On motivic and arithmetic refinements of Donaldson-Thomas invariants

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In recent years, a version of enumerative geometry over arbitrary fields has been developed and studied by Kass-Wickelgren, Levine, and others, in which the counts obtained are not integers but quadratic forms. Aiming to understand the relation to other "refined invariants", and especially their possible interpretation in quantum theory, we explain how to obtain a quadratic version of Donaldson-Thomas invariants from the motivic invariants defined in the work of Kontsevich and Soibelman and pose some questions. We calculate these invariants in a few simple examples that provide standard tests for these questions, including degree zero invariants of $\mathbb{A}^3$ and higher-genus Gopakumar-Vafa invariants recently studied by Liu and Ruan. The comparison with known real and complex counts plays a central role throughout.

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1. Introduction

Over the years, it has been a fruitful endeavour to elucidate the structure of enumerative invariants attached to various classes of problems stemming from algebraic geometry, symplectic geometry, and theoretical physics, see e.g. [31]. One of the common themes in this context has been the identification of problems attached to distinct geometric or algebraic setups ("duality"). Such an identification can uncover otherwise unexpected properties, such as regularity, recursions, or modularity, and often allows for exact solutions. More recently, it has been of interest to compare invariants attached to the same geometry, but defined in different theories, relying on higher structures or hidden symmetries ("refinement"). It is ultimately reasonable

to expect, if not outright necessary for coherence, that any refinement or structure present in one setup or theory can be ported to all the others, and any missing piece is as much an opportunity for further progress.

In this note, we will be concerned mainly with the youngest type of "arithmetic invariants" which are defined in $\mathbb{A}^1$-enumerative geometry and take values in the Grothendieck-Witt ring of quadratic forms, and the somewhat older "motivic invariants", which take values in the Grothendieck ring of varieties. We briefly recall their salient features here, and their more proper definition in the main text.

$\mathbb{A}^1$-enumerative geometry consists in applying the machinery of so-called $\mathbb{A}^1$-homotopy theory to classical enumerative problems. This machinery was introduced in the 90s, in the work of Morel and Voevodsky [29]. The main idea is to consider the affine line $\mathbb{A}^1$ rather than the interval $[0, 1] \subset \mathbb{R}$ as a parameter space and thereby construct a homotopy theory for schemes. This allows us to define an algebraic or $\mathbb{A}^1$-degree for maps between schemes, which naturally generalizes the classical degree from algebraic topology. While the natural ring where this degree is defined is not $\mathbb{Z}$ but rather the Grothendieck-Witt ring of quadratic forms, the theory has the highly attractive feature of being defined over arbitrary fields, rather than just the complex or real numbers. The latter, numerical, counts can be recovered as special cases from rank and signature of the quadratic form. Problems which have been solved in this theory include the classical enumeration of lines on a cubic surface [16], lines on a quintic threefold [32] and the enumerative geometry of twisted cubics [25]. The latter study in particular relies on Atiyah-Bott localization, which was extended to this context in [23]. The results for the quintic specialize to the real counts first obtained in [34] and [36]. Even more recently, a refined enumeration of rational curves through points in a del Pezzo surface was obtained in [14], which in particular refines the classical counts of degree d rational curves in $\mathbb{P}^2$ trough $3d - 1$ points.

Motivic Donaldson-Thomas (DT) invariants were introduced from a rather different direction by Kontsevich and Soibelman [18]. Their main motivation was to understand wall-crossing phenomena in the theory of BPS states and specifically the enumerative geometry of Calabi-Yau 3-folds. The most basic observation in this context is that enumerative invariants arise from a perfect obstruction theory on a moduli space (equipped with a stability condition), and the basic idea is to identify that algebraic obstruction class directly as a "motivic invariant", while the standard "numerical" DT-invariants can be recovered via the Euler Characteristic morphism (or motivic integration).

Apart from circumstantial evidence in the B-model [20], an early indication that the "refined" theory might be related to an enumerative geometry over fields other than the complex numbers appeared in work of Göttsche and Shende [13] on the enumerative geometry of curves on surfaces. The main observation in this regard was that the real count, defined via the compactly supported Euler characteristic, appears in a special limit of the refinement parameter that is different from the standard complex one.

Our goal here is to shed light on the relationship between these two generalizations of classical enumerative invariants, and on how they arise by composing the complex and real counts, which are either known or accessible by more standard methods. The basic mathematical idea is to utilize the compactly supported $\mathbb{A}^1$-Euler characteristic

$$\chi_c^{\mathbb{A}^1} : K_0(\mathit{Var}(k)) \to GW(k) \, ,$$

which is already considered for a similar purpose in the papers [2] and [33], to go from motivic to arithmetic invariants, in a way that is compatible with taking numerical degrees on the two sides. We illustrate the discussion by two examples which to our knowledge have not been studied in the arithmetic context before. The first is the Hilbert scheme of points on $\mathbb{A}^3$, for which the motivic DT invariants were computed in the paper [6]. Using their formula for the motivic partition function of the Hilbert scheme of points of $\mathbb{C}^3$ and passing to the arithmetic invariants, we get the expected result over $\mathbb{R}$ (as computed in [19]) and, as a byproduct, a count that would make sense over any field. Our second example is based on the recent calculation of Gopakumar-Vafa invariants at the Castelnuovo-Mumford bound by Liu-Ruan [27] and Alexandrov et al. in [1] and also leads to a prediction for a count over any field.

In all these examples, the knowledge of the real (and of course, the complex) count is sufficient to determine the arithmetic invariant (i.e., the value of the invariant over any field). On the other hand, the real enumerative invariants admit a physics interpretation, which mimics the relationship between real and open Gromov-Witten theory, as "open" or "relative" BPS invariants. This is indicative of a much deeper relationship between arithmetic refinements of enumerative geometry and the physics of BPS states. In fact, our interest in the arithmetic invariants arises in part from the observations, recorded in [37] and otherwise, that enumerative predictions of mirror symmetry depend in general on an arithmetic structure of the mirror manifold, and that this is especially true in the context of relative invariants. These observations have not been explained from the motivic point of view.

An initial idea for the physics interpretation of the quadratic form is the occurrence of the Eisenbud–Levine–Khimshiashvili class as local $\mathbb{A}^1$ degree [15]. In such examples, the function whose critical locus defines the enumerative invariant enters as superpotential into the action functional of the physical theory. A critical point defines a supersymmetric vacuum, and a refinement via the Hessian would correspond to keeping track of the masses of the physical fields.

In [22], Levine has given an alternative definition of DT invariants valued in $GW(k)$, based on obstruction theory. The construction is an $\mathbb{A}^1$-homotopy version of the intrinsic normal cone used by Behrend and Fantechi [7] and leads to a virtual class whose degree lies in $GW(k)$. The relationship between those invariants and the ones defined here is not entirely clear to us, but we expect them to be closely related, see section 3.4.

To summarize, the main purpose of this paper is not so much to claim any new fundamental insight, but merely to point out that the arithmetic and motivic refinements are too closely related for this to be an accident, that this applies specifically in a class of examples that are highly relevant in the context of BPS state counting and M-theory dualities, and thereby to pose the question for a physics interpretation of all these structures.

2. Motivic DT invariants

2.1. The Grothendieck ring of varieties

Let k be a field of characteristic zero.

Definition 2.1.1. *The Grothendieck ring of varieties is the abelian group generated by isomorphism classes of all varieties over k modulo relations*

$$[X] = [Y] + [X \setminus Y],$$

where Y is closed in X.

