

Spectral geometry of functional metrics on noncommutative tori

ASGHAR GHORBANPOUR* AND MASOUD KHALKHALI†

We introduce a new family of metrics, called functional metrics, on noncommutative tori and study their spectral geometry. We define a class of Laplace type operators for these metrics and study their spectral invariants obtained from the heat trace asymptotics. A formula for the second density of the heat trace is obtained. In particular, the scalar curvature density and the total scalar curvature of functional metrics are explicitly computed in all dimensions for certain classes of metrics including conformally flat metrics and twisted product of flat metrics. Finally a Gauss-Bonnet type theorem for a noncommutative two torus equipped with a general functional metric is proved.

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1	Introduction	182
2	Preliminaries and computational tools	187
2.1	Noncommutative d-tori and Connes' pseudodifferential calculus	187
2.2	Rearrangement lemma	189
2.3	Newton divided differences and closed derivations	191
3	Heat trace asymptotics for Laplace type h-differential operators	195

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3.1	Laplace type h -differential operators and their paramatrix	196
3.2	Coefficients of the asymptotic expansion of the heat trace	202
3.3	T -functions and their properties	204
4	Functional metrics and their scalar curvature density	209
4.1	Functional metrics and their Laplacian	209
4.2	Conformally flat metrics	212
4.3	Twisted product functional metrics	219
5	Total scalar curvature of functional metrics	224
5.1	Trace of the second density	225
5.2	Dimension two and a Gauss-Bonnet theorem	229
	Summary and outlook	232
	References	232

1. Introduction

Investigating the differential geometry of curved noncommutative tori started in [9], then [15, 8, 16] and was followed up in many publications. An incomplete list includes [12, 17, 3, 10, 29, 19, 11, 30]. In recent years, geometry, topology and analysis on noncommutative tori have attracted much attention due to its relevance to problems in solid state physics such as quantum Hall effect and topological phases of matter [36, 37]. This paper is a contribution in developing computational noncommutative geometry of higher noncommutative tori in presence of a class of non-conformally flat metrics and unraveling the topological significance of curvature terms obtained from heat trace asymptotics.

Unlike the Riemannian geometry of manifolds, which is defined by a metric tensor, the geometries of the noncommutative tori, and noncommutative spaces in general, are defined through operators which imitate the properties of geometric operators such as Dirac or Laplace type operators [5]. On noncommutative tori such operators can be constructed by analogy using basic facts from the complex geometry or the theory of de Rham complexes

on smooth manifolds. Inspired by spectral geometry, some of the spectral invariants of these operators can be used to define geometric quantities such as scalar curvature, total scalar curvature and even Ricci curvature [19, 11]. Similar ideas were worked out for noncommutative toric manifolds in [31, 33]. Tools such as Connes' pseudo-differential calculus for C^* -dynamical systems [4] and the rearrangement lemma [4, 28, 9, 8] play an essential role.

The geometries studied in the above works are mostly restricted to conformally flat metrics. The rationale behind the popularity of this specific class comes from the fact that by the uniformization theorem the conformally flat metrics cover all possible geometries on two tori. This is not true for the higher dimensional tori and for the two dimensional noncommutative torus the situation is much less understood as far as the uniformization theorem goes. Although the study of conformally flat geometries alone have shown many interesting new phenomenon, we believe that this is only the tip of the iceberg and much more remains to be discovered for non-conformal metrics and in higher dimensions.

In this paper we introduce a new class of metrics, called *functional metrics*. Let $C^\infty(\mathbb{T}_\theta^d)$ denotes the algebra of smooth elements of a d -dimensional noncommutative torus. Fix a selfadjoint element $h \in C^\infty(\mathbb{T}_\theta^d)$ and let the functions $g_{ij} : \mathbb{R} \rightarrow \mathbb{R}$, $1 \leq i, j \leq d$, be such that $(g_{ij}(t))$ is a positive definite symmetric matrix for all t in the spectrum of h . If $\theta = 0$, one can consider the metric tensor on $\mathbb{T}_\theta^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$ given by

$$g_{ij}(h)dx^i dx^j.$$

Although, the metric tensor is not a well defined notion in the noncommutative case, we can exploit the perspective explained in the earlier lines: find the geometric invariants using the spectral analysis of a good differential operator on the noncommutative tori. We construct an operator $\Delta_{0,g}$ which is antiunitary equivalent to the Laplacian on functions of \mathbb{T}_θ^d . To define the latter Laplacian, following the same lines as in [9], the entries $g_{ij}(h)$ are used to deform the inner product on functions and 1-forms.

Having a candidate for a geometric operator in hand, we go ahead and compute the densities of the heat trace asymptotic expansion $\text{Tr}(e^{-t\Delta_{0,g}})$ using Connes' pseudodifferential calculus on the noncommutative tori. It turns out that to handle this general case, we need to upgrade some of our tools. At this stage, we found the point of view taken in [28] very useful. The primary goal of this work was to offer a different proof for the rearrangement lemma and it provided a rigorous way to interpret the outcome elements of

the original rearrangement lemma. We call this type of elements as written in the *contraction form*:

$$F(h_{(0)}, \dots, h_{(n)})(b_1 \cdot b_2 \cdots b_n).$$

By a slight variation in one of the main results of [28], we become well-equipped to handle the difficulties. We also noticed that the symbol calculus can be applied for differential operators whose symbols can be written in the contraction form. We call these operators *h-differential operators*. We first find the way in which derivations act on elements written in the contraction form (Theorem 2.2). The Newton divided difference calculus plays an important role. For instance, we have

$$\begin{aligned} \delta_j(f(h_{(0)}, h_{(1)})(b_1)) &= f(h_{(0)}, h_{(1)})(\delta_j(b_1)) + [h_{(0)}, h_{(1)}; f(\cdot, h_{(2)})(\delta_j(h) \cdot b)] \\ &\quad + [h_{(1)}, h_{(2)}; f(h_{(0)}, \cdot)(b \cdot \delta_j(h))]. \end{aligned}$$

Using these facts and applying the pseudodifferential calculus, we compute the spectral densities for positive *h*-differential operators whose principal symbol is given by a functional metric. We call such an operator a *Laplace type h-differential operator*.

This change in order of the computations, i.e. writing symbols in the contraction form from the beginning, not only induced a smoother computation, at least symbolically, but also assisted us in a very fundamental way to consider general cases, e.g. finding the curvature in all higher dimensions for conformally flat and twisted product of flat metrics. Using the contraction form also enabled us to use Einstein summation convention.

While computing the heat trace densities of a positive Laplace type *h*-differential operator *P* with principal symbol $P_2^{ij}(h)\xi_i\xi_j$, we come across integrals of the form

$$\frac{-1}{\pi^{d/2}} \int_{\mathbb{R}^d} \xi_{n_1} \cdots \xi_{n_{2|\alpha|-4}} \frac{1}{2\pi i} \int_{\gamma} e^{-\lambda} B_0^{\alpha_0}(t_0) \cdots B_0^{\alpha_n}(t_n) d\lambda d\xi,$$

where $B_0(t) = (P_2^{ij}(t)\xi_i\xi_j - \lambda)^{-1}$. Such an integral is a smooth function of t_j 's and we call it a *T-function* and denote it by $T_{\mathbf{n};\alpha}(t_0, \dots, t_n)$. In Lemma 3.2, we find a formula for *T*-functions as an integral over the standard simplex which helps us to evaluate these functions in some cases (see Lemmas 4.2, 4.4 and 5.4). As an example, *T*-functions for conformally flat principal symbols $P_2^{ij}(t) = f(t)g^{ij}$ for $d \neq 2$, are given by

$$T_{\mathbf{n};\alpha}(t_0, \dots, t_n) =$$

$$\sqrt{|g|} \sum_n \prod g_{n_i n_{\sigma(i)}} \frac{(-1)^{|\alpha|-1} \Gamma(\frac{d}{2} - 1)}{\Gamma(\frac{d}{2} + |\alpha| - 2)} \partial_x^\beta [x_0, \dots, x_n; u^{1-\frac{d}{2}}] \Big|_{x_j=f(t_j)},$$

where $\beta = (\alpha_0 - 1, \dots, \alpha_n - 1)$.

In Theorem 4.3, we show that the scalar curvature of the d -dimensional noncommutative tori \mathbb{T}_θ^d equipped with a conformally flat metric $f(h)^{-1}g_{ij}$, where g_{ij} are constant numbers, is of the form

$$R = \sqrt{|g|} \left(K_d(h_{(0)}, h_{(1)}) (g^{ij} \delta_i \delta_j(h)) + H_d(h_{(0)}, h_{(1)}, h_{(2)}) (g^{ij} \delta_i(h) \cdot \delta_j(h)) \right),$$

A formula for the functions K_d and H_d is given for all d in Theorem 4.3. In particular, for $d \neq 2$, the function K_d is given by

$$K_d(t_0, t_1) = \frac{4 f(t_0)^{2-\frac{3d}{4}} f(t_1)^{2-\frac{3d}{4}}}{d(d-2)(f(t_0) - f(t_1))^3} [t_0, t_1; f] \times \\ \left((d-1)f(t_0)^{\frac{d}{2}} f(t_1)^{\frac{d}{2}-1} - (d-1)f(t_0)^{\frac{d}{2}-1} f(t_1)^{\frac{d}{2}} - f(t_0)^{d-1} + f(t_1)^{d-1} \right).$$

In Section 4.3, we study the geometry of the noncommutative tori \mathbb{T}_θ^d equipped with metrics of the form

$$f(t)^{-1}g \oplus \tilde{g}.$$

Following the differential geometry convention, we call such a metric a *twisted product functional metric* of (\mathbb{T}_θ^r, g) and $(\mathbb{T}_\theta^{d-r}, \tilde{g})$ with the twisting element $f(h)^{-1}$. The scalar curvature density of this metric is then obtained as

$$R = \sqrt{|g||\tilde{g}|} \left(K_r(h_{(0)}, h_{(1)}) (g^{ij} \delta_i \delta_j(h)) \right. \\ \left. + H_r(h_{(0)}, h_{(1)}, h_{(2)}) (g^{ij} \delta_i(h) \cdot \delta_j(h)) \right. \\ \left. + \tilde{K}_r(h_{(0)}, h_{(1)}) (\tilde{g}^{ij} \delta_i \delta_j(h)) \right. \\ \left. + \tilde{H}_r(h_{(0)}, h_{(1)}, h_{(2)}) (\tilde{g}^{ij} \delta_i(h) \cdot \delta_j(h)) \right).$$

The new functions \tilde{K}_r and \tilde{H}_r for $r \neq 2, 4$ are given in Theorem 4.5. For instance, we have

$$\tilde{K}_r(t_0, t_1) = \\ \frac{(2r-4)(f(t_0)^2 - f(t_1)^2) f(t_0)^{\frac{r}{2}} f(t_1)^{\frac{r}{2}} + 4f(t_0)^2 f(t_1)^r - 4f(t_0)^r f(t_1)^2}{(r-4)(r-2) f(t_0)^{\frac{3}{4}r} f(t_1)^{\frac{3}{4}r} (f(t_0) - f(t_1))^3} [t_0, t_1; f].$$

Finally the *total scalar curvature*, $\varphi(b_2(\Delta_{0,g}))$, of functional metrics on noncommutative tori is studied in Section 5. Note that the presence of the trace φ provides a ground to simplify the formula for the total scalar curvature. One then has Proposition 5.2 which asserts that

$$\varphi(R) = \varphi \left(F_S^{ij}(h_{(0)}, h_{(1)}) (\delta_i(h)) \delta_j(h) \right).$$

The functions F_S^{ij} are given by

$$\begin{aligned} F_S^{ij}(t_0, t_1) = & \frac{1}{2(t_0 - t_1)^2} \\ & \left(A^{ij} \sqrt{|g|(t_0)|g|(t_1)} - 2A^{ij}T_{;1,1}(t_0, t_1) \right. \\ & + T_{k,l;1,2}(t_0, t_1) (2A^{ik}g^{lj}(t_1) + 2A^{kj}g^{il}(t_1) - A^{ik}A^{lj}) \\ & \left. + T_{k,l;2,1}(t_0, t_1) (2A^{ik}g^{lj}(t_0) + 2A^{kj}g^{il}(t_0) - A^{ik}A^{lj}) \right), \end{aligned}$$

where $A^{ij} = |g|^{\frac{1}{4}}(t_0)|g|^{\frac{-1}{4}}(t_1)g^{ij}(t_0) + |g|^{\frac{1}{4}}(t_1)|g|^{\frac{-1}{4}}(t_0)g^{ij}(t_1)$. This formula is then used in Example 5.1 to generate the functions for the total scalar curvature for twisted product of flat metrics on \mathbb{T}_θ^d . As an another type of example, we compute functions F_S^{ij} for doubly twisted product metrics on \mathbb{T}_θ^4 in Example 5.2. On the other hand, using the Proposition 5.2 and evaluating the functions F_S^{ij} , we prove a Gauss-Bonnet theorem in dimension two, which generalizes the results obtained for conformally flat noncommutative two torus in [9, 15]. More precisely, Theorem 5.3 states that the total scalar curvature $\varphi(R)$ for any functional metrics g on \mathbb{T}_θ^2 vanishes.

This paper is organized as follows. In Section 2, we review preliminaries such as noncommutative tori, Connes’ pseudodifferential calculus and Newton divided difference calculus. Also in Subsection 2.2 a version of the rearrangement lemma is provided and used to find the derivation of an element written in the contraction form. Section 3 deals with the heat trace asymptotics for Laplace type h -differential operators. The T -functions are introduced in this section and their properties are studied. The class of functional metrics on noncommutative tori is introduced in Section 4 and the scalar curvature of some of the examples of such geometries is explicitly computed in all dimensions. In Section 5, we manage to track down further simplifications for the trace of the total scalar curvature for the functional metrics and prove a Gauss-Bonnet theorem for these metrics.

2. Preliminaries and computational tools

In [9, 15] to prove a version of the Gauss-Bonnet theorem for the noncommutative two torus and later in [8, 16] to compute the scalar curvature of a curved noncommutative two torus equipped with a conformally flat metric, the second density of the heat trace of the Laplacian D^2 of the Dirac operator was studied. This term was explicitly computed for the heat trace and the localized heat trace and the final results were written in a novel form which we will call the contraction form. These computations involved two main steps. First, the symbol of the parametrix of the differential operator were computed and several integrations were performed. The rearrangement lemma (cf. [8]) plays a central role in this step and it is the part that generates elements in the contraction form. In the second step, the outcome from the previous step, which are in terms of an auxiliary element $k = e^h$, were manipulated to be written in terms of the dilation h and then simplified to its best form. One of our goals is to show that we do not need to wait until the end of the step one to have elements in the contraction form; we can start off with operators whose symbol is originally written in the contraction form. In this section, in addition to recalling basics of the theory, we shall develop the required tools to be able to perform the symbol calculus on symbols in the contraction form.

2.1. Noncommutative d -tori and Connes' pseudodifferential calculus

In this section, we review the definitions and fix notations for the noncommutative d -dimensional torus. We also recall a pseudodifferential calculus on these C^* -algebras which is a special case of the calculus introduced by Connes [4] for C^* -dynamical systems.

Let θ be an anti-symmetric $d \times d$ real matrix. The noncommutative d -torus $C(\mathbb{T}_\theta^d)$ is the universal C^* -algebra generated by d unitary elements U_1, \dots, U_d which satisfy the commutation relations

$$U_k U_j = e^{2\pi i \theta_{jk}} U_j U_k.$$

When θ is the zero matrix, the C^* -algebra $C(\mathbb{T}_\theta^d)$ is isomorphic to the C^* -algebra $C(\mathbb{T}^d)$ of continuous functions on the d -dimensional torus \mathbb{T}^d .

The algebra of smooth elements of $C(\mathbb{T}_\theta^d)$, denoted by $C^\infty(\mathbb{T}_\theta^d)$, is a dense involutive subalgebra consisting of elements of the form

$$\sum_{\mathbf{n}=(n_1, \dots, n_d) \in \mathbb{Z}^d} a_{\mathbf{n}} U_1^{n_1} \cdots U_d^{n_d},$$

where $\{a_n\}_{n \in \mathbb{Z}^d}$ is a rapidly decaying sequence of complex numbers. There are d closed densely defined derivations on $C(\mathbb{T}_\theta^d)$ with the common domain $C^\infty(\mathbb{T}_\theta^d)$ whose values on U_k are given by

$$\delta_j(U_k) = \delta_{jk}U_k.$$

We note that $\delta_j(a)^* = -\delta_j(a^*)$, for any $a \in C^\infty(\mathbb{T}_\theta^d)$. The noncommutative d -tori possess a tracial state φ which acts on $C^\infty(\mathbb{T}_\theta^d)$ as

$$\varphi\left(\sum a_n U_1^{n_1} \cdots U_d^{n_d}\right) = a_0.$$

The trace φ is compatible with the derivations δ_j , i.e. $\varphi \circ \delta_j = 0$. This implies that a version of the integration by parts is valid on \mathbb{T}_θ^d ,

$$\varphi(a\delta_j(b)) = -\varphi(\delta_j(a)b), \quad a, b \in C^\infty(\mathbb{T}_\theta^d), \quad j = 1, \dots, d.$$

The subalgebra $C^\infty(\mathbb{T}_\theta^d)$ for $\theta = 0$, is indeed the algebra of smooth functions on the d -dimensional flat torus $\mathbb{T}^d = \mathbb{R}^d / (2\pi\mathbb{Z})^d$. In this case, the derivations δ_j are nothing but $\frac{1}{i} \frac{\partial}{\partial x_j}$ while the trace φ is the (normalized) integration with respect to the volume form $dx^1 \cdots dx^d$.

Connes' pseudodifferential calculus on the noncommutative d -tori can be constructed, as a special case of the general theory developed in [4] for C^* -dynamical systems, using the action of \mathbb{R}^d on $C(\mathbb{T}_\theta^d)$ by $*$ -automorphisms given by

$$\alpha_s(U_1^{n_1} \cdots U_d^{n_d}) = e^{is \cdot n} U_1^{n_1} \cdots U_d^{n_d}, \quad s = (s_1, \dots, s_d) \in \mathbb{R}^d.$$

We fix the following notations for a non-negative integer multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$:

$$\begin{aligned} \partial^\alpha &= \frac{\partial^{\alpha_1}}{\partial \xi_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial \xi_d^{\alpha_d}}, & \delta^\alpha &= \delta_1^{\alpha_1} \cdots \delta_d^{\alpha_d}, \\ |\alpha| &= \sum \alpha_j, & \alpha! &= \alpha_1! \cdots \alpha_d!. \end{aligned}$$

A smooth map $\rho : \mathbb{R}^d \rightarrow C^\infty(\mathbb{T}_\theta^d)$ is called a *symbol of order m* , if

- (i) For every multi-indices of non-negative integers α and β , there exists a constant C such that

$$\|\delta^\alpha \partial^\beta(\rho(\xi))\| \leq C(1 + |\xi|)^{m-|\beta|}.$$

(ii) There exists a smooth map $K : \mathbb{R}^d \rightarrow C^\infty(\mathbb{T}_\theta^d)$ such that

$$\lim_{\lambda \rightarrow \infty} \lambda^{-m} \rho(\lambda \xi_1, \dots, \lambda \xi_d) = K(\xi_1, \dots, \xi_d).$$

The space of all symbols of order m is denoted by $S^m(\mathbb{T}_\theta^d)$.

To every symbol $\rho \in S^m(\mathbb{T}_\theta^d)$, a pseudodifferential operator P_ρ is assigned which on the elements of $C^\infty(\mathbb{T}_\theta^d)$ is defined as

$$P_\rho(a) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{is\xi} \rho(\xi) \alpha_\xi(a) d\xi.$$

The composition $P_{\rho_1} \circ P_{\rho_2}$ of two pseudodifferential operators P_{ρ_1} and P_{ρ_2} with symbols $\rho_j \in S^{m_j}(\mathbb{T}_\theta^d)$, is again a pseudodifferential operator with the product symbol $\rho \in S^{m_1+m_2}(\mathbb{T}_\theta^d)$ asymptotically given by [4, Proposition 8]:

$$(1) \quad \rho \sim \sum_{\alpha \geq 0} \partial^\alpha(\rho_1) \delta^\alpha(\rho_2) / \alpha!.$$

For more details on Connes' pseudodifferential calculus we refer the reader to [4], for generalized cases to [29, 8] and for recent account on the subject to [21, 22].

