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Infinite families of quantum modular 3-manifold invariants*

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One of the first key examples of a quantum modular form, which unifies the Witten-Reshetikhin-Turaev (WRT) invariants of the Poincaré homology sphere, appears in work of Lawrence and Zagier. We show that the series they construct is one instance in an infinite family of quantum modular invariants of negative definite plumbed 3-manifolds whose radial limits toward roots of unity may be thought of as a deformation of the WRT invariants. We use a recently developed theory of Akhmechet, Johnson, and Krushkal (AJK) which extends lattice cohomology and BPS q-series of 3-manifolds. As part of this work, we provide the first calculation of the AJK series for an infinite family of 3-manifolds. Additionally, we introduce a separate but related infinite family of invariants which also exhibit quantum modularity properties.

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1. Introduction and statement of results

In 1999, Lawrence and Zagier established a connection between modular forms and invariants of 3-manifolds arising from quantum topology and physics [16]. The Witten-Reshetihkin-Turaev (WRT) invariants, conceptualized by Witten in terms of a path integral for $SU(2)$ Chern-Simons theory and made mathematically precise by Reshetikhin and Turaev, are a family of 3-manifold invariants indexed by roots of unity [23, 21].

Lawrence and Zagier unified these WRT invariants by defining a holomorphic function $A(q)$ on the unit disk whose limiting values at roots of unity recover the invariants of the Poincaré homology sphere. The function $A(q)$ is in some sense an integral of a theta function; its obstruction to modularity on the unit circle is given by a factor introduced by integrating. It is one of the first key examples of a *quantum modular form*, a term coined by Zagier in 2010 with this series in mind [24]. Zagier's seminal work has inspired an extensive and ongoing body of research on quantum modular forms; see Chapter 21 of [3] and the references listed therein.

Gukov, Pei, Putrov, and Vafa (GPPV) provided a physical interpretation for $A(q)$ which not only gave rise to the construction of analogous series $\widehat{Z}(q)$ for a large class of 3-manifolds equipped with a spin^c structure, but also motivated a conjecture that such series exist for all 3-manifolds [14, 15]. These unified invariants were further extended in [13, 8] and the quantum modularity of these series for certain classes of manifolds were established in [4, 5].

Recent work of Akhmechet, Johnson, and Krushkal generalizes $\widehat{Z}(q)$ to a two-variable series invariant $\widehat{\widehat{Z}}_Y(t,q)$ of a 3-manifold Y [1]. This series is defined using an extension of lattice cohomology, a theory developed by Némethi in [17]. More details on the topological construction and properties of this invariant appear in Section 2.

The aim of this paper is to show that this two-variable series gives rise to infinitely many quantum modular forms whose values at roots of unity can be considered deformations of the WRT invariants. These results realize the work of Lawrence and Zagier as a special case. As a first step in this process, we provide a calculation of $\hat{\hat{Z}}_\Sigma(t,q)$ where Σ is a Brieskorn homology sphere, generating the first known calculation of this invariant for an infinite family of manifolds.

In particular, we find explicit formulae for the coefficients $\varphi(n;t)$ of the q-series $\hat{\hat{Z}}$, which are Laurent polynomials in t. The result is, for each Brieskorn sphere Σ, a q-series of the form

$$\hat{\hat{Z}}_\Sigma(t,q) = q^\Delta\Big(C - \sum_{n\geq 0}\varphi(n;t)q^{\frac{n^2}{4p}}\Big), \tag{1}$$

where $\Delta \in \mathbb{Q}$, $p \in \mathbb{Z}$, and C is zero unless Σ is the Poincaré homology sphere, in which case it equals $q^{1/120}(t+t^{-1})$; see Section 3 for full definitions.

A priori, $\hat{\hat{Z}}_\Sigma(t,q)$ is convergent as a two-variable series for $t \in \mathbb{C}$ and $|q| < 1$. By leveraging the arithmetic properties of the coefficients $\varphi(n;t)$ when t is a root of unity, we are able to show the following:

Theorem 1.1. *Let ζ be a jth root of unity, ξ a Kth root of unity, and Σ a Brieskorn sphere. Define $\hat{\hat{Z}}_\Sigma(\zeta,\xi) := \lim_{t\searrow 0}\hat{\hat{Z}}_\Sigma(\zeta,\xi e^{-t})$. This limit exists and we have*

$$\hat{\hat{Z}}_\Sigma(\zeta,\xi) = \xi^\Delta\left(D + \sum_{n=1}^{2pjK}\left(\frac{n}{2pjK}-\frac{1}{2}\right)\varphi(n;\zeta)\xi^{\frac{n^2}{4p}}\right),$$

where $D = \xi^{1/120}Re(\zeta)$ when Σ is the Poincaré homology sphere and equals zero otherwise.

In general, these limit calculations give a novel family of "t-deformed" WRT invariants whose topological interpretation is an open question. However, using the above results we prove that for t a fixed root of unity, $\hat{\hat{Z}}_\Sigma(t,q)$ is, up to normalization, a quantum modular form:

Theorem 1.2. *Let $q = e^{2\pi i\tau}$. If ζ is a jth root of unity and Σ is a Brieskorn sphere, then*

$$\hat{\hat{Z}}_\Sigma(\zeta, q) = q^\Delta \left(C - A_\zeta(\tau)\right),$$

where $A_\zeta(\tau)$ is a quantum modular form of weight $1/2$ with respect to $\Gamma(4pj^2)$.

Remark 1.1. *The definition of a quantum modular form is deferred until Section 5, but we remark here that they are functions on $\mathbb{P}^1(\mathbb{Q})$ which have no nice modular or analytic properties, but whose obstructions to modularity are "nice" (in our case, extend to real-analytic functions on $\mathbb{R}$).*

The classical theory of theta functions involves forms of weight $1/2$ and $3/2$ that are related through differentiation of the Jacobi theta function. Because of the existence of the second variable, one can differentiate $\hat{\hat{Z}}_\Sigma(t, q)$, summand by summand, with respect to t and consider the new invariant that arises. This series, under specialization, also enjoys quantum modularity properties; the result is a sum of quantum modular forms of mixed weight:

Theorem 1.3. *Define $\hat{\hat{Z}}'_\Sigma(t, q) := t\dfrac{\partial}{\partial t}\hat{\hat{Z}}_\Sigma(t, q)$. Let ζ be a jth root of unity, and let C' equal $q^{1/120}(t - t^{-1})$ when Σ is the Poincaré homology sphere and equal zero otherwise. Then*

$$\hat{\hat{Z}}'_\Sigma(\zeta, q) = q^\Delta \left(C' - A'_\zeta(\tau)\right),$$

where $A'_\zeta(\tau)$ is a sum of quantum modular forms of weight $1/2$ and $3/2$ for $\Gamma(4pj^2)$.

Remark 1.2. *The quantum set is notably smaller for the quantum modular forms in Theorem 1.3. This is due to the fact that the weight $3/2$ quantum modular form need not correspond to the Eichler integral of a cusp form. For more details, see Section 6 and Theorem 1.1 of [11].*

The paper is organized as follows. In Section 2 we recall the necessary background from low-dimensional topology to understand and motivate the study of this two-variable series. In Section 3 we derive an explicit formula for the q-series coefficients of $\hat{\hat{Z}}_\Sigma(t, q)$ where Σ is a Brieskorn sphere and give two tables of example calculations for selected manifolds. Section 4 contains a proof of Theorem 1.1. In Section 5 we discuss the theory of modular and quantum modular forms as it pertains to $\hat{\hat{Z}}_\Sigma(t, q)$, and in Section 6 we offer proofs of Theorems 1.2 and 1.3.

