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Stability of N Solitary Waves for the Generalized BBM Equations

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Abstract. We consider the generalized BBM (Benjamin-Bona-Mahony) equations:

$$(1 - \partial_x^2)u_t + (u + u^p)_x = 0, \tag{0.1}$$

for $p \geqslant 2$ integer, and the family of solitary wave solutions $\varphi_c(x - x_0 - ct)$ of this equation. For any p, there exists a necessary and sufficient condition on the speed $c > 1$ so that a solitary wave solution is nonlinearly stable ([**21**], [**20**]). Following the approach of [**14**] for the generalized KdV equations, we prove that the sum of N sufficiently decoupled stable solitary wave solutions is also stable in the energy space. The proof combines arguments of [**21**] to prove the stability of a single solitary wave, and monotonicity results of [**6**]. We also obtain asymptotic stability results following [**6**]. Using the same tools, we then prove the existence and uniqueness of a solution behaving asymptotically in large time as the sum of N given solitary waves, following the method of [**11**].

Contents

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1. Introduction

We consider in this paper the generalized BBM equations (gBBM henceforth)

$$(1.1)\qquad \begin{cases} (1-\partial_x^2)u_t + (u+u^p)_x = 0, \ (t,x)\in\mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x), \ x\in\mathbb{R}, \end{cases}$$

where $p \in \mathbb{N}$, $p \geqslant 2$, as introduced by Peregrine [**19**] and Benjamin, Bona and Mahony [**2**]. The Cauchy problem associated to (1.1) is globally well posed in $H^1(\mathbb{R})$ (see [**2**]), and H^1 solutions are such that

$$(1.2)\qquad E(u(t)) = \frac{1}{2}\int u^2(t,x)dx + \frac{1}{p+1}\int u^{p+1}(t,x)dx = E(u_0),$$

$$(1.3)\qquad m(u(t)) = \frac{1}{2}\int \left(u^2(t,x)+u_x^2(t,x)\right)dx = m(u_0).$$

The quantity $\int u(t)$ is also formally conserved. However, there is no value of p such that the gBBM equation admits more conserved quantities. In particular, the gBBM equation is not completely integrable, for any value of p. As a consequence no inverse-scattering theory can be developed for this equation, see [**15**] and [**18**]. This situation is in contrast with the generalized Korteweg–de Vries equations (gKdV equations):

$$(1.4)\qquad u_t + (u_{xx}+u^p)_x = 0,$$

which is completely integrable for both $p=2$ and 3 (but not for other values of p).

As the gKdV, the gBBM equation has a two parameter family of solitary wave solutions: for any $c>1$ and $x_0\in\mathbb{R}$, $u(t,x) = \varphi_c(x-ct-x_0)$ is a traveling wave solution of (1.1) if φ_c is solution of

$$(1.5)\qquad -c\partial_x^2\varphi_c + (c-1)\varphi_c - \varphi_c^p = 0.$$

The unique even function going to zero at infinity which is solution of (1.5) is given by

$$\varphi_c(x) = (c-1)^{\frac{1}{p-1}} Q\left(\sqrt{\frac{c-1}{c}}x\right),$$

where

$$Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)}\right)^{\frac{1}{p-1}} \quad \text{satisfies} \quad Q'' + Q^p = Q.$$

The H^1 nonlinear stability of a solitary wave solution $\varphi_c(x-ct-x_0)$ of (1.1) was studied by Weinstein [**21**] and Souganidis and Strauss [**20**]. We say that φ_c is stable if:

For any $\gamma>0$, there exists $\delta>0$ such that $\|u_0-\varphi_c\|_{H^1}<\delta$ implies that there exists $r(t)$ such that for all $t\in\mathbb{R}$, $\|u(t,.-r(t))-\varphi_c\|_{H^1}\leqslant\gamma$.

From [**21**] and [**20**], it turns out that $\varphi_c(x-ct-x_0)$ is stable if the following condition is satisfied:

$$(1.6)\qquad \frac{d}{dc}m(\varphi_c) > 0,$$

and is unstable if $\frac{d}{dc}m(\varphi_c)<0$. Indeed, condition (1.6) was found to be a natural condition under which stability is true, not only for the Schrödinger, gKdV and gBBM equation ([**21**]) but also for other nonlinear dispersive equations. In the case of the gBBM equation, we have by straightforward calculations

$$m(\varphi_c) = \frac{(c-1)^{\frac{5-p}{2(p-1)}}}{\sqrt{c}}\left[c\left(\frac{p+1}{p-1}\right)-\frac{1}{2}\right]\int Q_x^2, \tag{1.7}$$

and thus it is easily checked that if we define, for $p \geqslant 6$,

$$c^\star(p) = \frac{(p-1)(2+\sqrt{2(p+3)})}{4(p+1)}, \tag{1.8}$$

and $c^\star(p) = 1$ for $p = 2, 3, 4$ or 5, then condition (1.6) is satisfied if and only if $c > c^\star(p)$. Therefore:

- If $p = 2, 3, 4, 5$ and $c > 1$, or $p \geqslant 6$ and $c > c^\star(p)$, then $\varphi_c(x - ct - x_0)$ is stable, see Weinstein [**21**].