2.2. Rearrangement lemma

The rearrangement lemma appeared in [9] as a tool to evaluate integrals of the form

$$\int_0^\infty (uk^2 + 1)^{-m} b(uk^2 + 1) u^m du,$$

where $k = e^{h/2}$ for some smooth selfadjoint element $h \in C^\infty(\mathbb{T}_\theta^2)$ and an element b which do not necessarily commute with h . A version of the lemma, dealing with more than one b , then appeared in [8, 15]. Later, a detailed study of this type of integrals has been carried out in [28], where a new proof for more general version of the lemma is offered and a different point of view of such computations is provided. In [28] for $\prod f_j(u, x)$ satisfying an integrability condition (see [28, Theorem 3.4]) a formula for the integral

$$\int_0^\infty f_0(u, k) b_1 f_1(u, k) \rho_2 \cdots b_n f_n(u, k) du,$$

is given. This approach uses the multiplication map $\mu : a_1 \otimes a_2 \otimes \cdots \otimes a_n \mapsto a_1 a_2 \cdots a_n$ from the projective tensor product $A^{\otimes_\gamma(n)}$ to A . The above

integrals are expressed as the contraction of the product of an element $F(k_{(0)}, \dots, k_{(n)})$ of $A^{\otimes, (n+1)}$, which is given by a multivariate functional calculus, and the element $b_1 \otimes b_2 \otimes \dots \otimes b_n \otimes 1$ which is

$$\mu\left(F(k_{(0)}, \dots, k_{(n)})(b_1 \otimes b_2 \otimes \dots \otimes b_n \otimes 1)\right).$$

We shall use the convention that such an element will be written as

$$(2) \quad F(k_{(0)}, \dots, k_{(n)})(b_1 \cdot b_2 \cdot \dots \cdot b_n).$$

An element in the form of (2) will be said is written in the *contraction form*. We call the first part $F(h_{(0)}, \dots, h_{(n+1)})$ the *commutative part* or *operator part* of the element, while the second part, $b_1 \cdot b_2 \dots b_n$, will be called the *noncommutative part* of the element.

The following lemma is a version of the rearrangement lemma that we will use in this work and it can be proved exactly as in [28, Theorem 3.4] with a slight change in the domain of integration from $[0, \infty)$ to any domain in \mathbb{R}^N .

Lemma 2.1 (Rearrangement lemma). *Let A be a unital C^* -algebra, $h \in A$ be a selfadjoint element, and Λ be an open neighborhood of the spectrum of h in \mathbb{R} . For a domain U in \mathbb{R}^N , let $f_j : U \times \Lambda \rightarrow \mathbb{C}$ for $0 \leq j \leq n$ be smooth functions such that $f(u, \lambda) = \prod_{j=0}^n f_j(u, \lambda_j)$ satisfies the following integrability condition: for any compact subset $K \subset \Lambda^{n+1}$ and every given integer multi-index α we have*

$$\int_U \sup_{\lambda \in K} |\partial_\lambda^\alpha f(u, \lambda)| du < \infty.$$

Then,

$$(3) \quad \int_U f_0(u, h) b_1 f_1(u, h) \cdot \dots \cdot b_n f_n(u, h) du = F(h_{(0)}, h_{(1)}, \dots, h_{(n)})(b_1 \cdot b_2 \cdot \dots \cdot b_n),$$

where $F(\lambda) = \int_U f(u, \lambda) du$.

For more information on the functional calculus $F(h_{(0)}, h_{(1)}, \dots, h_{(n)})$ on $A^{\otimes, (n+1)}$ and its relation to the Bochner integral $\int_U f(u, h_{(0)}, h_{(1)}, \dots, h_{(n)}) du$, we refer the reader to [28, Section 3.2].

Remark 2.1. Equation (3) shows that integration is a source to generate elements in the contraction form. Using lemma 2.1, as noted in [28, Remark

3.3], every expression in the contraction form with a Schwartz function $F \in \mathcal{S}(\mathbb{R}^{n+1})$ used in the operator part can indeed be obtained using an integral such as the one in the left hand side of equation (3). To see this, it is enough to set

$$f_n(\xi, \lambda) = \hat{f}(\xi)e^{i\xi_n\lambda}, \quad f_j(\xi, \lambda) = e^{i\xi_j\lambda}, \quad 0 \leq j \leq n - 1,$$

and $f(\xi, \lambda_0, \dots, \lambda_n) = \prod_{j=0}^n f_j(\xi, \lambda_j)$. By the Fourier inversion formula, we have $F(\lambda) = \int f(\xi, \lambda)d\xi$. Then, Lemma 2.1 gives the equality

$$(4) \quad F(h_{(0)}, \dots, h_{(n)})(b_1 \cdot b_2 \cdots b_n) = \int_{\mathbb{R}^n} e^{i\xi_0 h} b_1 e^{i\xi_1 h} b_2 \cdots b_n e^{i\xi_n h} \hat{f}(\xi) d\xi.$$

This fact will be used several times in this work.

2.3. Newton divided differences and closed derivations

As an application of the rearrangement lemma, we find a formula for the derivative of a smooth element written in contraction form in terms of the (partial) divided differences of the function used in the functional part and the derivatives of the noncommutative part. We first briefly review the Newton divided differences.

Let x_0, x_1, \dots, x_n be distinct points in an interval $I \subset \mathbb{R}$ and let f be a function on I . The n^{th} -order Newton divided difference of f , denoted by $[x_0, x_1, \dots, x_n; f]$, is the coefficient of x^n in the interpolating polynomial of f at the given points. In other words, if the interpolating polynomial is $p(x)$ then

$$p(x) = p_{n-1}(x) + [x_0, x_1, \dots, x_n; f](x - x_0) \cdots (x - x_{n-1}),$$

where $p_{n-1}(x)$ is a polynomial of degree at most $n - 1$. The following is the recursive formula for the divided difference which justifies the name, divided difference.

$$\begin{aligned} [x_0; f] &= f(x_0) \\ [x_0, x_1, \dots, x_n; f] &= \frac{[x_1, \dots, x_n; f] - [x_0, x_1, \dots, x_{n-1}; f]}{x_n - x_0}. \end{aligned}$$

The divided difference is also given by the explicit formula

$$[x_0, x_1, \dots, x_n; f] = \sum_{j=0}^n \frac{f(x_j)}{\prod_{j \neq l} (x_j - x_l)}.$$

It is clear that the n^{th} -order divided difference, as a function of x_j 's, is a symmetric function. There are different integral formulas for the divided differences from which we will use the *Hermite-Genocchi formula* (cf. e.g. [1, Theorem 3.3]). For an n times continuously differentiable function f , Hermite-Genocchi formula gives

$$(5) \quad [x_0, \dots, x_n; f] = \int_{\Sigma_n} f^{(n)}\left(\sum_{j=0}^n s_j x_j\right) ds,$$

where Σ_n denotes the standard n -simplex

$$\Sigma_n = \left\{ (s_0, \dots, s_n) \in \mathbb{R}^{n+1} \mid \sum_{j=0}^n s_j = 1 \text{ and } s_j \geq 0 \text{ for all } j \right\}.$$

We refer the reader to [35, Chapter 1] or [1, Section 3.2] for more details on the Newton divided differences.

Let δ be a (densely defined, unbounded) closed derivation on a C^* -algebra A . If $a \in \text{Dom}(\delta)$, then $e^{za} \in \text{Dom}(\delta)$ for any $z \in \mathbb{C}$ [38, p. 68] and

$$(6) \quad \delta(e^{za}) = z \int_0^1 e^{zsa} \delta(a) e^{z(1-s)a} ds.$$

Using the rearrangement lemma, one can go one step further and show that the derivation of $f(h)$ can be presented in the contraction form. In [9] this result is used for $f(x) = \exp(x)$ as a corollary of the expansional formula for e^{A+B} and in [28], the expansional formula is proved for general functions [28, Proposition 3.7]. The following theorem gives the result not only for $f(h)$ but for any element written in the contraction form.

Theorem 2.2. *Let δ be a closed derivation of a C^* -algebra A and $h \in \text{Dom}(\delta)$ be a selfadjoint element.*

(i) *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a smooth function. Then, $f(h) \in \text{Dom}(\delta)$ and*

$$\delta(f(h)) = [h_{(0)}, h_{(1)}; f](\delta(h)).$$

(ii) *Let $b_j \in \text{Dom}(\delta)$, $1 \leq j \leq n$, and let $f : \mathbb{R}^{n+1} \rightarrow \mathbb{C}$ be a smooth function. Then $f(h_{(0)}, \dots, h_{(n)})(b_1 \cdot b_2 \cdot \dots \cdot b_n)$ is in the domain of δ*

and

$$\begin{aligned} & \delta(f(h_{(0)}, \dots, h_{(n)})(b_1 \cdot b_2 \cdots b_n)) \\ &= \sum_{j=1}^n f(h_{(0)}, \dots, h_{(n)})(b_1 \cdots b_{j-1} \cdot \delta(b_j) \cdot b_{j+1} \cdots b_n) \\ & \quad + \sum_{j=0}^n f_j(h_{(0)}, \dots, h_{(n+1)})(b_1 \cdots b_j \cdot \delta(h) \cdot b_{j+1} \cdots b_n), \end{aligned}$$

where $f_j(t_0, \dots, t_{n+1})$, which we call the partial divided difference, is defined as

$$f_j(t_0, \dots, t_{n+1}) = [t_j, t_{j+1}; t \mapsto f(t_0, \dots, t_{j-1}, t, t_{j+2}, \dots, t_n)].$$

Proof. It is enough to assume that f is a Schwartz class function on \mathbb{R} . We have

$$f(h) = \int_{\mathbb{R}} e^{ih\xi} \hat{f}(\xi) d\xi,$$

where \hat{f} is the Fourier transform of the function f . By closedness of δ , we can then change the order of integration and derivation:

$$\begin{aligned} \delta(f(h)) &= \int_{\mathbb{R}} \delta(e^{ih\xi}) \hat{f}(\xi) d\xi \\ &= \int_{\mathbb{R}} i\xi \int_0^1 e^{i\xi sh} \delta(h) e^{i\xi(1-s)h} ds \hat{f}(\xi) d\xi \\ &= \mathcal{C} \left(\int_{\mathbb{R}} \int_0^1 e^{i\xi sh} \otimes e^{i\xi(1-s)h} i\xi \hat{f}(\xi) ds d\xi (\delta(h) \otimes 1) \right) \\ &= F(h_{(0)}, h_{(1)})(\delta(h)). \end{aligned}$$

Here $F(t_0, t_1)$ is the function given by the integral

$$\int_{\mathbb{R}} \int_0^1 e^{i\xi st_0} e^{i\xi(1-s)t_1} i\xi \hat{f}(\xi) ds d\xi.$$

Using Hermite-Genocchi formula (5) and the properties of the Fourier transform, we have

$$F(t_0, t_1) = \int_0^1 \int_{\mathbb{R}} e^{i\xi(st_0 + \xi(1-s)t_1)} i\xi \hat{f}(\xi) d\xi ds$$

$$\begin{aligned}
&= \int_0^1 f'(st_0 + \xi(1-s)t_1) ds \\
&= [t_0, t_1; f].
\end{aligned}$$

We prove the part *ii*) only for $n = 1$; the proof of the general case follows similarly. We first use Remark 2.1 to write $f(h_{(0)}, h_{(1)})(b_1)$ as an integral

$$f(h_{(0)}, h_{(1)})(b_1) = \int_{\mathbb{R}^2} e^{i\xi_1 h} b_1 e^{i\xi_2 h} \hat{f}(\xi) d\xi.$$

The Leibniz rule together with the fact that δ is closed imply that

$$\begin{aligned}
&\delta\left(f(h_{(0)}, h_{(1)})(b_1)\right) \\
&= \delta\left(\int_{\mathbb{R}^2} e^{i\xi_1 h} b_1 e^{i\xi_2 h} \hat{f}(\xi) d\xi\right) \\
&= \int_{\mathbb{R}^2} \delta\left(e^{i\xi_1 h}\right) b_1 e^{i\xi_2 h} \hat{f}(\xi) d\xi + \int_{\mathbb{R}^2} e^{i\xi_1 h} \delta\left(b_1\right) e^{i\xi_2 h} \hat{f}(\xi) d\xi \\
&\quad + \int_{\mathbb{R}^2} e^{i\xi_1 h} b_1 \delta\left(e^{i\xi_2 h}\right) \hat{f}(\xi) d\xi \\
&= \int_{\mathbb{R}^2} \int_0^1 i\xi_1 e^{is\xi_1 h} \delta(h) e^{i(1-s)\xi_1 h} b_1 e^{i\xi_2 h} \hat{f}(\xi) ds d\xi \\
&\quad + \int_{\mathbb{R}^2} e^{i\xi_1 h} \delta(b_1) e^{i\xi_2 h} \hat{f}(\xi) d\xi \\
&\quad + \int_{\mathbb{R}^2} \int_0^1 i\xi_2 e^{i\xi_1 h} b_1 e^{is\xi_2 h} \delta(h) e^{i(1-s)\xi_2 h} \hat{f}(\xi) ds d\xi \\
&= f_1(h_{(0)}, h_{(1)}, h_{(2)})(b_1 \cdot \delta(h)) + f(h_{(0)}, h_{(1)})(\delta(b_1)) \\
&\quad + f_2(h_{(0)}, h_{(1)}, h_{(2)})(\delta(h) \cdot b_1),
\end{aligned}$$

where

$$\begin{aligned}
f_1(t_0, t_1, t_2) &= \int_{\mathbb{R}^2} \int_0^1 i\xi_1 e^{is\xi_1 t_0} e^{i(1-s)\xi_1 t_1} e^{i\xi_2 t_2} \hat{f}(\xi) ds d\xi = [t_0, t_1; f(\cdot, t_2)], \\
f_2(t_0, t_1, t_2) &= \int_{\mathbb{R}^2} \int_0^1 i\xi_2 e^{i\xi_1 t_0} e^{is\xi_2 t_1} e^{i(1-s)\xi_2 t_2} \hat{f}(\xi) ds d\xi = [t_1, t_2; f(t_0, \cdot)].
\end{aligned}$$

□

Corollary 2.3. *Let h be a selfadjoint element in a C^* -algebra A and let δ_1 and δ_2 be two closed derivations. If $h \in \text{Dom}(\delta_2)$ and $\delta_2(h) \in \text{Dom}(\delta_1)$, then*

for any smooth function f we have

$$\begin{aligned} & \delta_1 \delta_2(f(h)) = \\ & [h_{(0)}, h_{(1)}, h_{(2)}; f] (\delta_2(h) \cdot \delta_1(h) + \delta_1(h) \cdot \delta_2(h)) + [h_{(0)}, h_{(1)}; f] (\delta_1 \delta_2(h)). \end{aligned}$$

Proof. Using Theorem 2.2, we have

$$\begin{aligned} \delta_1 \delta_2(f(h)) &= \delta_1 (\delta_2(f(h))) = \delta_1 ([h_{(0)}, h_{(1)}; f] (\delta_2(h))) \\ &= [h_{(0)}, h_{(1)}; f] (\delta_1 \delta_2(h)) \\ &\quad + [h_{(0)}, h_{(1)}; t \mapsto [t, h_{(2)}; f]] (\delta_1(h) \cdot \delta_2(h)) \\ &\quad + [h_{(1)}, h_{(2)}; t \mapsto [h_{(0)}, t; f]] (\delta_2(h) \cdot \delta_1(h)). \end{aligned}$$

The functional part of the last two terms, by the definition of second order divided difference and invariance of divided difference under the permutation of variables t_j 's, are equal to $[h_{(0)}, h_{(1)}, h_{(2)}; f]$. The proof is done using the fact that the contraction is linear with respect to the noncommutative part. \square

This corollary can also be deduced from the variational formula used in [28, Example 3.9]. In the case of the noncommutative two tori, the conditions of Corollary 2.3 are automatically satisfied for any smooth selfadjoint element $h = h^* \in C^\infty(\mathbb{T}_\theta^2)$. For $f(t) = e^{\frac{t}{2}}$, we have

$$\begin{aligned} [t_0, t_1, t_2; e^{\frac{t}{2}}] &= \frac{e^{t_0/2}}{(t_0 - t_1)(t_0 - t_2)} + \frac{e^{t_1/2}}{(t_1 - t_0)(t_1 - t_2)} + \frac{e^{t_2/2}}{(t_2 - t_0)(t_2 - t_1)} \\ &= e^{t_0/2} \frac{(t_1 - t_2) + (t_2 - t_0)e^{(t_1-t_0)/2} + (t_0 - t_1)e^{(t_2-t_0)/2}}{(t_0 - t_1)(t_0 - t_2)(t_1 - t_2)}, \\ [t_0, t_1; e^{\frac{t}{2}}] &= \frac{e^{t_0/2} - e^{t_1/2}}{t_0 - t_1} = e^{t_0/2} \frac{1 - e^{(t_1-t_0)/2}}{t_0 - t_1}. \end{aligned}$$

This is in complete agreement with the formula (6.8) in [8] by setting

$$s = t_1 - t_0, \quad t = t_2 - t_1, \quad \text{and } h = 2 \log(k).$$

3. Heat trace asymptotics for Laplace type h -differential operators

Following the strategy underlined in the previous section, we shall compute the first three terms of the symbol of the parametrix of an elliptic operator on the noncommutative tori with a positive definite principal symbol whose

symbol is given in the contraction form. These terms will be given in the contraction form as well.

3.1. Laplace type h -differential operators and their paramatrix

In the following definition we introduce a new class of differential operators on noncommutative tori that generalize the already studied classes.

Definition 3.1. Let $h \in C^\infty(\mathbb{T}_\theta^d)$ be a smooth selfadjoint element.

- (i) By an h -differential operator on \mathbb{T}_θ^d , we mean a differential operator $P = \sum_\alpha p_\alpha \delta^\alpha$, with $C^\infty(\mathbb{T}_\theta^d)$ -valued coefficients p_α which can be written in the contraction form

$$p_\alpha = P_{\alpha, \alpha_1, \dots, \alpha_k}(h_{(0)}, \dots, h_{(k)})(\delta^{\alpha_1}(h) \cdots \delta^{\alpha_k}(h)).$$

- (ii) We call a second order h -differential operator P a Laplace type h -differential operator if its symbol is a sum of homogeneous parts p_j of the form

$$\begin{aligned} p_2 &= P_2^{ij}(h)\xi_i\xi_j, \\ (7) \quad p_1 &= P_1^{ij}(h_{(0)}, h_{(1)})(\delta_i(h))\xi_j, \\ p_0 &= P_{0,1}^{ij}(h_{(0)}, h_{(1)})(\delta_i\delta_j(h)) + P_{0,2}^{ij}(h_{(0)}, h_{(1)}, h_{(2)})(\delta_i(h) \cdot \delta_j(h)), \end{aligned}$$

where the principal symbol $p_2 : \mathbb{R}^d \rightarrow C^\infty(\mathbb{T}_\theta^d)$ is a $C^\infty(\mathbb{T}_\theta^d)$ -valued quadratic form such that $p_2(\xi) > 0$ for all $\xi \in \mathbb{R}^d$.

We can allow the symbols to be matrix valued, that is $p_j : \mathbb{R}^d \rightarrow C^\infty(\mathbb{T}_\theta^d) \otimes M_n(\mathbb{C})$, provided that all $p_2(\xi) \in C^\infty(\mathbb{T}_\theta^d) \otimes I_n$ for all nonzero $\xi \in \mathbb{R}^d$.