2. The AJK series invariant

We begin with a motivating overview of the $\widehat{\widehat{Z}}$ invariant. This recently-developed two-variable series provides a common refinement of two existing invariants: the GPPV invariant $\widehat{Z}$ [14] and lattice cohomology [17]. As is the case for each of these theories, the $\widehat{\widehat{Z}}$ series is defined for negative-definite plumbed 3-manifolds equipped with spin^c structures. Complex spin structures, or spin^c structures, rose to prominence in low-dimensional topology through the advent of the Seiberg-Witten equations, which led to novel invariants of smooth 4-manifolds.

Using information from lattice cohomology, one can associate to a negative-definite plumbed 3-manifold and spin^c structure an object called a graded root. The authors of [1] assigned Laurent polynomial weights to the vertices of this root such that this new "weighted graded root" is still an invariant of the manifold. The series $\widehat{\widehat{Z}}_Y(t,q)$ results from taking the limit, in a precise sense, of these weights. Setting $t = 1$ recovers the GPPV invariant $\widehat{Z}(q)$. This paper calculates the two-variable series $\widehat{\widehat{Z}}$ for an infinite family of 3-manifolds, however the calculation of weighted graded roots remains an open problem. Below, we cover details of this construction necessary for our work.

2.1. Negative definite plumbed 3-manifolds

Let Γ be a finite graph with integer weights on its vertices. As in [1], we restrict to the case in which Γ is a tree. Let $m\colon v(\Gamma) \to \mathbb{Z}$ be the corresponding weight function and $s = |v(\Gamma)|$. Choosing an order on $v(\Gamma)$ enables us to write a weight vector $m \in \mathbb{Z}^s$ given by $m_i = m(v_i)$ and a degree vector $\delta \in \mathbb{Z}^s$ given by $\delta_i = \delta(v_i)$. With this ordering, we can associate to Γ a symmetric $s \times s$ matrix M given by

$$M_{i,j} = \begin{cases} m_i & i = j \\ 1 & i \neq j \text{ and } v_i \text{ and } v_j \text{ are connected by an edge} \\ 0 & \text{otherwise.} \end{cases}$$

We say Γ is negative definite whenever M is negative definite.

To obtain a 3-manifold from Γ, create a framed link $\mathcal{L}(\Gamma) \subset S^3$ by associating to each vertex v_i an unknot with framing m_i and Hopf linking unknots together whenever their corresponding vertices share an edge. The resulting linking matrix of $\mathcal{L}(\Gamma)$ is the plumbing matrix M. $Y(\Gamma)$ is defined

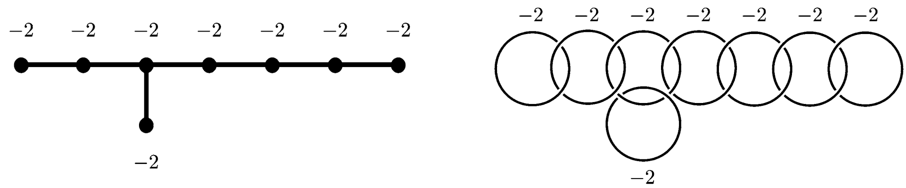

Figure 1: A plumbing tree and its associated link for the Poincaré homology sphere $\Sigma(2,3,5)$.

to be the 3-manifold obtained by Dehn surgery on $\mathcal{L}(\Gamma)$. Y also bounds the 4-manifold X which is obtained by adding 2-handles to $\mathbb{D}^4$ along $\mathcal{L}(\Gamma)$. From this perspective, M represents the intersection form of X.

In general, we say Y is a negative-definite plumbed 3-manifold if it is homeomorphic to $Y(\Gamma)$ for some negative-definite plumbing graph Γ. Two distinct plumbing trees may result in homeomorphic manifolds; in fact this is the case if and only if the trees can be related by a finite sequence of Neumann moves of type (a) and (b) [18]. Therefore, an invariant of a negative-definite plumbed manifold Y must be invariant under these two moves on its plumbing graph.

As with the GPPV invariant, $\widehat{\widehat{Z}}_Y(t,q)$ takes as inputs a negative-definite plumbed 3-manifold Y and a chosen spin^c structure. In the particular case of a negative-definite plumbed manifold, the set of spin^c structures can be given in terms of the plumbing data; it is known that

$$\text{spin}^c(Y) \cong \frac{m+2\mathbb{Z}^s}{2M\mathbb{Z}^s} \cong \frac{\delta+2\mathbb{Z}^s}{2M\mathbb{Z}^s}, \tag{2}$$

where the second isomorphism is given by $[k] \mapsto [k-(m+\delta)]$; for details, see [1, Section 2.2]. For negative definite plumbed 3-manifolds, one can also identify spin^c structures with abelian flat connections; see [12, Section 2]. More generally, the set of spin^c structures admits a free, transitive action by $H^2(Y;\mathbb{Z})$, giving a bijective correspondence between these sets. We note that both of these identifications are non-canonical.

2.1.1. Key example: Brieskorn homology spheres Brieskorn manifolds are well-studied, in part due to the fact they produce examples of exotic 7-spheres. In three dimensions, they are realized as the intersection of a singular complex surface with the unit sphere in $\mathbb{C}^3$. In particular, let (b_1, b_2, b_3) be pairwise relatively prime positive integers $b_1 < b_2 < b_3$. The corresponding Brieskorn sphere $\Sigma(b_1, b_2, b_3)$ is given by

$$\Sigma(b_1, b_2, b_3) := \{(z_1, z_2, z_3) \in \mathbb{C}^3 \ \mid \ z_1^{b_1} + z_2^{b_2} + z_3^{b_3} = 0\} \cap S^5 \subset \mathbb{C}^3.$$

Given (b_1, b_2, b_3), Neumann and Reymond provide an algorithm by which one can find a plumbing tree Γ for the associated Brieskorn sphere [19]. This process guarantees that Γ is always a star graph with one 3-valent vertex and 3 legs, as is the case in Figure 1.

As integral homology spheres, Brieskorn spheres have only one spin^c structure, and $\widehat{\widehat{Z}}$ is independent of choice of spin^c representative. Therefore, in calculations involving Brieskorn spheres we will drop the subscript indicating the spin^c structure.

The general formula for $\widehat{\widehat{Z}}_Y(t, q)$ depends on plumbing data, but in Section 3 we give a formula for Brieskorn spheres which only requires the triple (b_1, b_2, b_3). To achieve this, we use methods similar to those of Gukov and Manolescu, who provide a formula for the GPPV invariant in terms of (b_1, b_2, b_3); see Proposition 4.8 of [13].