- If $p \geqslant 6$ and $1 < c < c^\star(p)$ then $\varphi_c(x - ct - x_0)$ is unstable, see Souganidis and Strauss [**20**].

Numerical studies on the asymptotic behavior of solutions of (1.1) were performed by Bona and al. [**4**].

In this paper, we consider N solitary waves $\varphi_{c_j}(x - c_j^0 t - x_j^0)$ of (1.1) which are stable. We prove that their sum is also stable, in an appropriate sense, provided that the solitary waves are sufficiently decoupled. Our main result is the following.

THEOREM 1.1. *Let $p \geqslant 2$ be an integer and let $u_0 \in H^1(\mathbb{R})$. Fix N velocities: $1 \leqslant c^\star(p) < c_1^0 < \ldots < c_N^0$. There exist $\gamma_0, A_0, L_0, \alpha_0 > 0$ such that if for some $L > L_0$, $\alpha < \alpha_0$, and $x_1^0 < \ldots < x_N^0$,*

$$\left\| u_0 - \sum_{j=1}^{N} \varphi_{c_j^0}(\cdot - x_j^0) \right\|_{H^1} \leqslant \alpha, \quad \text{with } x_j^0 > x_{j-1}^0 + L, \text{ for all } j = 2, \ldots, N, \tag{1.9}$$

then, there exist $x_1(t), \ldots, x_N(t)$ such that the solution $u(t)$ of (1.1) satisfies:

$$\text{for any } t \geqslant 0, \quad \left\| u(t) - \sum_{j=1}^{N} \varphi_{c_j^0}(\cdot - x_j(t)) \right\|_{H^1} \leqslant A_0\left(\alpha + e^{-\gamma_0 L}\right). \tag{1.10}$$

Remark 1. A similar result was proved by Martel, Merle and Tsai [**14**] for the generalized KdV equations. Here, we combine a generalization of the argument of [**14**] with some tools developed by El Dika [**5**]-[**7**] for the gBBM equation. This paper is thus an illustration of the fact that the approach in [**14**] does not depend on specific calculations for the gKdV equation, but is a general method for proving the stability of the sum of N solitary waves of a nonlinear dispersive equation as a consequence of two basic properties:

- a dynamical proof of the stability of solitary waves solutions, as provided in [**21**] for several dispersive equations,

- a property of almost monotonicity of a local version of an invariant quantity, see Lemma 2.1.

We expect that these two properties hold not only for the gKdV and the gBBM equations, but also for several other nonlinear dispersive equations, for example : the fifth-order KdV equation, the Benjamin-Ono equation, and the ILW equation (see [**21**]). Let us give some details for the fifth-order KdV equation:

$$\begin{cases} u_t + (u_{xx} - u_{xxxx} + u^2)_x = 0, \ (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(0, x) = u_0(x), \ x \in \mathbb{R}. \end{cases} \tag{1.11}$$

For this equation, the monotonicity property is easily checked (as for the gKdV equation, see Corollary 1 in [**12**]), and a proof of stability of solitary waves is available (using numerical calculations), see Il'ichev and Semenov [**9**].

Now, we turn to the question of asymptotic stability. Recall that the first result of asymptotic stability of solitary waves for the gBBM equation (for $p = 2, 3$) in the energy space has been proved by the first author ([**5**], [**6**]) and independently by Mizumachi [**17**]. Their work is in the same spirit as asymptotic stability results for the generalized KdV equations by Martel and Merle [**12**] (see also [**13**] for a simplified proof).

A direct corollary of the asymptotic stability result for one solitary wave and the stability of the sums of N solitary waves (Theorem 1.1) is the following result of asymptotic stability of the sums of N solitary waves in the energy space.

THEOREM 1.2. *Let $p = 2$ or 3. There exists a set $E \subset (1, +\infty)$ without accumulation points (E may be empty) for which : given N velocities $1 < c_1^0 < \cdots < c_N^0$, such that for $j \in \{1, \cdots, N\}$, $c_j^0 \in (1, +\infty) \setminus E$, there exist $\gamma_0, A_0, L_0, \alpha_0 > 0$ such that if for some $L > L_0$, $\alpha < \alpha_0$, and $x_1^0 < \ldots < x_N^0$, $u_0 \in H^1(\mathbb{R})$ satisfies*

$$\left\| u_0 - \sum_{j=1}^{N} \varphi_{c_j^0}(\cdot - x_j^0) \right\|_{H^1} \leqslant \alpha, \quad \text{and } x_j^0 > x_{j-1}^0 + L, \text{ for all } j = 2, \ldots, N, \tag{1.12}$$

then, there exist $1 < c_1^{+\infty} < \cdots < c_N^{+\infty}$, $x_1(t), \ldots, x_N(t)$ such that the solution $u(t)$ of (1.1) satisfies:

$$u(t) - \sum_{j=1}^{N} \varphi_{c_j^{+\infty}}(\cdot - x_j(t)) \to 0 \quad \text{in } H^1(x > \tfrac{1+c_1^0}{2} t) \text{ as } t \to +\infty. \tag{1.13}$$

Remark 2. Note that the asymptotic stability in $H^1(\mathbb{R})$ (1.13) has a local in space sense. As for the gKdV equation, we cannot have in general convergence in $H^1(\mathbb{R})$, since almost all solutions have dispersion for $x < t$, see Remark 5.