Example 3.1. Many of the differential operators on noncommutative tori which were studied in the literature are Laplace type h -differential operators. For instance, the two differential operators on \mathbb{T}_θ^2 whose spectral invariants are studied in [16, 8] are indeed Laplace type h -differential operator. Using the notation of [8, Lemma 1.11], $k = e^{h/2}$, these operators are given by

$$k\Delta k = k\delta\delta^*k, \quad \Delta_\varphi^{(0,1)} = \delta^*k^2\delta,$$

where, $\delta = \delta_1 + \bar{\tau}\delta_2$ and $\delta^* = \delta_1 + \tau\delta_2$ for some complex number τ in the upper half plane. The operator $k\Delta k$ is antiunitary equivalent to the Laplacian on the functions on \mathbb{T}_θ^2 equipped with the conformally flat metric

$k^2 \begin{pmatrix} 1 & \Re(\tau) \\ \Re(\tau) & |\tau|^2 \end{pmatrix}$. We shall construct this operator in all dimensions for the more general class of metrics, which will be introduced in Section 4.1, and study it in details then. On the other hand, the symbol of the other operator $\Delta_\varphi^{(0,1)}$ is the sum of the following homogeneous terms [8, Lemma 6.1]:

$$\begin{aligned} p_2(\xi) &= k^2(\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2), \\ p_1(\xi) &= (\delta_1(k^2) + \tau\delta_2(k^2))\xi_1 + (\bar{\tau}\delta_1(k^2) + |\tau|^2\delta_2(k^2))\xi_2, \\ p_0(\xi) &= 0. \end{aligned}$$

Clearly, we have

$$p_2^{11}(t) = e^t, \quad p_2^{12}(t) = p_2^{21}(t) = \Re(\tau)e^t, \quad p_2^{22}(t) = |\tau|^2e^t,$$

and using Corollary 2.3, we obtain the other functions

$$\begin{aligned} P_1^{1,1}(t_0, t_1) &= [t_0, t_1; e^u], & P_1^{2,1}(t_0, t_1) &= \tau [t_0, t_1; e^u], \\ P_1^{1,2}(t_0, t_1) &= \bar{\tau} [t_0, t_1; e^u], & P_1^{2,2}(t_0, t_1) &= |\tau|^2 [t_0, t_1; e^u]. \end{aligned}$$

Remark 3.1. The Laplace type h -operators can also be made sense on smooth vector bundles on manifolds, where h can be taken (locally) as a function or an endomorphism of the bundle. If h is a function, then a Laplace type h -differential operator is indeed a Laplace type operator. However, if h is an endomorphism, these operators belong to a different class of elliptic operators of order two for which some aspects of its heat trace densities are studied in [23, 2]. The matrix-valued conformally perturbed Dirac operator on smooth manifolds, which is an h -differential operator, is also studied recently in [26].

Let P be a positive Laplace type h -differential operator. For any λ in $\mathbb{C} \setminus \mathbb{R}^{\geq 0}$, the operator $(P - \lambda)^{-1}$ is a classical pseudodifferential operator of order -2 , i.e. its symbol can be written as

$$\sigma((P - \lambda)^{-1}) = b_0(\xi, \lambda) + b_1(\xi, \lambda) + b_2(\xi, \lambda) + \dots,$$

where b_j is a homogeneous symbol of order $-2 - j$. The following recursive formula for b_j 's can be obtained from the product formula for symbols (1).

$$\begin{aligned} (8) \quad b_0(\xi, \lambda) &= (p_2(\xi) - \lambda)^{-1}, \\ b_n(\xi, \lambda) &= - \sum_{j=0}^{n-1} \sum_{k=0}^2 \sum_{|\alpha|=n-j+k-2} \partial^\alpha b_j(\xi, \lambda) \delta^\alpha(p_k(\xi)) b_0(\xi, \lambda) / \alpha! \quad n > 0. \end{aligned}$$

If we start off with symbols written in the contraction form, the formula (8) shows that all $b_n(\xi, \lambda)$'s can also be computed in the contraction form using Theorem 2.2. In the rest of this section, we compute the terms $b_0(\xi, \lambda)$, $b_1(\xi, \lambda)$ and $b_2(\xi, \lambda)$ in the contraction form as follows.

Term $b_0(\xi, \lambda)$: This term is defined as $b_0(\xi, \lambda) = (p_2(\xi) - \lambda)^{-1}$ and it can be written as

$$b_0(\xi, \lambda) = B_0(h), \quad B_0(t_0) = (P_2^{ij}(t_0)\xi_i\xi_j - \lambda)^{-1}.$$

We choose to overlook the dependence of B_0 on the variables ξ and λ in its notation.

Term $b_1(\xi, \lambda)$: Using (8), we have

$$b_1 = -b_0(\xi, \lambda)p_1(\xi)b_0(\xi, \lambda) - \sum \partial_i b_0(\xi, \lambda) \delta_i(p_2(\xi))b_0(\xi, \lambda).$$

We then write each term of the above expression in the contraction form:

$$\begin{aligned} -b_0(\xi, \lambda)p_1(\xi)b_0(\xi, \lambda) &= -\xi_j B_0(h_{(0)})P_1^{ij}(h_{(0)}, h_{(1)})B_0(h_{(1)})(\delta_i(h)), \\ \partial_i b_0(\xi, \lambda) &= -2P_2^{ij}(h)B_0^2(h)\xi_j, \\ \delta_i(p_2(\xi)) &= \xi_k\xi_\ell [h_{(0)}, h_{(1)}; P_2^{k\ell}](\delta_i(h)), \end{aligned}$$

and

$$\begin{aligned} & - \sum \partial_i b_0(\xi, \lambda) \delta_i(p_2(\xi))b_0(\xi, \lambda) = \\ & 2\xi_j\xi_k\xi_\ell P_2^{ij}(h_{(0)})B_0^2(h_{(0)})[h_{(0)}, h_{(1)}; P_2^{k\ell}]B_0(h_{(1)})(\delta_i(h)). \end{aligned}$$

Therefore, $b_1(\xi, \lambda) = B_1^i(\xi, \lambda, h_{(0)}, h_{(1)})(\delta_i(h))$, where

$$\begin{aligned} B_1^i(\xi, \lambda, t_0, t_1) &= \\ & -\xi_j B_0(t_0)P_1^{ij}(t_0, t_1)B_0(t_1) + 2\xi_j\xi_k\xi_\ell P_2^{ij}(t_0)B_0^2(t_0)[t_0, t_1; P_2^{k\ell}]B_0(t_1). \end{aligned}$$

Term $b_2(\xi, \lambda)$: Formula (8) for $n = 2$, gives $b_2(\xi, \lambda)$ in terms of $b_0(\xi, \lambda)$ and $b_1(\xi, \lambda)$ and $p_j(\xi)$'s:

$$\begin{aligned} (9) \quad b_2(\xi, \lambda) &= -b_0(\xi, \lambda)p_0(\xi)b_0(\xi, \lambda) - b_1(\xi, \lambda)p_1(\xi)b_0(\xi, \lambda) \\ & - \sum \partial_i b_0(\xi, \lambda) \delta_i(p_1(\xi))b_0(\xi, \lambda) - \sum \partial_i b_1(\xi, \lambda) \delta_i(p_2(\xi))b_0(\xi, \lambda) \\ & - \sum \frac{1}{2} \partial_i \partial_j b_0(\xi, \lambda) \delta_i \delta_j(p_2(\xi))b_0(\xi, \lambda). \end{aligned}$$

To compute $b_2(\xi, \lambda)$ we need $\partial_j b_1(\xi, \lambda)$. First, we should note that

$$\partial_i b_1(\xi, \lambda) = \partial_j (B_1^i(\xi, \lambda, h_{(0)}, h_{(1)})(\delta_i(h))) = \partial_j (B_1^i(\xi, \lambda, h_{(0)}, h_{(1)})) (\delta_i(h)).$$

Hence, we have

$$\begin{aligned} (10) \quad & \partial_j B_1^i(\xi, \lambda, t_0, t_1) = \\ & - B_0(t_0) P_1^{ij}(t_0, t_1) B_0(t_1) \\ & + 2\xi_k \xi_\ell P_2^{j\ell} B_0^2(t_0) P_1^{ik}(t_0, t_1) B_0(t_1) \\ & + 2\xi_k \xi_\ell B_0(t_0) P_1^{ik}(t_0, t_1) P_2^{j\ell} B_0^2(t_1) \\ & + 2\xi_\ell \xi_m P_2^{ij} B_0^2(t_0) [t_0, t_1; P_2^{\ell m}] B_0(t_1) \\ & + 4\xi_k \xi_\ell P_2^{ik}(t_0) B_0^2(t_0) [t_0, t_1; P_2^{j\ell}] B_0(t_1) \\ & - 8\xi_k \xi_\ell \xi_m \xi_n P_2^{ik}(t_0) P_2^{jn}(t_0) B_0^3(t_0) [t_0, t_1; P_2^{\ell m}] B_0(t_1) \\ & - 4\xi_k \xi_\ell \xi_m \xi_n P_2^{ik}(t_0) B_0^2(t_0) [t_0, t_1; P_2^{\ell m}] P_2^{jn}(t_1) B_0^2(t_1). \end{aligned}$$

We now compute each term of expression (9) separately:

$$\begin{aligned} & -b_0(\xi, \lambda) p_0(\xi) b_0(\xi, \lambda) = \\ & -B_0(h_{(0)}) P_{0,1}^{ij}(h_{(0)}, h_{(1)}) B_0(h_{(1)}) (\delta_i \delta_j(h)) \\ & -B_0(h_{(0)}) P_{0,2}^{ij}(h_{(0)}, h_{(1)}, h_{(2)}) B_0(h_{(2)}) (\delta_i(h) \cdot \delta_j(h)). \end{aligned}$$

The second term of the sum (9) is equal to

$$\begin{aligned} -b_1(\xi, \lambda) p_1(\xi) b_0(\xi, \lambda) & = \left(\xi_k \xi_\ell B_0(h_{(0)}) P_1^{ik}(h_{(0)}, h_{(1)}) B_0(h_{(1)}) P_1^{j\ell}(h_{(1)}, h_{(2)}) B_0(h_{(2)}) \right. \\ & \left. - 2\xi_k \xi_\ell \xi_m \xi_n P_2^{im} B_0^2(h_{(0)}) [h_{(0)}, h_{(1)}; P_2^{k\ell}] B_0(h_{(1)}) P_1^{jn}(h_{(1)}, h_{(2)}) B_0(h_{(2)}) \right) (\delta_i(h) \cdot \delta_j(h)). \end{aligned}$$

We need $\delta_i(p_1(\xi))$ for the next term;

$$\begin{aligned} \delta_i(p_1(\xi)) & = \xi_\ell P_1^{j\ell}(h_{(0)}, h_{(1)}) (\delta_i \delta_j(h)) \\ & + \xi_\ell [h_{(0)}, h_{(1)}; P_1^{j\ell}(\cdot, h_{(2)})] (\delta_i(h) \cdot \delta_j(h)) \\ & + \xi_\ell [h_{(1)}, h_{(2)}; P_1^{j\ell}(h_{(0)}, \cdot)] (\delta_j(h) \cdot \delta_i(h)). \end{aligned}$$

Then, the third term of (9) is given by

$$- \sum \partial_i b_0(\xi, \lambda) \delta_i(p_1(\xi)) b_0(\xi, \lambda)$$

$$\begin{aligned}
&= 2\xi_k\xi_\ell P_2^{ik} B_0^2(h_{(0)}) P_1^{j\ell}(h_{(0)}, h_{(1)}) B_0(h_{(1)}) (\delta_i\delta_j(h)) \\
&\quad + 2\xi_k\xi_\ell P_2^{ik} B_0^2(h_{(0)}) [h_{(0)}, h_{(1)}; P_1^{j\ell}(\cdot, h_{(2)})] B_0(h_{(2)}) (\delta_i(h) \cdot \delta_j(h)) \\
&\quad + 2\xi_k\xi_\ell P_2^{ik} B_0^2(h_{(0)}) [h_{(1)}, h_{(2)}; P_1^{j\ell}(h_{(0)}, \cdot)] B_0(h_{(2)}) (\delta_j(h) \cdot \delta_i(h)).
\end{aligned}$$

Considering (10), it can readily be seen that $-\sum_j \partial_j(b_1)\delta_j(p_2)b_0$ is of the form

$$F^{ij}(h_{(0)}, h_{(1)}, h_{(2)})(\delta_i(h) \cdot \delta_j(h)),$$

where

$$\begin{aligned}
F^{ij}(t_0, t_1, t_2) = & \xi_k\xi_\ell B_0(t_0) P_1^{ij}(t_0, t_1) B_0(t_1) [t_1, t_2; P_2^{k\ell}] B_0(t_2) \\
& - 2\xi_k\xi_\ell \xi_m \xi_n P_2^{j\ell}(t_0) B_0^2(t_0) P_1^{ik}(t_0, t_1) B_0(t_1) [t_1, t_2; P_2^{mn}] B_0(t_2) \\
& - 2\xi_k\xi_\ell \xi_m \xi_n B_0(t_0) P_1^{ik}(t_0, t_1) P_2^{j\ell}(t_1) B_0^2(t_1) [t_1, t_2; P_2^{mn}] B_0(t_2) \\
& - 2\xi_k\xi_\ell \xi_m \xi_n P_2^{ij}(t_0) B_0^2(t_0) [t_0, t_1; P_2^{lm}] B_0(t_1) [t_1, t_2; P_2^{kn}] B_0(t_2) \\
& - 4\xi_k\xi_\ell \xi_m \xi_n P_2^{ik}(t_0) B_0^2(t_0) [t_0, t_1; P_2^{j\ell}] B_0(t_1) [t_1, t_2; P_2^{mn}] B_0(t_2) \\
& + 8\xi_k\xi_\ell \xi_m \xi_n \xi_p \xi_q P_2^{ik}(t_0) P_2^{jn}(t_0) B_0^3(t_0) [t_0, t_1; P_2^{lm}] B_0(t_1) [t_1, t_2; P_2^{pq}] B_0(t_2) \\
& + 4\xi_k\xi_\ell \xi_m \xi_n \xi_p \xi_q P_2^{ik}(t_0) B_0^2(t_0) [t_0, t_1; P_2^{lm}] P_2^{jn}(t_1) B_0^2(t_1) [t_1, t_2; P_2^{pq}] B_0(t_2).
\end{aligned}$$

To compute the very last term $-\sum_{i,j} \frac{1}{2} \partial_i \partial_j b_0(\xi, \lambda) \delta_i \delta_j(p_2(\xi)) b_0(\xi, \lambda)$, we first write the terms $\frac{1}{2} \partial_i \partial_j b_0(\xi, \lambda)$ and $\delta_i \delta_j(p_2(\xi))$ in the contraction form:

$$\frac{1}{2} \partial_i \partial_j b_0(\xi, \lambda) = -P_2^{ij}(h) B_0^2(h) + 4\xi_k\xi_\ell P_2^{ik} P_2^{j\ell}(h) B_0^3(h),$$

and

$$\begin{aligned}
\delta_i \delta_j(p_2(\xi)) = & \xi_m \xi_n [h_{(0)}, h_{(1)}; P_2^{mn}] (\delta_i \delta_j(h)) \\
& + \xi_m \xi_n [h_{(0)}, h_{(1)}, h_{(2)}; P_2^{mn}] (\delta_i(h) \cdot \delta_j(h)) \\
& + \xi_m \xi_n [h_{(0)}, h_{(1)}, h_{(2)}; P_2^{mn}] (\delta_j(h) \cdot \delta_i(h)).
\end{aligned}$$

Therefore, the last term of (9) is given by the following formula:

$$\begin{aligned}
& - \sum_{i,j} \frac{1}{2} \partial_i \partial_j b_0(\xi, \lambda) \delta_i \delta_j(p_2(\xi)) b_0(\xi, \lambda) = \\
& + \xi_k \xi_\ell P_2^{ij} B_0^2(h_{(0)}) [h_{(0)}, h_{(1)}; P_2^{k\ell}] B_0(h_{(1)}) (\delta_i \delta_j(h))
\end{aligned}$$

$$\begin{aligned}
& - 4\xi_k\xi_\ell\xi_m\xi_n P_2^{ik} P_2^{j\ell}(h_{(0)}) B_0^3(h_{(0)}) [h_{(0)}, h_{(1)}; P_2^{mn}] B_0(h_{(1)}) (\delta_i \delta_j(h)) \\
& + 2\xi_k\xi_\ell P_2^{ij} B_0^2(h_{(0)}) [h_{(0)}, h_{(1)}, h_{(2)}; P_2^{k\ell}] B_0(h_{(2)}) (\delta_i(h) \cdot \delta_j(h)) \\
& - 8\xi_k\xi_\ell\xi_m\xi_n P_2^{ik} P_2^{j\ell} B_0^3(h_{(0)}) [h_{(0)}, h_{(1)}, h_{(2)}; P_2^{mn}] B_0(h_{(2)}) (\delta_i(h) \cdot \delta_j(h)).
\end{aligned}$$

Adding all the terms, $b_2(\xi, \lambda)$ is found to be

$$\begin{aligned}
b_2(\xi, \lambda) &= B_{2,1}^{ij}(\xi, \lambda, h_{(0)}, h_{(1)}) (\delta_i \delta_j(h)) \\
&+ B_{2,2}^{ij}(\xi, \lambda, h_{(0)}, h_{(1)}, h_{(2)}) (\delta_i(h) \cdot \delta_j(h)),
\end{aligned}$$

where the functions $B_{2,k}^{ij}$ are given by

$$\begin{aligned}
B_{2,1}^{ij}(\xi, \lambda, t_0, t_1) &= - B_0(t_0) P_{0,1}^{ij}(t_0, t_1) B_0(t_1) \\
&+ 2\xi_k\xi_\ell B_0^2(t_0) P_2^{ik}(t_0) P_1^{j\ell}(t_0, t_1) B_0(t_1) \\
&+ \xi_k\xi_\ell B_0^2(t_0) P_2^{ij}(t_0) [t_0, t_1; P_2^{k\ell}] B_0(t_1) \\
&- 4\xi_k\xi_\ell\xi_m\xi_n B_0^3(t_0) P_2^{ik}(t_0) P_2^{j\ell}(t_0) [t_0, t_1; P_2^{mn}] B_0(t_1),
\end{aligned}$$

and

$$\begin{aligned}
B_{2,2}^{ij}(\xi, \lambda, t_0, t_1, t_2) &= \\
&- B_0(t_0) P_{0,2}^{ij}(t_0, t_1, t_2) B_0(t_2) \\
&+ \xi_k\xi_\ell B_0(t_0) P_1^{ik}(t_0, t_1) B_0(t_1) P_1^{j\ell}(t_1, t_2) B_0(t_2) \\
&- 2\xi_k\xi_\ell\xi_m\xi_n P_2^{im} B_0^2(t_0) [t_0, t_1; P_2^{k\ell}] B_0(t_1) P_1^{jn}(t_1, t_2) B_0(t_2) \\
&+ 2\xi_k\xi_\ell P_2^{ik} B_0^2(t_0) [t_0, t_1; P_1^{j\ell}(\cdot, t_2)] B_0(t_2) \\
&+ 2\xi_k\xi_\ell P_2^{jk} B_0^2(t_0) [t_1, t_2; P_1^{i\ell}(h_{(0)}, \cdot)] B_0(t_2) \\
&+ \xi_k\xi_\ell B_0(t_0) P_1^{ij}(t_0, t_1) B_0(t_1) [t_1, t_2; P_2^{k\ell}] B_0(t_2) \\
&- 2\xi_k\xi_\ell\xi_m\xi_n P_2^{j\ell} B_0^2(t_0) P_1^{ik}(t_0, t_1) B_0(t_1) [t_1, t_2; P_2^{mn}] B_0(t_2) \\
&- 2\xi_k\xi_\ell\xi_m\xi_n B_0(t_0) P_1^{ik}(t_0, t_1) P_2^{j\ell}(t_1) B_0^2(t_1) [t_1, t_2; P_2^{mn}] B_0(t_2) \\
&- 2\xi_k\xi_\ell\xi_m\xi_n P_2^{ij} B_0^2(t_0) [t_0, t_1; P_2^{\ell m}] B_0(t_1) [t_1, t_2; P_2^{kn}] B_0(t_2) \\
&- 4\xi_k\xi_\ell\xi_m\xi_n P_2^{ik} B_0^2(t_0) [t_0, t_1; P_2^{j\ell}] B_0(t_1) [t_1, t_2; P_2^{mn}] B_0(t_2) \\
&+ 8\xi_k\xi_\ell\xi_m\xi_n \xi_p \xi_q P_2^{ik} P_2^{jn} B_0^3(t_0) [t_0, t_1; P_2^{\ell m}] B_0(t_1) [t_1, t_2; P_2^{pq}] B_0(t_2) \\
&+ 4\xi_k\xi_\ell\xi_m\xi_n \xi_p \xi_q P_2^{ik} B_0^2(t_0) [t_0, t_1; P_2^{\ell m}] P_2^{jn} B_0^2(t_1) [t_1, t_2; P_2^{pq}] B_0(t_2) \\
&+ 2\xi_k\xi_\ell P_2^{ij} B_0^2(t_0) [t_0, t_1, t_2; P_2^{k\ell}] B_0(t_2) \\
&- 8\xi_k\xi_\ell\xi_m\xi_n P_2^{ik} P_2^{j\ell} B_0^3(t_0) [t_0, t_1, t_2; P_2^{mn}] B_0(t_2).
\end{aligned}$$

Note that in the above formulas, $B_0(t)$ is always a scalar function and commutes with the other terms P_i^{jk} which can be matrix valued functions. Hence, we can bring all the powers of B_0 's in each summand up front and group them with the other scalar terms such as ξ_j 's. But, the order of the other terms must stay untouched to respect the noncommutativity of matrix product. This is what we will do in the next section while we find a formula for the integral of these terms.