2.2. The two-variable series

For a choice $k \in \mathbb{Z}^s$ of a spin^c representative $[k] \in \dfrac{m + 2\mathbb{Z}^s}{2M\mathbb{Z}^s}$, and for any $x \in \mathbb{Z}^s$, let

$$\chi_k(x) := \frac{-k \cdot x + \langle x, x\rangle}{2} \in \mathbb{Z},$$

where $(\cdot)$ denotes the Euclidean dot product and $\langle -, -\rangle$ denotes the bilinear form given by the plumbing matrix M. For $r \in \mathbb{Z}$ and $n \in \mathbb{N}$, let

$$\widehat{F}_n(r) := \begin{cases} \frac{1}{2}\mathrm{sgn}(r)^n \binom{\frac{n+|r|}{2}-2}{n-3} & \text{if } |r| \geq n-2,\ r \equiv n \pmod 2; \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

and define

$$\widehat{F}_{\Gamma,k}(x) := \prod_{v_i \in v(\Gamma)} \widehat{F}_{\delta_i}((2Mx + k - Mu)_i).$$

Note that $\widehat{F}_n(r)\colon \mathbb{Z} \to \mathbb{Q}$ describes the coefficient on z^{-r} in the expansion of $(z - z^{-1})^{2-n}$. To state the definition of $\widehat{\widehat{Z}}$ we will use for our calculations, we define $u\colon = (1, 1, \ldots)$ as well as

$$\Theta_k = \frac{k \cdot u - \langle u, u\rangle}{2}, \qquad \varepsilon_k = -\frac{(k - Mu)^2 + 3s + \sum_v m_v}{4} + 2\chi_k(x) + \langle x, u\rangle.$$

Then we have the following:

Theorem 2.1 (Theorems 6.3 and 7.6 of [1]). *Let Y be a negative-definite plumbed 3-manifold with spin^c structure $[k]$. The series*

$$\widehat{\widehat{Z}}_{Y,[k]}(t,q) := \sum_{x\in\mathbb{Z}^s} \widehat{F}_{\Gamma,k}(x) q^{\varepsilon_k(x)} t^{\Theta_k+\langle x,u\rangle} \tag{4}$$

is an invariant of the pair $(Y,[k])$, and

$$\widehat{Z}_a(q) = \widehat{\widehat{Z}}_{Y,[k]}(1,q),$$

where $\widehat{Z}_a(q)$ is the GPPV invariant for $(Y,[a])$ and a corresponds to k via (2).

Remark 2.1. *The family of functions $\{\widehat{F}_n\}_{n\in\mathbb{N}}$ is defined in [1] to be admissible in that it satisfies axioms that guarantee that the series (4) is an invariant. In this sense, $\widehat{\widehat{Z}}$ belongs to a family of two-variable series invariants developed in [1], parametrized by admissible families.*

3. Series analysis

We now develop an explicit formula for the coefficients of $\widehat{\widehat{Z}}_\Sigma(t,q)$ as a series in q whenever Σ is a Brieskorn sphere. The arithmetic properties of these coefficients will allow us to take limits toward roots of unity and establish quantum modularity properties in Sections 4 and 6. For a general negative-definite plumbed 3-manifold Y, one can use a program created by Peter Johnson[1] to calculate the first N coefficients of $\widehat{\widehat{Z}}_Y(t,q)$.

Let k be a spin^c representative for the unique spin^c structure $[k]$ of Σ, and set $a = k - Mu$. For $x \in \mathbb{Z}^s$, we let $\ell := a + 2Mx$. Using the fact established in [1] that $\frac{\ell^T M \ell}{4} = \frac{a^2}{4} - 2\chi_k(x) - \langle x,u\rangle$, we write

$$\widehat{\widehat{Z}}_\Sigma(t,q) = q^{-\frac{3s+\sum_v m_v}{4}} \sum_{x\in\mathbb{Z}^s} \prod_{v_i\in v(\Gamma)} \widehat{F}_{\delta_i}(\ell_i) q^{-\frac{\ell^T M^{-1}\ell}{4}} t^{\Theta_k+\langle x,u\rangle}.$$

Below is the plumbing graph of a Brieskorn sphere with the vertices ordered as needed for this section. The only $x \in \mathbb{Z}^s$ for which $\prod_{v_i} \widehat{F}_{\delta_i}(\ell_i) \neq 0$ are

[1] Available at https://github.com/peterkj1/plum

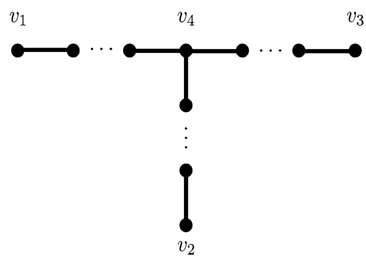

Figure 2: The plumbing graph for a Brieskorn sphere Σ.

those of the form $\ell = (\varepsilon_1, \varepsilon_2, \varepsilon_3, m, 0, \dots)$ for $\varepsilon_i \in \{\pm 1\}$ and m odd. In this case, we have that $\widehat{F}_1(\varepsilon_i) = -\varepsilon_i$ and $\widehat{F}_3(m) = \frac{1}{2}\mathrm{sign}(m)$, so

$$\prod_{v_i \in v(\Gamma)} \widehat{F}_{\delta_i}(\ell_i) = -\frac{1}{2}\varepsilon_1\varepsilon_2\varepsilon_3 \mathrm{sign}(m).$$

Since Brieskorn spheres have unimodular plumbing matrices M, every possible combination $(\varepsilon_1, \varepsilon_2, \varepsilon_3, m, 0, \dots)$ is in $a + 2M\mathbb{Z}^s$. Therefore we can write

$$\widehat{\widehat{Z}}_\Sigma(t,q) = \frac{-q^{-\frac{3s+\sum_v m_v}{4}}}{2} \sum_{\varepsilon_i \in \{\pm 1\}} \sum_{m \text{ odd}} \varepsilon_1\varepsilon_2\varepsilon_3 \mathrm{sign}(m) q^{-\frac{\ell^T M^{-1}\ell}{4}} t^{\Theta_k + \langle x, u\rangle}.$$

One can check that $\langle u, x\rangle = \frac{\varepsilon_1+\varepsilon_2+\varepsilon_3+m-a^T u}{2}$. Moreover, since $\widehat{\widehat{Z}}$ does not depend on a choice of spin^c representative, we make the convenient choice of $a = (1,1,1,1,0,\dots) \in \delta + 2M\mathbb{Z}^s$. In this case, $\Theta_k = 2$ and the exponent on t becomes $(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + m)/2$.