Remark 3. The restriction $c_j^0 \notin E$ is probably a technical condition, due to the use of a spectral result by Miller and Weinstein [**16**], as in the asymptotic stability result of [**6**]. In fact, the conclusion of Theorem 1.2 would be true in general provided that for any c_j^0, a linear rigidity condition related to $\varphi_{c_j^0}$ is satisfied (see [**6**], Theorem 6.1 and section 4.1 of this paper).

Finally, we state another result related to N-solitary waves for the gBBM equations. Being given $1 \leqslant c^\star(p) < c_1^0 < \cdots < c_N^0$, and $x_1^0, \cdots, x_N^0 \in \mathbb{R}$, we prove that there exists a unique solution $U(t)$ of (1.1) such that

$$\lim_{t \to +\infty} \left\| U(t) - \sum_{j=1}^{N} \varphi_{c_j^0}(\cdot - x_j^0 - c_j^0 t) \right\|_{H^1} = 0. \tag{1.14}$$

A similar result was proved by the second author for the generalized KdV equations, see [**11**].

THEOREM 1.3. *Let $p \geqslant 2$ be an integer. Let $1 \leqslant c^\star(p) < c_1^0 < \ldots < c_N^0$, and $x_1^0, \ldots, x_N^0 \in \mathbb{R}$. There exists a unique function $U \in C(\mathbb{R}, H^1(\mathbb{R}))$ which is solution* *of (1.1) and satisfies*

$$\Big\| U(t) - \sum_{j=1}^{N} \varphi_{c_j}(. - x_j^0 - c_j^0 t) \Big\|_{H^1} \to 0 \quad \text{as } t \to +\infty. \tag{1.15}$$

Moreover, $U(t)$ is such that, for any $s \geqslant 1$, for any $t \geqslant 0$,

$$\Big\| U(t) - \sum_{j=1}^{N} \varphi_{c_j}(. - x_j^0 - c_j^0 t) \Big\|_{H^s} \leqslant A_s e^{-\gamma t}, \tag{1.16}$$

where $\gamma > 0$, and $A_s > 0$.

Remark 4. The existence of such N-solitary wave solutions is a somewhat surprising phenomenon for a non integrable equation. For the KdV and modified KdV equations (the integrable cases) such N-solitary wave solutions are explicitly known, and describe a perfect interaction between several solitary waves. By Theorem 1.1 and the continuity of the flow of the gBBM equation, the family of solutions $U_{\{c_j^0, x_j^0\}}$ constructed in the above Theorem is stable in $H^1(\mathbb{R})$ for $t \geqslant 0$.

Recall that the first result of stability of N-solitary wave solutions of the KdV equation was proved in $H^N(\mathbb{R})$ by Maddocks and Sachs [**10**].

Remark 5. The uniqueness part implies that the result of asymptotic stability in Theorem 1.2 is in some sense optimal, since convergence in $H^1(\mathbb{R})$, that is (1.13), determines uniquely the solution. Note also that in Theorem 1.3, as in Theorem 1.1, there is no restriction on the c_j^0 nor on p: the proof does not use the spectral result of Miller and Weinstein [**16**].

2. Preliminaries

2.1. Modulation. The aim of this section is to prove that if u is a solution of the gBBM equation which remains close to the manifold of the sum of N solitary waves for $t \in [0, t_0]$, then for the same time interval we can decompose u as the sum of N modulated solitary waves plus a function $\varepsilon(t)$ which remains small in $H^1(\mathbb{R})$:

$$u(t,x) = \sum_{j=1}^{N} \varphi_{c_j(t)}(x - x_j(t)) + \varepsilon(t,x), \tag{2.1}$$

with $\varepsilon(t)$ orthogonal to $(1-\partial_x^2)\varphi_{c_j(t)}(\cdot - x_j(t))$ and $(1-\partial_x^2)\partial_x\varphi_{c_j(t)}(\cdot - x_j(t))$ in L^2, for $t \in [0, t_0]$.