3.2. Coefficients of the asymptotic expansion of the heat trace

Using the Cauchy integral formula, we have

$$e^{-tP} = \frac{-1}{2\pi i} \int_{\gamma} e^{-t\lambda} (P - \lambda)^{-1} d\lambda, \quad t > 0.$$

Here γ is a contour around the non-negative real line with the counter clockwise orientation. The expansion of the symbol of the paramatrix $\sigma((P - \lambda)^{-1})$ can be used to find the asymptotic expansion of the localized heat trace $\text{Tr}(ae^{-tP})$ as $t \rightarrow 0^+$ for any $a \in C^\infty(\mathbb{T}_\theta^d)$:

$$\text{Tr}(ae^{-tP}) \sim \sum_{n=0}^{\infty} c_n(a)t^{(n-d)/2}.$$

Here, each $c_n(a)$ can be written as $\varphi(ab_n)$ for some $b_n = b_n(P) \in C^\infty(\mathbb{T}_\theta^d)$, where

$$(11) \quad b_n(P) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{-1}{2\pi i} \int_{\gamma} b_n(\xi, \lambda) d\lambda d\xi.$$

For more details see e.g. [9] for the noncommutative tori or [20] for a similar computation in the commutative case. Using the rearrangement lemma (Lemma 2.1) and the fact that the contraction map and integration commute, we can readily see that

$$\begin{aligned} b_2(P) &= \left(\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{-1}{2\pi i} \int_{\gamma} B_{2,1}^{ij}(\xi, \lambda, h_{(0)}, h_{(1)}) e^{-\lambda} d\lambda d\xi \right) (\delta_i \delta_j(h)) \\ &+ \left(\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{-1}{2\pi i} \int_{\gamma} B_{2,2}^{ij}(\xi, \lambda, h_{(0)}, h_{(1)}, h_{(2)}) e^{-\lambda} d\lambda d\xi \right) (\delta_i(h) \cdot \delta_j(h)). \end{aligned}$$

The dependence of $B_{2,k}^{ij}$ on λ comes only from different powers of B_0 in its terms, while its dependence on ξ_j 's is the result of appearance of ξ_j as well

as of B_0 in the terms. Therefore, the contour integral will only contain $e^{-\lambda}$ and product of powers of $B_0(t_j)$. Hence, we need to deal with a certain kind of contour integral for which we shall use the following notation and will call them T -functions:

$$(12) \quad T_{\mathbf{n};\boldsymbol{\alpha}}(t_0, \dots, t_n) := \frac{-1}{\pi^{\frac{d}{2}}} \int_{\mathbb{R}^d} \xi_{n_1} \cdots \xi_{n_{2|\boldsymbol{\alpha}|-4}} \frac{1}{2\pi i} \int_{\gamma} e^{-\lambda} B_0^{\alpha_0}(t_0) \cdots B_0^{\alpha_n}(t_n) d\lambda d\xi,$$

where $\mathbf{n} = (n_1, \dots, n_{2|\boldsymbol{\alpha}|-4})$ and $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_n)$. We study T -functions and their properties in Subsection 3.3. Now, we write a formula for $b_2(P)$.

Proposition 3.1. *For a positive Laplace type h -differential operator P with the symbol given by (7), the term $b_2(P)$ in the contraction form is given by*

$$b_2(P) = (4\pi)^{-d/2} \left(B_{2,1}^{ij}(h_{(0)}, h_{(1)}) (\delta_i \delta_j(h)) + B_{2,2}^{ij}(h_{(0)}, h_{(1)}, h_{(2)}) (\delta_i(h) \cdot \delta_j(h)) \right),$$

where the functions are defined by

$$\begin{aligned} B_{2,1}^{ij}(t_0, t_1) &= -T_{;1,1}(t_0, t_1) P_{0,1}^{ij}(t_0, t_1) + 2T_{k\ell;2,1}(t_0, t_1) P_2^{ik}(t_0) P_1^{j\ell}(t_0, t_1) \\ &\quad + T_{k\ell;2,1}(t_0, t_1) P_2^{ij}(t_0) [t_0, t_1; P_2^{k\ell}] \\ &\quad - 4T_{k\ell mn;3,1}(t_0, t_1) P_2^{ik}(t_0) P_2^{j\ell}(t_0) [t_0, t_1; P_2^{mn}], \end{aligned}$$

and

$$\begin{aligned} B_{2,2}^{ij}(t_0, t_1, t_2) &= \\ &- T_{;1,1}(t_0, t_2) P_{0,2}^{ij}(t_0, t_1, t_2) \\ &+ T_{k\ell;1,1,1}(t_0, t_1, t_2) P_1^{ik}(t_0, t_1) P_1^{j\ell}(t_1, t_2) \\ &- 2T_{k\ell mn;2,1,1}(t_0, t_1, t_2) P_2^{im}(t_0) [t_0, t_1; P_2^{k\ell}] P_1^{jn}(t_1, t_2) \\ &+ 2T_{k\ell;2,1}(t_0, t_2) P_2^{ik}(t_0) [t_0, t_1; P_1^{j\ell}(\cdot, t_2)] \\ &+ 2T_{k\ell;2,1}(t_0, t_2) P_2^{jk}(t_0) [t_1, t_2; P_1^{i\ell}(t_0, \cdot)] \\ &+ T_{k\ell;1,1,1}(t_0, t_1, t_2) P_1^{ij}(t_0, t_1) [t_1, t_2; P_2^{k\ell}] \\ &- 2T_{k\ell mn;2,1,1}(t_0, t_1, t_2) P_2^{j\ell}(t_0) P_1^{ik}(t_0, t_1) [t_1, t_2; P_2^{mn}] \\ &- 2T_{k\ell mn;1,2,1}(t_0, t_1, t_2) P_1^{ik}(t_0, t_1) P_2^{j\ell}(t_1) [t_1, t_2; P_2^{mn}] \\ &- 2T_{k\ell mn;2,1,1}(t_0, t_1, t_2) P_2^{ij}(t_0) [t_0, t_1; P_2^{\ell m}] [t_1, t_2; P_2^{kn}] \\ &- 4T_{k\ell mn;2,1,1}(t_0, t_1, t_2) P_2^{ik}(t_0) [t_0, t_1; P_2^{j\ell}] [t_1, t_2; P_2^{mn}] \\ &+ 8T_{k\ell mnpq;3,1,1}(t_0, t_1, t_2) P_2^{ik}(t_0) P_2^{jn}(t_0) [t_0, t_1; P_2^{\ell m}] [t_1, t_2; P_2^{pq}] \end{aligned}$$

$$\begin{aligned}
 &+ 4T_{k\ell mnpq;2,2,1}(t_0, t_1, t_2) P_2^{ik}(t_0) [t_0, t_1; P_2^{\ell m}] P_2^{jn}(t_1) [t_1, t_2; P_2^{pq}] \\
 &+ 2T_{k\ell;2,1}(t_0, t_2) P_2^{ij}(t_0) [t_0, t_1, t_2; P_2^{k\ell}] \\
 &- 8T_{k\ell mn;3,1}(t_0, t_2) P_2^{ik}(t_0) P_2^{j\ell}(t_0) [t_0, t_1, t_2; P_2^{mn}].
 \end{aligned}$$

The computation of the higher heat trace densities for a Laplace type h -operator can be similarly carried out; expecting more terms in the results. This would give a way to generalize results obtained for the conformally flat noncommutative two torus in [6], where b_4 of the Laplacian D^2 of the Dirac operator D is computed.

3.3. T -functions and their properties

Evaluating T -functions (12), are the only parts of formulas for $B_{2,1}^{i,j}$ and $B_{2,2}^{i,j}$ that need to be evaluated, is not always an easy task. We give a concise integral formula for T -functions and study their properties in this section and later on show that in some cases, they can be computed explicitly.

We start with the contour integral in (12). It is evident that there are functions $f_{\alpha_1, \dots, \alpha_n}$ such that

$$\frac{-1}{2\pi i} \int_{\gamma} e^{-\lambda} B_0^{\alpha_0}(t_0) \cdots B_0^{\alpha_n}(t_n) d\lambda = f_{\alpha_0, \dots, \alpha_n} (\|\xi\|_{t_0}^2, \dots, \|\xi\|_{t_n}^2).$$

Here, we denoted $P_2^{ij}(t_k)\xi_i\xi_j$ by $\|\xi\|_{t_k}^2$. Examples of such functions are

$$\begin{aligned}
 f_{1,1}(x_0, x_1) &= -\frac{e^{-x_0}}{x_0 - x_1} - \frac{e^{-x_1}}{x_1 - x_0}, \\
 f_{2,1}(x_0, x_1) &= -\frac{e^{-x_0}}{x_0 - x_1} - \frac{e^{-x_0}}{(x_0 - x_1)^2} + \frac{e^{-x_1}}{(x_1 - x_0)^2}.
 \end{aligned}$$

We can indeed find a general formula for f_{α} in terms of derivatives of the divided differences of the function $\exp(-\lambda)$:

$$\begin{aligned}
 f_{\alpha}(x_0, \dots, x_n) &= \frac{-1}{2\pi i} \int_{\gamma} e^{-\lambda} \prod_{j=0}^n (x_j - \lambda)^{-\alpha_j} d\lambda \\
 &= \frac{(-1)^{|\alpha|+1}}{2\pi i} \int_{\gamma} e^{-\lambda} \prod_{j=0}^n (\lambda - x_j)^{-\alpha_j} d\lambda \\
 &= \frac{(-1)^{|\alpha|+1}}{\beta!} \partial_x^{\beta} \left(\frac{1}{2\pi i} \int_{\gamma} e^{-\lambda} \prod_{j=0}^n (\lambda - x_j)^{-1} d\lambda \right)
 \end{aligned}$$

$$= \frac{(-1)^{|\alpha|+1}}{\beta!} \partial_x^\beta \left[x_0, \dots, x_n; \lambda \mapsto e^{-\lambda} \right],$$

where β is the multi-index $(\alpha_0 - 1, \dots, \alpha_n - 1)$. In the last formula we used a contour integral formula for divided differences of an analytic function which can be found in [35, Section 1.7.]. Now, we can apply Hermite-Genocchi formula (5) for $f(\lambda) = \exp(-\lambda)$ and rewrite f_α as an integral $\frac{1}{\beta!} \int_{\Sigma_n} e^{-(\sum_{j=0}^n s_j x_j)} \prod_{j=0}^n s_j^{\alpha_j-1} ds$ over the standard simplex. We have

$$\begin{aligned} T_{n;\alpha}(t_0, \dots, t_n) &= \pi^{-\frac{d}{2}} \int_{\mathbb{R}^d} \xi_{n_1} \cdots \xi_{n_{2|\alpha|-4}} f_{\alpha_0, \dots, \alpha_n} (\|\xi\|_{t_0}^2, \dots, \|\xi\|_{t_n}^2) d\xi \\ &= \pi^{-\frac{d}{2}} \int_{\mathbb{R}^d} \xi_{n_1} \cdots \xi_{n_{2|\alpha|-4}} \frac{1}{\beta!} \int_{\Sigma_n} e^{-\langle \sum_{j=0}^n s_j P_2(t_j) \xi, \xi \rangle} \prod_{j=0}^n s_j^{\alpha_j-1} ds d\xi \\ &= \frac{\pi^{-\frac{d}{2}}}{\beta!} \int_{\Sigma_n} \prod_{j=0}^n s_j^{\alpha_j-1} \int_{\mathbb{R}^d} \xi_{n_1} \cdots \xi_{n_{2|\alpha|-4}} e^{-\langle \sum_{j=0}^n s_j P_2(t_j) \xi, \xi \rangle} d\xi ds \\ &= \frac{1}{2^{|\alpha|-2} \beta!} \int_{\Sigma_n} \prod_{j=0}^n s_j^{\alpha_j-1} \frac{\sum \prod \left(\sum_{j=0}^n s_j P_2(t_j) \right)^{-1}_{n_i n_{\sigma(i)}}}{\sqrt{\det \left(\sum_{j=0}^n s_j P_2(t_j) \right)}} ds. \end{aligned}$$

In the last step, we applied the Wick’s theorem (See e.g. [27]) for the inner integral. We use $\sum \prod$ to denoted $\sum_{\sigma \in \Pi_{|\alpha|-2}} \prod_{i \in S/\sigma} \Pi_k$, where Π_k is the set of all pairings on $S = \{1, \dots, 2k\}$. For instance,

$$\sum_{(k,\ell)} \prod A_{n_i n_{\sigma(i)}} = A_{k\ell} \quad \text{and} \quad \sum_{(k,\ell,m,n)} \prod A_{n_i n_{\sigma(i)}} = A_{k\ell} A_{mn} + A_{km} A_{\ell n} + A_{kn} A_{\ell m}.$$

Putting it all together, we have proved the following lemma.

Lemma 3.2. *Let $P_2(t)$ be a positive definite $d \times d$ matrix of smooth real functions. Then*

$$T_{n;\alpha}(t_0, \dots, t_n) = \frac{1}{2^{|\alpha|-2} \beta!} \int_{\Sigma_n} \prod_{j=0}^n s_j^{\alpha_j-1} \frac{\sum \prod P^{-1}(s)^{n_i n_{\sigma(i)}}}{\sqrt{\det P(s)}} ds,$$

where $P(s) = \sum_{j=0}^n s_j P_2(t_j)$ and $\beta = (\alpha_0 - 1, \dots, \alpha_n - 1)$.

For our purposes, we need only two types of T -functions:

$$T_{\mathbf{n};\alpha,1} = \frac{1}{2^{\alpha-1}(\alpha-1)!} \int_0^1 s^{\alpha-1} \frac{\sum \prod P^{-1}(s)_{n_i n_{\sigma(i)}}}{\sqrt{\det P(s)}} ds,$$

$$T_{\mathbf{n};\alpha_0,\alpha_1,1} = \frac{1}{2^{\alpha_0+\alpha_1-1}\Gamma(\alpha_0)\Gamma(\alpha_1)} \int_0^1 \int_0^{1-s_1} s_0^{\alpha_0-1} s_1^{\alpha_1-1} \frac{\sum \prod P^{-1}(s_0, s_1)_{n_i n_{\sigma(i)}}}{\sqrt{\det P(s_0, s_1)}} ds_0 ds_1.$$

Remark 3.2. We list some basic properties of T -functions below:

- (i) $T_{n_{\sigma(1)}, \dots, n_{\sigma(k)}; \alpha} = T_{\mathbf{n}; \alpha}, \quad \sigma \in S(n), \quad \sigma \in S_k,$
- (ii) $T_{\mathbf{n}; \alpha_{\sigma(0)}, \alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}}(t_{\sigma(0)}, t_{\sigma(1)}, \dots, t_{\sigma(n)}) = T_{\mathbf{n}; \alpha_0, \alpha_1, \dots, \alpha_n}(t_0, t_1, \dots, t_n), \quad \sigma \in S_{n+1}$
- (iii) $\lim_{t_n \rightarrow t_0} T_{\mathbf{n}; \alpha_0, \dots, \alpha_{n-1}, 1}(t_0, t_1, \dots, t_n) = T_{\mathbf{n}; \alpha_0+1, \dots, \alpha_{n-1}}(t_0, t_1, \dots, t_{n-1}),$
- (iv) $T_{\mathbf{n}; \alpha_0, \dots, \alpha_{j-1}, 0, \alpha_{j+1}, \dots, \alpha_n}(t_0, \dots, t_n) = T_{\mathbf{n}; \alpha_0, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_n}(t_0, \dots, t_{j-1}, t_{j+1}, \dots, t_n),$
- (v) $\frac{\partial}{\partial t_j} T_{\mathbf{n}; \alpha_0, \dots, \alpha_n}(t_0, \dots, t_n) = -\alpha_j P'(t_j)^{k, \ell} T_{\mathbf{n}, k, \ell; \alpha_0, \dots, \alpha_{j+1}, \dots, \alpha_n}(t_0, \dots, t_n).$

Finally, we would like to point out a recursive formula for T -functions.

Lemma 3.3. *Let $\mathbf{n} = (n_1, \dots, n_\ell)$ and $\tilde{\mathbf{n}} = (n_1, \dots, n_\ell, m, n)$. Then, we have*

$$T_{\tilde{\mathbf{n}}; \alpha, 1}(t_0, t_1)(P_2(t_0) - P_2(t_1))^{mn} = \begin{cases} T_{\mathbf{n}; \alpha-1, 1}(t_0, t_1) - \frac{\sum \prod g(t_0)_{n_i n_{\sigma(i)}}}{2^{\alpha-2}\Gamma(\alpha)\sqrt{\det P_2(t_0)}}, & \alpha \geq 2, \\ \frac{2\sum \prod g(t_1)_{n_i n_{\sigma(i)}}}{\sqrt{\det P_2(t_1)}} - \frac{2\sum \prod g(t_0)_{n_i n_{\sigma(i)}}}{\sqrt{\det P_2(t_0)}}, & \alpha = 1, \end{cases}$$

and

$$T_{\tilde{\mathbf{n}}; \alpha_0, \alpha_1, 1}(t_0, t_1, t_2)(P_2(t_1) - P_2(t_2))^{mn} = \begin{cases} T_{\mathbf{n}; \alpha_0, \alpha_1-1, 1}(t_0, t_1, t_2) - T_{\mathbf{n}; \alpha_0, \alpha_1}(t_0, t_1), & \alpha_1 \geq 2, \\ T_{\mathbf{n}; \alpha_0, 1}(t_0, t_2) - T_{\mathbf{n}; \alpha_0, 1}(t_0, t_1), & \alpha_1 = 1. \end{cases}$$

Proof. The key point here lies in the following formula which is a direct

result of Jacobi's formula: $d \det(A) = \det(A) \operatorname{tr}(A^{-1} dA)$.

$$\frac{d}{ds} \left(\frac{\sum \prod P^{-1}(s)_{n_i n_{\sigma(i)}}}{\sqrt{\det(P(s))}} \right) = -\frac{1}{2} \frac{\sum \prod P^{-1}(s)_{n_i n_{\sigma(i)}}}{\sqrt{\det(P(s))}} (P_2(t_0) - P_2(t_1))^{mn}.$$

Then, an integration by parts will complete the proof:

$$\begin{aligned} & T_{\tilde{\mathbf{n}}; \alpha, 1}(t_0, t_1) (P_2(t_0) - P_2(t_1))^{mn} \\ &= \frac{1}{2^{\alpha-1} \Gamma(\alpha)} \int_0^1 s^{\alpha-1} \frac{\sum \prod P^{-1}(s)_{n_i n_{\sigma(i)}}}{\sqrt{\det P(s)}} ds (P_2(t_0) - P_2(t_1))^{mn} \\ &= \frac{-1}{2^{\alpha-2} \Gamma(\alpha)} \int_0^1 s^{\alpha-1} \frac{d}{ds} \left(\frac{\sum \prod P^{-1}(s)_{n_i n_{\sigma(i)}}}{\sqrt{\det P(s)}} \right) ds \\ &= \frac{-1}{2^{\alpha-2} \Gamma(\alpha)} \left(\left[s^{\alpha-1} \frac{\sum \prod P^{-1}(s)_{n_i n_{\sigma(i)}}}{\sqrt{\det P(s)}} \right]_0^1 - (\alpha-1) \int_0^1 s^{\alpha-2} \frac{\sum \prod P^{-1}(s)_{n_i n_{\sigma(i)}}}{\sqrt{\det P(s)}} ds \right). \end{aligned}$$

Similar steps works to prove the second equation. \square

This lemma can be used to further simplify the expression for $B_{2,1}^{i,j}$ and $B_{2,2}^{i,j}$ given in Proposition 3.1. As an example, we use it here to simplify the third term of $B_{2,1}^{i,j}$:

$$\begin{aligned} & T_{k\ell; 2, 1}(t_0, t_1) P_2^{ij}(t_0) [t_0, t_1; P_2^{k\ell}] \\ &= (T_{; 1, 1}(t_0, t_1) - T_{; 2}(t_0)) P_2^{ij}(t_0) \frac{1}{t_0 - t_1} \\ &= T_{; 1, 1}(t_0, t_1) P_2^{ij}(t_0) \frac{1}{t_0 - t_1} - \frac{1}{\sqrt{\det P_2(t_0)}} P_2^{ij}(t_0) \frac{1}{t_0 - t_1}. \end{aligned}$$

Applying the lemma on the other terms, we can find a new and simpler formula for $B_{2,1}^{i,j}$ given by

$$B_{2,1}^{ij}(\xi, \lambda, t_0, t_1) = T_{; 1, 1}(t_0, t_1) \left(-P_{0,1}^{ij}(t_0, t_1) + P_2^{ij}(t_0) \frac{1}{t_0 - t_1} \right)$$

$$+ 2T_{k\ell;2,1}(t_0, t_1)P_2^{ik}(t_0) \left(P_1^{j\ell}(t_0, t_1) - 2P_2^{j\ell}(t_0) \frac{1}{t_0 - t_1} \right).$$

In this new form, each term of expression may not be smooth anymore and this is the reason that we won't write down the simplified formula for $B_{2,2}^{ij}$. However, we shall use this lemma in Section 5 where we want to give a formula for the total curvature of the curved noncommutative tori.