Remark 3.1. *Following Section 4.6 of [13] we can can rewrite*

$$\frac{-\ell^t M^{-1}\ell}{4} = \frac{b_1b_2b_3}{4}\left(m + \sum_i \frac{\varepsilon_i}{b_i}\right)^2 - \frac{b_1b_2b_3}{4}\sum \frac{1}{b_i^2} + \frac{\sum_i h_i}{4},$$

where h_i refers to the cardinality of $H_1(\Sigma')$ for Σ' the plumbed manifold that results from removing the ith vertex of the plumbing graph for Σ. Setting

$$\Delta := \frac{1}{4}\left(\sum_i h_i - 3s - \sum_v m_v - \frac{b_2b_3}{b_1} - \frac{b_1b_3}{b_2} - \frac{b_1b_2}{b_3}\right),$$

and rewriting $\varepsilon := \varepsilon_1 + \varepsilon_2 + \varepsilon_3$*, we now have*

$$(5) \qquad \widehat{\widehat{Z}}_\Sigma(t,q) = \frac{-q^\Delta}{2} \sum_{m \text{ odd}} \sum_{\varepsilon_i \in \{\pm 1\}} \varepsilon_1\varepsilon_2\varepsilon_3 sign(m) q^{\frac{b_1b_2b_3}{4}(m+\sum_i \frac{\varepsilon_i}{b_i})^2} t^{\frac{\varepsilon+m}{2}}.$$

Now, set $p := b_1b_2b_3$ and

$$\begin{aligned}
\alpha_1 &:= b_1b_2b_3 - b_1b_2 - b_1b_3 - b_2b_3;\\
\alpha_2 &:= b_1b_2b_3 + b_1b_2 - b_1b_3 - b_2b_3;\\
\alpha_3 &:= b_1b_2b_3 - b_1b_2 + b_1b_3 - b_2b_3;\\
\alpha_4 &:= b_1b_2b_3 + b_1b_2 + b_1b_3 - b_2b_3.
\end{aligned}$$

Theorem 3.1. *Let* $\Sigma(b_1, b_2, b_3)$ *be a Brieskorn sphere. Then*

$$\widehat{\widehat{Z}}_\Sigma(t,q) = q^\Delta\Big(C - \sum_{n\geq 1} \varphi(n;t) q^{\frac{n^2}{4p}}\Big),$$

where C *is nonzero and equals* $q^{\frac{1}{120}}(t+t^{-1})$ *only when* $(b_1,b_2,b_3) = (2,3,5)$ *and*

$$\varphi(n;t) = \begin{cases} \mp\frac{1}{2}(t^{\frac{\mp n + (\alpha_1+2p)}{2p}} + t^{\frac{\pm n - (\alpha_1+2p)}{2p}}) & n \equiv \pm\alpha_1 \pmod{2p},\\ \pm\frac{1}{2}(t^{\frac{\mp n+\alpha_k}{2p}} + t^{\frac{\pm n-\alpha_k}{2p}}) & n \equiv \pm\alpha_k \pmod{2p},\ k = 2,3\\ \mp\frac{1}{2}(t^{\frac{\mp n + (\alpha_4-2p)}{2p}} + t^{\frac{\pm n - (\alpha_4-2p)}{2p}}) & n \equiv \pm\alpha_4 \pmod{2p},\\ 0 & \textit{otherwise}. \end{cases}$$

Remark 3.2. *Note that when* $t = 1$*, this collapses back to the GPPV invariant as calculated by Gukov and Manolescu in [13]. Fixing* $\Sigma = \Sigma(2,3,5)$*, the function* $\varphi(n;1)$ *is equal to* $\chi_+(n)$ *as defined in [16].*

Proof. Begin with the calculation given by (5). Using the fact that $(\varepsilon_1)(\varepsilon_2)(\varepsilon_3)(\mathrm{sign}(m)) = (-\varepsilon_1)(-\varepsilon_2)(-\varepsilon_3)(\mathrm{sign}(-m))$, replacing m odd with $2n+1$, and setting $\varepsilon' := \frac{\varepsilon+2n+1}{2}$, we write

$$\widehat{\widehat{Z}}_\Sigma(t,q) = \frac{-q^\Delta}{2} \sum_{\varepsilon_i\in\{\pm1\}} \sum_{n\geq 0} \varepsilon_1\varepsilon_2\varepsilon_3 q^{p(n^2+n+\frac{1}{4}+(n+\frac{1}{2})\sum_i \frac{\varepsilon_i}{b_i} + \frac{1}{4}(\sum_i \frac{\varepsilon_i}{b_i})^2)} (t^{\varepsilon'} + t^{-\varepsilon'}).$$

Following [13], fix ε_2 and ε_3 and split into two cases based on the value of ε_1. If $\varepsilon_1 = -1$, observe that $b_1b_2b_3(1+\sum_i \frac{\varepsilon_i}{b_i}) = \alpha_k$ for some $k \in \{1,2,3,4\}$.

The corresponding summation over n for this triple of ε_i's is

$$-\varepsilon_2\varepsilon_3 \sum_{n\geq 0} q^{pn^2+\alpha_k n+\frac{\alpha_k^2}{4p}} (t^{\frac{\varepsilon_2+\varepsilon_3+2n}{2}} + t^{\frac{-(\varepsilon_2+\varepsilon_3+2n)}{2}}). \tag{6}$$

On the other hand, when $\varepsilon_1 = 1$, we can replace n with $n-1$ in the corresponding sum to get

$$\varepsilon_2\varepsilon_3 \sum_{n\geq 1} q^{pn^2-\alpha_j n+\frac{\alpha_j^2}{4p}} (t^{\frac{\varepsilon_2+\varepsilon_3+2n}{2}} + t^{\frac{-(\varepsilon_2+\varepsilon_3+2n)}{2}}), \tag{7}$$

where for each k the corresponding j is given by

k	1	2	3	4
j	4	3	2	1.

Remark 3.3. *In [13], it is incorrectly claimed that $j = k$ for each k. This fact does not change the outcome of their calculations, but it does affect ours.*

Summing over all four possible values of $(\varepsilon_2, \varepsilon_3)$ gives eight sums, each of which has exponent on q of the form $pn^2 \pm \alpha_k n + \frac{n^2}{4p}$ as in (6) and (7). The sums involving $+\alpha_k n$ begin at $n = 0$ and the sums involving $-\alpha_k n$ begin at $n = 1$. The four values of $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ for which $\varepsilon_2 = -\varepsilon_3$ contribute

$$\begin{aligned}
&\sum_{n\geq 0} q^{pn^2+\alpha_2 n+\frac{\alpha_2^2}{4p}} (t^n + t^{-n}) - \sum_{n\geq 1} q^{pn^2-\alpha_2 n+\frac{\alpha_2^2}{4p}} (t^n + t^{-n});\\
&\sum_{n\geq 0} q^{pn^2+\alpha_3 n+\frac{\alpha_3^2}{4p}} (t^n + t^{-n}) - \sum_{n\geq 1} q^{pn^2-\alpha_3 n+\frac{\alpha_3^2}{4p}} (t^n + t^{-n}),
\end{aligned} \tag{8}$$

whereas when $\varepsilon_2 = \varepsilon_3$ we have

$$\begin{aligned}
&-\sum_{n\geq 0} q^{pn^2+\alpha_4 n+\frac{\alpha_4^2}{4p}} (t^{n+1} + t^{-(n+1)}) + \sum_{n\geq 1} q^{pn^2-\alpha_4 n+\frac{\alpha_4^2}{4p}} (t^{n-1} + t^{-(n-1)});\\
&-\sum_{n\geq 0} q^{pn^2+\alpha_1 n+\frac{\alpha_1^2}{4p}} (t^{n-1} + t^{-(n-1)}) + \sum_{n\geq 1} q^{pn^2-\alpha_1 n+\frac{\alpha_1^2}{4p}} (t^{n+1} + t^{-(n+1)}).
\end{aligned} \tag{9}$$