Henceforth we fix an integer $p \geqslant 2$, and N velocities

$$1 \leqslant c^\star(p) < c_1^0 < c_2^0 < \cdots < c_N^0,$$

where $c^\star(p)$ is, as noted in the Introduction, the critical speed for stability. We also fix

$$\sigma_0 = \frac{1}{2}\min\left(2, \sqrt{\frac{c_1^0 - 1}{c_1^0}}, c_1^0 - c^\star(p), c_2^0 - c_1^0, \cdots, c_N^0 - c_{N-1}^0\right) > 0. \tag{2.2}$$

We denote by $\mathcal{U}(\alpha, L)$ the neighborhood of size α of all the sum of N solitary waves of speed c_j^0 such that the distance between their spatial shifts x_j is larger then L, i.e.

$$\mathcal{U}(\alpha, L) = \Big\{ u \in H^1(\mathbb{R});\ \inf_{x_j > x_{j-1}+L} \Big\| u - \sum_{j=1}^N \varphi_{c_j^0}(\cdot - x_j) \Big\|_{H^1} < \alpha \Big\}. \tag{2.3}$$

Proposition 2.1. *There exists $L_1, \alpha_1, K_1 > 0$ such that if for some $L > L_1$, $0 < \alpha < \alpha_1$, $t_0 > 0$,*

$$u(t) \in \mathcal{U}(\alpha, L) \text{ for all } t \in [0, t_0],$$

then there exist unique C^1 functions

$$c_j \ : \ [0, t_0] \to (c^\star(p), +\infty), \quad x_j \ : \ [0, t_0] \to \mathbb{R},$$

such that if we define ε by

$$\varepsilon(t) = u(t) - \sum_{j=1}^N R_j(t), \ \text{ where } R_j(t) = \varphi_{c_j(t)}(\cdot - x_j(t)),$$

then the following properties are satisfied for all $j \in \{1, \cdots, N\}$, for all $t \in [0, t_0]$:

$$\int \varepsilon(t)(1 - \partial_x^2) R_j(t) dx = \int \varepsilon(t)(1 - \partial_x^2)\partial_x R_j(t) dx = 0, \tag{2.4}$$

$$\|\varepsilon(t)\|_{H^1} + \sum_{j=1}^N |c_j(t) - c_j^0| \leqslant K_1 \alpha, \tag{2.5}$$

$$|\dot{c}_j(t)| + |\dot{x}_j(t) - c_j(t)| \leqslant K_1 \left(\int e^{-\sigma_0 |x - x_j(t)|} \varepsilon^2(t) dx \right)^{\frac{1}{2}} + K_1 e^{-\frac{\sigma_0 (L + \sigma_0 t)}{4}}, \tag{2.6}$$

for some constant $K_1 > 0$.

Proof. First, we prove the decomposition result for general function $u \in \mathcal{U}(\alpha, L)$, i.e., with no time dependency. Let $L > 0$, $X^0 = (x_j^0) \in \mathbb{R}^N$ such that $x_j^0 > x_{j-1}^0 + L$, and set $R_{X^0} = \sum_{j=1}^N \varphi_{c_j^0}(\cdot - x_j^0)$. We denote by $B(R_{X^0}, \alpha)$ the ball in $H^1(\mathbb{R})$ of center R_{X^0} and radius α, and we define the mapping :

$$\mathcal{Y} : \prod_{j=1}^N (c_j^0 - \alpha, c_j^0 + \alpha) \times \prod_{j=1}^N (-\alpha, \alpha) \times B(R_{X^0}, \alpha) \longrightarrow \mathbb{R}^N \times \mathbb{R}^N$$

by $\mathcal{Y} = (\mathcal{Y}^{1,1}, \cdots, \mathcal{Y}^{1,N}, \mathcal{Y}^{2,1}, \cdots, \mathcal{Y}^{2,N})$, where

$$\begin{aligned} &\mathcal{Y}^{1,j}(c_1, \cdots, c_N, y_1, \cdots, y_N, u) \\ &= \int \Big(u(x) - \sum_{k=1}^N \varphi_{c_k^0}(x - x_k^0 - y_k) \Big) (1 - \partial_x^2) \varphi_{c_j^0}(x - x_j^0 - y_j) dx, \end{aligned}$$

and

$$\begin{aligned} &\mathcal{Y}^{2,j}(c_1, \cdots, c_N, y_1, \cdots, y_N, u) \\ &= \int \Big(u(x) - \sum_{k=1}^N \varphi_{c_j^0}(x - x_k^0 - y_k) \Big) (1 - \partial_x^2) \partial_x \varphi_{c_j^0}(x - x_j^0 - y_j) dx. \end{aligned}$$

By the dominated convergence theorem and the smoothness of φ_c, it can be seen that $\mathcal{Y}$ is a C^1-mapping. In view of applying the implicit function theorem, let us compute the partial derivatives of $\mathcal{Y}$ at the point $M_0 = (c_1^0, \cdots, c_N^0, 0, \cdots, 0, R_{X^0})$, for all $j = 1, \cdots, N$:

$$\frac{\partial \mathcal{Y}^{1,j}}{\partial c_j}(M_0) = -\int \partial_c \varphi_{c_j^0}(\cdot - x_j^0)(1-\partial_x^2)\varphi_{c_j^0}(\cdot - x_j^0)dx.$$