We will, in some cases, evaluate T -functions. For conformally flat metrics, we shall find a closed formula for T -functions in terms of divided differences (Lemma 4.2). A similar formula for the twisted product metrics is computed in Lemma 4.4. We also have a formula for some of the T -functions in dimension two (see Lemma 5.4).

Example 3.2. In this example, we find some T -functions for a Laplace type h -differential operator on \mathbb{T}_θ^4 with the principal symbol given by entries of the matrix

$$f(t)g^{-1} \oplus \tilde{f}(t)\tilde{g}^{-1},$$

where g and \tilde{g} are two by two positive definite symmetric real matrices and f and \tilde{f} are positive smooth functions. First we note that

$$\sqrt{\det P(s)} = \frac{1}{\sqrt{|g||\tilde{g}|}}(sf(t_0) + (1 - s)f(t_1))(s\tilde{f}(t_0) + (1 - s)\tilde{f}(t_1)).$$

The integral formula from Lemma 3.2 for $T_{n;\alpha,1}$ turns out to be a basic integral and we have

$$T_{;1,1}(t_0, t_1) = \frac{\sqrt{|g||\tilde{g}|}}{f(t_0)\tilde{f}(t_1) - f(t_1)\tilde{f}(t_0)} \log \left(\frac{f(t_0)\tilde{f}(t_1)}{f(t_1)\tilde{f}(t_0)} \right),$$

and for $1 \leq k, l \leq 2$, we have

$$T_{k,l;2,1}(t_0, t_1) = \frac{g_{kl}\sqrt{|g||\tilde{g}|} \left(f(t_1)\tilde{f}(t_0) + f(t_0)\tilde{f}(t_1) \left(\log \left(\frac{f(t_0)\tilde{f}(t_1)}{f(t_1)\tilde{f}(t_0)} \right) - 1 \right) \right)}{2f(t_0) \left(f(t_1)\tilde{f}(t_0) - f(t_0)\tilde{f}(t_1) \right)^2}.$$

The functions $T_{k,l;2,1}$ for $3 \leq k, l \leq 4$ are given by the same formula after we interchange f with \tilde{f} and g with \tilde{g} . These T -functions can be computed in the same manner in cases with more than two metrics and functions, called multiply twisted product metrics of two tori.

Further exploration of the properties of T -functions is an important problem. In particular, finding an explicit formula of T -functions for more general functional metrics is an open problem. Along these lines, The following point may be noticed. The T -functions for conformally flat metrics are given by derivatives of divided differences which appear as coefficient of the polynomial interpolation. Similarly, integrals over the standard simplex, such as the one found in Lemma 3.3 for T -functions, appear as the coefficients of a multivariate polynomial interpolation called Kergin interpolation (see e.g. [34]).

4. Functional metrics and their scalar curvature density

In this section, we apply Proposition 3.1 to a geometric operator on the noncommutative tori equipped with a new class of metrics, called functional metrics, which are introduced in this work for the first time. First, we construct the Laplacian on functions of (\mathbb{T}_θ^d, g) and find an h -differential operator $\Delta_{0,g}$ which is antiunitary equivalent to the Laplacian. We then write the symbol of $\Delta_{0,g}$ in the contraction form and apply Proposition 3.1 to find $b_2(\Delta_{0,g})$.

The geometric importance of the second density of the heat trace of the Laplacian on functions (or Laplacian of the Dirac operator) originates from the fact in spectral geometry that on manifolds [20, Section] this density is a multiple of the scalar curvature of the metric. This fact provided a way to define the scalar curvature R , called scalar curvature density, for noncommutative spaces; an idea which was initiated by [8, 16] see also [7, Section 11]. Here, we define the *scalar curvature density* of a given functional metric to be

$$(13) \quad R = (4\pi)^{\frac{d}{2}} b_2(\Delta_{0,g}).$$

We compute the scalar curvature density for two classes of examples in all dimensions: conformally flat metrics and twisted product flat metrics.

4.1. Functional metrics and their Laplacian

We first introduce a new class of metrics on the noncommutative tori and then a Laplace type h -differential operator to study the spectral invariants of the geometries defined by these metrics.

Definition 4.1. Let h be a selfadjoint smooth element of the noncommutative d -tori and let $g_{ij} : \mathbb{R} \rightarrow \mathbb{R}$, for $1 \leq i, j \leq d$, be smooth functions

such that the matrix $(g_{ij}(t))$ is a positive definite matrix for every t in a neighborhood of the spectrum of h . We shall refer to $g_{ij}(h)$ as a *functional metric* on $C(\mathbb{T}_\theta^d)$.

The condition is equivalent to saying that $g_{ij}(h)\xi^i\xi^j \in C(\mathbb{T}_\theta^d)^+$ for every $(\xi^1, \dots, \xi^d) \in \mathbb{R}^d$.

Without loss of generality, we always assume that functions g_{ij} are Schwartz class functions. Since the spectrum of h is a compact subset of \mathbb{R} , we can always replace $g_{ij}(t)$ with Schwartz or even compactly supported functions \tilde{g}_{ij} such that $\tilde{g}_{ij}(h) = g_{ij}(h)$. In our notations we frequently use the property of functional calculus that $F \circ g_{ij}(h) = F(g_{ij})(h)$ for any given smooth function $F : \mathbb{R}^{d^2} \rightarrow \mathbb{R}$. For example, we shall denote the determinant of $(g_{ij}(h))$ by either $|g(h)|$ or $|g|(h)$. Also, since the matrix $(g_{ij}(x))$ is invertible on a neighborhood of the spectrum of h , we use $g^{ij}(h)$ to denote the entries of $(g_{ij}(h))^{-1}$.

To construct the Laplacian on \mathbb{T}_θ^d equipped with a functional metric $g_{ij}(h)$, we first consider the exterior derivative $\delta : C^\infty(\mathbb{T}_\theta^d) \rightarrow \Omega^1\mathbb{T}_\theta^d = C^\infty(\mathbb{T}_\theta^d) \otimes \mathbb{C}^d$ which is defined by

$$\delta(a) = (i\delta_1(a), i\delta_2(a), \dots, i\delta_d(a)).$$

The key role of the functional metric $g_{ij}(h)$ then is in turning $C^\infty(\mathbb{T}_\theta^d)$ and $\Omega^1\mathbb{T}_\theta^d$ into Hilbert spaces and finding the adjoint of δ . Analogous to the classical theory, we define the inner product on functions $C^\infty(\mathbb{T}_\theta^d)$ to be

$$\langle a, b \rangle_0 = \varphi(b^* a \sqrt{|g|(h)}), \quad a, b \in C^\infty(\mathbb{T}_\theta^d).$$

In addition, the inner product on 1-forms $\Omega^1\mathbb{T}_\theta^d$ is defined to be

$$\langle \mathbf{a}, \mathbf{b} \rangle_1 = \varphi \left(g^{ij}(h) b_i^* a_j \sqrt{|g|(h)} \right).$$

Here $\mathbf{a} = (a_1, a_2, \dots, a_d)$, $\mathbf{b} = (b_1, b_2, \dots, b_d)$ are elements of $\Omega^1\mathbb{T}_\theta^d$. The Hilbert space completion of functions $C^\infty(\mathbb{T}_\theta^d)$ and 1-forms $\Omega^1\mathbb{T}_\theta^d$ with respect to the corresponding inner products will be denoted by $\mathcal{H}_{0,g}$ and $\mathcal{H}_{1,g}$ respectively.

The exterior derivative, as a densely defined operator from $\mathcal{H}_{0,g}$ to $\mathcal{H}_{1,g}$, is a closed unbounded operator from \mathcal{H}_0 to \mathcal{H}_1 and its formal adjoint $\delta^* : \mathcal{H}_{1,g} \rightarrow \mathcal{H}_{0,g}$ on elements of $\Omega^1\mathbb{T}_\theta^d$ is given by

$$\delta^*(\mathbf{b}) = -ib_j g^{jk}(h) \delta_k \left(\sqrt{|g|(h)} \right) |g|^{-\frac{1}{2}}(h) - ib_j \delta_k \left(g^{jk}(h) \right) - i\delta_k(b_j) g^{jk}(h).$$

Therefore, $\delta^* \delta : \mathcal{H}_{0,g} \rightarrow \mathcal{H}_{0,g}$ on elements of $C^\infty(\mathbb{T}_\theta^d)$ is given by

$$\delta_j(a)g^{jk}(h)\delta_k(|g|^{\frac{1}{2}}(h))|g|^{-\frac{1}{2}}(h) + \delta_j(a)\delta_k(g^{jk}(h)) + i\delta_k(\delta_j(a))g^{jk}(h).$$

To carry the spectral analysis of the Laplacian $\delta^* \delta : \mathcal{H}_{0,g} \rightarrow \mathcal{H}_{0,g}$, we switch to the antiunitary equivalent setting as follows. Let \mathcal{H}_0 be the Hilbert space obtained by the GNS construction from $C(\mathbb{T}_\theta^d)$ using the nonperturbed tracial state φ .

Proposition 4.1. *The operator $\delta^* \delta : \mathcal{H}_{0,g} \rightarrow \mathcal{H}_{0,g}$ is antiunitary equivalent to a Laplace type h -differential operator $\Delta_{0,g} : \mathcal{H}_0 \rightarrow \mathcal{H}_0$ whose symbol, when expressed in the contraction form, has the functional parts given by*

$$\begin{aligned} P_2^{jk}(t_0) &= g^{jk}(t_0), \\ P_1^{jk}(t_0, t_1) &= |g|^{-\frac{1}{4}}(t_0)[t_0, t_1; |g|^{\frac{1}{4}}]g^{jk}(t_1) + [t_0, t_1; g^{jk}] \\ &\quad + |g|^{\frac{1}{4}}(t_0)g^{jk}(t_0)[t_0, t_1; |g|^{-\frac{1}{4}}], \\ P_{0,1}^{jk}(t_0, t_1) &= |g|^{\frac{1}{4}}(t_0)g^{jk}(t_0)[t_0, t_1; |g|^{-\frac{1}{4}}], \\ P_{0,2}^{jk}(t_0, t_1, t_2) &= |g|^{-\frac{1}{4}}(t_0)[t_0, t_1; g^{jk}|g|^{\frac{1}{2}}][t_1, t_2; |g|^{-\frac{1}{4}}] \\ &\quad + 2|g|^{\frac{1}{4}}(t_0)g^{jk}(t_0)[t_0, t_1, t_2; |g|^{-\frac{1}{4}}]. \end{aligned}$$

Proof. The right multiplication by $|g|^{-1/4}$ extends to a unitary operator $W : \mathcal{H}_0 \rightarrow \mathcal{H}_{0,g}$ and the map $a \mapsto a^*$ gives an antiunitary map on \mathcal{H}_0 which we denote by J . Then, $\delta^* \delta$ is antiunitary equivalent to the operator $JW^* \delta^* \delta W J$, which we denote by $\Delta_{0,g}$. Using the basic property of $*$ and its relation with the derivations, $\delta_j(a^*) = -\delta_j(a)^*$, we immediately obtain

$$\begin{aligned} \Delta_{0,g}(a) &= g^{jk}(h)\delta_k\delta_j(a) + \left(|g|^{-\frac{1}{4}}\delta_k(|g|^{\frac{1}{4}}g^{jk}(h)) + |g|^{\frac{1}{4}}g^{jk}(h)\delta_k(|g|^{-\frac{1}{4}})\right)\delta_j(a) \\ &\quad + |g|^{-\frac{1}{4}}\delta_k\left(|g|^{\frac{1}{2}}g^{jk}(h)\delta_j(|g|^{-\frac{1}{4}})\right)a. \end{aligned} \tag{14}$$

The operator $\Delta_{0,g}$ is clearly a differential operator and the homogeneous parts of its symbol reads

$$\begin{aligned} p_2(\xi) &= g^{jk}(h)\xi_j\xi_k, \\ p_1(\xi) &= \left(|g|^{-\frac{1}{4}}\delta_k(|g|^{\frac{1}{4}})g^{jk}(h) + \delta_k(g^{jk}(h)) + |g|^{\frac{1}{4}}g^{jk}(h)\delta_k(|g|^{-\frac{1}{4}})\right)\xi_j, \\ p_0(\xi) &= |g|^{-\frac{1}{4}}\delta_k(|g|^{\frac{1}{2}})g^{jk}(h)\delta_j(|g|^{-\frac{1}{4}}) + |g|^{\frac{1}{4}}\delta_k(g^{jk}(h))\delta_j(|g|^{-\frac{1}{4}}) \\ &\quad + |g|^{\frac{1}{4}}g^{jk}(h)\delta_k\delta_j(|g|^{-\frac{1}{4}}). \end{aligned}$$

We then apply Theorem 2.2 and express the above terms in the contraction form. \square

Remark 4.1. The analysis of functional metrics on manifolds is an interesting topic to be further investigated. In particular, classifying Riemannian metrics that can be written locally or globally as a functional metric, with h being a smooth function on the manifold, seems a nice problem.

The scalar curvature of a functional metric $g_{ij}(h)dx^i dx^j$ on an open subset U of \mathbb{R}^d is given by

$$\begin{aligned} R = & \left(-g^{-1}\text{tr}(g^{-1}g'') - \frac{1}{4}g^{-1}\text{tr}(g^{-1}g')^2 + \frac{3}{4}g^{-1}\text{tr}(g^{-1}g'g^{-1}g') \right. \\ & \left. + \text{tr}(g^{-1}g')g^{-1}g'g^{-1} + g^{-1}g''g^{-1} - \frac{3}{2}g^{-1}g'g^{-1}g'g^{-1} \right)^{\mu\nu} \frac{\partial h}{\partial x^\mu} \frac{\partial h}{\partial x^\nu} \\ & + \left(g^{-1}g'g^{-1} - g^{-1}\text{tr}(g^{-1}g') \right)^{\mu\nu} \frac{\partial^2 h}{\partial x^\mu \partial x^\nu}. \end{aligned}$$

4.2. Conformally flat metrics

An important case of the functional metric is the conformally flat metric

$$(15) \quad g_{ij}(t) = f(t)^{-1}g_{ij},$$

where f is a positive smooth function and g_{ij} 's are the entries of a constant metric on \mathbb{R}^d . The formula (14) for the operator $\Delta_{0,g}$ is simplified for conformally flat metric in general dimension d

$$\begin{aligned} \Delta_{0,g}(a) = & f(h)g^{jk}\delta_k\delta_j(a) \\ & + g^{jk}\left(f(h)^{\frac{d}{4}}\delta_k(f(h)^{1-\frac{d}{4}}) + f(h)^{1-\frac{d}{4}}\delta_k(f(h)^{\frac{d}{4}})\right)\delta_j(a) \\ & + g^{jk}f(h)^{\frac{d}{4}}\delta_k\left(f(h)^{1-\frac{d}{2}}\delta_j(f(h)^{\frac{d}{4}})\right)a, \end{aligned}$$

The functions given by Proposition 4.1, for the conformally flat metrics, gives us the following:

$$(16)$$

$$P_2^{jk}(t_0) = g^{jk}f(t_0),$$

$$P_1^{jk}(t_0, t_1) = g^{jk}\left(f(t_0)^{\frac{d}{4}}[t_0, t_1; f^{1-\frac{d}{4}}] + f(t_0)^{1-\frac{d}{4}}[t_0, t_1; f^{\frac{d}{4}}]\right),$$

$$P_{0,1}^{jk}(t_0, t_1) = g^{jk}f(t_0)^{1-\frac{d}{4}}[t_0, t_1; f^{\frac{d}{4}}],$$

$$P_{0,2}^{jk}(t_0, t_1, t_2) = g^{jk}\left(f(t_0)^{\frac{d}{4}}[t_0, t_1; f^{1-\frac{d}{2}}][t_1, t_2; f^{\frac{d}{4}}] + 2f(t_0)^{1-\frac{d}{4}}[t_0, t_1, t_2; f^{\frac{d}{4}}]\right).$$

A careful examination of formula (16) shows that for any function P_{\bullet}^{ij} there exist a function P_{\bullet} such that $P_{\bullet}^{ij} = g^{ij} P_{\bullet}$. We have similar situation with the T -functions for conformally flat metrics.

Lemma 4.2. *Let α and $\mathbf{n} = (n_1, \dots, n_{2|\alpha|-4})$ be two multi-indices. Then the T -function $T_{\mathbf{n},\alpha}$ for the conformally flat metric (15) is of the form*

$$T_{\mathbf{n},\alpha}(t_0, \dots, t_n) = \sqrt{|g|} \sum_{\mathbf{n}} \prod g_{n_i n_{\sigma(i)}} T_{\alpha}(t_0, \dots, t_n).$$

The function T_{α} in dimension $d \neq 2$ is given by

$$(17) \quad T_{\alpha}(t_0, \dots, t_n) = \frac{(-1)^{|\alpha|-1} \Gamma(\frac{d}{2} - 1)}{\Gamma(\frac{d}{2} + |\alpha| - 2)} \partial_x^{\beta} [x_0, \dots, x_n; u^{1-\frac{d}{2}}] \Big|_{x_j=f(t_j)},$$

where $\beta = (\alpha_0 - 1, \dots, \alpha_n - 1)$.