The product structure that makes $K_0(Var(k))$ a ring is given by the usual Cartesian product $[X] \cdot [Y] = [X \times Y]$. The class of the affine line $[\mathbb{A}^1_k]$ is denoted by $\mathbb{L}$ in $K_0(Var(k))$ and we set

$$\mathcal{M}_k := K_0(Var(k))[\mathbb{L}^{-\frac{1}{2}}],$$

which will be called ring of motivic weights.

Remark 2.1.2. *The following statements are true:*

- If $f : X \to S$ is a Zariski locally trivial fibration with fiber F, then $[X] = [S] \cdot [F]$.
- If $f : X \to Y$ is a bijective morphism, it is true that $[X] = [Y]$.

With this notation, the classes of cellular varieties can be computed in terms of $\mathbb{L}$, for example,

$$[\mathbb{P}^n] = 1 + \mathbb{L} + \mathbb{L}^2 + \cdots + \mathbb{L}^n$$

and a similar expression can be written for the Grassmanians:

$$[Gr(n,k)] = \frac{(\mathbb{L}^n - 1)(\mathbb{L}^{n-1} - 1)\cdots(\mathbb{L} - 1)}{(\mathbb{L}^{n-k})(\mathbb{L}^{n-k-1})\cdots(\mathbb{L} - 1) \cdot (\mathbb{L}^k - 1)(\mathbb{L}^{k-1} - 1)\cdots(\mathbb{L} - 1)}.$$

Remark 2.1.3. *Notice that, over non algebraically closed fields, besides the rational points, varieties have also points which correspond to Galois orbits over the algebraic closure. This gives rise to different classes of points in $K_0(Var(k))$, which are the classes of $\mathrm{Spec}(\mathrm{L})$, with $L \supset k$.*

Example 2.1.4. *Over $\mathbb{R}$, there are two classes of points: $[\mathrm{Spec}\mathbb{R}]$ and $[\mathrm{Spec}\mathbb{C}]$. Therefore, although $\mathbb{P}^1(\mathbb{R})$ is a circle, there are non rational points which correspond to pairs of complex conjugate points. Indeed, $\mathbb{P}^1$ is not isomorphic to $X = \{x^2 + y^2 = 1\} \subset \mathbb{A}^2$, since there are two complex conjugate points "at infinity". In terms of the Grothendieck ring of varieties, we get $[X] = \mathbb{L} + 1 - [\mathrm{Spec}\mathbb{C}] \in K_0(Var(\mathbb{R}))$.*

Over $k = \mathbb{C}$, there is a morphism to $\mathbb{Z}$ given by the Euler characteristic with compact support.

$$\chi_c : K_0(Var(\mathbb{C})) \to \mathbb{Z}$$

which can be extended to a morphism

$$\chi_c^{\mathbb{C}} : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}$$

if we send $\mathbb{L}^{\frac{1}{2}}$ to -1 (since $\mathbb{L}$ goes to 1). The same can be done for considering the Euler characteristic of the real points. In Section 3, we will introduce the $\mathbb{A}^1$-Euler characteristic, which is defined for varieties over any field. Notice that we have to be careful when extending to $\mathcal{M}_{\mathbb{R}}$. As $\chi_c^{\mathbb{R}}(\mathbb{L}) = -1$, the extended Euler characteristic takes values in $\mathbb{Z}[i]$, where $i = \sqrt{-1}$.

$$\chi_c^{\mathbb{R}} : \mathcal{M}_{\mathbb{R}} \to \mathbb{Z}[i]$$

If one does not want to consider compactly supported Euler characteristics, one could actually only consider projective varieties. In fact, the abelian group generated by projective varieties with relations given by

$$[X] - [Y] = [\mathrm{Bl}_Y X] - [E] \tag{1}$$

$$[\varnothing] = 0 \tag{2}$$

where $\mathrm{Bl}_Y X$ is the blow up of X along Y and E is the exceptional divisor, is isomorphic to $K_0(Var(k))$. For details, see Theorem 3.1 of [8].

2.2. Donaldson-Thomas invariants

We consider Donaldson-Thomas invariants following [28]. Those are simply numbers[1] obtained by integration against virtual classes of moduli spaces of ideal sheaves in the complex setting. If M is a smooth Calabi-Yau threefold, any ideal sheaf defines a subscheme N. Then, the moduli spaces we are interested in are the ones of the form $X = I_n(M, \beta)$ and consist of sheaves for which N represents the homology class β and $\chi(\mathcal{O}_N) = n$. Usually, we are interested in the cases for which N has dimension at most one, which implies that β is in $H_2(M)$. $I_n(M, \beta)$ is, then, isomorphic to a Hilbert Scheme of curves in M. In particular, for the case $\beta = 0$, we are considering exactly the situation in which N has dimension 0, that is, $I_n(M, 0)$ is the Hilbert Scheme of points of M.

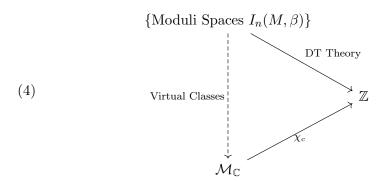The DT invariants of M are defined by integrals:

$$DT(n, \beta) = \int_{[I_n(M,\beta)]^{vir}} 1 \tag{3}$$

where $[I_n(M, \beta)]^{vir}$ is a virtual class in the homology defined via obstruction theory. It was proved, in [5], to depend only on the scheme structure of the moduli space and not on the chosen obstruction theory. In particular, if the moduli space ends up being smooth, this virtual class is, up to a factor $(-1)^{\dim I_n(M,\beta)}$, simply the Poincaré dual of the top Chern class, which means that the integral above is given, up to sign, by the Euler characteristic with compact support of the moduli space.

A natural way of refining such invariants is, therefore, to consider these virtual classes as elements in the Grothendieck ring of varieties, since the Euler characteristic gives us a natural morphism from this ring to $\mathbb{Z}$. If the

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[1]For emphasis, one sometimes refers to them as "numerical DT-invariants", to distinguish them from their "motivic" version, defined below.

moduli space is smooth these virtual classes should be given by the classes of $I_n(M, \beta)$ in the ring $\mathcal{M}_\mathbb{C}$, since the DT invariants coincide with the Euler characteristic. Notice, though, that, if the moduli space is singular, it is usually not straightforward to define such virtual classes in the Grothendieck ring of varieties. Pictorially, we want a commutative diagram as below.

(4)

$$\{\text{Moduli Spaces } I_n(M, \beta)\}$$

Virtual Classes

DT Theory

χ_c

$\mathbb{Z}$

$\mathcal{M}_\mathbb{C}$

The idea of refining DT invariants via the Grothendieck ring of varieties was first considered in [18]. For us, the interest behind this is that, as already pointed out in the introduction and in [13], by considering real and complex Euler characteristics, we can, in some cases, recover real DT invariants from the same formulas in the Grothendieck ring. In general, the virtual classes are not in $\mathcal{M}_k$ (or $\mathcal{M}_\mathbb{C}$), but in a slightly larger, equivariant version of this ring. In the next section, we present a way to define virtual classes over a general field k of characteristic zero when the moduli spaces are defined over k and can be represented as a critical locus of a smooth function, based mainly on the work of Denef and Loser [11]. In practice, this is the case in many interesting examples in which $D^b(X) \cong D^b(\mathbf{Q} - \text{mod})$, that is, in which we can represent the moduli spaces as quiver varieties.