Using the identity

$$\partial_c \varphi_{c_0}(x) = \frac{1}{(c_0-1)(p-1)}\varphi_{c_0}(x) + \frac{x}{2c_0(c_0-1)}\partial_x \varphi_{c_0}(x), \tag{2.7}$$

and integrating by parts, we find that

$$\frac{\partial \mathcal{Y}^{1,j}}{\partial c_j}(M_0) = \frac{-4c_j^0 - 1 + p}{4c_j^0(c_j^0-1)}\int \varphi_{c_j^0}^2 dx - \frac{4c_j^0+1}{4c_j^0(c_j^0-1)}\int (\partial_x \varphi_{c_j^0})^2 dx.$$

Remark that $\frac{\partial \mathcal{Y}^{1,j}}{\partial c_j}(M_0) = -\frac{d}{dc}m(\varphi_c)|_{c=c_j^0} < 0$, since $c^\star(p) < c_j^0$, see the Introduction and [**20**], [**21**]. Moreover, we deduce from the above identity that $\frac{\partial \mathcal{Y}^{1,j}}{\partial c_j}(M_0) \leqslant -C_1$, where $C_1 > 0$ depends only on the (c_j^0). We also integrate by parts to compute :

$$\frac{\partial \mathcal{Y}^{1,j}}{\partial y_j}(M_0) = \int \partial_x \varphi_{c_j^0}(\cdot - x_j^0)(1-\partial_x^2)\varphi_{c_j^0}(\cdot - x_j^0)dx = 0,$$

$$\frac{\partial \mathcal{Y}^{2,j}}{\partial c_j}(M_0) = -\int \partial_c \varphi_{c_j^0}(\cdot - x_j^0)(1-\partial_x^2)\partial_x\varphi_{c_j^0}(\cdot - x_j^0)dx = 0,$$

and

$$\begin{aligned}\frac{\partial \mathcal{Y}^{2,j}}{\partial y_j}(M_0) &= \int \partial_x \varphi_{c_j^0}(\cdot - x_j^0)(1-\partial_x^2)\partial_x\varphi_{c_j^0}(\cdot - x_j^0)dx,\\ &= \|\partial_x \varphi_{c_j^0}\|_{H^1}^2 \geqslant C_2,\end{aligned}$$

where $C_2 > 0$ depends only on the (c_j^0). Remark now that there exists $C > 0$, such that for all $j = 1 \cdots N$,

$$|\varphi_{c_j^0}(x)| + |\partial_x \varphi_{c_j^0}(x)| + |\partial_x^2 \varphi_{c_j^0}(x)| \leqslant Ce^{-\sqrt{\frac{c_j^0-1}{c_j^0}}|x|} \leqslant Ce^{-2\sigma_0|x|},$$

this allows one to compute, for $j \neq k$

$$\begin{aligned}&\left|\frac{\partial \mathcal{Y}^{1,j}}{\partial c_k}(M_0)\right| = \left|\int \partial_c \varphi_{c_k^0}(\cdot - x_k^0)(1-\partial_x^2)\varphi_{c_j^0}(\cdot - x_j^0)dx\right|\\ &\leqslant \left|\int \left(\frac{\varphi_{c_k^0}(x-x_k^0)}{(c_k^0-1)(p-1)} + \frac{(x-x_k)\partial_x\varphi_{c_k^0}(x-x_k^0)}{2c_k^0(c_k^0-1)}\right)(1-\partial_x^2)\varphi_{c_j^0}(\cdot - x_j^0)dx\right|\\ &\leqslant C\int e^{-\sigma_0(|x-x_j^0|+|x-x_k^0|)}dx \leqslant Ce^{-\sigma_0|x_j^0-x_k^0|/2} \leqslant Ce^{-\sigma_0 L/2}.\end{aligned}$$

In the same way we compute, for $j \neq k$

$$\left|\frac{\partial \mathcal{Y}^{1,j}}{\partial y_k}(M_0)\right| = \left|\int \partial_x \varphi_{c_k^0}(\cdot - x_k^0)(1-\partial_x^2)\varphi_{c_j^0}(\cdot - x_j^0)dx\right| \leqslant Ce^{-\sigma_0 L/2},$$

$$\left|\frac{\partial \mathcal{Y}^{2,j}}{\partial c_k}(M_0)\right| = \left|\int \partial_c \varphi_{c_k^0}(\cdot - x_k^0)(1-\partial_x^2)\partial_x\varphi_{c_j^0}(\cdot - x_j^0)dx\right| \leqslant Ce^{-\sigma_0 L/2},$$

and

$$\left|\frac{\partial \mathcal{Y}^{2,j}}{\partial y_k}(M_0)\right| = \left|\int \partial_x \varphi_{c_k^0}(\cdot - x_k^0)(1-\partial_x^2)\partial_x \varphi_{c_j^0}(\cdot - x_j^0)dx\right| \leqslant Ce^{-\sigma_0 L/2}.$$