Proof. First note that the terms involving $P(\mathbf{s}) = \sum_{j=0}^n s_j f(t_j)$ are given by

$$P^{-1}(\mathbf{s})_{kl} = g_{kl} \left(\sum_{j=0}^n s_j f(t_j) \right)^{-1} \quad \text{and} \quad \sqrt{\det P(\mathbf{s})} = \frac{1}{|g|} \left(\sum_{j=0}^n s_j f(t_j) \right)^{\frac{d}{2}}.$$

This implies that the T -function $T_{\mathbf{n},\alpha}$ can be written as $\sqrt{|g|} \sum_{\mathbf{n}} \prod g_{n_i n_{\sigma(i)}} T_{\alpha}$, where T_{α} has an integral formula which we evaluate below. Let x_j , for $j = 0, \dots, \leq n$, be positive and distinct real numbers. Then for $d \neq 2$, we have

$$\begin{aligned} & \int_{\Sigma_n} \prod_{j=0}^n s_j^{\alpha_j-1} \left(\sum_{j=0}^n s_j x_j \right)^{-\frac{d}{2}-|\alpha|+2} ds \\ &= \frac{(-1)^{|\alpha|-n-1} \Gamma(\frac{d}{2} + n - 1)}{\Gamma(\frac{d}{2} + |\alpha| - 2)} \partial_x^{\beta} \int_{\Sigma_n} \left(\sum_{j=0}^n s_j x_j \right)^{-\frac{d}{2}-n+1} ds \\ &= \frac{(-1)^{|\alpha|-1} \Gamma(\frac{d}{2} - 1)}{\Gamma(\frac{d}{2} + |\alpha| - 2)} \partial_x^{\beta} [x_0, \dots, x_n; u^{1-\frac{d}{2}}]. \end{aligned}$$

Here again $\beta = (\alpha_0 - 1, \dots, \alpha_n - 1)$. Replacing x_j by $f(t_j)$, we find the formula (17) for functions T_{α} . □

As an example, we have

$$T_{\alpha,1}(t_0, t_1) = \frac{(-1)^\alpha \Gamma(\frac{d}{2} - 1)}{2^{\alpha-1} \Gamma(\frac{d}{2} + \alpha - 1)} \times \left(\frac{f(t_1)^{1-\frac{d}{2}}}{(f(t_1) - f(t_0))^\alpha} - \sum_{m=0}^{\alpha-1} \frac{(-1)^m \Gamma(\frac{d}{2} + m - 1)}{\Gamma(\frac{d}{2} - 1)m!} \frac{f(t_0)^{-\frac{d}{2}-m+1}}{(f(t_1) - f(t_0))^{\alpha-m}} \right).$$

Note that for dimension two, $T_{\alpha,1}(t_0, t_1)$ can be obtained by taking the limit of (17) as d approaches 2. When $f(t) = t$, we have

$$T_{\alpha,1}(t_0, t_1) = \frac{(-1)^{\alpha-1}}{2^{\alpha-1} \Gamma(\alpha)^2} \partial_{t_0}^{\alpha-1} [t_0, t_1; \log(u)].$$

We shall not consider dimension two separately, but just as the limit $d \rightarrow 2$.

Remark 4.2. It was first noted in [32] that for conformally flat metrics, the integral that defines functions $T_{\mathbf{n},\alpha}$ can be written in terms of Gauss hypergeometric ${}_2F_1$ and Appell hypergeometric F_1 functions. More precisely, in our notation, we have

$$T_{\mathbf{n};\alpha,1}(t_0, t_1) = \frac{\Gamma(\frac{d}{2}) \sum_{\mathbf{n}} \prod g_{n_i n_{\sigma(i)}}}{2^{\alpha-1} (\alpha-1)! \Gamma(\frac{d}{2} + \alpha - 1)} f(t_1)^{-\frac{d}{2}-\alpha+1} {}_2F_1^{(\alpha-1)} \left(\frac{d}{2}, 1; 2; 1 - \frac{f(t_0)}{f(t_1)} \right),$$

and

$$T_{\mathbf{n};\alpha_0,\alpha_1,1}(t_0, t_1, t_2) = \frac{\Gamma(\frac{d}{2} + 1) \sum_{\mathbf{n}} \prod g_{n_i n_{\sigma(i)}}}{2^{\alpha_0+\alpha_1} \Gamma(\frac{d}{2} + \alpha_0 + \alpha_1 + 1) \Gamma(\alpha_0) \Gamma(\alpha_1)} \times f(t_2)^{-\frac{d}{2}-\alpha_0-\alpha_1+1} F_1^{(\alpha_0-1,\alpha_1-1)} \left(\frac{d}{2} + 1; 1, 1; 3; 1 - \frac{f(t_0)}{f(t_2)}, 1 - \frac{f(t_1)}{f(t_2)} \right).$$

What we found in (17) is that these hypergeometric functions for the specific parameters, can be written in terms of divided differences of the function $u^{1-\frac{d}{2}}$.

Finally, we can substitute (16) and T -functions (17) in the formulas from Proposition 3.1 and obtain the curvature of conformally flat noncommutative torus of dimension d .

Theorem 4.3. *The scalar curvature of the d -dimensional noncommutative tori \mathbb{T}_θ^d equipped with the metric $f(h)^{-1}g_{ij}$ is given by*

$$R = \sqrt{|g|} \left(K_d(h_{(0)}, h_{(1)}) (\Delta(h)) + H_d(h_{(0)}, h_{(1)}, h_{(2)}) (\square(h)) \right),$$

where $\Delta(h) = g^{ij}\delta_i\delta_j(h)$, $\square(h) = g^{ij}\delta_i(h) \cdot \delta_j(h)$. The functions K_d and H_d are given by

$$(18) \quad \begin{aligned} K_d(t_0, t_1) &= K_d^t(f(t_0), f(t_1)) [t_0, t_1; f], \\ H_d(t_0, t_1, t_2) &= H_d^t(f(t_0), f(t_1), f(t_2)) [t_0, t_1; f] [t_1, t_2; f] \\ &\quad + 2K_d^t(f(t_0), f(t_2)) [t_0, t_1, t_2; f], \end{aligned}$$

Where K_d^t and H_d^t are the functions K_d and H_d when $f(t) = t$. For $d \neq 2$, they can be computed to be

$$K_d^t(x, y) = \frac{4x^{\frac{3d}{4}}y^{\frac{3d}{4}}}{d(d-2)(x-y)^3} \left((d-1)x^{\frac{d}{2}}y^{\frac{d}{2}-1} - (d-1)x^{\frac{d}{2}-1}y^{\frac{d}{2}} - x^{d-1} + y^{d-1} \right),$$

and

$$\begin{aligned} H_d^t(x, y, z) &= \\ &= \frac{2x^{-\frac{3d}{4}}y^{-d}z^{-\frac{3d}{4}}}{(d-2)d(x-y)^2(x-z)^3(y-z)^2} \times \\ &\left(x^d y^d z^2 (x-y) (3x^2 y - 2x^2 z - 4xy^2 + 4xyz - 2xz^2 + yz^2) \right. \\ &\quad + x^d y^{\frac{d}{2}+1} z^{\frac{d}{2}+1} (x-z)^2 (z-y) (dx + (1-d)y) \\ &\quad + x^d y^3 z^d (z-x)^3 + x^{\frac{d}{2}+1} y^{\frac{3d}{2}} z^2 (x-y) (x-z)^2 \\ &\quad + 2(d-1)x^{\frac{d}{2}+1} y^d z^{\frac{d}{2}+1} (x-y) (x-z) (z-y) (x-2y+z) \\ &\quad - x^{\frac{d}{2}+1} y^{\frac{d}{2}+1} z^d (x-y) (x-z)^2 ((1-d)y + dz) - x^2 y^{\frac{3d}{2}} z^{\frac{d}{2}+1} (x-z)^2 (z-y) \\ &\quad \left. + x^2 y^d z^d (y-z) (x^2 y - 2x^2 z + 4xyz - 2xz^2 - 4y^2 z + 3yz^2) \right). \end{aligned}$$

These functions for the dimension two are given by

$$K_2^t(x, y) = -\frac{\sqrt{x}\sqrt{y}}{(x-y)^3} ((x+y) \log(x/y) + 2(y-x)),$$

and

$$\begin{aligned} H_2^t(x, y, z) &= \frac{2\sqrt{x}\sqrt{z}}{(x-y)^2(x-z)^3(y-z)^2} \times \\ &\left(-(x-y)(x-z)(y-z)(x-2y+z) + y(x-z)^3 \log(y) \right. \\ &\quad \left. + (y-z)^2 (-2x^2 + xy + yz) \log(x) - (x-y)^2 (xy + zy - 2z^2) \log(z) \right). \end{aligned}$$

Proof. A closer look at the formulas of Proposition 3.1 reveals that for conformally flat metrics, the dependence of $B_{2,1}^{ij}$ and $B_{2,2}^{ij}$ on the indices is simply through a factor of g^{ij} . In other words, there exist functions $K_d(t_0, t_1)$ and $H_d(t_0, t_1, t_2)$ such that

$$\begin{aligned} B_{2,1}^{ij}(h_{(0)}, h_{(1)})(\delta_i \delta_j(h)) &= \sqrt{|g|} g^{ij} K_d(h_{(0)}, h_{(1)})(\delta_i \delta_j(h)) \\ &= \sqrt{|g|} K_d(h_{(0)}, h_{(1)})(g^{ij} \delta_i \delta_j(h)), \\ B_{2,2}^{ij}(h_{(0)}, h_{(1)}, h_{(2)})(\delta_i(h) \cdot \delta_j(h)) &= \sqrt{|g|} H_d(h_{(0)}, h_{(1)}, h_{(2)})(\delta_i(h) \cdot \delta_j(h)) \\ &= \sqrt{|g|} K_d(h_{(0)}, h_{(1)}, h_{(2)})(g^{ij} \delta_i(h) \cdot \delta_j(h)). \end{aligned}$$

To reach this result, we applied identities such as

$$\begin{aligned} \sum_{(k,\ell)} \prod g_{n_i n_{\sigma(i)}} g^{k\ell} g^{ij} &= dg^{ij}, \\ \sum_{(k,\ell,m,n)} \prod g_{n_i n_{\sigma(i)}} g^{ij} g^{\ell m} g^{kn} &= (d^2 + 2d)g^{ij}, \\ \sum_{(k,\ell,m,n)} \prod g_{n_i n_{\sigma(i)}} g^{ik} g^{j\ell} g^{mn} &= (d + 2)g^{ij}, \\ \sum_{(k,\ell,m,n,p,q)} \prod g_{n_i n_{\sigma(i)}} g^{ik} g^{jn} g^{\ell m} g^{pq} &= (d^2 + 6d + 8)g^{ij}. \end{aligned}$$

To continue it is enough to take $f(t) = t$ and we denote the functions found for $f(t) = t$ by K_d^t and H_d^t . Note that for $f(t) = t$ we have $[t_0, t_1; P_2^{ij}] = 1$ and $[t_0, t_1, t_2; P_2^{ij}] = 0$ which simplify the formula. Then for a general function f , we simply take $\tilde{h} = f(h)$ and then the result is given as

$$K_d^t(\tilde{h}_0, \tilde{h}_1)(g^{ij} \delta_i \delta_j(\tilde{h})) + H_d^t(\tilde{h}_0, \tilde{h}_1, \tilde{h}_2)(g^{ij} \delta_i(\tilde{h}) \delta_j(\tilde{h})).$$

By Theorem 2.2 and equation (2.3), we can turn everything in terms of original element h and get the formulas given in (18).

To find the functions K_2^t and H_2^t , as mentioned before, we should evaluate the limit of K_d^t and H_d^t as $d \rightarrow 2$. □

Note that the Function $K_d^t(x, y)$ is the symmetric part of the function

$$\frac{8x^{2-\frac{d}{4}} y^{2-\frac{3d}{4}}}{d(d-2)(x-y)^3} \left((d-1)y^{\frac{d}{2}-1} - x^{\frac{d}{2}-1} \right).$$

Similarly, $H_d^t(x, y, z)$ is equal to $(F_d(x, y, z) + F_d(z, y, x))/2$ where

$$\begin{aligned}
 F_d(x, y, z) &= \frac{4x^{-\frac{3d}{4}}y^{-d}z^{-\frac{3d}{4}}}{d(d-2)(x-y)^2(x-z)^3(y-z)^2} \times \\
 &\left(x^d y^d z^2 (x-y)(3x^2y - 2x^2z - 4xy^2 + 4xyz - 2xz^2 + yz^2) \right. \\
 &+ x^d y^{\frac{d}{2}+1} z^{\frac{d}{2}+1} (x-z)^2 (z-y)(dx + (1-d)y) + \frac{1}{2} x^d y^3 z^d (z-x)^3 \\
 &\left. + x^{\frac{d}{2}+1} y^{\frac{3d}{2}} z^2 (x-y)(x-z)^2 + 2(d-1)x^{\frac{d}{2}+1} y^d z^{\frac{d}{2}+1} (x-y)^2 (x-z)(z-y) \right).
 \end{aligned}$$

In low dimensions, the curvature of the conformally flat metrics was studied in [16, 8, 25, 17, 11]. Here we show that those results can be recovered from our general formula. We should first note that the functions found in all the aforementioned works are written in terms of $[h, \cdot]$, denoted by Δ , rather than h itself. To produce those functions from our result, we are only required to apply a linear substitution on the variables t_j in terms of new variables s_j (cf. [28]). On the other hand, it is important to note that the functions $K_d^t(x, y)$ and $H_d^t(x, y, z)$ are homogeneous rational functions of order $-\frac{d}{2}$ and $-\frac{d}{2} - 1$ respectively. Using formula (18), it is clear that the functions $K_d(t_0, t_1)$ and $H_d(t_0, t_1, t_2)$ are homogeneous of order $1 - \frac{d}{2}$ in $f(t_j)$'s. This is the reason that for function $f(t) = e^t$ and a linear substitution such as $t_j = \sum_{m=0}^j s_m$, a factor of some power of e^{s_0} comes out. This term can be replaced by a power of e^h multiplied from the left to the final outcome. This explains how the functions in the aforementioned papers have one less variable than our functions. In other words, we have

$$\begin{aligned}
 K_d(s_0, s_0 + s_1) &= e^{(1-\frac{d}{2})s_0} K_d(s_1), \\
 H_d(s_0, s_0 + s_1, s_0 + s_1 + s_2) &= e^{(1-\frac{d}{2})s_0} H_d(s_1, s_2).
 \end{aligned}$$

For instance, function $K_d(s)$ is given by

$$K_d(s_1) = \frac{8e^{\frac{d+2}{4}s_1} \left((d-1) \sinh\left(\frac{s_1}{2}\right) + \sinh\left(\frac{(1-d)s_1}{2}\right) \right)}{d(d-2)d(e^{s_1} - 1)^2 s_1}.$$

Now, we can obtain functions in dimension two:

$$H_2(s_1) = -\frac{e^{\frac{s_1}{2}} (e^{s_1} (s_1 - 2) + s_1 + 2)}{(e^{s_1} - 1)^2 s_1},$$

$$K_2(s_1, s_2) = \left(s_1(s_1 + s_2) \cosh(s_2) - s_2(s_1 + s_2) \cosh(s_1) \right)$$

$$\begin{aligned}
 & - (s_1 - s_2) (s_1 + s_2 + \sinh(s_1) + \sinh(s_2) - \sinh(s_1 + s_2)) \Big) \\
 & \times \operatorname{csch}\left(\frac{s_1}{2}\right)\operatorname{csch}\left(\frac{s_2}{2}\right)\operatorname{csch}^2\left(\frac{s_1 + s_2}{2}\right)/(4s_1s_2(s_1 + s_2)).
 \end{aligned}$$

we have $-4H_2 = H$ and $-2K_2 = K$ where K and H are the functions found in [8, 16]. The difference is coming from the fact that the noncommutative parts of the results in [16, Section 5.1] are $\Delta(\log(e^{h/2})) = \frac{1}{2}\Delta(h)$ and $\square(\log(e^{h/2})) = \frac{1}{4}\square(h)$.

The functions for dimension four, with the same conformal factor $f(t) = e^t$ and substitution $t_j = \sum_{m=0}^j s_m$, gives the following which up to a negative sign are in complete agreement with the results from [17]:

$$\begin{aligned}
 H_4(s_1) &= \frac{1 - e^{s_1}}{2e^{s_1}s_1}, \\
 K_4(s_1, s_2) &= \frac{(e^{s_1} - 1)(3e^{s_2} + 1)s_2 - (e^{s_1} + 3)(e^{s_2} - 1)s_1}{4e^{s_1+s_2}s_1s_2(s_1 + s_2)}.
 \end{aligned}$$

To recover the functions for curvature of a noncommutative three torus equipped with a conformally flat metric [25, 11], we need to set $f(t) = e^{2t}$ and $t_0 = s_0$, $t_1 = s_0 + s_1/3$ and $t_2 = s_0 + (s_1 + s_2)/3$. Then up to a factor of e^{-s_0} , we have

$$\begin{aligned}
 H_3(s_1) &= \frac{4 - 4e^{\frac{s_1}{3}}}{e^{\frac{s_1}{6}}(s_1e^{\frac{s_1}{3}} + 1)}, \\
 K_3(s_1, s_2) &= \frac{6(e^{\frac{s_1}{3}} - 1)(3e^{\frac{s_2}{3}} + 1)s_2 - 6(e^{\frac{s_1}{3}} + 3)(e^{\frac{s_2}{3}} - 1)s_1}{e^{\frac{s_1+s_2}{6}}(e^{\frac{s_1+s_2}{3}} + 1)s_1s_2(s_1 + s_2)}.
 \end{aligned}$$

Finally, we would like to check the classical limit of our results as $\theta \rightarrow 0$. In the commutative case, the scalar curvature of a conformally flat metric $\tilde{g} = e^{2h}g$ on a d -dimensional space reads

$$\tilde{R} = -2(d - 1)e^{-2h}g^{jk}\partial_j\partial_k(h) - (d - 2)(d - 1)e^{-2h}g^{jk}\partial_j(h)\partial_k(h).$$

If we let $f(t) = e^{-2t}$, we have the limit

$$\begin{aligned}
 \lim_{t_0, t_1 \rightarrow t} K_d(t_0, t_1) &= \frac{1}{3}(d - 1)e^{(d-2)t}, \\
 \lim_{t_0, t_1, t_2 \rightarrow t} H_d(t_0, t_1, t_2) &= \frac{1}{6}(d - 2)(d - 1)e^{(d-2)t}.
 \end{aligned}$$

We should also add that since $\delta_j \rightarrow -i\partial_j$ as $\theta \rightarrow 0$, we have $\Delta(h) \rightarrow -g^{jk}\partial_j\partial_k(h)$ and $\square(h) \rightarrow -g^{jk}\partial_j(h)\partial_k(h)$. Therefore, our result recovers the classical result up to the factor of $\sqrt{|g|}e^{dh}/6$. The factor $\sqrt{|g|}e^{dh}$ represents the volume form in the scalar curvature density and the factor $1/6$ is due to our choice of normalization in (13).

Remark 4.3. The rationality of the heat trace densities in different situations [14, 18] has been studied. Here, we have a rationality of the second density for the conformally flat metric on a noncommutative tori and it can be expected to be true in the higher terms of the heat trace densities. On the other hand, finding relations between the functions K_d and H_d in different dimensions can shed more light on the structure of these functions.

4.3. Twisted product functional metrics

In this subsection, we shall compute the scalar curvature density of a noncommutative d -torus equipped with a class of functional metrics, which following the standard convention in the differential geometry will be called a twisted product metric (see [24] and references therein).

Definition 4.2. Let g be an $r \times r$ and \tilde{g} be a $(d-r) \times (d-r)$ positive definite real symmetric matrices and assume f is a positive function on the real line. We call the functional metric

$$(19) \quad G = f(t)^{-1}g \oplus \tilde{g},$$

the *twisted product functional metric* with the twisting element $f(h)^{-1}$.

Some examples of the twisted product metrics on noncommutative tori were already studied. The asymmetric two torus whose Dirac operator and spectral invariants are studied in [10] is a twisted product metric for $r = 1$. The scalar and Ricci curvature of noncommutative three torus of twisted product metrics with $r = 2$ are studied in [11]. It is worth mentioning that conformally flat metrics as well as warped metrics are two special cases of twisted product functional metrics.

