13265837355895808
12	43	136	17880	816224	3983508192	−2920617786752	550836611504760
13	49	0	2242304	107984640	−791226604800	201252013167104	−24393882174586624
14	57	184	19352	517696	5502562160	−4239252796968	892029516487568
15	64	0	−3009408	−57591808	1080060791808	−307623836581376	42224741744709120
16	73	240	17832	−7064	7283098000	−5881013303280	1377169141402320
17	81	0	3894528	−55349504	−1411208698624	450913093594624	−69905571017188608
18	91	304	11880	−637720	9339141568	−7892176820432	2051229771888392
19	100	0	−4897664	253498368	1780270216704	−639463383246336	111688644307754752
20	111	376	−136	−1103312	11680881536	−10326002693808	2969507704650056
21	121	0	6018816	−562346240	−2180065252608	882336333453824	−173179252180073216
22	133	456	−20040	−932336	14301420112	−13242624843432	4199274885440864
23	144	0	−7257984	1010216960	2599854822400	−1189005528876544	261596095595733504
24	157	544	−49848	602936	17155557680	−16707780101408	5821612261875808
25	169	0	8615168	−1628266752	−3024477174528	1568911793583616	−386029168134457600

Despite this discouraging result, one can proceed assuming that for some coefficients Eq. (4.19) still holds for $k = k_0 - 1$. This assumption will be justified *a posteriori* by matching the predictions of (4.19) and modularity for many more coefficients. For this choice of the spectral flow parameter, one finds

$$
\begin{aligned}
\text{(B.6)} \\
h_0 &\overset{?}{=} \mathrm{q}^{-\frac{9}{4}} \left(\underline{2 - 234\mathrm{q} + 35415\mathrm{q}^2 + 19018272\mathrm{q}^3 + \tfrac{523497643503}{7}\,\mathrm{q}^4 + \dots} \right), \\
h_1 &\overset{?}{=} \mathrm{q}^{-\frac{9}{4}+\frac{7}{12}} \left(\underline{0 + \left(5832 + \tfrac{40}{11}\mathrm{GV}_{13}^{(21)} \right) \mathrm{q} - \left(544320 + 78\mathrm{GV}_{13}^{(21)} \right) \mathrm{q}^2} \right. \\
&\qquad + \left(3919923072 + \tfrac{9880}{9}\,\mathrm{GV}_{13}^{(21)} \right) \mathrm{q}^3 \\
&\qquad \left. + \left(2506521907872 - \tfrac{45695}{4}\,\mathrm{GV}_{13}^{(21)} \right) \mathrm{q}^4 + \dots \right), \\
h_2 &\overset{?}{=} \mathrm{q}^{-\frac{9}{4}+\frac{1}{3}} \left(\underline{0 + 0\mathrm{q}} - \tfrac{1}{12} \left(\mathrm{GV}_{14}^{(22)} + 44\mathrm{GV}_{14}^{(23)} + 1035\mathrm{GV}_{14}^{(24)} \right) \mathrm{q}^2 + \dots \right), \\
h_3 &\overset{?}{=} \mathrm{q}^{-\frac{9}{4}+\frac{1}{4}} \left(\underline{0 + 0\mathrm{q} + 0\mathrm{q}^2} + \tfrac{1}{13}\,\mathrm{PT}(15,-23)\mathrm{q}^3 + \dots \right),
\end{aligned}
$$

where we expressed the result in terms of GV invariants and put question marks to emphasize that these expansions need not be correct. For example, the first two terms in h_0 clearly disagree with the rigorous result (B.5). Nonetheless, let us assume that all other polar terms, except the $\mathcal{O}(q)$ term in h_2, are correctly computed by (B.6). Comparing with (B.5), this implies the vanishing of $\mathrm{PT}(13, -20) = \mathrm{GV}_{13}^{(21)}$ and $\mathrm{PT}(14, -23) = \mathrm{GV}_{14}^{(24)}$, which allows to further simplify (B.6). In particular, all polar terms, except the $\mathcal{O}(q)$ term in h_2, are now fixed.

To get a sufficient number of conditions to fix the modular form, let us further assume that the $\mathcal{O}(q^3)$ term in h_0 is also correctly computed by (B.6). This assumption provides the missing condition and allows to find a unique modular form matching all coefficients

$$
\begin{aligned}
h_\mu = \frac{1}{\eta^{72}} \Bigg[&\frac{19161576 E_4^7 E_6 - 86969808 E_4^4 E_6^3 - 36701208 E_4 E_6^5}{17199267840} \\
&+ \frac{29888136 E_4^8 + 147874032 E_4^5 E_6^2 + 16326792 E_4^2 E_6^4}{716636160} D \\
&- \frac{4751784 E_4^6 E_6 + 9532080 E_4^3 E_6^3 + 646056 E_6^5}{5971968} D^2 \\
&- \frac{1686312 E_4^7 + 10686384 E_4^4 E_6^2 + 2557224 E_4 E_6^4}{1244160} D^3 \Bigg] \vartheta_\mu^{(6)},
\end{aligned}
$$
(B.7)

with the following expansion:

(B.8)
$$
\begin{aligned}
h_0 = q^{-\frac{9}{4}} \Big(&5 - 624q + \underline{35415q^2} + 19018272q^3 + 74785378407q^4 \\
&+ 23744184704784q^5 + 2912626940217084q^6 \\
&+ 201892603398250080q^7 + \dots \Big),
\end{aligned}
$$

$$
\begin{aligned}
h_1 = q^{-\frac{9}{4}+\frac{7}{12}} \Big(&0 + 5832q - 544320q^2 + 3919923072q^3 + 2506521907872q^4 \\
&+ 426826821029328q^5 + 36510169956413184q^6 \\
&+ 1975570599744644544q^7 + \dots \Big),
\end{aligned}
$$

$$
\begin{aligned}
h_2 = q^{-\frac{9}{4}+\frac{1}{3}} \Big(&0 + 81q - \underline{455544q^2} + 418794867q^3 + 589406293224q^4 \\
&+ 127700521014312q^5 + 12611391702441624q^6 \\
&+ 754527616229888955q^7 + \dots \Big),
\end{aligned}
$$

$$h_3 = q^{-\frac{9}{4}+\frac{1}{4}} \left(\underline{0 + 0q} - \underset{\cdots\cdots}{322496q^2} + 154768800q^3 + 356674019472q^4 \right.$$
$$+ 84550767361152q^5 + 8789804684886144q^6$$
$$\left. + 544775594940872640q^7 + \dots \right).$$

Remarkably, we find that all terms up to q^4 in h_1 turn out to coincide with those in (B.6), which provides strong support for the above assumptions leading to (B.7). Furthermore, only the $\mathcal{O}(q^2)$ term in h_0 differs from the value 35334 given by the naive ansatz (4.8), while all other polar terms as well as the $\mathcal{O}(q^2)$ coefficients in h_2 and h_3 perfectly match (4.8).