For $t = 1$ and $\alpha_k \geq 0$, each of the above collapse to the false theta functions $\tilde{\Psi}_p^{(\alpha_k)}$ into which $\widehat{Z}$ is decomposed in [13]. The only case in which $\alpha_k < 0$ for some k is $\Sigma(2, 3, 5)$, for which $\alpha_1 = -1$. We momentarily postpone this case and take $(b_1, b_2, b_3) \neq (2, 3, 5)$. Focusing on the case given by (8), we write $pn^2 \pm n\alpha_3 + \frac{a_3^2}{4p} = p(n \pm \frac{\alpha_3}{2p})^2$ and perform the changes of variables $m = 2pn \pm \alpha_3$. This gives

$$\sum_{\substack{m\geq 0\\ m\equiv \alpha_3\ (2p)}} q^{\frac{m^2}{4p}}(t^{\frac{m-\alpha_3}{2p}} + t^{-\frac{m-\alpha_3}{2p}}) - \sum_{\substack{m\geq 0\\ m\equiv -\alpha_3\ (2p)}} q^{\frac{m^2}{4p}}(t^{\frac{m+\alpha_3}{2p}} + t^{-\frac{m+\alpha_3}{2p}}).$$

The calculation is the same when α_3 is replaced with α_2. When $\varepsilon_1 = \varepsilon_3 = 1$, we get the sums

$$-\sum_{\substack{m\geq 0\\ m\equiv \alpha_4\ (2p)}} q^{\frac{m^2}{4p}}(t^{\frac{m-\alpha_4}{2p}+1}+t^{-(\frac{m-\alpha_4}{2p}+1)})+\sum_{\substack{m\geq 0\\ m\equiv -\alpha_4\ (2p)}} q^{\frac{m^2}{4p}}(t^{\frac{m+\alpha_4}{2p}-1}+t^{-(\frac{m+\alpha_4}{2p}-1)}),$$

and when $\varepsilon_2 = \varepsilon_3 = -1$ we get

$$-\sum_{\substack{m\geq 0\\ m\equiv \alpha_1\ (2p)}} q^{\frac{m^2}{4p}}(t^{\frac{m-\alpha_1}{2p}-1}+t^{-(\frac{m-\alpha_1}{2p}-1)})+\sum_{\substack{m\geq 0\\ m\equiv -\alpha_1\ (2p)}} q^{\frac{m^2}{4p}}(t^{\frac{m+\alpha_1}{2p}+1}+t^{-(\frac{m+\alpha_1}{2p}+1)}).$$

If $(b_1, b_2, b_3) \neq (2, 3, 5)$ we are done. We conclude with the special case of the Poincaré homology sphere. The argument is the same up through the calculation of (9). In this case, we have that

$$\begin{aligned}
&-\sum_{n\geq 1} q^{30n^2-n+\frac{1}{120}}(t^{n-1} + t^{-(n-1)}) + \sum_{n\geq 0} q^{30n^2+n+\frac{1}{120}}(t^{n+1} + t^{-(n+1)}) \\
&= -\sum_{\substack{m\geq 0\\ m\equiv -1\ (60)}} q^{\frac{m^2}{120}}(t^{\frac{m-59}{60}} + t^{-(\frac{m-59}{60})}) + \sum_{\substack{m\geq 0\\ m\equiv 1\ (60)}} q^{\frac{m^2}{120}}(t^{\frac{m+59}{60}} + t^{-(\frac{m+59}{60})}),
\end{aligned}$$

and the bounds on the sums on the left hand side do not agree with those in (9). The solution is to subtract $2q^{\frac{1}{120}}(t+t^{-1})$ from (9), as they only disagree in the sign of their constant term. □

Remark 3.4. *In the course of the above proof, it becomes apparent that $\hat{\hat{Z}}_\Sigma(t, q)$ can be written terms of false theta functions of the form*

$$\sum_{n\in\mathbb{Z}} sgn(n) q^{p\left(n+\frac{\alpha}{2p}\right)} t^{2n+\alpha},$$

for $\alpha \in \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. This observation, along with Theorem 1.2, motivate the question of whether $\hat{\hat{Z}}_\Sigma(t, q)$ is a quantum Jacobi form; see, e.g. [2].

3.1. Example calculations

Below are calculations of $\hat{\hat{Z}}_\Sigma(\zeta, q)$ for various specializations of ζ for the Brieskorn spheres $\Sigma(2,3,5)$ and $\Sigma(2,7,15)$.

Table 1: Examples of $\hat{\hat{Z}}_\Sigma(\zeta, q)$ for $\Sigma(2,3,5)$.

ζ	$\hat{\hat{Z}}_\Sigma(\zeta, q)$
1	$2q^{-3/2} - q^{-3/2}(1 + q + q^3 + q^7 - q^8 - q^{14} - q^{20} - q^{29} + q^{31} + q^{42} + q^{52} + \dots)$
-1	$-2q^{-3/2} - q^{-3/2}(-1 + q + q^3 + q^7 + q^8 + q^{14} + q^{20} - q^{29} + q^{31} - q^{42} + \dots)$
$e^{\frac{2\pi i}{3}}$	$-q^{-3/2} - q^{-3/2}(-\frac{1}{2} + q + q^3 + q^7 + \frac{1}{2}q^8 + \frac{1}{2}q^{14} + \frac{1}{2}q^{20} - q^{29} - \frac{1}{2}q^{31} + \dots)$
i	$-q^{-3/2}(q + q^3 + q^7 - q^{29} - q^{31} + q^{69} + q^{85} + q^{99} - q^{143} - q^{161} - q^{185} + \dots)$

Table 2: Examples of $\hat{\hat{Z}}_\Sigma(\zeta, q)$ for $\Sigma(2,7,15)$.

ζ	$\hat{\hat{Z}}_\Sigma(\zeta, q)$
1	$-q^{5/2}(-q^4 + q^9 + q^{17} - q^{26} + q^{87} - q^{106} - q^{130} + q^{153} - q^{275} + q^{308} + \dots)$
-1	$-q^{5/2}(q^4 + q^9 + q^{17} + q^{26} + q^{87} + q^{106} + q^{130} + q^{153} - q^{275} - q^{308} + \dots)$
$e^{\frac{2\pi i}{3}}$	$-q^{5/2}(\frac{1}{2}q^4 + q^9 + q^{17} + \frac{1}{2}q^{26} + \frac{1}{2}q^{30} - \frac{1}{2}q^{153} - q^{275} - \frac{1}{2}q^{308} - \frac{1}{2}q^{348} + \dots)$
i	$-q^{5/2}(q^9 + q^{17} + q^{87} - q^{153} - q^{275} + q^{385} + q^{615} + q^{671} - q^{1027} - q^{1099} + \dots)$

In the above examples, we factor out a rational power of q so that all other powers are integral. This can be done in general, and is explicitly realized for Brieskorn spheres in the following lemma:

Lemma 3.2. *Define α_k for $1 \leq k \leq 4$ as on page 8 preceding Theorem 3.1. Then $\alpha_1^2 \equiv \alpha_2^2 \equiv \alpha_3^2 \equiv \alpha_4^2 \pmod{4p}$.*

We will denote this common congruence class modulo $4p$ by w for the remainder of the paper.

4. Radial limits at roots of unity

In this section, we analyze the arithmetic properties of the coefficients $\varphi(n;t)$ which will ultimately allow for the calculation of radial limits at roots of unity in terms of particular L-series. We first check that the coefficients of $\varphi(n;t)$ have the necessary properties for our method of calculation.