2.3. The motivic nearby class

Let X be a smooth variety and $f : X \to \mathbb{A}^1$ be a regular function. Denote the central fibre $f^{-1}(0)$ by X_0. Our goal is to define a class $[Z]_{vir}$ for $Z = \{df = 0\}$, the critical locus of f, which encodes also the scheme structure of Z.

Definition 2.3.1. *Let μ_n be the group of n-th roots of unity in $\bar{k}$. Notice that it has a structure of algebraic variety over k. We have maps $\mu_{nd} \to \mu_n$ given by $x \mapsto x^d$. This gives us a projective system. We denote the limit by $\hat{\mu}$. A good μ_n action on X is a group action $\mu_n \times X \to X$ which is a morphism*

*of varieties and such that each orbit is contained in an affine subvariety of
X. A good $\hat{\mu}$-action is a group action $\hat{\mu} \times X \to X$ which factors through a
good action of μ_n for some n.*

Definition 2.3.2. *Consider the abelian group generated by symbols $[X, \hat{\mu}] =
[X]$, where X is a variety with a good $\hat{\mu}$-action, modulo isomorphisms com-
patible with the action. The relations are the same scissor relations $[X] =
[Y] + [X \setminus Y]$ that we have for $K_0(Var(k))$ but such that the action on Y
is induced by the one on X. Finally, there is one more relation given by
$[X \times V] = [X \times \mathbb{A}^n]$, where V is the affine space of dimension n with a
good $\hat{\mu}$-action and, on the other side, $\mathbb{A}^n$ has the trivial action. The product
is given by the usual Cartesian product with the product action. The class
of the affine line with the trivial action is denoted by $\mathbb{L}$. This ring will be
denoted $K_0(Var^{\hat{\mu}}(k))$ and its localization $K_0(Var^{\hat{\mu}}(k))[\mathbb{L}^{-\frac{1}{2}}]$ by $\mathcal{M}_k^{\hat{\mu}}$.*

Take a resolution of $f : X \to \mathbb{A}^1$, that is, a map $h : Y \to X$, with Y
smooth and irreducible such that $Y_0 = h^{-1}(X_0)$ has only normal crossings
and the restriction $h : Y - Y_0 \to X - X_0$ is an isomorphism. We fix the
notation:

- E_i, $i \in J$ denote the components of Y_0 and N_i their multiplicities;
- E_I, $I \subset J$ denote the intersections $\bigcap_{i \in I} E_i$;
- E_I°, $I \subset J$ denote $E_I - \bigcup_{j \notin I} E_j \cap E_I$

To take multiplicities into account, we define natural covering spaces $\tilde{E}_I^\circ \to
E_I^\circ$, which are unramified and Galois with Galois groups given by μ_{m_I}, where
m_I is the greatest common divisor of the multiplicities of all $E_i, i \in I$. Take
$U \subset Y$ an open set such that $f \circ h = uv^{m_I}$, where u is a unit in $\mathcal{O}_Y(U)$ and
v is a morphism $v : U \to \mathbb{A}^1$. We can define $\tilde{E}_I^\circ$ by gluing the sets

$$\{(x, t) \in (U \cap E_I^\circ) \times \mathbb{A}^1 | t^{m_I} = u^{-1}(x)\}.$$

The considerations above show that $\tilde{E}_I^\circ$ is a Galois covering and that there
exists a natural good action of μ_n (and thus, of $\hat{\mu}$) on it.

Definition 2.3.3. *The motivic nearby class of f is defined as follows:*

(5) $$S_f := \sum_{\varnothing \neq I} (1 - \mathbb{L})^{|I|-1} [\tilde{E}_I^o] \in \mathcal{M}_k^{\hat{\mu}}$$

Among other remarks, the belief is stated in [11] that this is a motivic
incarnation of the complex of nearby cycles of X_0, which was defined in
Exposé XIII of [10]. Restricting S_f to a point $x \in X_0$, we get the local

version, which the authors believe to be a motivic version of the classical Milnor fibre of f over x.

$$(6) \qquad S_{f,x} := \sum_{\varnothing \neq I} (1 - \mathbb{L})^{|I|-1} [\tilde{F}_I^o] \in \mathcal{M}_k^{\hat{\mu}}$$

where F_i is the fiber of E_i over x.

Example 2.3.4. *Let $f : \mathbb{A}^2 \to \mathbb{A}^1$ be given by $f(x,y) = x^2 - y^2$. Then X_0 has already only normal crossings, which are the two lines E_1 and E_2 intersecting at the origin. We have:*

$$E_1^\circ = E_1 - \{0\}$$
$$E_2^\circ = E_2 - \{0\}$$
$$E_{1,2}^\circ = \{0\}$$

As all multiplicities are equal to 1, we do not need to worry about the covering spaces. Computing S_f, we get:

(7) $\ S_f = [E_1 - 0] + [E_2 - 0] + (1 - \mathbb{L})[0] = \mathbb{L} - 1 + \mathbb{L} - 1 + 1 - \mathbb{L} = \mathbb{L} - 1$

For $S_{f,0}$, we can write:

(8) $\ S_{f,x} = [(E_1-0) \cap h^{-1}(0)] + [(E_2-0) \cap h^{-1}(0)] + (1-\mathbb{L})[0 \cap h^{-1}(0)] = 1-\mathbb{L}$

It is instructive to compare these results with what happens over $\mathbb{R}$ and $\mathbb{C}$. The Milnor fibre over a point $x \in X_0$, over $\mathbb{C}$, is defined as the intersection of the preimage $f^{-1}(w)$, for small $w \in \mathbb{C}$, with a small ball around x. This gives us a cylinder around x, which has Euler characteristic 0: this is what we get by making $\mathbb{L} = 1$ in equation 8. Considering the same definition over $\mathbb{R}$ (which cannot be done in all cases, considering that the topology of the fibre $f^{-1}(w)$ can change depending on w), we get two hyperbola segments. As we are intersecting with a ball, those are compact and the Euler characteristic should be 2, which is compatible with making $\mathbb{L} = -1$ in equation 8. If we do not intersect with the ball (that is, consider, in some sense, a global fibre), we would get an infinite cylinder ($\chi_c = 0$) or two non compact branches of hyperbola($\chi_c = -2$). This is compatible with equation 7.

As we will point out in Section 3, $\mathbb{A}^1$-enumerative geometry has an arithmetic incarnation of the Milnor number for isolated singularities, which is the EKL class of the gradient of f. Although, in order to compute DT invariants, we use the definition above in the case of non isolated singularities, an interesting problem would be to understand the relationship between Milnor numbers and the class S_f. We elaborate on this in Section 3.

2.4. Virtual class of a critical locus

Definition 2.4.1. *Let X be a smooth variety and $f : X \to \mathbb{A}^1$ be a regular function. If Z is the critical locus of f and $X_0 = f^{-1}(0)$, the virtual class of the critical locus Z is defined as*

$$(9) \qquad [Z]_{vir} = -\mathbb{L}^{-\frac{\dim X}{2}}(S_f - [X_0]) \in \mathcal{M}_k^{\hat{\mu}}$$

The idea behind this definition is that, over the smooth points of X_0, the Milnor fibre should correspond to X_0 itself. This means that this virtual class encodes exactly what is happening on the singular locus of X_0. Notice that this definition depends on the function f chosen, that is, it depends on how the variety is presented as a critical locus.