Let us find the terms to be substituted in the formulas from Proposition 3.1, for the twisted product functional metric (19). First, we note that there are functions P_\bullet and \tilde{P}_\bullet such that the functions P_\bullet^{ij} of functional parts (7) of the symbol of $\Delta_{0,G}$ can be written as

$$(20) \quad P_\bullet^{ij} = \begin{cases} g^{ij} P_\bullet & 1 \leq i, j \leq r \\ \tilde{g}^{ij} \tilde{P}_\bullet & r < i, j \leq d \\ 0 & \text{otherwise.} \end{cases}$$

The functions P_\bullet are exactly the functions in (16) obtained for the conformally flat metrics in dimension r . However, the other functions \tilde{P}_\bullet are given by

$$\begin{aligned} \tilde{P}_2(t_0) &= 1, \\ \tilde{P}_1(t_0, t_1) &= f(t_0)^{\frac{r}{4}} [t_0, t_1; f^{-\frac{r}{4}}] + f(t_0)^{-\frac{r}{4}} [t_0, t_1; f^{\frac{r}{4}}], \\ \tilde{P}_{0,1}(t_0, t_1) &= f(t_0)^{-\frac{r}{4}} [t_0, t_1; f^{\frac{r}{4}}], \\ \tilde{P}_{0,2}(t_0, t_1, t_2) &= f(t_0)^{\frac{r}{4}} [t_0, t_1; f^{-\frac{r}{2}}] [t_1, t_2; f^{\frac{r}{4}}] + 2f(t_0)^{-\frac{r}{4}} [t_0, t_1, t_2; f^{\frac{r}{4}}]. \end{aligned}$$

On the other hand, while evaluating T -functions for metric (19), we first find that

$$\begin{aligned} P^{-1}(\mathbf{s})_{kl} &= \begin{cases} g_{kl} \left(\sum_{j=0}^n s_j f(t_j) \right)^{-1} & 1 \leq i, j \leq r \\ \tilde{g}_{kl} & r < i, j \leq d \end{cases} \\ \sqrt{\det P(\mathbf{s})} &= \frac{1}{\sqrt{|g|\tilde{g}|}} \left(\sum_{j=0}^n s_j f(t_j) \right)^{\frac{r}{2}}. \end{aligned}$$

This implies that when exactly k of the entries of \mathbf{n} are less than equal to r , then we have a function T_α^k such that $T_{\mathbf{n},\alpha} = \sqrt{|g|\tilde{g}|} \sum_{\mathbf{n}} \prod g_{n_i n_{\sigma(i)}} T_\alpha^k$. The function T_α^k is given by

$$T_\alpha^k(t_0, \dots, t_n) = \frac{1}{2^{|\alpha|-2} \beta!} \int_{\Sigma_n} \prod s_j^{\alpha_j-1} \left(\sum_{j=0}^n s_j f(t_j) \right)^{-\frac{r+k}{2}}.$$

Similar to the conformal case, the function T_α^k can be computed as derivatives of the divided difference of explicit functions.

Lemma 4.4. *Let α and \mathbf{n} be multi-indices such that exactly k entries of \mathbf{n} are less than equal to r . Then, for the twisted product metric (19) with $f(t) = t$, the T -function $T_{\mathbf{n},\alpha}$ can be written as*

$$T_{\mathbf{n},\alpha}(t_0, \dots, t_n) = \sqrt{|g|\tilde{g}|} \sum_{\mathbf{n}} \prod g_{n_i n_{\sigma(i)}} T_\alpha^k(t_0, \dots, t_n).$$

where,

$$(21) \quad T_{\alpha}^k(t_0, \dots, t_n) = \begin{cases} \frac{2^{2-|\alpha|}}{\Gamma(\beta) \prod_{j=1}^{|\alpha|-1} (j - \frac{r+k}{2})} \partial_t^{\beta} [t_0, \dots, t_n; u^{-\frac{r+k}{2} + |\alpha|-1}] & |\alpha| \neq \frac{r+k}{2} + 1, \\ \frac{(-2)^{2-|\alpha|}}{\Gamma(\beta) \Gamma(\frac{r+k}{2})} \partial_t^{\beta} [t_0, \dots, t_n; \log(u)] & |\alpha| = \frac{r+k}{2} + 1. \end{cases}$$

It can readily be seen that for $k = 2|\alpha| - 4$ the function T_{α}^k is equal to T_{α} found in (17) for the conformally flat case.

Theorem 4.5. *The scalar curvature density of the d -dimensional noncommutative tori \mathbb{T}_{θ}^d equipped with the twisted product functional metric (19) with the twisting element $f(h)^{-1}$ is given by*

$$R = \sqrt{|g||\tilde{g}|} \left(K_r(h_{(0)}, h_{(1)}) (\Delta(h)) + H_r(h_{(0)}, h_{(1)}, h_{(2)}) (\square(h)) \right. \\ \left. + \tilde{K}_r(h_{(0)}, h_{(1)}) (\tilde{\Delta}(h)) + \tilde{H}_r(h_{(0)}, h_{(1)}, h_{(2)}) (\tilde{\square}(h)) \right),$$

where $\tilde{\Delta}(h) = \sum_{r < i, j} \tilde{g}^{ij} \delta_i \delta_j(h)$ and $\tilde{\square}(h) = \sum_{r < i, j} \tilde{g}^{ij} \delta_i(h) \delta_j(h)$ and Δ , \square , K_r and H_r are given by Theorem 4.3. The functions \tilde{K}_r and \tilde{H}_r for $r \neq 2, 4$ are given by

$$\tilde{K}_r(t_0, t_1) = \tilde{K}_r^t(f(t_0), f(t_1)) [t_0, t_1; f],$$

and

$$\tilde{H}_r(t_0, t_1, t_2) = \tilde{H}_r^t(f(t_0), f(t_1), f(t_2)) [t_0, t_1; f] [t_1, t_2; f] + 2\tilde{K}_r^t(f(t_0), f(t_2)) [t_0, t_1, t_2; f].$$

The functions \tilde{K}_r^t and \tilde{H}_r^t are

$$\tilde{K}_r^t(x, y) = \frac{(2r-4)(x^2-y^2)x^{\frac{r}{2}}y^{\frac{r}{2}} + 4x^2y^r - 4x^ry^2}{(r-4)(r-2)x^{\frac{3}{4}r}y^{\frac{3}{4}r}(x-y)^3},$$

and

$$\tilde{H}_r^t(x, y, z) = \frac{2x^{-\frac{3r}{4}}y^{-r}z^{-\frac{3r}{4}}}{(r-4)(r-2)(x-y)^2(x-z)^3(y-z)^2} \times \\ \left(x^ry^rz^2(x-y)(x^2+2x(y-2z)-4y^2+6yz-z^2) \right. \\ \left. + x^ry^{\frac{r}{2}}z^{\frac{r}{2}}(x-z)^2(y-z)((r-3)yz-x((r-3)z+y)) \right. \\ \left. - x^ry^2z^r(x-z)^3 + x^{\frac{r}{2}}y^{\frac{3r}{2}}z^2(x-y)(x-z)^2 \right)$$

$$\begin{aligned}
 & -x^{\frac{r}{2}}y^r z^{\frac{r}{2}}(x-y)(x-z)(y-z) \times \\
 & \quad \left((r-3)x^2 - 2x((r-2)y + (1-r)z) + z((r-3)z - 2(r-2)y) \right) \\
 & + x^{\frac{r}{2}}y^{\frac{r}{2}}z^r(y-x)(x-z)^2(yz - (r-3)x(y-z)) \\
 & + x^2y^{\frac{3r}{2}}z^{\frac{r}{2}}(x-z)^2(y-z) - x^2y^r z^r(y-z) \times \\
 & \quad \left(x^2 - 6xy + 4xz + 4y^2 - 2yz - z^2 \right).
 \end{aligned}$$

Proof. We first separate the terms in the formulas from Proposition 3.1 in two parts: the part which contains no tilde term and the rest of the terms. For example the formula for B_{21}^{ij} can be rewritten as

$$\begin{aligned}
 B_{21}^{ij}(t_0, t_1) &= \sqrt{|g||\tilde{g}|} \times \\
 & \left(g^{ij} \left(-T_{1,1}^0(t_0, t_1)P_{0,1}(t_0, t_1) + T_{2,1}^2(t_0, t_1)t_0(2P_1(t_0, t_1) + r) \right. \right. \\
 & \quad \left. \left. - 4(r+2)T_{3,1}^4(t_0, t_1)t_0^2 \right) \right. \\
 & \quad \left. + \tilde{g}^{ij} \left(-T_{1,1}^0(t_0, t_1)\tilde{P}_{0,1}(t_0, t_1) + 2T_{2,1}^0(t_0, t_1)\tilde{P}_1(t_0, t_1) \right. \right. \\
 & \quad \left. \left. + rT_{2,1}^2(t_0, t_1) - 4rT_{3,1}^2(t_0, t_1) \right) \right).
 \end{aligned}$$

The second part, multiples of \tilde{g}^{ij} , will be denoted by \tilde{K}_r^t , while the first part, multiple of g^{ij} , is nothing but $K_r^t(t_0, t_1)$; the function found for the conformally flat case in Theorem 4.3. In other words, we have

$$\begin{aligned}
 & B_{2,1}^{ij}(h_{(0)}, h_{(1)})(\delta_i\delta_j(h)) = \\
 & \sqrt{|g||\tilde{g}|} \left(K_r^t(h_{(0)}, h_{(1)})(\Delta(h)) + \tilde{K}_r^t(h_{(0)}, h_{(1)})(\tilde{\Delta}(h)) \right),
 \end{aligned}$$

where $\Delta(h) = \sum_{i,j \leq r} g^{ij} \delta_i \delta_j(h)$ and $\tilde{\Delta}(h) = \sum_{i,j > r} \tilde{g}^{ij} \delta_i \delta_j(h)$. Similarly, this can be done for B_{22}^{ij} , i.e. there are functions \tilde{H}_r^t and H_r^t such that

$$\begin{aligned}
 & B_{2,2}^{ij}(h_{(0)}, h_{(1)}, h_{(2)})(\delta_i(h) \cdot \delta_j(h)) = \\
 & \sqrt{|g||\tilde{g}|} \left(H_r(h_{(0)}, h_{(1)}, h_{(2)})(\square(h)) + \tilde{H}_r(h_{(0)}, h_{(1)}, h_{(2)})(\tilde{\square}(h)) \right).
 \end{aligned}$$

Here we use the notation

$$\square(h) = \sum_{i,j \leq r} g^{ij} \delta_i(h) \cdot \delta_j(h) \text{ and } \tilde{\square}(h) = \sum_{i,j > r} \tilde{g}^{ij} \delta_i(h) \cdot \delta_j(h).$$

The rest is just substitution of each term into the formula for B_{21}^{ij} and B_{22}^{ij} . □

When the selfadjoint element $h \in C^\infty(\mathbb{T}_\theta^d)$ has the property that $\delta_j(h) = 0$ for $1 \leq j \leq r$, we call the twisted product functional metric (19) a *warped functional metric* with the warping element $1/f(h)$.

Corollary 4.6. *The scalar curvature density of a warped product of \tilde{g} and g with the warping element $1/f(h)$ is given by*

$$R = \sqrt{|g||\tilde{g}|} \left(\tilde{K}_r(h_{(0)}, h_{(1)}) (\tilde{\Delta}(h)) + \tilde{H}_r(h_{(0)}, h_{(1)}, h_{(2)}) (\tilde{\square}(h)) \right).$$

Proof. It is enough to see that $\Delta(h)$ and $\square(h)$ vanish for the warped metric. □

For $r = 2$ and $r = 4$, functions \tilde{H}_r and \tilde{K}_r are the limit of the functions given in Theorem 4.5 as r approaches 2 or 4. This is because of the fact that for these values of r , some of T_α^k functions are the limit case of formulas found in (21). The functions for $r = 2$ are given by

$$\tilde{K}_2(x, y) = \frac{-x^2 + y^2 + 2xy \log(\frac{x}{y})}{\sqrt{xy}(x - y)^3},$$

and

$$\begin{aligned} \tilde{H}_2(x, y, z) = & \frac{1}{2y\sqrt{xz}(x - y)^2(x - z)^3(y - z)^2} \left(-y(x + y)(x - z)^3(y + z) \log(y) \right. \\ & - z(x - y)^2(-3x^2y + x^2z - 8xy^2 + 10xyz - 2xz^2 + yz^2 + z^3) \log(z) \\ & + x(y - z)^2(x^3 + x^2y - 2x^2z + 10xyz + xz^2 - 8y^2z - 3yz^2) \log(x) \\ & \left. + 2y(y - x)(x - z)(x + z)(z - y)(x - 2y + z) \right). \end{aligned}$$

For $r = 4$, on the other hand, we have

$$\tilde{K}_4(x, y) = \frac{x^2 - y^2 - (x^2 + y^2) \log(\frac{x}{y})}{xy(x - y)^3},$$

and

$$\begin{aligned} \tilde{H}_4(x, y, z) = & \frac{1}{2x(x - y)^2y^2(x - z)^3(y - z)^2z} \times \\ & \left((x^2 + y^2)(x - z)^3(y^2 + z^2) \log(y) \right. \\ & \left. + \log(x)(y - z)^2(x^4y + x^4z - 6x^3y^2 - 2x^3yz - 2x^3z^2 + 3x^2y^3 + x^2y^2z) \right) \end{aligned}$$

$$\begin{aligned}
 & +x^2yz^2 + x^2z^3 + 2xy^3z - 4xy^2z^2 + 3y^3z^2 + y^2z^3) \\
 - \log(z)(x - y)^2 & (x^3y^2 + x^3z^2 + 3x^2y^3 - 4x^2y^2z + x^2yz^2 - 2x^2z^3 + 2xy^3z \\
 & +xy^2z^2 - 2xyz^3 + xz^4 + 3y^3z^2 - 6y^2z^3 + yz^4) \\
 - 2(x - y)(x - z)(y - z) & (x^3z + x^2y^2 - 2x^2z^2 - 2xy^3 + 2xy^2z \\
 & +xz^3 - 2y^3z + y^2z^2) \Big).
 \end{aligned}$$

In [11, section 4.1], the scalar curvature density of twisted product functional metric on noncommutative three torus for $f(t) = e^{2t}$ and $r = 2$ is found. This result can be recovered from our formulas given in Theorem 4.5 by setting $t_0 = s_0$, $t_1 = s_0 + s_1/2$ and $t_2 = s_0 + s_1/2 + s_2/2$.

We would like to conclude this section by a remark on multiply twisted metrics and commuting families of metrics. Let $[g^{ij}(t)]$ be a commuting family of positive definite real symmetric matrices. Such a family diagonalizes simultaneously; i.e. there exist an orthogonal matrix O such that

$$[g^{ij}(t)] = O^{-1} \text{diag}(\lambda_1(t), \dots, \lambda_d(t))O.$$

Such a family is included in the class of so called *multiply twisted product metrics*. Let f_j 's be positive functions and take g_j 's to be n_j dimensional constant metrics. Then

$$G = \bigoplus_j f_j(h)g_j$$

defines a functional metric on \mathbb{T}_θ^d , where $d = \sum n_j$. The T -functions computed in Example 3.2 are for metrics on four tori with two twisting factors f and \tilde{f} . Very similar computations would work for multiply twisted product of even dimensional tori equipped with constant metrics. We shall use the T -functions found for the doubly twisted metric on \mathbb{T}_θ^4 to compute its total scalar curvature in the next section.

5. Total scalar curvature of functional metrics

The goal of this section is to show how the trace of the second density of the heat trace of a Laplace type h -differential operator P on \mathbb{T}_θ^d can be simplified. When $P = \Delta_{0,g}$ for some functional metric $g(h)$, we shall call

$$\varphi(R) = (4\pi)^{\frac{d}{2}} \varphi(b_2(\Delta_{0,g}))$$

the *total scalar curvature* of the metric $g(h)$ on the noncommutative torus. In this section, we show that the total scalar curvature for all functional

metrics on \mathbb{T}_θ^2 vanishes. This generalizes the Gauss-Bonnet theorem proved in [9, 15] for the conformally flat metrics on \mathbb{T}_θ^2 .

5.1. Trace of the second density

We start with the following lemma which essentially is an avatar of the tracial property $\phi(ab) = \phi(ba)$ when the trace acts on an element written in the contraction form. Versions of this can be found in [32].

Lemma 5.1. *Let h be a selfadjoint element of a C^* -algebra A and let ϕ be a trace on A . For every smooth function $F : \mathbb{R}^{n+1} \rightarrow \mathbb{C}$, we have*

$$\begin{aligned} (i) \quad & \phi \left(F(h_{(0)}, \dots, h_{(n)}) (b_1 \cdots b_n) \right) = \\ & \phi \left(F(h_{(0)}, \dots, h_{(n-1)}, h_{(0)}) (b_1 \cdots b_{(n-1)}) b_n \right), \\ (ii) \quad & \phi \left(F(h_{(0)}, \dots, h_{(n)}) (b_1 \cdots b_n) b_{n+1} \right) = \\ & \phi \left(b_1 F(h_{(1)}, \dots, h_{(n)}, h_{(0)}) (b_2 \cdots b_{n+1}) \right). \end{aligned}$$

Proof. To prove the identities, first we write the element in the contraction form as an integral given in Remark 2.1. After applying the trace property to the integrand, we can then use the rearrangement lemma 2.1 to put the element back into the contraction form. \square

For a Laplace type h -differential operator P on \mathbb{T}_θ^d , this lemma implies that

$$(22) \quad \varphi(b_2(P)) = (4\pi)^{-d/2} \varphi \left(\tilde{B}_{2,1}^{ij}(h) \delta_i \delta_j(h) + \tilde{B}_{2,2}^{ij}(h_{(0)}, h_{(1)}) (\delta_i(h)) \delta_j(h) \right),$$

where

$$\tilde{B}_{2,1}^{ij}(t) = B_{2,1}^{ij}(t, t), \quad \tilde{B}_{2,2}^{ij}(t_0, t_1) = B_{22}(t_0, t_1, t_0).$$

We can hence compute and simplify the functions $\tilde{B}_{2,1}^{ij}(t)$ and $\tilde{B}_{2,2}^{ij}(t_0, t_1)$ for a Laplace type h -differential operator P :

$$\begin{aligned} \tilde{B}_{21}(t) = \\ \frac{1}{\sqrt{\det P_2(t)}} \left(-P_{0,1}^{ij}(t, t) + \frac{1}{2} P_1^{ji}(t, t) + \frac{1}{12} \text{tr} \left(P^{-1}(t) P_2'(t) \right) P_2^{ij}(t) - \frac{1}{3} P'^{ij}(t) \right), \end{aligned}$$

and

$$\begin{aligned}
\tilde{B}_{2,2}(t_0, t_1) = & \frac{1}{\sqrt{\det P_2(t_0)}} \left(-P_{0,2}^{ij}(t_0, t_1, t_0) + [t_0, t_1; P_1^{ij}(t_0, \cdot)] - \frac{2}{3} P_2^{ij}(t_0) \frac{1}{t_0 - t_1} \right. \\
& \left. + P_2^{ij}(t_0) \frac{1}{(t_0 - t_1)^2} + \frac{\text{tr}(P_2(t_0)^{-1} P_2'(t_0))}{6(t_0 - t_1)} P_2^{ij}(t_0) \right) \\
& + T_{k\ell;2,1}(t_0, t_1) \left(P_1^{ik}(t_0, t_1) - 2P_2^{ik}(t_0) \frac{1}{t_0 - t_1} \right) \left(P_1^{j\ell}(t_1, t_0) - 2[t_0, t_1; P_2^{j\ell}] \right) \\
& - 2T_{k\ell;1,2}(t_0, t_1) \left(P_1^{ik}(t_0, t_1) - 2P_2^{ik}(t_0) \frac{1}{t_0 - t_1} \right) P_2^{j\ell}(t_1) \frac{1}{t_0 - t_1} \\
& + T_{;1,1}(t_0, t_1) \left(P_1^{ij}(t_0, t_1) - 2P_2^{ij}(t_0) \frac{1}{t_0 - t_1} \right) \frac{1}{t_0 - t_1}.
\end{aligned}$$

To write these functions in the present form, we also used Lemma 3.3 and some of the basic properties of T -functions from Remark 3.2. To illustrate these steps, we show the complete work required to be done on the tenth term of B_{22}^{ij} from Proposition 3.1.