Lemma 4.1. *If ζ is a jth root of unity, then $\varphi(n;\zeta)$ is $2pj$-periodic and has mean value zero.*

In order to calculate these radial limits, we will make use of a general result that allows for the computation of full asymptotic expansions of our series as q radially approaches roots of unity. We are interested in asymptotic expansions of functions $f(t)\colon \mathbb{R}_{>0} \to \mathbb{C}$ as $t \to 0^+$. In particular, we will be interested in expansions of the form

$$f(t) \sim \sum_{n=0}^{\infty} a_n t^n, \qquad (t \to 0^+),$$

meaning that the difference $f(t) - \sum_{n=0}^{N-1} a_n t^n = O(t^N)$ as $t \to 0^+$. Note that $\sum_{n=0}^{\infty} a_n t^n$ need not be convergent for any nonzero t for this definition to be sensible, but in the following case there is indeed a positive radius of convergence. To realize these radial limits, we make use of the following general proposition.

Proposition 4.2. *Let $C\colon \mathbb{Z} \to \mathbb{C}$ be a periodic function with mean value zero. Then the associated L-series $L(s,C) := \sum_{n\geq 1} \frac{C(n)}{n^s}$, $\Re(s) > 1$, extends holomorphically to all of $\mathbb{C}$ and the function $\sum_{n\geq 1} C(n)e^{-n^2 t}$, $t > 0$, has the asymptotic expansion*

$$\sum_{n\geq 1} C(n)e^{-n^2 t} \sim \sum_{r\geq 0} L(-2r, C)\frac{(-t)^r}{r!}$$

as $t \to 0^+$. Then numbers $L(-r,C)$ are given explicitly by

$$L(-r,C) = -\frac{M^r}{r+1}\sum_{n=1}^{M} C(n)B_{r+1}\left(\frac{n}{M}\right), \quad (r = 0, 1, \dots),$$

where $B_k(x)$ is the kth Bernoulli polynomial and M is any period of the function $C(n)$.

For details, see e.g. [16] p. 98. Note the slight abuse of notation where C may refer to either an arithmetic function or the extra term C which appears in (1) in the case were the 3-manifold in question is the Poincaré homology sphere. We will specify when unclear from context.

4.1. Proof of Theorem 1.1

Let ξ be a root of unity and set $C(n) := \varphi(n;\zeta)\xi^{\frac{n^2}{4p}}$. If K is a period of ξ, then $C(n)$ is $2pjK$-periodic and has mean value zero since $C(2pjK-n) = -C(n)$. Let

$$A_\zeta(q) := \sum_{n\geq 0} \varphi(n;\zeta) q^{\frac{n^2}{4p}}, \tag{10}$$

and observe that

$$A_\zeta(\xi e^{-t}) = \sum_{n=1}^{\infty} C(n) e^{-n^2(t/4p)}.$$

By the previous proposition, the above has asymptotic expansion

$$\sum_{r=0}^{\infty} L(-2r, C)\frac{(-t/4p)^r}{r!}$$

as $t \to 0^+$ and limiting value

$$A_\zeta(\xi) := \lim_{t\to 0^+} A_\zeta(\xi e^{-t}) = L(0, C).$$

The analytic continuation of this L-series to $s = 0$ is given by the sum

$$-\sum_{n=1}^{2pjK} \left(\frac{n}{2pjK} - \frac{1}{2}\right) \varphi(n;\zeta)\xi^{\frac{n^2}{4p}}.$$

Evaluating both q^{Δ} the extra term $C = q^{\frac{1}{120}}(t + t^{-1})$ (which appears only when Σ is the Poincaré homology sphere) at (ζ, ξ) gives the desired formula. $\square$

5. Modular and quantum modular forms

We begin with a brief introduction to the theory of modular forms of half-integral weight. For a more thorough discussion, see [20, 22]. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ act on $\mathbb{H}$ by the linear fractional transformation

$$\gamma\tau := \frac{a\tau + b}{c\tau + d}.$$

We are interested in the action of particular congruence subgroups Γ of $\mathrm{SL}_2(\mathbb{Z})$. Define

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \ a \equiv d \equiv 1 \pmod{N},\ c \equiv 0 \pmod{N} \right\};$$
$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \ a \equiv d \equiv 1 \pmod{N},\ b \equiv c \equiv 0 \pmod{N} \right\}.$$

The above are congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ of level N. Note the containment $\Gamma(N) \subseteq \Gamma_1(N)$. The equivalence classes in $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{i\infty\}$ under the action of a congruence subgroup Γ are called the *cusps* of Γ.

To state the appropriate transformation law for half-integral weight modular forms, we need the following definitions. For odd d, define

$$\varepsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}; \\ i & \text{if } d \equiv 3 \pmod{4}, \end{cases}$$

and let $\left(\frac{\cdot}{\cdot}\right)$ denote the Jacobi symbol. Throughout, we let $\sqrt{z}$ be the branch of the square root with argument in $(-\pi/2, \pi/2]$. For functions $f \colon \mathbb{H} \to \mathbb{C}$, the Petersson slash operator of weight $k \in \frac{1}{2}\mathbb{Z}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is defined by

$$f|_k\gamma(\tau) := \begin{cases} (c\tau + d)^{-k} f(\gamma\tau) & \text{if } k \in \mathbb{Z}; \\ \varepsilon_d^{2k} \left(\frac{c}{d}\right) (c\tau + d)^{-k} f(\gamma\tau) & \text{if } k \in \frac{1}{2} + \mathbb{Z}, \end{cases}$$

where additionally one must require that γ is contained in a congruence subgroup of level 4 when k is not an integer. We can now state the following:

Definition 5.1. Let Γ be a congruence subgroup of level N such that $4 \mid N$. We say that a holomorphic function $f \colon \mathbb{H} \to \mathbb{C}$ is a modular form (resp. cusp form) of weight $k \in \frac{1}{2}\mathbb{Z}$ with multiplier χ for Γ if:

1. for all $\gamma \in \Gamma$, the function f satisfies $f - \overline{\chi}(\gamma) f|_k\gamma = 0$, and
2. for all $\gamma \in SL_2(\mathbb{Z})$, $(c\tau + d)^{-k} f(\gamma\tau)$ is bounded (resp. vanishes) as $\tau \to i\infty$.

While our work makes contact with modular forms as described above, the modular objects of primary interest will be quantum modular forms. This term, coined by Zagier in 2010, was inspired in part by the examples arising from quantum field theory and quantum invariants of 3-manifolds such as the WRT invariants [24].

Fix a congruence subgroup Γ of level N such that $4 \mid N$, and suppose $\mathcal{Q} = \mathbb{Q}\backslash S$ where S is discrete and $\mathcal{Q}$ is closed under the action of Γ. We define a *quantum modular form* of weight k with multiplier χ for Γ to be a function $f\colon \mathcal{Q} \to \mathbb{C}$ such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, the functions $h_\gamma\colon \mathcal{Q}\backslash\{\gamma^{-1}(i\infty)\}) \to \mathbb{C}$,

$$h_\gamma(x) := f(x) - \overline{\chi}(\gamma) f|_k\gamma(x) \tag{11}$$

extends to some "nice" function on $\mathbb{R}$ (away from possibly a bad set of points). The set $\mathcal{Q}$ is called the quantum set of f.