Remark 2.4.2. *Definition 2.4.1 was given in the paper [6]. The extension by the squareroot is related to the existence of a sign that appears when relating DT invariants to the Euler characteristic (see [5]).*

Example 2.4.3. *If $f = 0$ is the zero function, then $Z = X_0 = X$ (every point is critical). We can compute $S_f = 0$ using the fact that X_0 can be seen as the zero divisor. Therefore:*

$$[Z]_{vir} = -\mathbb{L}^{-\frac{\dim X}{2}}(0 - [X_0]) = \mathbb{L}^{-\frac{\dim X}{2}}[X]$$

This example shows that the virtual class of a smooth variety is its class in $\mathcal{M}_k$ up to multiplication by a factor. In terms of DT invariants, this factor correspond to the sign difference that we get with respect to the Euler characteristics.

The virtual class is not always in $\mathcal{M}_k$ but, over $\mathbb{C}$, Behrend, Bryan and Szendrői showed, in [6], that, when f is equivariant with respect to a torus action on X, $[Z]_{vir} \in \mathcal{M}_k$ and can be computed from the difference between the zero fiber and the generic fiber.

Proposition 2.4.4 (cf. Thm. B.1 in [6])**.** *Let $f : X \to \mathbb{C}$ be a regular morphism on a smooth quasi-projective variety. Let Z be the critical locus of f. Assume that there exists an action of a connected complex torus on X such that f is equivariant with respect to a primitive character. If there is a one parameter subgroup $\mathbb{C}^\times \subset T$ such that the induced action is circle compact, that is, the set of fixed points is compact and the limit $\lim_{\lambda \to 0} \lambda x$ exists for all $x \in X$, then, the virtual class $[Z]_{vir}$ is given by*

$$[Z]_{vir} = -\mathbb{L}^{-\frac{\dim X}{2}}([X_1] - [X_0]) \in \mathcal{M}_\mathbb{C},$$

where $[X_1]$ is the class of the fibre $f^{-1}(1)$.

The result above strongly uses the fact that the varieties are defined over $\mathbb{C}$, but our computations in section 4 show that, at least in the case of degree zero invariants of $\mathbb{A}^3$, the real DT invariants are also encoded in the formula. Inspired by the work of Levine on localization formulas in $\mathbb{A}^1$-enumerative geometry [23], one may think that it could be possible to extend these arguments to more general fields.

3. $\mathbb{A}^1$-Enumerative geometry

3.1. Rings of quadratic forms

In this section, k continues to be a field of characteristic zero.

Definition 3.1.1. *The Grothendieck-Witt ring is the group completion of the set of all quadratic forms over k up to isometries with the operations*

$$(10) \qquad q + q' : V \oplus W \to k \qquad q + q'(x,y) = q(x) + q'(y)$$

$$(11) \qquad qq' : V \otimes W \to k \qquad qq'(x \otimes y) = q(x)q'(y)$$

where $q : V \to k$ and $q' : W \to k$ are quadratic forms representing isometry classes.

There are two important natural maps from $GW(k)$ to $\mathbb{Z}$ which are the rank and the signature.

- rk : $GW(k) \to \mathbb{Z}$ computes the dimension of the vector space in which the quadratic form is defined.
- sgn : $GW(k) \to \mathbb{Z}$ computes the signature of the quadratic form.

Remark 3.1.2. *If we have a field extension $k \subset L$, we can define a map $GW(L) \to GW(k)$ by simply considering composition with the trace map $Tr_{L/k} : L \to k$.*

Remark 3.1.3. *Since every quadratic form can be diagonalized, $GW(k)$ is generated by elements of rank 1. These are represented by forms $q(x) = ax^2$, with $a \in k^\times$. They are denoted $\langle a \rangle \in GW(k)$. If a is a square, $\langle a \rangle = \langle 1 \rangle$ in $GW(k)$, and therefore the generators of $GW(k)$ are the elements in $k^\times/(k^\times)^2$. Using this notation, we write $\mathbb{H} = \langle 1 \rangle + \langle -1 \rangle$. This is called the hyperbolic form. It has the property that $\langle a \rangle \mathbb{H} = \mathbb{H}$ for any $a \in k$.*

The Grothendieck-Witt ring is suitable to refine the classical $\mathbb{Z}$-valued enumerative invariants (counts) defined over algebraically closed fields. Very roughly, the intuition is that we count points considering local "orientations" (as it is done over $\mathbb{R}$), which correspond to quadratic forms. Counting with such orientations allows us to get invariant counts over any field k. The ranks of these quadratic forms correspond to the size of the Galois orbit of the point and, therefore, they recover the counts over $\bar{k}$. The signature recovers the signed counts over $\mathbb{R}$ (when $k \subset \mathbb{R}$), since the two square classes over $\mathbb{R}$ will correspond to the two possible local orientations (signs). The definition of the $\mathbb{A}^1$-degree of a map from $\mathbb{A}^n \to \mathbb{A}^n$ is a good example.

Definition 3.1.4. *Let $P : \mathbb{A}^n \to \mathbb{A}^n$ be an étale morphism. The local degree of P at a closed point x with rational image $y = P(x)$ and which is isolated in its fiber is given by:*

$$(12) \qquad \deg_x^{\mathbb{A}^1}(P) = \mathrm{Tr}_{k(x)/k}\langle \det J(x)\rangle$$

where $J(x)$ denotes the derivative (Jacobian) of P at the point x and $k(x)$ denotes the residue field of x. The degree of P is simply the sum over all the preimages.

$$(13) \qquad \deg^{\mathbb{A}^1}(P) = \sum_{x \in P^{-1}(y)} \mathrm{Tr}_{k(x)/k}\langle \det J(x)\rangle$$

Remark 3.1.5. *When f is not étale, the degree can still be defined using more involved machinery from homotopy theory. We refer the reader to section 2 of [15] for a more complete exposition and to [29] for technical details.*

Equation (12) corresponds to our local "orientation" and, in (13), we "count" all the points. The simplest example is the function $P(x) = x^2$ from $\mathbb{A}^1$ to $\mathbb{A}^1$.

Example 3.1.6. *Let $k = \mathbb{Q}$ and $P(x) = x^2$ in $\mathbb{A}^1$. The derivative is given by $J(x) = 2x$. Therefore, using equation (13) we can write, choosing $y = 1$ and $y = -1$, the following formulas.*

$$\deg^{\mathbb{A}^1}(P) = \mathrm{Tr}_{\mathbb{Q}(1)/\mathbb{Q}}\langle 2\rangle + \mathrm{Tr}_{\mathbb{Q}(-1)/\mathbb{Q}}\langle -2\rangle = \langle 2\rangle + \langle -2\rangle = \mathbb{H}$$

$$\deg^{\mathbb{A}^1}(P) = \mathrm{Tr}_{\mathbb{Q}(\sqrt{-1})/\mathbb{Q}}\langle 2\sqrt{-1}\rangle = \mathrm{Tr}_{\mathbb{Q}(\sqrt{-1})/\mathbb{Q}}\langle 1\rangle = \mathbb{H}$$

In the second formula we used that $2\sqrt{-1} = (1 + \sqrt{-1})^2$ and that the trace is simply the quadratic form $a + b\sqrt{-1} \mapsto a^2 - b^2$.

Notice that $\mathrm{rk}(\mathbb{H}) = 2$, *which is the number of preimages in the algebraic closure and that* $\mathrm{sgn}(\mathbb{H}) = 0$, *which is the real topological degree.*

In the next section, we will introduce Chow-Witt groups, which are a generalization of the classical Chow groups to this context. The idea is to do intersection theory taking such "orientations" into account.