We can apply a similar procedure to provide additional constraints on GV invariants and additional checks on the modular function (B.7). First, taking into account that $\mathrm{PT}(14, -22) = \mathrm{GV}_{14}^{(23)} + 46\mathrm{GV}_{14}^{(24)} = \mathrm{GV}_{14}^{(23)}$ and matching the $\mathcal{O}(q)$ term in h_2 between (B.5) and (B.8), one obtains $\mathrm{GV}_{14}^{(23)} = 1053$. To get a constraint at genus 22, we further assume that the $\mathcal{O}(q^2)$ term in h_2 is correctly captured by (B.6). Comparing it with (B.8) and taking into account the previous findings for GV invariants, one gets $\mathrm{GV}_{14}^{(22)} = 5420196$. With all these constraints, it is possible to compute GV invariants up to genus 23 and check that the coefficients of q^3 and q^4 in h_2 computed using (4.19) with $k = k_0 - 1$ match those in (B.8), which can be considered as a verification of the above assumption.

To go to even higher genus, the predictions of modularity based on the rigorous use of (4.19) are again insufficient because GV invariants at genus 24 turn out to depend on unknown PT invariants. In particular, it can be shown that $\mathrm{GV}_{15}^{(24)} = \mathrm{PT}(15, -23) - 216717312$. Fortunately, we can apply the same trick as above: let us assume that the $\mathcal{O}(q^3)$ term in h_3 is computed correctly by (B.6). This fixes the required PT invariant and gives $\mathrm{GV}_{15}^{(24)} = 1795277088$. As a result, the direct integration method can be pushed up to genus 24, while the maximal genus attainable by the standard direct integration method is only 20. One can also check that the coefficients of q^4 and q^5 in h_3 computed using (4.19) with $k = k_0 - 1$ match those in (B.8), which supports the above assumption. Finally, using modularity, we can predict GV invariants close to the Castelnuovo bound to arbitrary genus (see Table 8).

B.4. $X_{6,4}$

We now consider the complete intersection of degree $(6, 4)$ in $\mathbb{P}^5_{3,2,2,1,1,1}$, In this case, $\kappa = 2$, $n_1^p = 3$ and $n_1^c = 0$ so the modular form h_μ is uniquely

Table 8: GV invariants $\mathrm{GV}_Q^{(g_C(Q)-\delta)}$ for $X_{4,3}$, assuming modularity. A question mark indicates that the result depends on as yet unknown PT invariants

Q	g_C	$\delta = 0$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 4$	$\delta = 5$
1	1	0	1944				
2	2	0	27	223560			
3	3	0	0	161248	64754568		
4	4	0	81	227448	381704265	27482893704	
5	5	0	5832	155520	3896917776	638555324400	14431471821504
6	7	10	816	26757	−40006768	75047188236	20929151321496
7	8	0	−23328	−1358856	−7825332240	2609489667744	1159250594105376
8	10	0	405	1815696	1246578255	−1193106464964	169353267859971
9	12	0	0	−1612480	−590680416	1077388111920	−185398224083488
10	14	0	567	2719656	2033988975	−2396370890772	426751496255367
11	16	0	−46656	−2503872	−23437746576	13091629897584	−1992347003533392
12	19	55	4260	139245	−159384576	528260763000	−157181565397200
13	21	0	64152	2908224	35118682704	−21249125934480	3648284023741704
14	24	0	1053	5420196	4378100382	−6073970861304	1376630062962426
15	27	0	0	4514944	?	?	?
16	30	0	1377	7211592	5890376457	?	?
17	33	0	110808	2927664	66253494456	−43956428447664	?
18	37	115	5448	68415	−405033180	1447081995873	−519150013281888
19	40	0	−139968	−1881792	−85700360016	58822283187000	−12434745915614736
20	44	0	2187	11660436	9506092041	−14945171094720	4071759470600148
21	48	0	0	−9352384	−3491811840	9974934265584	−3041698928528400
22	52	0	2673	14310108	11567018943	−18861070782672	5421323164985343
23	56	0	−209952	3623616	−132448385088	96512552546400	−22792919002464096
24	61	205	360	−124995	−768252196	2853693391443	−1169466146662224
25	65	0	250776	−8736336	159825290616	−119774566448496	29839722776131176
26	70	0	3807	20437272	16072226307	−28239088327452	9015178386188196

fixed by 3 coefficients. This model was first considered in [31], assuming the naive Ansatz (4.8) for the polar terms. Using GV invariants up to genus 14, Eq. (4.19) predicts

(B.9)
$$h_0 = \mathrm{q}^{-\frac{34}{24}} \left(\underline{3 - 304\mathrm{q}} + \tfrac{1}{13}\,\mathrm{PT}(8, -18)\mathrm{q}^2 - \tfrac{1}{12}\,\mathrm{PT}(8, -17)\mathrm{q}^3 + \dots \right),$$
$$h_1 = \mathrm{q}^{-\frac{34}{24}+\frac{3}{4}} \left(\underline{-16} - \tfrac{1}{10}\,\mathrm{PT}(7, -14)\mathrm{q} + \tfrac{1}{9}\,(\mathrm{PT}(7, -13) + 192)\,\mathrm{q}^2 + \dots \right).$$

In particular, the polar part of h_0 differs from the value $3 - 312\mathrm{q}$ predicted by the naive Ansatz (4.8). There is a unique modular form that matches these polar terms, namely

Table 9: GV invariants $\mathrm{GV}_Q^{(g_C(Q)-\delta)}$ for $X_{6,4}$, assuming modularity

Q	g_C	$\delta=0$	$\delta=1$	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$
1	1	8	15552				
2	3	3	128	258344	27904176		
3	4	−48	−64432	36976576	5966034472	133884554688	
4	7	15	1036	800065	−272993052	15929894952	4502079839576
5	9	96	160128	−148759496	14847229472	−592538522344	42148996229312
6	13	27	1784	1846330	−838903420	76751964798	−3326821152316
7	16	−176	−318240	338189520	−43591449792	2519386074032	−86921827226312
8	21	45	2456	3387175	−1727130716	?	?
9	25	288	536160	−610236992	91763910544	−6449197272904	?
10	31	69	2548	5409137	−2981186776	388162502583	−26079491452172
11	36	−432	−810912	970636496	−166948527648	13842057435472	−721713847987144
12	43	99	1412	7922463	−4655472528	697407486327	−55249696746420
13	49	608	1138656	−1426615872	278955967328	−26973362355200	1666567885265984
14	57	135	−1744	10981213	−6816859292	1167120353936	−107938440865312
15	64	−816	−1514784	1986583568	−440060692768	49244924907392	−3567835755931072
16	73	177	−7856	14707727	−9547461076	1857133031696	−198887588738688
17	81	1056	1933920	−2659986752	665252326368	−85577571342976	7205538528304192
18	91	225	−18004	19321425	−12950149776	2843375759861	−349939554154236
19	100	−1328	−2389920	3457146192	−972514271520	142892550610016	−13868257921375616
20	111	279	−33412	25171927	−17157902216	4221401587493	−592607749008964

$$(B.10) \qquad h_\mu = \frac{1}{\eta^{40}} \left[-\frac{85E_4^3 E_6 + 23E_6^3}{432} - \frac{13E_4^4 + 23E_4 E_6^2}{6} D \right] \vartheta_\mu^{(2)},$$

with the following expansion:

(B.11)
$$h_0 = \mathrm{q}^{-\frac{34}{24}} \left(3 - \underline{304\mathrm{q}} + 270431\mathrm{q}^2 + 133585104\mathrm{q}^3 + 12401092398\mathrm{q}^4 + \dots \right),$$
$$h_1 = \mathrm{q}^{-\frac{34}{24}+\frac{3}{4}} \left(\underline{-16} + 32352\mathrm{q} + 36578048\mathrm{q}^2 + 4364892672\mathrm{q}^3 \right.$$
$$\left. + 226014399392\mathrm{q}^4 + \dots \right).$$

The term of order q in h_1 is correctly reproduced by (4.19) with $k = k_0 - 1$. Assuming that (B.19) is correct, one can produce additional boundary conditions for the direct integration method (see Table 9), allowing to reach genus 17, beyond the genus 14 available by standard methods. Note that to get a boundary condition at genus 17, one uses the fact the $\mathcal{O}(\mathrm{q}^4)$ coefficient in h_0 is subject to Prop. 1.

B.5. $X_{3,3}$

Next, we consider the bicubic in $\mathbb{P}^5$, first studied in [20]. In this case, $\kappa = 9$, $n_1^p = 14$ and $n_1^c = 1$ (as first noted in [52]) so the modular form h_μ is uniquely fixed by 13 coefficients. Using the basis (5.5), the generating function proposed in [20] reads

(B.12)
$$
h_\mu = \frac{1}{\eta^{90}} \left[\frac{47723 E_4^9 E_6 + 25095 E_4^6 E_6^3 - 68943 E_4^3 E_6^5 - 3875 E_6^7}{107495424} \right.
$$
$$
+ \frac{289326 E_4^{10} + 415189 E_4^7 E_6^2 - 3458324 E_4^4 E_6^4 - 729839 E_4 E_6^6}{334430208} D
$$
$$
+ \frac{2261629 E_4^8 E_6 + 3219046 E_4^5 E_6^3 - 6371 E_4^2 E_6^5}{30965760} D^2
$$
$$
- \frac{94271 E_4^9 + 1496733 E_4^6 E_6^2 + 1342665 E_4^3 E_6^4 + 52315 E_6^6}{5160960} D^3
$$
$$
\left. - \frac{162167 E_4^7 E_6 + 300338 E_4^4 E_6^3 + 35159 E_4 E_6^5}{286720} D^4 \right] \vartheta_\mu^{(9)},
$$

and has the following expansion

(B.13)
$$
h_0 = q^{-\frac{63}{24}} \left(\underline{-6 + 720q} - 40032q^2 - 678474q^3 + 30885198768q^4 \right.
$$
$$
\left. + 35708825468142q^5 + \dots \right),
$$
$$
h_1 = q^{-\frac{63}{24}+\frac{5}{9}} \left(\underline{0 - 4212q + 448578q^2} + 374980104q^3 \right.
$$
$$
\left. + 2020724648442q^4 + \dots \right),
$$
$$
h_2 = q^{-\frac{63}{24}+\frac{2}{9}} \left(\underline{0 + 0q + 158436q^2} - 12471246q^3 + 174600085086q^4 + \dots \right),
$$
$$
h_3 = q^{-\frac{63}{24}} \left(\underline{0 + 0q + 10206q^2} - 13828428q^3 + 24425287884q^4 + \dots \right),
$$
$$
h_4 = q^{-\frac{63}{24}+\frac{8}{9}} \left(\underline{0 + 0q} - 11040786q^2 + 6769752552q^3 \right.
$$
$$
\left. + 17629606262268q^4 + \dots \right).
$$

Unfortunately, with GV invariants being known up to genus 29, Eq. (4.19) only allows to confirm the coefficients $-6+720q$ in h_0 and $0\,q^0$ in h_3. Applying (4.19) with $k = k_0 - 1$, we find evidence that the coefficients of all terms

Table 10: GV invariants $\mathrm{GV}_Q^{(g_C(Q)-\delta)}$ for $X_{3,3}$, assuming modularity. A question mark indicates that the result depends on as yet unknown PT invariants

Q	$g_C(Q)$	$\delta=0$	$\delta=1$	$\delta=2$	$\delta=3$	$\delta=4$	$\delta=5$
1	1	0	1053				
2	2	0	0	52812			
3	3	0	0	3402	6424326		
4	3	0	0	5520393	1139448384		
5	4	0	0	5520393	4820744484	249787892583	
6	6	0	0	10206	6852978	23395810338	3163476682080
7	7	0	0	158436	−484542	174007524240	42200615912499
8	8	0	6318	372762	−784819773	2028116431098	785786604262830
9	10	15	1170	39033	−5412348	−61753761036	36760497856020
10	11	0	−21060	−1421550	1150458714	−4055688274977	1055748342473838
11	13	0	0	792180	42487254	523544632866	−277740359622189
12	15	0	0	−61236	−67672476	−96817818078	107933688748656
13	16	0	0	66244716	32180134734	−71248361250798	17551409134469472
14	18	0	0	−77285502	−38299950252	89193730254030	−23552769634742655
15	21	0	0	91854	107320680	167270244048	−217376516354913
16	23	0	0	−1584360	−48866814	−1393793916300	990222417035712
17	25	0	50544	2609334	−3916924776	18349298486658	−7855011831413205
18	28	−90	−5220	−120186	23305068	338860808028	−372702765685392
19	30	0	−63180	−2653560	5125104738	−24509155811472	11014900785838314
20	33	0	0	−2534976	−3497030	−2437828042176	1882564212119436
21	36	0	0	−183708	−222958548	?	?
22	38	0	0	−209774934	−107171300556	?	?
23	41	0	0	−231856506	−117773956584	?	?
24	45	0	0	−244944	−297478548	−512109217728	784094829426108
25	48	0	0	−4119336	58959090	−4179092501304	3448552834527066
26	51	0	−122148	−589680	10761641532	−53529168000492	27142164772551882
27	55	−198	−1656	78588	61819596	890324824482	−1089181917906228
28	58	0	143208	−1236222	−12753357660	63994621219614	−33427151297813844
29	62	0	0	−5703696	236542518	−5924469211524	5107223091368232
30	66	0	0	398034	475327980	841660464438	−1378633833342540
31	69	0	0	−452672226	−211037368248	718903306166688	−291661896939934680
32	73	0	0	485794584	223088117976	−775956404598264	320699529577227510
33	78	0	0	−489888	−576452916	−1032773712696	1747571748926544
34	82	0	0	8080236	−653114988	8557070940234	−7761488275449180
35	86	0	−231660	14695668	20684191104	−108491328034740	62714086906118814

up to q^5 in h_0 and all the vanishing coefficients in other components are indeed correct. Moreover, the $\mathcal{O}(q^3)$ coefficient in h_3 and $\mathcal{O}(q^2)$ coefficient in h_4 as well as all polar terms turn out to agree with the ansatz (4.8). In addition, we observe that the coefficients -4212 in h_1 and 10206 in h_3 are given by $\frac{1}{5}\,\mathrm{PT}(10,-9)$ and $-\frac{1}{6}\,\mathrm{PT}(12,-12)$, even though the corresponding values of (Q,m) do not satisfy the optimality conditions. Thus, there is strong evidence that (B.13) is correct.