Remark 5.1. *This definition is intentionally vague, as it was built to fit varied examples naturally arising from disparate areas of study. A variety of specializations of this definition are considered to fall under the "quantum modular" umbrella; however, for the purposes of this paper, it suffices to replace "nice" with "real-analytic."*

Ultimately, we need the following lemma in order to renormalize the powers of q that appear in Section 6. Using the definitions, one can verify the following:

Lemma 5.2. *If $\psi(\tau)$ is a quantum modular form of weight $1/2$ for $\Gamma(4pj)$ with multiplier χ, then $\psi(j\tau)$ is a quantum modular form of weight $1/2$ for $\Gamma(4pj^2)$ with the same multiplier.*

5.1. Eichler integrals

The utilization of Eichler integrals to construct quantum modular forms has its roots in the work of Lawrence and Zagier previously discussed [16]. Many authors have since extended and generalized this procedure to systematically construct families of quantum modular forms: see e.g. [10, 7, 11]. Here we sketch the procedure by which the authors of [7] construct quantum modular forms, modified to fit our context. For references that reflect these arguments with our particular congruence subgroups and multiplier systems, see e.g. [6, 11].

Suppose a function $F(\tau)$ for $\tau \in \mathbb{H}$ may be written as

$$F(\tau) = \sum_{n\geq 0} a(n) q^{\frac{n^2}{4pj}}, \qquad (q = e^{2\pi i\tau}),$$

for some integers $p,\ j$. Further suppose that

$$f(\tau) := \sum_{n\geq 0} n a(n) q^{\frac{n^2}{4pj}}$$

is a cusp form of weight 3/2 for $\Gamma_1(N)$. We consider the *non-holomorphic Eichler integral* of f, given by

$$F^*(\tau) := \int_{\overline{\tau}}^{i\infty} \frac{f(\omega)}{\sqrt{-i(\omega-\tau)}}\, d\omega, \qquad (\tau \in \mathbb{H}^-).$$

Bringmann and Rolen show that, after suitable renormalization, the functions $F(\tau)$ and $F^*(\tau)$ "agree to infinite order" at any $x \in \mathbb{Q}$. That is, for any x there exists a sequence $\beta(0), \beta(1), \ldots$ such that as $t \to 0^+$,

$$F\left(x + \frac{it}{2\pi}\right) \sim \sum_{r\geq 0} \beta(r) \frac{(-t)^r}{r!} \quad \text{and} \quad F^*\left(x - \frac{it}{2\pi}\right) \sim \sum_{r\geq 0} \beta(r) \frac{t^r}{r!}.$$

This is accomplished by first integrating F^* term-by-term to obtain a series expansion for $F^*(\tau)$ involving Γ-factors. Then using Proposition 4.2 and more general tools for studying the Mellin transform of error functions, they obtain asymptotic expansions of both of these functions which agree in the above sense.

The function $F^*(\tau)$ admits an explicit obstruction to modularity from its definition; for $\tau \in \mathbb{H}^-$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$, we have

$$F^*(\tau) - \left(\tfrac{-4}{d}\right) F^*|_k \gamma\,(\tau) = r_\gamma(\tau),$$

where

$$r_\gamma(\tau) := \int_{\gamma^{-1}(i\infty)}^{i\infty} \frac{f(\omega)}{\sqrt{-i(\omega-\tau)}}\, d\omega.$$

which extends to a C^∞ function on $\partial\mathbb{H}^- = \mathbb{R}$ which is real-analytic on $\mathbb{R}\backslash\{\gamma^{-1}(i\infty)\}$ and gives h_γ for the resulting quantum modular form.

6. Proofs of main theorems

Below we prove two results regarding the quantum modularity of $\widehat{\widehat{Z}}_\Sigma(t,q)$. The main novelty of these results is our uniform treatment of these series as two-parameter families in t and Σ. This is only possible once we have explicitly computed these series by Theorem 3.1.

6.1. Proof of Theorem 1.2

In light of the work summarized in the preceding section, it suffices to show that

$$\sum_{n\geq 0} n\varphi(n;\zeta)q^{\frac{n^2}{4pj}} \tag{12}$$

is a cusp form for $\Gamma(4pj)$. Then the results of Section 5.1 used in conjunction with Lemma 5.2 will imply

$$A_\zeta(\tau) := \sum_{n\geq 0} \varphi(n;\zeta)q^{\frac{n^2}{4p}}$$

is a quantum modular form. We further have that $A_\zeta(\tau)$ is a "strong" quantum modular form in the sense of [24]. We begin with an elementary lemma which will be useful for simplifying our expressions later.

Lemma 6.1. *Let $0 \leq n < 2pj$ be such that $n \equiv \pm\alpha_k \pmod{2p}$ for some k. Then we have $n^2 \equiv w + 4pi \pmod{4pj}$ for some $0 \leq i < j$, where w is the common congruence class modulo $4p$ of the α_k^2's coming from Lemma 3.2.*

We are now ready to analyze (12). Since $\varphi(n;\zeta)$ is $2pj$-periodic, we have

$$\sum_{n\geq 0} n\varphi(n;\zeta)q^{\frac{n^2}{4pj}} = \sum_{0\leq\alpha<2pj} \varphi(\alpha;\zeta)\sum_{n\geq 0}(2pjn+\alpha)q^{\frac{(2pjn+\alpha)^2}{4pj}}.$$

Every α for which $\varphi(\alpha;\zeta)$ is nonzero satisfies $\alpha \equiv \pm\alpha_k \pmod{2p}$ for some k. Thus, we can write this sum as

$$\sum_{\substack{0\leq\alpha<2pj\\ \alpha\equiv\alpha_k(2p)}} \varphi(\alpha,\zeta)\sum_{n\geq 0}(2pjn+\alpha)q^{\frac{(2pjn+\alpha)^2}{4pj}}$$
$$+\sum_{\substack{0<\alpha\leq 2pj\\ \alpha\equiv\alpha_k(2p)}} \varphi(2pj-\alpha,\zeta)\sum_{n\geq 0}(2pjn+(2pj-\alpha))q^{\frac{(2pjn+(2pj-\alpha))^2}{4pj}}.$$

Using the fact that $\varphi(n;\zeta)$ is odd and $2pj$-periodic, the second set of sums can be rewritten as

$$\sum_{\substack{0<\alpha\leq 2pj\\ \alpha\equiv\alpha_k(2p)}} -\varphi(\alpha,\zeta)\sum_{n\geq 0}(2pj(n+1)-\alpha)q^{\frac{(2pj(n+1)-\alpha)^2}{4pj}}.$$

Reindexing by $n+1 \mapsto -n$ and combining with the first set of sums gives

$$\sum_{n\geq 0} n\varphi(n;\zeta) q^{\frac{n^2}{4pj}} = \sum_{\substack{0\leq \alpha<2pj\\ \alpha\equiv\alpha_k(2p)}} \varphi(\alpha;\zeta) \sum_{\substack{n\in\mathbb{Z}\\ n\equiv\alpha(2pj)}} nq^{\frac{n^2}{4pj}}. \tag{13}$$