We deduce that $\mathcal{D}_{(c_1,\cdots,c_N,y_1,\cdots,y_N)}\mathcal{Y}(M_0) = D + P$, where D is an invertible diagonal matrix with $||D|| \geqslant C_3$, where $C_3 > 0$ depends only on the (c_j^0), and $||P|| \leqslant Ce^{-\sigma_0 L/2}$. Hence there exists $L_1 > 0$ such that if $L \geqslant L_1$, $\mathcal{D}_{(c_1,\cdots,c_N,y_1,\cdots,y_N)}\mathcal{Y}(M_0)$ is invertible and its norm is larger than $C_3/2$. The implicit function theorem implies the existence of $\alpha_0 > 0$, and C^1 functions $(c_j, y_j)_{j=1}^N$ from $B(R_{X^0}, \alpha_0)$ in a neighborhood of $(c_1^0, \cdots, c_N^0, 0, \cdots, 0)$ such that $\mathcal{Y}(c_1(u), \cdots, c_N(u), y_1(u), \cdots, y_N(u), u) = 0$ for all $u \in B(R_{X^0}, \alpha_0)$. Moreover there exists $K_1 > 0$ such that if $u \in B(R_{X^0}, \alpha)$, where $0 < \alpha \leqslant \alpha_0$, then

$$\sum_{j=1}^N |c_j(u) - c_j^0| + \sum_{j=1}^N |y_j(u)| \leqslant K_1 \alpha. \tag{2.8}$$

It is crucial, for the next step, to note that α_0 and K_1 are independent from $X^0 = (x_j^0) \in \mathbb{R}^N$ provided that $x_j^0 > x_{j-1}^0 + L$, with $L \geqslant L_1$. For $u \in B(R_{X^0}, \alpha_0)$, we set $x_j(u) = x_j^0 + y_j(u)$, thus x_j is a C^1 function on $B(R_{X^0}, \alpha_0)$ such that

$$x_j(u) \geqslant x_{j-1}(u) + L - 2K_1\alpha_0. \tag{2.9}$$

We are now able to define the modulation of $u \in \mathcal{U}(\alpha, L)$ for $L \geqslant L_1$ and $0 < \alpha \leqslant \alpha_1$, α_1 to be chosen later. Indeed, for $\alpha \leqslant \alpha_1$ one can cover $\mathcal{U}(\alpha, L)$ as follows :

$$\mathcal{U}(\alpha, L) \subset \bigcup_{X \in \mathbb{R}^N, x_j > x_{j-1} + L} B(R_X, \rho_0),$$

where $\alpha_1 \leqslant \rho_0 \leqslant \alpha_0$, and ρ_0 is chosen such that if $u \in B(R_X, \rho_0) \cap B(R_{\tilde{X}}, \rho_0)$, then the modulation of u is uniquely defined thanks to the uniqueness in the implicit function theorem.

Now, we define the modulation of u solution of (gBBM) such that $u(t) \in \mathcal{U}(\alpha, L)$ for all $t \in [0, t_0]$, by setting for $j = 1, \cdots, N$ and $t \in [0, t_0]$

$$c_j(t) = c_j(u(t)) \text{ and } x_j(t) = x_j(u(t)),$$
$$\varepsilon(t) = u(t) - \sum_{j=1}^N \varphi_{c_j(t)}(\cdot - x_j(t)).$$

These functions clearly satisfy properties (2.4) and (2.5). To establish estimates (2.6) we argue as in the case of a single solitary wave ([**6**]). Indeed, substituting $u(t,x) = \sum_{j=1}^N \varphi_{c_j(t)}(x - x_j(t)) + \varepsilon(t,x)$ in the gBBM equation and using the equation of $\varphi_{c_j(t)}$, we find that $\varepsilon(t)$ satisfies for all $t \in [0, t_0]$,

$$\begin{aligned}(1-\partial_x^2)\varepsilon_t + \varepsilon_x \quad &+ \quad \sum_{j=1}^N \dot{c}_j (1-\partial_x^2)\partial_c R_j - \sum_{j=1}^N (\dot{x}_j - c_j)(1-\partial_x^2)\partial_x R_j \\ &+ \Big(\big(\varepsilon + \sum_{j=1}^N R_j\big)^p - \sum_{j=1}^N R_j^p\Big)_x = 0. \end{aligned} \tag{2.10}$$

Now remark that thanks to estimate (2.5), one can choose α_1 sufficiently small such that for all j,

$$\sigma_0 \leqslant \frac{4}{5}\min\Big(\sqrt{\frac{c_j(t)-1}{c_j(t)}}, c_1(t)-c^\star(p), c_2(t)-c_1(t), \cdots, c_N(t)-c_{N-1}(t)\Big), \tag{2.11}$$

this implies that

$$|R_j(t,x)| \leqslant Ce^{-\sigma_0|x-x_j(t)|}. \tag{2.12}$$

Taking the inner product in $L^2(\mathbb{R})$ of equation (2.10) with R_j and $\partial_x R_j$, integrating by parts and using the decay of R_j and its derivatives, we find

$$\begin{aligned} |\dot{c}_j(t)| + |\dot{x}_j(t) - c_j(t)| \quad &\leqslant \quad C\left(\int e^{-\sigma_0|x-x_j(t)|}\varepsilon^2(t)dx\right)^{1/2} \\ &\quad +C\sum_{k\neq j} e^{-\frac{\sigma_0}{2}|x_k(t)-x_j(t)|}. \end{aligned} \tag{2.13}$$

Using this inequality, the choice of σ_0, estimates (2.9) and (2.8), one can take α_1 small enough and L_1 large enough such that $|x_k(t) - x_j(t)| \geqslant L/2 + \sigma_0 t$; this and estimate (2.13) imply (2.6) and achieve the proof of Proposition 2.1.