Assuming that it is, one can produce additional boundary conditions for the direct integration method (see Table 10), allowing to reach genus 33, beyond the genus 29 available using standard boundary conditions.

B.6. $X_{4,4}$

We now consider the complete intersection of degree $(4, 4)$ in $\mathbb{P}^5_{2,2,1,1,1}$, In this case, $\kappa = 4$, $n_1^p = 6$ and $n_1^c = 1$ so the modular form h_μ is uniquely fixed by 5 coefficients. This model was first considered in [31], assuming the naive Ansatz (4.8) for the polar terms. Using GV invariants up to genus 26, Eq. (4.19) predicts

(B.14)
$$h_0 = q^{-\frac{44}{24}} \left(\underline{-4 + 432q - 10032q^2 + 148611456q^3} - \tfrac{1}{24} \operatorname{PT}(16, -36)_4 q^4 + \dots \right),$$
$$h_1 = q^{-\frac{44}{24}+\frac{5}{8}} \left(\underline{0} + \tfrac{1}{17} \operatorname{PT}(13, -26)q - \tfrac{1}{16} \operatorname{PT}(13, -25)q^2 + \dots \right),$$
$$h_2 = q^{-\frac{44}{24}+\frac{1}{2}} \left(\underline{0 - 2816q} + \tfrac{1}{19} \operatorname{PT}(14, -29)q^2 + \dots \right).$$

This is sufficient information to fully determine the generating series:

(B.15)
$$h_\mu = \frac{1}{\eta^{56}} \left[\frac{319E_4^5 E_6 + 113E_4^2 E_6^3}{11664} - \frac{146E_4^6 + 1025E_4^3 E_6^2 + 125E_6^4}{972} D \right.$$
$$\left. - \frac{566E_4^4 E_6 + 298E_4 E_6^3}{81} D^2 \right] \vartheta_\mu^{(4)},$$

with the following expansion:

(B.16)
$$h_0 = q^{-\frac{44}{24}} \left(\underline{-4 + 432q - 10032q^2 + 148611456q^3} + 53495321332q^4 \right.$$
$$+ 5858228664240q^5 + 338470263518000q^6 + +12773210724578176q^7$$
$$\left. + 352882974651781356q^8 \dots \right),$$
$$h_1 = q^{-\frac{44}{24}+\frac{5}{8}} \left(\underline{0 - 7424q} + 7488256q^2 + 7149513728q^3 + 1104027086592q^4 \right.$$
$$\left. + 78370863237632q^5 + 3411805769659904q^6 + \dots \right),$$
$$h_2 = q^{-\frac{44}{24}+\frac{1}{2}} \left(\underline{0 - 2816q} + 2167680q^2 + 3503031296q^3 + 619015800576q^4 \right.$$
$$\left. + 47430532268544q^5 + 2174342476769792q^6 + \dots \right).$$

In particular, the polar part agrees with the naive Ansatz (4.8) that was assumed in [31]. Assuming that (B.15) is correct, one can produce additional

Table 11: GV invariants $\mathrm{GV}_Q^{(g_C(Q)-\delta)}$ for $X_{4,4}$, assuming modularity

Q	g_C	$\delta = 0$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 4$	$\delta = 5$
1	1	0	3712				
2	2	0	1408	982464			
3	3	0	3712	6953728	683478144		
4	5	6	384	−12432	148208928	26841854688	699999511744
5	6	0	−22272	−14802048	7282971392	2161190443904	88647278203648
6	8	0	11264	6367872	−7046285440	773557598272	362668189458048
7	10	0	−37120	−29359104	21832649216	−2470237776768	278617066306304
8	13	32	2256	1728	742436816	−227235799678	21187753811008
9	15	0	59392	50769664	−44144389120	6476935523072	−435143766495232
10	18	0	28160	18608000	−28596423936	5125410035840	−407275256652416
11	21	0	89088	78985472	−74401243136	12415279501056	−983151655520000
12	25	64	3408	−88512	1931209232	−723035097878	91699351475728
13	28	0	−126208	−113249280	113072299008	−20742352242176	1847677262046464
14	32	0	56320	37806720	−65347039488	14118281042560	−1421834838533888
15	36	0	−170752	−153130496	160635374080	−32024183351808	3157850965939456
16	41	112	2704	−304000	3721068368	−1558763217664	236934426952368
17	45	0	222720	198005504	−217632888320	46941587427584	−5089550372194304
18	50	0	95744	62342016	−118393743616	29795537375872	−3617592752039168
19	55	0	282112	247161600	−284657253888	66297765350656	−7874526931335680
20	61	176	−2544	−535104	6123275152	−2835063806944	506679656992912
21	66	0	−348928	−299796992	362338625536	−91027164419584	11813373104231424
22	72	0	146432	90051200	−189003413760	55119444471424	−7968454696971008
23	78	0	−423168	−355020800	451332651008	−122204523786752	17290687427825664
24	85	256	−15792	−347712	9139883728	−4682106927504	976043266192272
25	91	0	504832	411853056	−552306662400	161054029205248	−24792539565154304
26	98	0	208384	118230912	−278503055616	93943868100224	−16042635201490176

boundary conditions for the direct integration method (see Table 11) allowing to reach genus 32. Furthermore, the term of order q^8 in h_0 is subject to Prop. 1 and provides an additional boundary condition at genus 33 that allows to push the direct integration up to genus 34. With this new data, we can further check that (B.16) is consistent up to orders q^3, q^4 and q^5 with (4.19), and even reproduce the coefficients of order q^5 in h_1 and q^6 in h_2 by applying (4.19) with $k = k_0 - 1$. Thus, there is overwhelming evidence that (B.15) (first conjectured in [31]) is indeed correct.

B.7. $X_{6,6}$

We now consider the complete intersection of degree $(6,6)$ in $\mathbb{P}^5_{3,3,2,2,1,1}$, In this case, $\kappa = 1$, $n_1^p = 1$ and $n_1^c = 0$ so the scalar modular form $h = h_0$ is uniquely fixed by a single coefficient. Since the leading coefficient is known,

Table 12: GV invariants $\mathrm{GV}_Q^{(g_C(Q)-\delta)}$ for $X_{6,6}$, assuming modularity

Q	g_C	$\delta = 0$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 4$	$\delta = 5$
1	2	1	360	67104			
2	4	−6	−928	291328	40692096	847288224	
3	7	−10	−1807	867414	−39992931	1253312442	254022248925
4	11	16	3054	−1752454	111434794	−3192574724	53221926192
5	16	24	4582	−2962836	226181014	−8162501599	181541450026
6	22	−34	−6284	4516784	−401198640	17316022722	−470838831620
7	29	−46	−8028	6434962	−657358676	33294527348	−1078394245876
8	37	60	9658	−8736900	1020136914	−59981343076	2281585927834
9	46	76	10994	−11438612	1519838840	−102904929012	4549405838854
10	56	−94	−11832	14549836	−2191738688	169716778670	−8644920617316
11	67	−114	−11944	18070914	−3076075680	270744962214	−15768167683888
12	79	136	11078	−21989312	4217848666	−419620144388	27750103729188
13	92	160	8958	−26275780	5666343644	−633965979716	47309465417064
14	106	−186	−5284	30880152	−7474321920	936139362212	−78385363446040
15	121	−214	268	35726786	−9696789948	1353997949560	−126558933123332

the generating series is necessarily [31]