The inner sum of the above equation is a theta function which is modular of weight 3/2. More precisely, define

$$\Theta(\tau;k,M) := \sum_{\substack{n\in\mathbb{Z}\\ n\equiv k(M)}} nq^{\frac{n^2}{2M}}.$$

By Proposition 2.1 of [22], we have that for $\gamma\in\Gamma_1(2M)$ that

$$\Theta(\gamma\tau;k,M) = e^{\frac{\pi iabk^2}{M}} \varepsilon_d^{-3} \left(\frac{2Mc}{d}\right) (c\tau+d)^{3/2} \Theta(\tau;ak,M),$$

and since $k\equiv ak \pmod{M}$, we have

$$\Theta(z;ak,M) = \Theta(z;k,M).$$

By Lemma 6.1, every n for which the coefficient of $q^{\frac{n^2}{4pj}}$ in (13) is nonzero satisfies $n^2\equiv w+4pi \pmod{4pj}$ for some $0\leq i<j$. Then

$$e^{\frac{\pi iabn^2}{2pj}} = e^{\frac{\pi iab(w+4pi)}{2pj}}$$

for some $0\leq i<j$. Then one may group the n's based on the corresponding i to get j functions $f_i(\tau)$ which for $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_1(4pj)$ satisfy

$$f_i(\gamma\tau) = e^{\frac{\pi iab(w+4pi)}{2pj}} \varepsilon_d^{-3} \left(\frac{4pjc}{d}\right) (c\tau+d)^{3/2} f_i(\tau).$$

Note the dependence of this transformation law on w. If one restricts to $\gamma\in\Gamma(4pj)\subset\Gamma_1(4pj)$, the multipliers for each f_i become identical. Thus the sum of the f_i's transform together as a cusp form on $\Gamma(4pj)$. □

6.2. Proof of Theorem 1.3

As in the study of the Jacobi Triple Product formula, one is often able to generate a modular object of dual-weight by differentiating with respect to one variable; see e.g. [9]. Following this approach, we find a second infinite family of quantum invariants by differentiating $\widehat{\widehat{Z}}_{\Sigma}(t,q)$, summand by summand, with respect to t. Our contribution to this principle is Theorem 1.3. Here we offer of proof of this result.

Fix ζ a jth root of unity. Consider the series

$$A'_{\zeta}(\tau) := \sum_{n\geq 0} \varphi'(n;\zeta) q^{\frac{n^2}{4p}},$$

where $\varphi'(n;\zeta)$ is the derivative of $\varphi(n;t)$ evaluated at $t=\zeta$. By Theorem 3.1, this is

$$\varphi'(n;\zeta) := \begin{cases} \frac{n\mp(\alpha_1+2p)}{4p}\left(\zeta^{\frac{\mp n+(\alpha_1+2p)}{2p}} - \zeta^{\frac{\pm n-(\alpha_1+2p)}{2p}}\right) & n\equiv\pm\alpha_1 \pmod{2p};\\ -\frac{n\mp\alpha_k}{4p}\left(\zeta^{\frac{\mp n+\alpha_k}{2p}} - \zeta^{\frac{\pm n-\alpha_k}{2p}}\right) & n\equiv\pm\alpha_k \pmod{2p}, k=2,3;\\ \frac{n\mp(\alpha_4-2p)}{4p}\left(\zeta^{\frac{\mp n+(\alpha_4-2p)}{2p}} - \zeta^{\frac{\pm n-(\alpha_4-2p)}{2p}}\right) & n\equiv\pm\alpha_4 \pmod{2p};\\ 0 & \textit{otherwise}. \end{cases}$$

Note that we may write $A'_{\zeta}(\tau)$ as

$$\sum_{n\geq 0} n\psi(n;\zeta) q^{\frac{n^2}{4p}} + \sum_{n\geq 0} \chi(n;\zeta) q^{\frac{n^2}{4p}},$$

where

$$\psi(n;\zeta) := \begin{cases} \frac{1}{4p}\left(\zeta^{\frac{\mp n+(\alpha_1+2p)}{2p}} - \zeta^{\frac{\pm n-(\alpha_1+2p)}{2p}}\right) & n\equiv\pm\alpha_1 \pmod{2p};\\ -\frac{1}{4p}\left(\zeta^{\frac{\mp n+\alpha_k}{2p}} - \zeta^{\frac{\pm n-\alpha_k}{2p}}\right) & n\equiv\pm\alpha_k \pmod{2p},\ k=2,3;\\ \frac{1}{4p}\left(\zeta^{\frac{\mp n+(\alpha_4-2p)}{2p}} - \zeta^{\frac{\pm n-(\alpha_4-2p)}{2p}}\right) & n\equiv\pm\alpha_4 \pmod{2p};\\ 0 & \textit{otherwise}; \end{cases}$$

$$\chi(n;\zeta) := \begin{cases} \mp\frac{(\alpha_1+2p)}{4p}\left(\zeta^{\frac{\mp n+(\alpha_1+2p)}{2p}} - \zeta^{\frac{\pm n-(\alpha_1+2p)}{2p}}\right) & n\equiv\pm\alpha_1 \pmod{2p};\\ \pm\frac{\alpha_k}{4p}\left(\zeta^{\frac{\mp n+\alpha_k}{2p}} - \zeta^{\frac{\pm n-\alpha_k}{2p}}\right) & n\equiv\pm\alpha_k \pmod{2p} k=2,3;\\ \mp\frac{(\alpha_4-2p)}{4p}\left(\zeta^{\frac{\mp n+(\alpha_4-2p)}{2p}} - \zeta^{\frac{\pm n-(\alpha_4-2p)}{2p}}\right) & n\equiv\pm\alpha_4 \pmod{2p};\\ 0 & \textit{otherwise}. \end{cases}$$

Then $\psi(n;\zeta)$ is even and $2pj$-periodic and $\chi(n;\zeta)$ is odd and $2pj$-periodic. Following the same style of argument as Theorem 1.2, one concludes that $\sum_{n\geq 0}\chi(n;\zeta)q^{\frac{n^2}{4p}}$ is a quantum modular form of weight $1/2$ on $\Gamma(4pj^2)$.

To analyze $\sum_{n\geq 0} n\psi(n;\zeta)q^{\frac{n^2}{4p}}$, note that the series $\sum_{n\geq 0} \psi(n;\zeta)q^{\frac{n^2}{4p}}$ is modular but may not be a cusp form. This requires us to appeal to a more general result of Goswami and Osburn (Theorem 1.1 of [11]) which gives a careful treatment of this more general case. Their result tells us that

$$\sum_{n\geq 0} n\psi(n;\zeta)q^{\frac{n^2}{4pj}}$$

is a quantum modular form on Q_{2pj} with respect to $\Gamma_1(4pj)$, where

$$Q_{2pj} := \{x \in \mathbb{Q} \ : \ x \text{ is } \Gamma_1(4pj)\text{-equivalent to } i\infty\}.$$

Note that one must still utilize Lemmas 5.2 and 6.1 in order to contend with the supports of these series. This ultimately allows us to conclude that $\sum_{n\geq 0} n\psi(n;\zeta)q^{\frac{n^2}{4p}}$ is a quantum modular form of weight $3/2$ as desired. □

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