3.2. Chow-Witt groups and Euler characteristics

We start by recalling that, over $\bar{k}$, one can define sheaves $\mathcal{K}_n^M$, known as Milnor K-sheaves, for which

$$(14) \qquad\qquad H^n(X, \mathcal{K}_n^M) \cong CH^n(X).$$

The idea is to replace this Chow group by the Chow-Witt groups, defined in such a way that the group associated to $\mathrm{Spec}(k)$ equals $GW(k)$.

The Chow-Witt groups are denoted $\widetilde{CH}(X, L)$ (for L a line bundle), and were first defined in [4]. The easiest way to define them is to generalize Equation 14, i.e, write

$$H^n(X, \mathcal{K}_n^{MW}(L)) \cong \widetilde{CH}^n(X, L),$$

for a suitable family of sheaves $\mathcal{K}_n^{MW}$ twisted by lines bundles.

The line bundles can be considered up to squares, that is, $\widetilde{CH}^n(X, L') \cong \widetilde{CH}^n(X, L' \otimes L^{\otimes 2})$ for any L, L'. This is related to the fact that $GW(k)$ is generated by classes in $k^\times/(k^\times)^2$. We have pullbacks and pushforwards for Chow-Witt groups, but we have to be careful about the line bundles. This new technical complication is a result of considering "orientations".

- For $f : X \to Y$, relative dimension d:
    - Pullback: $f^* : \widetilde{CH}^n(Y, L) \to \widetilde{CH}^n(X, f^*L)$
    - Pushfoward (for f proper): $f_* : \widetilde{CH}^n(X, \omega_X \otimes f^*\omega_Y \otimes f^*L) \to \widetilde{CH}^{n-d}(Y, L)$

These definitions give us:

- a map $\int_X : \widetilde{CH}^n(X, \omega_X) \to GW(k)$, given by pushforwarding to the point $\mathrm{Spec}(k)$;
- the Euler class of a vector bundle $\varepsilon(V) \in \widetilde{CH}^r(X, \det^{-1}(V))$, via pushfowarding the class 1_X by a section and then pulling it back.

Remark 3.2.1. *Notice that we can only compute the integral of an Euler class (that is, count the zeros of a generic section), if the determinant of the bundle is, up to squares, given by ω_X. This gives a notion of (relatively) orientable vector bundles. For the tangent bundle TX, we have that $\det^{-1}(TX) = \det^{-1}((\Omega^1_X)^{-1}) = \omega_X$. This implies that we can compute $\int_X \varepsilon(TX)$, for any X.*

Definition 3.2.2. *The Euler characteristic of a smooth and projective variety X can be defined as*

$$\chi^{\mathbb{A}^1}(X) := \int_X \varepsilon(TX)$$

If X is not projective, one can still define the Euler characteristic. We could consider the motivic stable homotopy category and use the fact that the infinite suspension spectrum of X is strongly dualizable and gives rise to an endomorphism of the sphere spectrum, which corresponds to an element of $GW(k)$. The relationship between these two definitions is explained in [21] and was first proved in [26]. The same idea can extend this map to a compactly supported Euler characteristic.

We can now consider the map defined on $K_0(Var(k))$ that takes any variety and evaluates its compactly supported Euler characteristic. Even tough we have not defined this for non projective varieties, as stated in the introduction, we only need to consider projective varieties. This was already pointed out in [2] and they stated the following:

Proposition 3.2.3 (cf. [2] Thm. 1.13)**.** *Let k be a field with $char(k) = 0$. Then the compactly supported $\mathbb{A}^1$-Euler characteristic is well defined and the following map:*

$$\chi_c^{\mathbb{A}^1} : K_0(Var(k)) \to GW(k)$$

is a homomorphism of rings.

The proof of the above proposition simply follows from the usual properties of Euler characteristics, which are compatible with the relations in $K_0(Var(k))$. The morphism from Proposition 3.2.3 can be extended to the localization of $K_0(Var(k))$ in the same way as the topological Euler characteristics (see Section 2.1) after adjoining a square root of $\langle -1 \rangle$ to $GW(k)$.

$$(15) \qquad\qquad \chi_c^{\mathbb{A}^1} : \mathcal{M}_k \to GW(k)(\alpha)$$

where α is such that $\alpha^2 = \chi_c^{\mathbb{A}^1}(\mathbb{L}) = \langle -1 \rangle$

The morphism above allows us to get a "numerical" version of DT invariants over any field, given that the motivic version defined in 2.4.1 is in $\mathcal{M}_k$ (seen as a subring of $\mathcal{M}_k^{\hat{\mu}}$). We call these invariants *arithmetic DT invariants.* One could also consider an equivariant version of the $\mathbb{A}^1$-Euler characteristic in order to define arithmetic invariants even when the virtual classes are not in $\mathcal{M}_k$.

An interesting question is whether there is a direct definition of such invariants which is not related to the Grothendieck ring of varieties but defined directly using the tools of $\mathbb{A}^1$-homotopy theory.

3.3. Local $\mathbb{A}^1$-degree and Eisenbud-Khimshiashvili-Levine classes

We now discuss some aspects of the Eisenbud-Khimshiashvili-Levine (EKL) classes, which, as we stated in the introduction and in section 2.3, are an important ingredient for possible physical interpretations and might be key to finding a relationship between the motivic nearby cycles and the $\mathbb{A}^1$-version of the Milnor number.

In the papers [12] and [17], the idea was to find a way of computing the local topological degree of $P : \mathbb{R}^n \to \mathbb{R}^n$ from algebraic information in the local ring of P at 0. They showed that there was a quadratic form (the EKL form) defined on the local algebra whose signature corresponded to the local degree. The same problem was considered for holomorphic maps, and the rank of the EKL form ended up being equal to the local degree.

The rank and signature appearing as above is already a hint that, over an arbitrary field k, the class of the EKL form (EKL class) in $GW(k)$ should be equal to the refinement of the local topological degree introduced in definition 3.1.4. Using such refinement, J. Kass and K. Wickelgren showed that the class of the EKL form in $GW(k)$ correspond to the local $\mathbb{A}^1$-degree [15]. We now give the definition of the EKL form.

Definition 3.3.1. *Consider a morphism $P : \mathbb{A}^n \to \mathbb{A}^n$ with $P(0) = 0$. Assume that this zero is isolated. We can write P as $(P_1, \ldots, P_n)$, with $P_i = \sum_{j=1}^n a_{ij} x_j$. Let $A = k[x_1, \ldots, x_n]_{(x_1, \ldots, x_m)}/(P)$ be the local algebra of P at 0. Define $E = det(a_{ij})$, where a_{ij} are such that $f_i = \sum_j a_{ij} x_j$.*

Let further $\phi : A \to k$ be any k-linear map and define the bilinear form $\beta_\phi(p, q) = \phi(pq)$. The Eisenbud-Khimshiashvili-Levine (EKL) class of P is the class of $\beta_\phi \in GW(k)$ for any ϕ with $\phi(E) = 1$. It is denoted $w_0(P)$.

Remark 3.3.2. *E basically carries the same information as the Jacobian determinant $\det J$ of P. Specifically, $\det J = \dim_k A \cdot E$. In particular, in characteristic zero, one can consider J instead of E.*

Remark 3.3.3. *If P has a simple zero at the origin, then $w_0(f)$ is simply the class $\langle E \rangle = \langle \det J \rangle$. This follows from the fact that in this case $A \cong k$. In particular, it corresponds to the local degree in this case, since P having a simple zero implies P is étale.*

EKL classes are related to critical loci and Milnor fibres for isolated singularities. For $f : \mathbb{A}^n \to k$, the derivative of f gives us a map $P := df : \mathbb{A}^n \to \mathbb{A}^n$ as we considered above. Then, the EKL class $w_0(P)$ refines the Milnor number of the singularity. Indeed, over $\mathbb{C}$, the Milnor number is the vector space dimension of the quotient A, which is, by definition, the rank of the EKL quadratic form. This refinement was introduced in [15, Section 6].