(B.17)
$$
h = -\frac{2E_4 E_6}{\eta^{23}} = \mathrm{q}^{-\frac{23}{24}} \Big(-\underline{2} + 482\mathrm{q} + 282410\mathrm{q}^2 + 16775192\mathrm{q}^3 + 460175332\mathrm{q}^4 \\
+ 8112401426\mathrm{q}^5 + 106227128612\mathrm{q}^6 \\
+ 1118140132310\mathrm{q}^7 + \dots \Big).
$$

Using GV invariants up to genus 18, we can use Eq. (4.19) to confirm all terms up to (and including) q^3. The $\mathcal{O}(\mathrm{q}^2)$ coefficient can also be verified independently using Prop. 1. We note that the coefficient $J_1 = 482$ differs from the naive prediction $\chi_{\mathfrak{Y}}(\chi_D - 1) = -120$, due to the singular curve where the two degree-one homogeneous coordinates vanish simultaneously. Assuming (B.17) is correct, one can produce additional boundary conditions for the direct integration method (see Table 12), allowing to reach genus 22, beyond the genus 18 available by standard methods.

B.8. $X_{6,2}$

Finally, we consider the complete intersection of degree $(6,2)$ in $\mathbb{P}^5_{3,1,1,1,1}$, In this case, $\kappa = 4$, $n_1^p = 7$ and $n_1^c = 0$ so the modular form h_μ is uniquely fixed by 7 coefficients. This model was first considered in [31], assuming the

naive Ansatz (4.8) for the polar terms. Using GV invariants up to genus 47, Eq. (4.19) and Prop. 1 predict

(B.18)
$$
h_0 = q^{-\frac{56}{24}} \Big(\underline{5 - 1024q + 96390q^2} + 2412544q^3 + 79408559682q^4
$$
$$
+ 34353222823936q^5 + 4968007484511900q^6 + 389580600939126784q^7
$$
$$
+ 20087040094321343657q^8 + \dots \Big),
$$
$$
h_1 = q^{-\frac{56}{24}+\frac{5}{8}} \Big(\underline{0 + 14976q} - 2520960q^2 + 2887376128q^3 + 3893723178368q^4
$$
$$
+ 809149241398912q^5 + 78688042019771776q^6
$$
$$
+ 4713543813612260224q^7 + 19877072034145544 0256q^8 + \dots \Big),
$$
$$
h_2 = q^{-\frac{56}{24}+\frac{1}{2}} \Big(\underline{6 - 1536q} - 4647736q^2 + 621617152q^3 + 1986721226130q^4
$$
$$
+ 453923870489088_3 q^5 + \tfrac{1}{30}(-2242806300 - \mathrm{PT}(18,-43))q^6 + \dots \Big).
$$

In particular, the polar coefficients in h_2 differ from the values $16 - 4608q$ predicted by the naive Ansatz (4.8). There is indeed a unique modular form which fits this vastly overdetermined set of coefficients:

(B.19)
$$
h_\mu = \frac{1}{\eta^{68}} \Bigg[-\frac{5(727E_4^8 + 3322E_4^5 E_6^2 + 1135E_4^2 E_6^4)}{559872}
$$
$$
+ \frac{2409E_4^6 E_6 + 5830E_4^3 E_6^3 + 401E_6^5}{5184} D
$$
$$
+ \frac{2519E_4^7 + 17978E_4^4 E_6^2 + 5423E_4 E_6^4}{1944} D^2 \Bigg] \vartheta_\mu^{(4)}.
$$

Assuming that (B.19) is correct, one can produce additional boundary conditions for the direct integration method (see Table 13), allowing in principle to reach genus 78, beyond the genus 63 available by standard methods.

Acknowledgments

Table 13: GV invariants $\mathrm{GV}_Q^{(g_C(Q)-\delta)}$ for $X_{6,2}$, assuming modularity

Q	g_C	$\delta = 0$	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 4$	$\delta = 5$
1	1	0	4992				
2	2	−4	−504	2388768			
3	3	0	14976	1228032	2732060032		
4	5	10	1456	87376	−13098688	79275664800	4599616564224
5	6	0	−59904	−7098624	−5731751168	3921835430016	633074010435840
6	8	−36	−7176	18680344	1776341072	−3978452463012	482407033529880
7	10	0	−89856	−11017344	−11354017792	11762488063616	−1739233315959552
8	13	45	7112	505904	−26309632	397727244436	−143734260919104
9	15	0	134784	16463616	19737387904	−23804240518144	4517211164682496
10	18	−72	−14448	46979932	4192707384	−16032046818880	3592452810930880
11	21	0	194688	22853376	30801879680	−40000460268544	8544691377500288
12	25	85	12304	783986	−82475800	1034830295100	−453574307495648
13	28	0	−269568	−28838784	−44399031040	60578209825920	−14063973371548160
14	32	−132	−24072	94648652	6368703752	−36273186323128	9662743440528720
15	36	0	−359424	−33541248	−60489784576	85729492457728	−21374438244874240
16	41	145	17072	812714	−187672824	1995257447554	−967713951224848
17	45	0	464256	35702784	79038018176	−115677339047680	30834343468656896
18	50	−216	−32592	162178504	5450058896	−64779929898136	19857403612132080
19	55	0	584064	33885696	100058488192	−150674169631488	42863166970805504
20	61	225	18056	407662	−336445208	3284718881588	−1740135148941248
21	66	0	−718848	−26472576	−123640960768	191002891152640	−57945747882503680
22	72	−324	−35400	249881836	−2285410296	−101366274190332	35739934943323496
23	78	0	−868608	−11666304	−149974972160	236979848907520	−76636691924665088
24	85	325	10936	−301906	−502103864	4910704750656	−2840815244088832
25	91	0	1033344	−12509952	179377684096	−288959785268736	99565363235000576

References

[1] A. Strominger and C. Vafa, "Macroscopic Entropy of $N = 2$ Extremal Black Holes," *Phys. Lett.* **B383** (1996) 39–43, hep-th/9602111. MR1399716

[2] M. Kontsevich and Y. Soibelman, "Stability structures, motivic Donaldson-Thomas invariants and cluster transformations," 0811.2435. MR2681792

[3] D. Joyce and Y. Song, "A theory of generalized Donaldson-Thomas invariants," *Memoirs of the Am. Math. Soc.* **217** (2012), no. 1020, 0810.5645. MR2951762

[4] Y. Toda, "Bogomolov-Gieseker type inequality and counting invariants," 1112.3411. MR3029426

[5] S. Feyzbakhsh and R. P. Thomas, "Curve counting and S-duality," *Épijournal de Géométrie Algébrique* **7** (2023) 2007.03037. MR4603486

[6] S. Feyzbakhsh and R. P. Thomas, "Rank r DT theory from rank 0," 2103.02915. MR4583775

[7] S. Feyzbakhsh and R. P. Thomas, "Rank r DT theory from rank 1," 2108.02828. MR4583775

[8] S. Feyzbakhsh, "Explicit formulae for rank zero DT invariants and the OSV conjecture," 2203.10617.