2.2. Monotonicity property. We introduce in this section a main tool in the proof of the stability result. It is an adaptation of the monotonicity result in the case of single perturbed solitary wave (Proposition 3.1 in [**6**]) or H^1-localized solutions of the gBBM equation (Lemma 2.1 in [**7**]) to the case of solutions near the sum of N solitary waves.

Before introducing this tool, we recall two fundamental identities (see proofs of Propositions 3.1 and 4.2 in [**6**] and proof of Lemma 2.1 in [**7**]) which are based on the conservation laws: *For any solution $u(t)$ of* (1.1), *and any C^1 function $g = g(x)$, the following holds:*

$$\begin{aligned} \frac{d}{dt}\int (u^2(t)+u_x^2(t))g(x)dx \quad &= \quad -\int u^2(t)g'dx - \frac{2}{p+1}\int u^{p+1}(t)g'dx \\ &\quad +2\int uhg'dx, \end{aligned} \tag{2.14}$$

and

$$\frac{d}{dt}\int (u^2(t)+\frac{2}{p+1}u^{p+1}(t))g(x)dx = \int h^2(t)g'dx - \int h_x^2(t)g'dx, \tag{2.15}$$

where $h = (1-\partial_x^2)^{-1}(u+u^p)$.

Consider the function ψ :

$$\psi(x) = \frac{\sigma_0}{3\int Q}\int_{-\infty}^x Q\Big(\frac{\sigma_0 y}{3}\Big)dy,$$

where Q is defined in the Introduction. Note that ψ is positive, increasing, $\psi(x)$ goes to 1 when x goes to $+\infty$, and ψ and its derivatives satisfy an exponential decay on the left : $\psi(x) + \psi'(x) + |\psi''(x)| \leqslant Ce^{\frac{\sigma_0 x}{3}}$ for $x \leqslant 0$. We introduce for all $j \in \{2, \cdots, N\}$:

$$\mathcal{I}_j(t) = \frac{1}{2}\int \big(u^2(t,x)+u_x^2(t,x)\big)\,\psi_j(t,x)dx,$$

$$\text{(2.16)} \qquad \text{where} \quad \psi_j(t,x) = \psi(x - y_j(t)) \quad \text{and} \quad y_j(t) = \frac{x_{j-1}(t) + x_j(t)}{2}.$$

Note that $\mathcal{I}_j(t)$ is close to $\frac{1}{2}\|u(t)\|^2_{H^1(x>y_j(t))}$. The following lemma claims that for a solution u of the gBBM equation such as in Proposition 2.1, the function $\mathcal{I}_j$ is almost decreasing with respect to time :

LEMMA 2.1. *Consider u solution of gBBM in $C(\mathbb{R}, H^1(\mathbb{R}))$ as in Proposition 2.1. There exist $\alpha_2 > 0$, $K_2 > 0$ and $L_2 > 0$, all depending only on σ_0, such that if $0 < \alpha < \alpha_2$, and for all $j \in \{2, \cdots, N\}$, $x_j(t) - x_{j-1}(t) \geqslant L$, for some $L \geqslant L_2$, then*

$$\text{(2.17)} \qquad \mathcal{I}_j(t) - \mathcal{I}_j(0) \leqslant K_2 e^{-\frac{\sigma_0 L}{12}}, \text{ for all } t \in [0, t_0].$$

Proof. Since the proof is very similar to the one of Lemma 2.1 in [**7**], we only give the main steps. From (2.11), we deduce that for all $j \in \{1, \cdots, N\}$, $\frac{5}{4}\sigma_0^2 \leqslant c_j(t) - 1$. Thus, using estimate (2.6), one can choose in Proposition 2.1 a value of α_1 sufficiently small and L_1 sufficiently large such that

$$\text{(2.18)} \qquad \forall j \in \{1, \cdots, N\}, \quad 1 + \sigma_0^2 \leqslant \dot{x}_j(t) \text{ for all } t \in [0, t_0].$$

Using this estimate and following the same steps as in the proof of Lemma 2.1 in [**7**], we compute :

$$\begin{aligned} \mathcal{I}_j'(t) \quad &\leqslant \quad -\Big(1 + \frac{\sigma_0^2}{2}\Big) \int u^2 \psi'(x - y_j(t))dx + \int u\psi'(x - y_j(t))(1 - \partial_x^2)^{-1} u dx \\ &\quad - \int \frac{u^{p+1}}{p+1} \psi'(x - y_j(t))dx + \int u\psi'(x - y_j(t))(1 - \partial_x^2)^{-1}(u^p)dx. \end{aligned}$$