Over $\mathbb{C}$, the Milnor number of f is closely related to the topology of the Milnor fibre, which is homotopic to a bouquet of μ spheres S^n, where μ is the Milnor number of f. This gives us the classical formula:

$$\chi(F) = 1 + (-1)^{n-1}\mu(f),$$

where F is the Milnor fibre of f. We suspect that this formula generalizes.

Example 3.3.4. *In the case considered in 2.3.4, we had $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $x^2 - y^2$. Its derivative is given by*

$$P := df : \mathbb{A}^2 \to \mathbb{A}^2$$
$$(x, y) \mapsto (2x, -2y).$$

The Jacobian matrix, in this context, is simply the Hessian of f. This gives us:

$$J(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix},$$

for any (x, y).

Now, the $\mathbb{A}^1$-Milnor number can be easily computed by the consideration on Remark 3.3.3 and Equation 3.3.

$$\mu^{\mathbb{A}^1} = \langle \det J(0) \rangle = \langle -4 \rangle = \langle -1 \rangle \in GW(k).$$

Finally,

$$\chi_c^{\mathbb{A}^1}(S_{f,0}) = \chi_c^{\mathbb{A}^1}(1 - \mathbb{L}) = \langle 1 \rangle - \langle -1 \rangle = \langle 1 \rangle + (-\langle 1 \rangle)^{2-1} \cdot \langle -1 \rangle$$
$$= \langle 1 \rangle + (-\langle 1 \rangle)^{n-1}\mu^{\mathbb{A}^1}(f).$$

Of course, in general, one has to consider an equivariant version of the $\mathbb{A}^1$-Euler characteristic, which takes into account the action of $\hat{\mu}$, since the classes $S_{f,x}$ are in $\mathcal{M}_k^{\hat{\mu}}$. Many interesting questions can be asked regarding this topic: what is the general relationship between the $\mathbb{A}^1$-Milnor number of a function f and the $\mathbb{A}^1$-Euler characteristic of the motivic class S_f? Is there anything that can be said for non isolated singularities?

3.4. Comparison with related work

As we mentioned in the introduction, Levine [22] has given an alternative definition of "arithmetic DT invariants", independently of the motivic invariants. This construction relies on the definition of DT invariants for general Deligne-Mumford stacks due to Behrend and Fantechi [7]. The idea is to construct a motivic version of the intrinsic normal cone, called "intrinsic stable normal cone". For a scheme Z over B, this cone is an object in the motivic stable homotopy category from $\mathbb{A}^1$-homotopy theory and, given any perfect obstruction theory $\phi : E_\bullet \to \mathbb{L}_{Z/B}$, it can be used to construct a class $[Z, \phi]^{vir} \in \widetilde{CH}_r(Z; \det E_\bullet)$. Now, if $\det E$ is isomorphic to ω_Z up to squares, integration gives us an invariant in $GW(k)$, as we described in 3.2.1 (of course, one needs certain assumptions on Z for push-forwards and Chow-Witt groups to be defined). The choice of an obstruction theory such that E is relatively orientable is therefore essential for this theory.

In the case we considered above, i.e., Z given as a critical locus of a function f on a smooth variety M, we have that Z is given by zeros of a section of the cotangent bundle. In this case, there is a canonical obstruction theory induced by the surjective morphism $df : \mathcal{T}_M \to \mathcal{I}_Z \to 0$, where $\mathcal{I}_Z$ is the ideal corresponding to Z. After tensoring by $\mathcal{O}_Z$, we see that df induces a map $F : \mathcal{T}_M \otimes \mathcal{O}_Z \to \mathcal{I}_Z/\mathcal{I}_Z^2$. Then, by considering the composition with the derivative in the first term of the complex, we get the obstruction theory

$$\phi = (F, \mathrm{Id}) : \left(\mathcal{T}_M \otimes \mathcal{O}_Z \overset{\partial_\phi}{\to} \Omega_{M/B} \otimes \mathcal{O}_Z \right) \to \left(\mathcal{I}_Z/\mathcal{I}_Z^2 \overset{d}{\to} \Omega_{M/B} \otimes \mathcal{O}_Z \right),$$

which induces a virtual class $[Z]^{vir}$ in the Chow-Witt group of Z twisted by $\det(T_M) \otimes O_Z$, which is clearly orientable.

A more detailed comparison can be made based on the works of Azouri [3] and Levine-Pepin-Lehalleur-Srinivas [24]. Specifically, the conductor formulas for the nearby fiber functor given in these works show that our invariants would agree with the ones in [22] over $\mathbb{R}$, but not necessarily in general. It is also worth mentioning that such formulas also give partial

answers to the questions at the end of the last section, with the formula $\langle 1 \rangle + (-\langle 1 \rangle)^{n-1} \mu^{\mathbb{A}^1}(f)$ for the Euler charactersitic of the motivic Milnor fibre being correct for the case of mappings $\mathbb{A}^n \to \mathbb{A}^1$ given by quasi-homogenous singularities. Finally, we point out that adjoining α to $GW(k)$ in (15) is not related to orientation issues, but simply due to the appearance of the class $\mathbb{L}^{\frac{1}{2}}$ in $K_0(Var(k))$ needed to make up for the sign differences (see remark 2.4.2).

4. Degree zero invariants of $\mathbb{A}^3$

4.1. The Hilbert scheme of points of the affine space as a critical locus

We recall how to realize the Hilbert scheme of n points of $\mathbb{A}^3$ as a critical locus of a function.

Fix the notation $\mathrm{Hilb}^n(\mathbb{A}^r)$ for the Hilbert scheme of n points of $\mathbb{A}^r$ and

$$\mathbb{A}^r = \mathrm{Spec}\ k[x_1, \ldots, x_r] = \mathrm{Spec}\ k[\mathbf{x}].$$

The following considerations can be found in the notes by Nakajima [30].

In the affine space, a subscheme $B \subset \mathbb{A}^r$ of dimension 0 and degree n correspond to a quotient of $k[\mathbf{x}]$ of dimension n. Therefore, if we fix a vector space V_n of dimension n, the structure we need to add in order to get a module consists of an action of $k[\mathbf{x}]$ and an element $1 \in V_n$ which generates the whole space under the action. An action of $k[\mathbf{x}]$ is the choice of r elements of $\mathrm{Hom}(V, V)$ which commute. This allows us consider the set:

$$B(V) = \{(A_1, \ldots, A_r, v) \in \mathrm{Hom}^r(V, V) \times V \mid [A_i, A_j] = 0,$$
$$v \text{ generates } V \text{ under the action}\}.$$

To get the Hilbert Space from $B(V)$, we need to mod by the action of $GL(V)$ (by conjugation on the A_i). This implies that:

Proposition 4.1.1. *Let $V \cong k^n$ be an n-dimensional k-vector space. The Hilbert scheme of points of $\mathbb{A}^r$ can be represented as*

$$(16) \quad \mathrm{Hilb}^n(\mathbb{A}^r) \cong \left\{ (A_1, \ldots, A_r, v) \middle| \right.$$

$$\begin{array}{l} \textit{(i) } [A_i, A_j] = 0 \textit{ for all } i, j \\ \textit{(ii) } v \textit{ generates } V \textit{ under the action of the } A_j \end{array} \left. \vphantom{\begin{array}{l}a\\b\end{array}} \right\} \Big/ \mathrm{GL}_n(k),$$

Proof. Each point on $\mathrm{Hilb}^n(\mathbb{A}^r)$ can be represented by an ideal $I \subset k[x]$. Take any isomorphism $\frac{k[\mathbf{x}]}{I} \cong V$. Let A_i be given by multiplication by x_i and let $v = \bar{1} \in \frac{k[\mathbf{x}]}{I}$. With such choices, *(i)* and *(ii)* are obvious. Notice that this map does not depend on the choice of the isomorphism since any two isomorphisms are related by conjugation by elements $GL_n(r)$ on V. On the other hand, given an element $(A_1, \ldots, A_r, v)$ satisfying *(i)* and *(ii)*, we can define a k-algebra structure on V via the map from $k[\mathbf{x}] \to V$ given by sending x_i to $A_i v$ and 1 to v. This map is well defined by *(i)* and it is surjective by *(ii)*. By taking the kernel, we get and ideal $I \subset k[\mathbf{x}]$ which represents an element of the Hilbert scheme. Those considerations imply that those two maps are inverses and therefore define an isomorphism. $\square$

Let us now restrict ourselves to the case $r = 3$. In this situation the above the result from Proposition 4.1.1 allows us to write $\mathrm{Hilb}^n(\mathbb{A}^3)$ as critical locus of a regular function on a smooth space. This puts us in the situation described in Section 2.3.

Proposition 4.1.2. *Fix a vector space $V \cong k^n$ of dimension n. Consider the quotient*

$$M_n = \left\{ (A, B, C, v) \, \middle| \, \begin{array}{l} v \text{ generates } V \text{ under the action of the} \\ \text{algebra generated by } A, B \text{ and } C \end{array} \right\} \Big/ \mathrm{GL}_n(V),$$

and the function

$$f_n : M_n \to k$$
$$(A, B, C, v) \mapsto \mathrm{Tr}([A, B]C)$$

Then M_n is a smooth variety, f_n is a regular map and $\mathrm{Hilb}^n(\mathbb{A}^3) = \{df_n = 0\} \subset M_n$.

Proof. Most of our argument comes from [6] and [35]. To see that M_n is an algebraic variety, it suffices to notice that M_n is a quotient of an open subset of a vector space by a free action. Indeed, the condition of the v generating the whole V under the action is an open condition. To see that the action is free, we have:

$$g(A, B, C, v) = (gAg^{-1}, gBg^{-1}, gCg^{-1}, gv) = (A, B, C, v),$$

which implies that $W = \ker(g - id)$ is stable under the action of A, B and C but contains v. By definition, v generates V under the action and so $W = V$. This shows $g = id$. Now we use a construction from Geometric

Invariant Theory (GIT). We consider the action on $\mathrm{Hom}(V,V)^3 \times V \times \mathbb{C}$ given by

$$g(A, B, C, v, z) = (gAg^{-1}, gBg^{-1}, gCg^{-1}, gv, det(g)^{-1}z).$$

The semistable points of this action are given exactly by points whose orbit closures are disjoint from $\mathrm{Hom}(V,V)^3 \times V \times 0$. We see that, if v generates V under the action, this condition is satisfied. GIT, then, gives us that the quotient is a smooth variety.

Now, by 4.1.1, it is enough to show that the condition $df_n = 0$ correspond to commutativity of the A_i. Indeed, we have:

$$\mathrm{Tr}([A, B]C) = \sum_i \sum_k \sum_j (a_{ij}b_{jk} - b_{ij}a_{jk})c_{ki}.$$

Therefore, if the derivatives with respect to each entry of C are all zero, we get that $[A, B] = 0$. As $\mathrm{Tr}([A, B]C) = \mathrm{Tr}(A[B, C]) = \mathrm{Tr}(B[C, A])$, the vanishing of the other derivatives implies that the other pairs of matrices commute. □

4.2. The virtual classes of the Hilbert scheme of points

To compute the virtual classes over $\mathbb{C}$, the authors of [6] used Proposition 2.4.4 and the fact that there is a natural toric action on M which descends given by

$$(t_1, t_2, t_3) \cdot (A, B, C, v) \mapsto (t_1 A, t_2 B, t_3 C, t_1 t_2 t_3 v)$$

which satisfies all the hypotheses. For details, see [6, Lemma 2.4].

This implies that the virtual class of $\mathrm{Hilb}^n(\mathbb{C}^3)$ can be computed by the difference $\left[f_n^{-1}(1)\right] - \left[f_n^{-1}(0)\right]$ multiplied by the factor $\mathbb{L}^{\frac{1}{2}}$ (see Definition 2.4.1). This difference was computed in [6] to correspond to a generating series

$$(17) \qquad Z_{\mathbb{C}^3}(t) = \sum_{n=0}^{\infty}[\mathrm{Hilb}(\mathbb{C}^3)]_{vir}t^n = \prod_{m=1}^{\infty}\prod_{k=0}^{m-1}(1 - \mathbb{L}^{k+2-m/2}t^m)^{-1}.$$

Remark 4.2.1. *The computation itself mainly relies on the motivic classes of Grassmanians, general linear groups and of the variety of commuting matrices, which are the same over any field. The only difficulty in generalizing this computation would be to prove that the virtual class is given by the*

difference of the fibers. Sample targets for generalization include examples studied by Choi-Katz-Klemm in [9].

Applying the morphism in equation 15, we can get arithmetic DT invariants as defined in Section 3.

Proposition 4.2.2. *The degree zero DT invariants for $\mathbb{A}^3$ can be refined over the ring $GW(k)(\alpha)$, that is, there are natural expressions in $GW(k)(\alpha)$ for which the rank gives us the classical (complex) DT invariants. The refined invariants are given in the generating series:*

$$(18) \qquad \prod_{n=1}^{\infty} (\langle 1 \rangle - (\alpha t)^{2n-1})^{-1} \prod_{n=1}^{\infty} (\langle 1 \rangle - (\alpha t)^n H + \langle -1 \rangle (\alpha t)^{2n})^{-\lfloor \frac{n}{2} \rfloor}$$

Proof. Applying the morphism, we first just send $\mathbb{L}$ to $\langle -1 \rangle$ and $\mathbb{L}^{\frac{1}{2}}$ to α.

$$\prod_{m=1}^{\infty} \prod_{k=0}^{m-1} (\langle 1 \rangle - \langle -1 \rangle^{k+2} \alpha^{-m} t^m)^{-1}$$

Notice that

$$(\langle 1 \rangle - \langle -1 \rangle \alpha^m t^m)(\langle 1 \rangle - \alpha^m t^m) = (\langle 1 \rangle - \alpha^m t^m H + \langle -1 \rangle \alpha^{2m} t^{2m})$$

where H is the hyperbolic form given by $\langle 1 \rangle + \langle -1 \rangle$.