$$\begin{aligned}
& \lim_{t_2 \rightarrow t_0} -4T_{k\ell mn;2,1,1}(t_0, t_1, t_2) P_2^{ik}(t_0) [t_0, t_1; P_2^{j\ell}] [t_1, t_2; P_2^{mn}] \\
& = -4T_{k\ell mn;2,1,1}(t_0, t_1, t_0) P_2^{ik}(t_0) [t_0, t_1; P_2^{j\ell}] [t_1, t_0; P_2^{mn}] \\
& = -4T_{k\ell mn;3,1}(t_0, t_1) [t_1, t_0; P_2^{mn}] P_2^{ik}(t_0) [t_0, t_1; P_2^{j\ell}] \\
& = -4(T_{k\ell;2,1}(t_0, t_1) - T_{k\ell;3}(t_0)) P_2^{ik}(t_0) [t_0, t_1; P_2^{j\ell}] \frac{1}{t_0 - t_1} \\
& = -4T_{k\ell;2,1}(t_0, t_1) P_2^{ik}(t_0) [t_0, t_1; P_2^{j\ell}] \frac{1}{t_0 - t_1} \\
& \quad + 4 \frac{P_2^{-1}(t_0)_{k\ell}}{4\sqrt{\det P_2(t_0)}} P_2^{ik}(t_0) (P_2(t_0)^{j\ell} - P_2(t_1)^{j\ell}) \frac{1}{(t_0 - t_1)^2} \\
& = -4T_{k\ell;2,1}(t_0, t_1) P_2^{ik}(t_0) (P_2^{j\ell}(t_0) - P_2^{j\ell}(t_1)) \frac{1}{(t_0 - t_1)^2} \\
& \quad + \frac{P_2^{-1}(t_0)_{k\ell}}{\sqrt{\det P_2(t_0)}} P_2^{ik}(t_0) P_2(t_0)^{j\ell} \frac{1}{(t_0 - t_1)^2} \\
& \quad - \frac{P_2^{-1}(t_0)_{k\ell}}{\sqrt{\det P_2(t_0)}} P_2^{ik}(t_0) P_2(t_1)^{j\ell} \frac{1}{(t_0 - t_1)^2} \\
& = -4T_{k\ell;2,1}(t_0, t_1) P_2^{ik}(t_0) P_2^{j\ell}(t_0) \frac{1}{(t_0 - t_1)^2} \\
& \quad + 4T_{k\ell;2,1}(t_0, t_1) P_2^{ik}(t_0) P_2^{j\ell}(t_1) \frac{1}{(t_0 - t_1)^2}
\end{aligned}$$

$$+ \frac{1}{\sqrt{\det P_2(t_0)}} P_2(t_0)^{ji} \frac{1}{(t_0 - t_1)^2} - \frac{1}{\sqrt{\det P_2(t_0)}} P_2(t_1)^{ji} \frac{1}{(t_0 - t_1)^2}.$$

Further, we use compatibility of the trace and the derivations, $\varphi \circ \delta_j = 0$, and the Leibniz rule to turn the term $\tilde{B}_{2,1}^{ij}(h)\delta_i\delta_j(h)$ into the form of the other summand,

$$\begin{aligned} \varphi\left(\tilde{B}_{2,1}^{ij}(h)\delta_i\delta_j(h)\right) &= \varphi\left(\delta_i(\tilde{B}_{2,1}^{ij}(h)\delta_j(h))\right) - \varphi\left(\delta_i(\tilde{B}_{2,1}^{ij}(h))\delta_j(h)\right) \\ &= -\varphi\left(\delta_i(\tilde{B}_{2,1}^{ij}(h))\delta_j(h)\right) \\ &= -\varphi\left([h_{(0)}, h_{(1)}; \tilde{B}_{2,1}^{ij}](\delta_i(h))\delta_j(h)\right). \end{aligned}$$

Then, we put both of the terms together and rewrite (22) as

$$\varphi(b_2(P)) = (4\pi)^{-d/2} \varphi(F^{ij}(h_{(0)}, h_{(1)})(\delta_i(h))\delta_j(h)),$$

where $F^{ij}(t_0, t_1) = [t_0, t_1; \tilde{B}_{2,1}^{ij}] + \tilde{B}_{2,2}^{ij}(t_0, t_1)$. Applying identity *ii*) of Lemma 5.1, we find that

$$\varphi((F^{ij}(h_{(0)}, h_{(1)}) - F^{ji}(h_{(1)}, h_{(0)}))(\delta_i(h))\delta_j(h)) = 0.$$

Therefore, only the expression $\frac{1}{2}(F^{ij}(t_0, t_1) + F^{ji}(t_1, t_0))$, denoted by $F_S^{ij}(t_0, t_1)$, may contribute to the value of $\varphi(b_2(P))$. This fact was first noticed in [9] for the conformally flat metrics and used in other works [16, 10] to prove a Gauss-Bonnet theorem in dimension two.

In the case of conformally flat metrics, since we have $F^{ij} = F^{ji}$, the function F_S^{ij} is indeed the symmetric part of the function F^{ij} and this is the rationale behind our notation F_S^{ij} .

Example 5.1. In this example, we examine what we have discussed so far for the total scalar curvature of twisted product of flat (\mathbb{T}_θ^r, g) and $(\mathbb{T}_\theta^{d-r}, \tilde{g})$ with twisting factor $f(t)^{-1}$ (see the Section 4.3). We assume that $f(t) = t$ and from Theorem 4.5 we have

$$\tilde{B}_{2,1}^{ij}(x) = \begin{cases} -\frac{1}{6}(r-1)x^{-\frac{r}{2}} & 1 \leq i, j \leq r \\ -\frac{1}{6}rx^{-\frac{r}{2}-1} & r < i, j \leq d \\ 0 & \text{otherwise.} \end{cases}$$

The $\tilde{B}_{22}^{ij}(x, y)$ for $1 \leq i, j \leq r$ is given by

$$\left(-6x^{\frac{3r}{2}}y^3 + 3x^ry^{\frac{r}{2}+1}((r-2)y - rx)((r-1)y - (r-2)x)\right)$$

$$+ x^{\frac{r}{2}} y^r (-r(r-1)(r-2)(x-y)^3 + 6xy((r-1)x - (r-2)y)) \\ + 3x^2 y^{\frac{3r}{2}} (rx - (r-2)y)) / (3(r-2)r(x-y)^4 x^r y^r).$$

Also, $\tilde{B}_{22}^{ij}(x, y)$ for $r < i, j \leq d$ is given by

$$\left(-6x^{\frac{3r}{2}+1} y^2 \right. \\ + 3x^{r+1} y^{r/2} ((r-3)(r-2)x^2 - (r-4)(2r-3)xy + (r^2 - 6r + 4)y^2) \\ + x^{\frac{r}{2}} y^r (r(r-4)(r-2)y^3 - 3(r^3 - 6r^r + 8r + 2)xy^2 \\ + 3(r^3 - 6r^2 + 6r + 8)x^2y - (r-2)(r^2 - 4r - 6)x^3) \\ \left. + 3x^2 y^{\frac{3r}{2}} ((r-2)x - (r-4)y) \right) / (3(r-4)(r-2)(x-y)^4 x^{r+1} y^r).$$

Therefore, there are two symmetric functions F_S and \tilde{F}_S such that

$$\varphi(R) = \sqrt{|g||\tilde{g}|} \varphi \left(g^{ij} F_S(f(h_{(0)}, h_{(1)})) [h_{(0)}, h_{(1)}; f] (\delta_i(h)) \delta_j(h) \right) \\ + \sqrt{|g||\tilde{g}|} \varphi \left(\tilde{g}^{ij} \tilde{F}_S(f(h_{(0)}, h_{(1)})) [h_{(0)}, h_{(1)}; f] (\delta_i(h)) \delta_j(h) \right).$$

These functions are given by

$$F_S(x, y) = \frac{x^{-r} y^{-r}}{2r(x-y)^3} (yx^{\frac{r}{2}} + xy^{\frac{r}{2}}) ((r-1)x^{\frac{r}{2}} y^{\frac{r}{2}} (x-y) - yx^r + xy^r), \\ \tilde{F}_S(x, y) = \frac{x^{-r} y^{-r}}{2(r-2)(x-y)^3} \times \\ \left(x^{\frac{r}{2}} y^r (rx + (1-r)y) - yx^{\frac{3r}{2}} + x^r y^{\frac{r}{2}} ((r-1)x - ry) + xy^{\frac{3r}{2}} \right).$$

Applied to the Laplacian of a functional metric g on \mathbb{T}_θ^d , what we have covered so far in this section gives a simplified term whose trace is equal to the total scalar curvature of (\mathbb{T}_θ^d, g) , which we summarize in the following proposition.

Proposition 5.2. *The total scalar curvature of \mathbb{T}_θ^d equipped with a functional metric g is given by*

$$\varphi(R) = \varphi \left(F_S^{ij}(h_{(0)}, h_{(1)}) (\delta_i(h)) \delta_j(h) \right),$$

where $F_S^{ij}(t_0, t_1)$ is given by

$$F_S^{ij}(t_0, t_1) = \frac{1}{2(t_0 - t_1)^2} \left(A^{ij} \sqrt[4]{|g|(t_0)|g|(t_1)} - 2A^{ij}T_{;1,1}(t_0, t_1) \right. \\ \left. + T_{k,l;1,2}(t_0, t_1)(2A^{ik}g^{lj}(t_1) + 2A^{kj}g^{il}(t_1) - A^{ik}A^{lj}) \right. \\ \left. + T_{k,l;2,1}(t_0, t_1)(2A^{ik}g^{lj}(t_0) + 2A^{kj}g^{il}(t_0) - A^{ik}A^{lj}) \right),$$

and $A^{ij} = |g|^{\frac{1}{4}}(t_0)|g|^{\frac{-1}{4}}(t_1)g^{ij}(t_0) + |g|^{\frac{1}{4}}(t_1)|g|^{\frac{-1}{4}}(t_0)g^{ij}(t_1)$.

Example 5.2. We conclude this section by computing the functions F_S^{ij} for the doubly twisted product functional metric $f(h)^{-1}g \oplus \tilde{f}(h)^{-1}\tilde{g}$ on \mathbb{T}_θ^4 whose T -functions was found in Example 3.2. Using Proposition 5.2, we have in this case

$$A = (\tilde{f}(t_0) + \tilde{f}(t_1)) \sqrt{\frac{f(t_0)f(t_1)}{\tilde{f}(t_0)\tilde{f}(t_1)}} g^{-1} \oplus (f(t_0) + f(t_1)) \sqrt{\frac{\tilde{f}(t_0)\tilde{f}(t_1)}{f(t_0)f(t_1)}} \tilde{g}^{-1}.$$

Then by substituting all the terms in Proposition 5.2, we find that for $1 \leq i, j \leq 2$

$$F_S^{ij}(t_0, t_1) = \frac{\sqrt{|g||\tilde{g}|}(\tilde{f}(t_0)^2 - \tilde{f}(t_1)^2)g^{ij}}{4(t_0 - t_1)^2(f(t_1)\tilde{f}(t_0) - f(t_0)\tilde{f}(t_1))^2} \times \\ \left(\frac{f(t_1)}{\tilde{f}(t_1)} \left(f(t_0) \left(\log\left(\frac{f(t_0)\tilde{f}(t_1)}{f(t_1)\tilde{f}(t_0)}\right) + 1 \right) + f(t_1) \right) \right. \\ \left. - \frac{f(t_0)}{\tilde{f}(t_0)} \left(f(t_1) \left(\log\left(\frac{f(t_1)\tilde{f}(t_0)}{f(t_0)\tilde{f}(t_1)}\right) + 1 \right) + f(t_0) \right) \right).$$

The formula for functions $F_S^{ij}(t_0, t_1)$ for $3 \leq i, j \leq 4$, can be obtained by interchanging f with \tilde{f} and g^{-1} with \tilde{g}^{-1} in the above formula.

5.2. Dimension two and a Gauss-Bonnet theorem

In this subsection, we study the functions F_S^{ij} in dimension two. We show that these functions vanish for the noncommutative two torus equipped with a functional metric g . This means that the total scalar curvature of (\mathbb{T}_θ^2, g) is independent of g . Similar to the conformally flat metrics [9, 15, 13], we call this result the Gauss-Bonnet theorem for functional metrics.

Theorem 5.3 (Gauss-Bonnet Theorem). *The total scalar curvature $\varphi(R)$ of the noncommutative two tori equipped with a functional metric vanishes, hence it is independent of the metric.*

Before we give the proof of the theorem, we make some comments and prove a lemma. To prove the theorem, we evaluate the functions F_S^{ij} for dimension two. What makes it possible for us to do this is that T -functions present in the formula for the total scalar curvature from Proposition 5.2 can be explicitly evaluated in dimension two for all functional metrics. In particular, the function $T_{;1,1}$ in dimension two is given by a basic integral; that is the integral of an inverse square root of a quadratic function, and we have

$$T_{;1,1}(t_0, t_1) = \begin{cases} \frac{1}{\sqrt{a}} \ln \left(\frac{2a+b+2\sqrt{a^2+ab+ac}}{b+2\sqrt{ac}} \right) & a > 0 \\ \frac{1}{\sqrt{-a}} \sin^{-1} \left(\frac{-2a+b}{\sqrt{b^2-4ac}} \right) - \frac{1}{\sqrt{-a}} \sin^{-1} \left(\frac{b}{\sqrt{b^2-4ac}} \right) & a < 0 \end{cases}$$

where $c = \det(P_2(t_1))$, $b = -\det(P_2(t_1))\text{tr}(I - P_2(t_0)P_2(t_1)^{-1})$ and $a = \det(P_2(t_1) - P_2(t_0))$.

The form of $T_{;1,1}(t_0, t_1)$ depends on the sign of a . In the conformal case, $a = (f(t_1) - f(t_0))^2$ which is always positive. In general, however, there are functional metrics for which a is negative. For instance, consider the diagonal functional metric with the diagonal entries t and a positive decreasing function $f(t)$. For all nonzero t , then $\det(P_2(t_1) - P_2(t_0)) < 0$. On the other hand, the case $a = 0$, which is the limit of the other cases, can also happen. The diagonal functional metric with one constant diagonal entry is an instance of such a case. While all three cases are possible at the same time, we will deal with them separately. This is simply because the conditions $a > 0$ or $a < 0$ are open conditions and we can always choose to work with a selfadjoint element h whose spectrum is either in $a^{-1}(0, +\infty) > 0$ or $a^{-1}(-\infty, 0) < 0$.

By the property $T_{k,l;2,1}(t_0, t_1) = T_{k,l;1,2}(t_1, t_0)$, we only need to find $T_{k,l;2,1}$. This is done in the following lemma in which we write $T_{k,l;2,1}$ in terms of $T_{1,1}$.

Lemma 5.4. *In dimension two, when $\det(P_2(t_0) - P_2(t_1)) \neq 0$, we have*

$$T_{k,l;2,1}(t_0, t_1) = \frac{1}{2|g(t_0) - g(t_1)|} T_{1,1}(t_0, t_1) \left(\frac{g_{kl}(t_0)}{|g|(t_0)} - \frac{g_{kl}(t_1)}{|g|(t_1)} \right) + \frac{|g|(t_0)^{3/2}|g|(t_1)(\sqrt{|g|(t_0)}\sqrt{|g|(t_1)}+|g|(t_1)-|g(t_0)-g(t_1)|)}{|g(t_0)-g(t_1)|(|g(t_0)-g(t_1)|-(\sqrt{|g|(t_0)}+\sqrt{|g|(t_1)})^2)} \left(\frac{g_{kl}(t_0)}{|g|(t_0)} - \frac{g_{kl}(t_1)}{|g|(t_1)} \right)$$

$$+ \frac{|g|(t_0)^{3/2}}{\left(\sqrt{|g|(t_0)} + \sqrt{|g|(t_1)}\right)^2 - |g(t_0) - g(t_1)|} g_{kl}(t_1).$$

Moreover, for $\det(P_2(t_0) - P_2(t_1)) = 0$, we have

$$T_{k,l;2,1}(t_0, t_1) = \frac{|g|(t_0)^{3/2}}{\left(\sqrt{|g|(t_0)} + \sqrt{|g|(t_1)}\right)^2} g_{kl}(t_1),$$

$$T_{;1,1}(t_0, t_1) = \frac{\sqrt{|g|(t_0)|g|(t_1)}}{\sqrt{|g|(t_0)} + \sqrt{|g|(t_1)}}.$$

Proof. We substitute the formula of T -functions from Lemma 5.4 in the formula for the total scalar curvature $\varphi(R)$ found in Proposition 5.2. We find that F_S^{ij} is the (i, j) component of the matrix

$$\left(2|g(t_0) - g(t_1)||g|^{\frac{1}{4}}(t_0)|g|^{\frac{1}{4}}(t_1)(t_0 - t_1)^2(|g(t_0) - g(t_1)| - (|g|^{\frac{1}{2}}(t_0) + |g|^{\frac{1}{2}}(t_1))^2)\right)^{-1}$$

$$\left(2T_{;1,1}(t_0, t_1) \left((|g|^{\frac{1}{2}}(t_0) + |g|^{\frac{1}{2}}(t_1))^2 - |g(t_0) - g(t_1)|\right) - 4|g|^{\frac{1}{2}}(t_0)|g|(t_1)\right.$$

$$\left. + |g(t_0) - g(t_1)||g|^{\frac{1}{4}}(t_0)|g|^{\frac{1}{4}}(t_1) - 4|g|^{\frac{1}{2}}(t_1)|g|(t_0)\right) (\alpha(t_0, t_1) + \alpha(t_1, t_0)),$$

where $\alpha(t_0, t_1)$ is a matrix valued function given by

$$\alpha(t_0, t_1) =$$

$$|g|^{\frac{1}{2}}(t_0) \left(|g(t_0) - g(t_1)| - |g|(t_0) - |g|(t_1) + |g|(t_0)g^{-1}(t_0)g(t_1) \right) g^{-1}(t_0)$$

$$+ |g|^{\frac{1}{2}}(t_1)|g|(t_0)g^{-1}(t_0).$$

Using the Cayley–Hamilton theorem and the identity $\det(A)A^{-1} = \text{tr}(A) - A$ which holds for every two by two matrix A , we have

$$\left(|g(t_0) - g(t_1)| - |g|(t_0) - |g|(t_1) + |g|(t_0)g^{-1}(t_0)g(t_1) \right) g^{-1}(t_0) =$$

$$-|g|(t_1)g^{-1}(t_1).$$

This implies that

$$\alpha(t_0, t_1) = |g|^{\frac{1}{2}}(t_1)|g|(t_0)g^{-1}(t_0) - |g|^{\frac{1}{2}}(t_0)|g|(t_1)g^{-1}(t_1).$$

Hence α is an anti-symmetric function and the functions F_S^{ij} vanish for all i and j , in dimension two. This completes the proof of the theorem. \square

Summary and outlook

In this paper we introduced a new family of metrics on noncommutative tori, called functional metrics, and studied their spectral geometry. We defined the Laplacian of these metrics and computed the heat trace asymptotics of these Laplacians. A formula for the second density of the heat trace is obtained. In fact our formula covers a class larger than these Laplacians. We call the latter class, introduced in this paper, Laplace type h -differential operators. As a result, the scalar curvature density and the total scalar curvature are explicitly computed in all dimensions for certain classes of functional metrics that include conformally flat metrics and twisted product of flat metrics. In dimension two our computations cover the total scalar curvature of all functional metrics. Finally a Gauss-Bonnet type theorem for a noncommutative two torus equipped with a general functional metric is proved. Extending noncommutative curvature computations beyond dimensions 2, 3, and 4, to all dimensions, and beyond conformally flat metrics has been an open problem that is effectively addressed in this paper for the first time. The moduli space of noncommutative metrics, even for noncommutative two tori, is poorly understood at present. It is thus important to treat wider classes of metrics by heat equation and spectral geometry techniques in search of finding common patterns.

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ASGHAR GHORBANPOUR
DEPARTMENT OF MATHEMATICS
WESTERN UNIVERSITY
LONDON, ONTARIO
CANADA, N6A 5B7
TEL.: +1-519-661-2111 x86540
E-mail address: aghorba@uwo.ca

MASOUD KHALKHALI
DEPARTMENT OF MATHEMATICS
WESTERN UNIVERSITY
LONDON, ONTARIO
CANADA, N6A 5B7
TEL.: +1-519-661-2111 x86524
E-mail address: masoud@uwo.ca

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