[9] A. Bayer, E. Macrì, and Y. Toda, "Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities," *Journal of Algebraic Geometry* **23** (2014) 117–163, 1103.5010. MR3121850

[10] A. Bayer, E. Macrì, and P. Stellari, "The space of stability conditions on abelian threefolds, and on some Calabi-Yau threefolds," *Inventiones mathematicae* **206** (2016), no. 3, 869–933. MR3573975

[11] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, "Gromov-Witten theory and Donaldson-Thomas theory. I," *Compos. Math.* **142** (2006), no. 5, 1263–1285. MR2264664

[12] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Holomorphic anomalies in topological field theories," *Nucl. Phys.* **B405** (1993) 279–304, hep-th/9302103. MR1240687

[13] M.-x. Huang, A. Klemm, and S. Quackenbush, *Topological string theory on compact Calabi-Yau: Modularity and boundary conditions*, pp. 1–58. Springer, 2008. hep-th/0612125. MR2596635

[14] T. W. Grimm, A. Klemm, M. Marino, and M. Weiss, "Direct Integration of the Topological String," *JHEP* **08** (2007) 058, hep-th/0702187.

[15] H. Ooguri, A. Strominger, and C. Vafa, "Black hole attractors and the topological string," *Phys. Rev.* **D70** (2004) 106007, hep-th/0405146. MR2123156

[16] F. Denef and G. W. Moore, "Split states, entropy enigmas, holes and halos," *JHEP* **1111** (2011) 129, hep-th/0702146. MR2913216

[17] J. M. Maldacena, A. Strominger, and E. Witten, "Black hole entropy in M-theory," *JHEP* **12** (1997) 002, `hep-th/9711053`. MR1608487

[18] J. de Boer, M. C. N. Cheng, R. Dijkgraaf, J. Manschot, and E. Verlinde, "A farey tail for attractor black holes," *JHEP* **11** (2006) 024, `hep-th/0608059`. MR2270426

[19] D. Gaiotto, A. Strominger, and X. Yin, "The M5-brane elliptic genus: Modularity and BPS states," *JHEP* **08** (2007) 070, `hep-th/0607010`. MR2342043

[20] D. Gaiotto and X. Yin, "Examples of M5-Brane Elliptic Genera," *JHEP* **11** (2007) 004, `hep-th/0702012`. MR2362145

[21] J. Manschot and G. W. Moore, "A Modern Fareytail," *Commun. Num. Theor. Phys.* **4** (2010) 103–159, `0712.0573`. MR2679378

[22] S. Alexandrov, S. Banerjee, J. Manschot, and B. Pioline, "Multiple D3-instantons and mock modular forms I," *Commun. Math. Phys.* **353** (2017), no. 1, 379–411, `1605.05945`. MR3638318

[23] S. Alexandrov, S. Banerjee, J. Manschot, and B. Pioline, "Multiple D3-instantons and mock modular forms II," *Commun. Math. Phys.* **359** (2018), no. 1, 297–346, `1702.05497`. MR3781452

[24] S. Alexandrov and B. Pioline, "Black holes and higher depth mock modular forms," *Commun. Math. Phys.* **374** (2019), no. 2, 549–625, `1808.08479`. MR4072224

[25] J. Manschot, "Stability and duality in N=2 supergravity," *Commun. Math. Phys.* **299** (2010) 651–676, `0906.1767`. MR2718927

[26] M. Alim, B. Haghighat, M. Hecht, A. Klemm, M. Rauch, and T. Wotschke, "Wall-crossing holomorphic anomaly and mock modularity of multiple M5-branes," *Commun. Math. Phys.* **339** (2015), no. 3, 773–814, `1012.1608`. MR3385984

[27] A. Dabholkar, S. Murthy, and D. Zagier, "Quantum Black Holes, Wall Crossing, and Mock Modular Forms," `1208.4074`.

[28] M. C. N. Cheng, J. F. R. Duncan, S. M. Harrison, J. A. Harvey, S. Kachru, and B. C. Rayhaun, "Attractive Strings and Five-Branes, Skew-Holomorphic Jacobi Forms and Moonshine," *JHEP* **07** (2018) 130, `1708.07523`. MR3858593

[29] A. Collinucci and T. Wyder, "The Elliptic genus from split flows

and Donaldson-Thomas invariants," *JHEP* **05** (2010) 081, 0810.4301.
MR2721126

[30] W. Van Herck and T. Wyder, "Black Hole Meiosis," *JHEP* **04** (2010)
047, 0909.0508.

[31] S. Alexandrov, N. Gaddam, J. Manschot, and B. Pioline, "Modular
bootstrap for D4-D2-D0 indices on compact Calabi-Yau threefolds,"
to appear in *Adv. Theo. Math. Phys.* (2024), 2204.02207.

[32] "One Parameter Calabi-Yau higher genus data," http://www.th.
physik.uni-bonn.de/Groups/Klemm/data.php.

[33] A. Joshi and A. Klemm, "Swampland Distance Conjecture for One-
Parameter Calabi-Yau Threefolds," *JHEP* **08** (2019) 086, 1903.00596.
MR4013358

[34] M. R. Douglas, B. Fiol, and C. Romelsberger, "Stability and BPS
branes," *JHEP* **0509** (2005) 006, hep-th/0002037. MR2171358

[35] M. R. Douglas, "D-branes, categories and N = 1 supersymmetry," *J.
Math. Phys.* **42** (2001) 2818–2843, hep-th/0011017. MR1840318

[36] P. S. Aspinwall, "D-branes on Calabi-Yau manifolds," *Theoretical Ad-
vanced Study Institute in Elementary Particle Physics (TASI 2003):
Recent Trends in String Theory* (2004) 1–152, hep-th/0403166.