In the case m is even, the product above will appear exactly $\frac{m}{2}$ times. For m odd, it appears $\frac{m-1}{2} = \lfloor \frac{m}{2} \rfloor$ times and we get an extra factor of the form $(\langle 1 \rangle - \alpha^m t^m)^{-1}$.

$$\prod_{m=1}^{\infty} \prod_{k=0}^{m-1} (\langle 1 \rangle - \langle -1 \rangle^{k+2} \alpha^m t^m)^{-1} =$$

$$\prod_{m \text{ odd}} (\langle 1 \rangle - \alpha^m t^m)^{-1} \prod_{m=1}^{\infty} (\langle 1 \rangle - (\alpha t)^m H + \langle -1 \rangle (\alpha t)^{2m})^{-\lfloor \frac{m}{2} \rfloor}$$

By making $m = 2n - 1$ in the first product and $m = n$ in the second, we get:

$$(19) \qquad \prod_{n=1}^{\infty} (\langle 1 \rangle - (\alpha t)^{2n-1})^{-1} \prod_{n=1}^{\infty} (\langle 1 \rangle (\alpha t)^n H + \langle -1 \rangle (\alpha t)^{2n})^{-\lfloor \frac{n}{2} \rfloor}$$

$\square$

Remark 4.2.3. *The above refinement is compatible with previous results over $\mathbb{R}$ and $\mathbb{C}$.*

Taking $k = \mathbb{C}$, we have $GW(k) = \mathbb{Z}$ and $\alpha = -1$, which results in the classical MacMahon generating function for the number of plane partitions:

$$\prod_{n=1}^{\infty}(1 - (-t)^{2n-1})^{-1} \prod_{n=1}^{\infty}(1 - 2(-t)^n + (-t)^{2n})^{-\lfloor \frac{n}{2} \rfloor} =$$

$$\prod_{n=1}^{\infty}(1 - (-t)^{2n-1})^{-1} \prod_{n=1}^{\infty}(1 - (-t)^n)^{-2\lfloor \frac{n}{2} \rfloor} = \prod_{n=1}^{\infty}(1 - (-t)^n)^{-n} = M(-t)$$

For $k = \mathbb{R}$, after computing the signature morphism (i.e., sending $\langle 1 \rangle$ to 1, $\langle -1 \rangle$ to -1 and taking $\alpha = i$), we get the symmetric MacMahon function, counting symmetric plane partitions, which correspond to the real count as computed by Pasquetti-Krefl-Walcher in [19].

$$\prod_{n=1}^{\infty}(1 - (-it)^{2n-1})^{-1} \prod_{n=1}^{\infty}(1 - 0(-it)^n + (-it)^{2n})^{-\lfloor \frac{n}{2} \rfloor} =$$

$$\prod_{n=1}^{\infty}(1 - (-it)^{2n-1})^{-1} \prod_{n=1}^{\infty}(1 - (-it)^{2n})^{-\lfloor \frac{n}{2} \rfloor} = M^{\text{sym}}(-it)$$

4.3. Real and refined Gopakumar-Vafa invariants at the Castelnuovo bound

Our interest now turns to the computation of Gopakumar-Vafa invariants for M a smooth quintic hypersurface in $\mathbb{P}^4$, through their relation to DT invariants. In this case, the moduli spaces of interest are $\mathcal{M}_{n,d} := I_n(M, d)$, which correspond to the Hilbert scheme parameterizing subschemes of M with Hilbert polynomial given by $dt + n$, that is, curves of degree d and arithmetic genus $1 - n$. The GV invariants n_g^d correspond to DT invariants $I_{1-g,d}$. Recent work by Liu-Ruan [27] and Alexandrov-Feyzbakhsh-Klemm-Pioline-Schimannek [1] has established the famous Castelnuovo bound for Gopakumar-Vafa invariants, which was predicted in physics,

$$n_g^d = I_{1-g,d} = 0, \text{ for any } d \text{ and } g \text{ with } g > \frac{d^2 + 5d + 10}{10} =: B(d).$$

and led to a computation of the numbers $n_{B(d)}^d = I_{1-B(d),d}$. That is, the numbers n_g^d when the pair (g, d) is on the bound. Here, we write formulae

for the motivic and arithmetic refinements of such numbers at the bound. This was done using the fact that, for $n = B(d)$, $\mathcal{M}_{n,d}$ is not only smooth but a projective bundle over a projective space (see [27, Prop. 6.2], where they prove it over $\mathbb{C}$).We believe that this is true over any k of characteristic zero.

Proposition 4.3.1. *Let $g = B(d)$. This implies that $B(d)$ is an integer, which means that d can be written as $d = 5m$ for some m. Assuming that we can write the moduli space as a projective bundle as above, motivic and arithmetic refinements of the invariants $I_{n,d}$ for $n = 1 - g = 1 - B(d)$, are given by the formulae:*

$$[\mathcal{M}_{n,d}]_{vir} = \mathbb{L}^{\frac{N}{2}+2}\frac{(\mathbb{L}^{N+1} - 1)(\mathbb{L}^5 - 1)}{(\mathbb{L} - 1)^2} \in \mathcal{M}_k$$

and, applying the morphism $\chi^{\mathbb{A}^1}$:

$$\chi^{\mathbb{A}^1}\left([\mathcal{M}_{n,d}]_{vir}\right) = \begin{cases} \alpha \cdot \left(\dfrac{6 + 5N}{2}\langle 1 \rangle + \dfrac{4 + 5N}{2}\langle -1 \rangle\right), & \text{for } m = 0, 1 \bmod 4 \\ \dfrac{5(N + 1)}{2}H, & \text{for } m = 2, 3 \bmod 4 \end{cases}$$

where $N = \binom{m+3}{3} - \binom{m-2}{3} - 1$.

Proof. As stated above, $\mathcal{M}_{n,d}$ is a projective bundle with base $\mathbb{P}^4$ and the fibre $\mathbb{P}^N$, where $N = \binom{m+3}{3} - \binom{m-2}{3} - 1$. The first part of the result is simply given by using the fact that, for smooth varieties, the motivic virtual class is simply the class of the variety times $\mathbb{L}^{-\dim \mathcal{M}_{n,d}/2}$ (see Definition 2.4.1 and Example 2.4.3). To compute $[\mathcal{M}_{n,d}]$, we use that the class of a fibre bundle is the product of the fibre and the base.

Using that

$$[\mathbb{P}^r] = \mathbb{L}^r + \mathbb{L}^{r-1} + \ldots \mathbb{L} + 1 = \frac{\mathbb{L}^{r+1}}{\mathbb{L} - 1}$$

and that $\dim \mathcal{M}_{d,n} = N + 4$, we get

$$[\mathcal{M}_{n,d}]_{vir} = \mathbb{L}^{-\frac{\dim \mathcal{M}_{n,d}}{2}}[\mathcal{M}_{n,d}] =$$

$$\mathbb{L}^{\frac{N}{2}+2}[\mathbb{P}^N] \cdot [\mathbb{P}^4] = \mathbb{L}^{\frac{N}{2}+2}\frac{(\mathbb{L}^{N+1} - 1)(\mathbb{L}^5 - 1)}{(\mathbb{L} - 1)^2} \in \mathcal{M}_k.$$

Finally, applying the morphism, we only need to keep track of whether N is even or odd. If it is even, the $\mathbb{A}^1$-Euler characteristic of $\mathbb{P}^N$ is $\frac{N+1}{2}H$ and if it is odd it is given by $\frac{N+1}{2}\langle 1 \rangle + \frac{N-1}{2}\langle -1 \rangle$. $\qquad\square$

The formula above, specially the one in $GW(k)(\alpha)$, gives a prediction of what should be the $\mathbb{A}^1$-count of curves of higher genus on the quintic over any field. It would be interesting to check if this result can be reached with direct methods, without making use of motivic DT invariants.

Acknowledgments

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