[37] G. V. Ravindra and V. Srinivas, "The Grothendieck-Lefschetz theorem
for normal projective varieties," *J. Algebraic Geom.* **15** (2006), no. 3,
563–590. MR2219849

[38] S. Alexandrov, D. Persson, and B. Pioline, "Fivebrane instantons,
topological wave functions and hypermultiplet moduli spaces," *JHEP*
1103 (2011) 111, 1010.5792. MR2821100

[39] T. Bridgeland, "Stability conditions on triangulated categories," *Ann.
of Math. (2)* **166** (2007), no. 2, 317–345. MR2373143

[40] K. Behrend, "Donaldson-thomas type invariants via microlocal ge-
ometry," *Annals of Mathematics* (2009) 1307–1338, math/0507523.
MR2600874

[41] V. Bouchard, T. Creutzig, D.-E. Diaconescu, C. Doran, C. Quigley,
and A. Sheshmani, "Vertical D4–D2–D0 Bound States on K3 Fibra-
tions and Modularity," *Commun. Math. Phys.* **350** (2017), no. 3, 1069–
1121, 1601.04030. MR3607470

[42] S. Galkin, V. Golyshev, and H. Iritani, "Gamma classes and quantum cohomology of fano manifolds: gamma conjectures," *Duke Mathematical Journal* **165** (2016), no. 11, 2005–2077. MR3536989

[43] S. Feyzbakhsh and R. P. Thomas, "An application of wall-crossing to Noether-Lefschetz loci," *Q. J. Math.* **72** (2021), no. 1-2, 51–70. with an appendix by C. Voisin. MR4271380

[44] C. Li, "On stability conditions for the quintic threefold," *Inventiones mathematicae* **218** (2019), no. 1, 301–340, 1810.03434. MR3994590

[45] S. Liu, "Stability condition on Calabi-Yau threefold of complete intersection of quadratic and quartic hypersurfaces," in *Forum of Mathematics, Sigma*, vol. 10, p. e106, Cambridge University Press. 2022. 2108.08934. MR4519060

[46] N. Koseki, "Stability conditions on calabi-yau double/triple solids," in *Forum of Mathematics, Sigma*, vol. 10, p. e63, Cambridge University Press. 2022. 2007.00044. MR4466641

[47] V. V. Batyrev and L. A. Borisov, "On Calabi-Yau complete intersections in toric varieties," alg-geom/9412017. MR1463173

[48] S. Alexandrov, J. Manschot, and B. Pioline, "D3-instantons, Mock Theta Series and Twistors," *JHEP* **1304** (2013) 002, 1207.1109. MR3065975

[49] S. Alexandrov, J. Manschot, and B. Pioline, "S-duality and refined BPS indices," *Commun. Math. Phys.* **380** (2020), no. 2, 755–810, 1910.03098. MR4170291

[50] M. C. N. Cheng and J. F. R. Duncan, "Rademacher Sums and Rademacher Series," *Contrib. Math. Comput. Sci.* **8** (2014) 143–182, 1210.3066. MR3559204

[51] P. Bantay and T. Gannon, "Vector-valued modular functions for the modular group and the hypergeometric equation," *Commun. Num. Theor. Phys.* **1** (2007) 651–680, 0705.2467. MR2412268

[52] J. Manschot, "On the space of elliptic genera," *Commun. Num. Theor. Phys.* **2** (2008) 803–833, 0805.4333. MR2492199

[53] R. P. Thomas, "A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on $K3$ fibrations," *J. Differential Geom.* **54** (2000), no. 2, 367–438. MR1818182

[54] R. Pandharipande and R. P. Thomas, "13/2 ways of counting curves," *Lond. Math. Soc. Lect. Note Ser.* **411** (2014) 282–333, `1111.1552`. MR3221298

[55] R. Pandharipande and R. P. Thomas, "Curve counting via stable pairs in the derived category," *Inventiones mathematicae* **178** (2009), no. 2, 407–447. MR2545686

[56] Y. Toda, "Curve counting theories via stable objects I. DT/PT correspondence," *Journal of the American Mathematical Society* **23** (2010), no. 4, 1119–1157. MR2669709

[57] T. Bridgeland, "Hall algebras and curve-counting invariants," *Journal of the American Mathematical Society* **24** (2011), no. 4, 969–998. MR2813335

[58] R. Gopakumar and C. Vafa, "M theory and topological strings. 1.," `hep-th/9809187`.

[59] R. Gopakumar and C. Vafa, "M theory and topological strings. 2.," `hep-th/9812127`.

[60] D. Maulik and Y. Toda, "Gopakumar–vafa invariants via vanishing cycles," *Inventiones mathematicae* **213** (Apr., 2018) 1017–1097. MR3842061

[61] M.-X. Huang, S. Katz, and A. Klemm, "Towards refining the topological strings on compact Calabi-Yau 3-folds," *JHEP* **03** (2021) 266, `2010.02910`. MR4260962

[62] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes," *Commun. Math. Phys.* **165** (1994) 311–428, `hep-th/9309140`. MR1301851

[63] E.-N. Ionel and T. Parker, "The Gopakumar-Vafa formula for symplectic manifolds," *Ann. Math. (2)* **187** (2018), no. 1, 1–64, `1306.1516`. MR3739228

[64] A. Doan, E.-N. Ionel, and T. Walpuski, "The Gopakumar-Vafa finiteness conjecture," `2103.08221`.

[65] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, "Gromov-Witten theory and Donaldson-Thomas theory. II," *Compos. Math.* **142** (2006), no. 5, 1286–1304. MR2264665

[66] A. Iqbal, N. Nekrasov, A. Okounkov, and C. Vafa, "Quantum foam and topological strings," *JHEP* **04** (2008) 011, `hep-th/0312022`. MR2425292

[67] R. Dijkgraaf, C. Vafa, and E. Verlinde, "M-theory and a topological string duality," `hep-th/0602087`. MR1935579

[68] R. Pandharipande and A. Pixton, "Gromov-Witten/Pairs correspondence for the quintic 3-fold," *J. Am. Math. Soc.* **30** (2017), no. 2, 389–449, `1206.5490`. MR3600040

[69] C. C. Cadogan, "The Möbius function and connected graphs," *Journal of Combinatorial Theory, Series B* **11** (1971) 193–200. MR0284377

[70] G. Castelnuovo, "Ricerche di geometria sulle curve algebriche," *Atti della Accademia delle Scienze di Torino* **24** (1889) 346–373.

[71] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*. John Wiley & Sons, Inc., Aug., 1994. MR1288523

[72] E. Macrì and B. Schmidt, "Derived categories and the genus of space curves," *Algebr. Geom.* **7** (2020), no. 2, 153–191. MR4061326

[73] Z. Liu and Y. Ruan, "Castelnuovo bound and higher genus Gromov-Witten invariants of quintic 3-folds," `2210.13411`.

[74] S. Yamaguchi and S.-T. Yau, "Topological string partition functions as polynomials," *JHEP* **07** (2004) 047, `hep-th/0406078`. MR2095047

[75] M. Alim and J. D. Lange, "Polynomial Structure of the (Open) Topological String Partition Function," *JHEP* **10** (2007) 045, `0708.2886`. MR2357933

[76] S. Hosono, "BCOV ring and holomorphic anomaly equation," *Adv. Stud. Pure Math.* **59** (2008) 79, `0810.4795`. MR2683207

[77] M. Alim, J. D. Lange, and P. Mayr, "Global Properties of Topological String Amplitudes and Orbifold Invariants," *JHEP* **03** (2010) 113, `0809.4253`. MR2653440

[78] C. Vafa, "A Stringy test of the fate of the conifold," *Nucl. Phys. B* **447** (1995) 252–260, `hep-th/9505023`. MR1349476

[79] A. Klemm, "The B-model approach to topological string theory on Calabi-Yau n-folds," in *B-model Gromov-Witten theory*, Trends Math., pp. 79–397. Birkhäuser/Springer, Cham, 2018. MR3965409

[80] S. Katz, A. Klemm, T. Schimannek, and E. Sharpe, "Topological Strings on Non-Commutative Resolutions," `2212.08655`. MR4709101

[81] S. H. Katz, A. Klemm, and C. Vafa, "M theory, topological strings and spinning black holes," *Adv. Theor. Math. Phys.* **3** (1999) 1445–1537, `hep-th/9910181`. MR1796683

[82] Y.-H. Kiem and J. Li, "Categorification of Donaldson-Thomas invariants via perverse sheaves," `1212.6444`.

[83] S. Hosono, M.-H. Saito, and A. Takahashi, "Relative Lefschetz action and BPS state counting," *Int. Math. Res. Not.* **15** (2001) 783–816, `math/0105148`. MR1849482

[84] Y. Toda, "Gopakumar-Vafa invariants and wall-crossing," *Journal of Differential Geometry* **123** (2023), no. 1, 141–193, `1710.01843`. MR4564413

[85] S. H. Katz, "Genus zero Gopakumar-Vafa invariants of contractible curves," *J. Diff. Geom.* **79** (2008), no. 2, 185–195, `math/0601193`. MR2420017

[86] S.-T. Yau and E. Zaslow, "BPS states, string duality, and nodal curves on K3," *Nucl. Phys. B* **471** (1996) 503–512, `hep-th/9512121`. MR1398633

[87] R. Pandharipande and R. P. Thomas, "Stable pairs and BPS invariants," *Electron. Res. Announ. AMS* **23** (2010) 267–297, `0711.3899`. MR2552254

[88] J. Choi, S. Katz, and A. Klemm, "The refined BPS index from stable pair invariants," *Commun. Math. Phys.* **328** (2014) 903–954, `1210.4403`. MR3201216

[89] N. Gaddam, "Elliptic genera from multi-centers," *JHEP* **05** (2016) 076, `1603.01724`. MR3521795

[90] F. Rodriguez Villegas and D. Zagier, "Square roots of central values of hecke l-series.," in *Advances in number theory. The proceedings of the third conference of the Canadian Number Theory Association, held at Queen's University, Kingston, Canada, August 18–24, 1991*, pp. 81–99, Oxford: Clarendon Press. 1993. MR1368412

[91] L. Göttsche, "Theta functions and Hodge numbers of moduli spaces of sheaves on rational surfaces," *Comm. Math. Phys.* **206** (1999) 105, `9808007`. MR1736989

[92] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces.* No. 18. American Mathematical Soc., 1999. MR1711344

[93] S. Alexandrov, S. Feyzbakhsh, A. Klemm, and B. Pioline, "Quantum geometry and mock modularity," 2312.12629.

[94] A. Klemm, J. Manschot, and T. Wotschke, "Quantum geometry of elliptic Calabi-Yau manifolds," 1205.1795. MR3068410

[95] M. Alim and E. Scheidegger, "Topological Strings on Elliptic Fibrations," *Commun. Num. Theor. Phys.* **08** (2014) 729–800, 1205.1784. MR3318389

[96] M.-x. Huang, S. Katz, and A. Klemm, "Topological String on elliptic CY 3-folds and the ring of Jacobi forms," *JHEP* **10** (2015) 125, 1501.04891. MR3435514

[97] R. Dijkgraaf, J. M. Maldacena, G. W. Moore, and E. P. Verlinde, "A Black hole Farey tail," hep-th/0005003.

[98] A. Dabholkar, F. Denef, G. W. Moore, and B. Pioline, "Exact and asymptotic degeneracies of small black holes," *JHEP* **08** (2005) 021, hep-th/0502157. MR2166317

[99] A. Dabholkar, F. Denef, G. W. Moore, and B. Pioline, "Precision counting of small black holes," *JHEP* **10** (2005) 096, hep-th/0507014. MR2180982

[100] A. Dabholkar, J. Gomes, and S. Murthy, "Localization & Exact Holography," *JHEP* **04** (2013) 062, 1111.1161. MR3065907

[101] A. Dabholkar, J. Gomes, and S. Murthy, "Nonperturbative black hole entropy and Kloosterman sums," *JHEP* **03** (2015) 074, 1404.0033. MR3340530

[102] L. V. Iliesiu, S. Murthy, and G. J. Turiaci, "Black hole microstate counting from the gravitational path integral," 2209.13602.

[103] F. Ferrari and V. Reys, "Mixed Rademacher and BPS Black Holes," *JHEP* **07** (2017) 094, 1702.02755. MR3686704

[104] G. Lopes Cardoso, A. Kidambi, S. Nampuri, V. Reys, and M. Rosselló, "The Gravitational Path Integral for $N = 4$ BPS Black Holes from Black Hole Microstate Counting," *Annales Henri Poincare* **24** (2023), no. 10, 3305–3346, 2211.06873. MR4642264

[105] S. Murthy and V. Reys, "Quantum black hole entropy and the holomorphic prepotential of N=2 supergravity," *JHEP* **10** (2013) 099, 1306.3796. MR3117718

[106] S. Murthy and V. Reys, "Functional determinants, index theorems, and exact quantum black hole entropy," *JHEP* **12** (2015) 028, 1504.01400. MR3464774

[107] J. Gomes, H. Het Lam, and G. Mathys, "On the exact entropy of $\mathcal{N} = 2$ black holes," 1912.00029.

[108] S. Feyzbakhsh, "An effective restriction theorem via wall-crossing and Mercat's conjecture," *Math. Z.* **301** (2022), no. 4, 4175–4199, 1608.07825. MR4449744

[109] A. Bayer, S. Beentjes, S. Feyzbakhsh, G. Hein, D. Martinelli, F. Rezaee, and B. Schmidt, "The desingularization of the theta divisor of a cubic threefold as a moduli space," *to appear in Geometry & Topology* (2022) 2011.12240. MR4711835

SERGEI ALEXANDROV
LABORATOIRE CHARLES COULOMB (L2C)
UNIVERSITÉ DE MONTPELLIER, CNRS
F-34095, MONTPELLIER
FRANCE
E-mail address: sergey.alexandrov@umontpellier.fr

SOHEYLA FEYZBAKHSH
IMPERIAL COLLEGE
LONDON SW7 2AZ
UNITED KINGDOM
E-mail address: s.feyzbakhsh@imperial.ac.uk

ALBRECHT KLEMM
BETHE CENTER FOR THEORETICAL PHYSICS
AND HAUSDORFF CENTER FOR MATHEMATICS
UNIVERSITÄT BONN
D-53115, GERMANY
E-mail address: aoklemm@th.physik.uni-bonn.de

BORIS PIOLINE
LABORATOIRE DE PHYSIQUE THÉORIQUE ET HAUTES ENERGIES
CNRS AND SORBONNE UNIVERSITÉ
CAMPUS PIERRE ET MARIE CURIE
4 PLACE JUSSIEU, F-75005, PARIS
FRANCE
E-mail address: pioline@lpthe.jussieu.fr

Thorsten Schimannek
Laboratoire de Physique Théorique et Hautes Energies
CNRS and Sorbonne Université
Campus Pierre et Marie Curie
4 place Jussieu, F-75005, Paris
France
E-mail address: schimannek@lpthe.jussieu.fr