Next we introduce the function $\tilde{h} = (1 - \partial_x^2)^{-1} u \in H^2(\mathbb{R})$, this change of variable and the estimate

$$|\psi'''(x)| = \frac{\sigma_0^2}{9} \frac{\sigma_0}{3\int Q(y)dy} \left|\partial_x^2 Q\left(\frac{\sigma_0 x}{3}\right)\right| \leqslant \frac{\sigma_0^2}{9}\psi'(x),$$

imply

$$\begin{aligned} \mathcal{I}_j'(t) \quad &\leqslant \quad -\frac{\sigma_0^2}{4} \int u^2 \psi'(x - y_j(t))dx \\ &\quad - \int \frac{u^{p+1}}{p+1} \psi'(x - y_j(t))dx + \int u\psi'(x - y_j(t))(1 - \partial_x^2)^{-1}(u^p)dx, \end{aligned}$$

(see [**7**]). It remains to deal with the nonlinear terms, the idea is to decompose each of them as the sum of two integrals, one of them being over a region where u is small. To do this, we set $I = [x_{j-1}(t) + L/4, x_j(t) - L/4]$ and $I^C = \mathbb{R} \setminus I$. Remark that for $x \in I$, taking $L_2 = L_2(\sigma_0)$ sufficiently large, and $a_0 = a_0(\sigma_0)$ sufficiently small, we have by the expression of $\varphi_c(x)$,

$$|u(t,x)|^{p-1} = \Big|\sum_{j=1}^N R_j(t,x) + \varepsilon(t,x)\Big|^{p-1} \leqslant C(e^{-\frac{\sigma_0 L}{4}} + \|\varepsilon(t)\|_{H^1}^{p-1}) \leqslant (p+1)\frac{\sigma_0^2}{8}.$$

This implies that

$$\text{(2.19)} \qquad \Big|\int_I \frac{u^{p+1}}{p+1}\psi'(x - y_j(t))dx\Big| \leqslant \frac{\sigma_0^2}{8} \int u^2 \psi'(x - y_j(t))dx.$$

Remark that from (2.6) and (2.11), we can choose in Proposition 2.1 the parameter α_0 sufficiently small so that $\dot{x}_j(t) - \dot{x}_{j-1}(t) \geqslant \sigma_0$. Thus, for $x \in I^C$, we have

$$|x - y_j(t)| \geqslant (x_j(t) - x_{j-1}(t))/2 - L/4 \geqslant \sigma_0 t/2 + L/4.$$

This estimate and the exponential decay of ψ' imply that

$$\Big|\int_{I^C} \frac{u^{p+1}}{p+1}\psi'(x - y_j(t))dx\Big| \leqslant Ce^{-\frac{\sigma_0}{6}(\sigma_0 t + L/2)}. \tag{2.20}$$

The second nonlinear term, $\int u\psi'(x - y_j(t))(1-\partial_x^2)^{-1}(u^p)dx$, is also decomposed as above, using the same estimates as the ones of the proof of Lemma 2.1 in [**7**], we get

$$\begin{aligned} \int u\psi'(x - y_j(t))(1-\partial_x^2)^{-1}(u^p)dx \quad &\leqslant \quad \frac{\sigma_0^2}{8}\int u^2\psi'(x - y_j(t))dx \\ &\quad + Ce^{-\frac{\sigma_0}{6}(\sigma_0 t + L/2)}. \end{aligned} \tag{2.21}$$

Hence, gathering estimates (2.19), (2.20) and (2.21), we obtain

$$\mathcal{I}_j'(t) \leqslant Ce^{-\frac{\sigma_0}{6}(\sigma_0 t + L/2)}). \tag{2.22}$$

This implies, after integration between 0 and t, that

$$\mathcal{I}_j(t) - \mathcal{I}_j(0) \leqslant Ce^{-\frac{\sigma_0 L}{12}},$$

where C is independent of t. Thus, Lemma 2.1 is proved.

2.3. Linearization of the energy and coercivity.

Lemma 2.2. *There exists $K_3 > 0$ and $L_3 > 0$ such that the decomposition of u given in Proposition* 2.1 *satisfies the following : if for all j, $x_j(t) - x_{j-1}(t) \geqslant L \geqslant L_3$, then for all $t \in [0, t_0]$,*

$$\begin{aligned} &\Big|\sum_{j=1}^N \big[E(R_j(t)) - E(R_j(0))\big] + \frac{1}{2}\int \big(\varepsilon^2(t) + pR^{p-1}(t)\varepsilon^2(t)\big)dx\Big| \\ &\leqslant K_3\left(\|\varepsilon(0)\|_{H^1}^2 + \|\varepsilon(t)\|_{H^1}^3 + e^{-\sigma_0 L/2}\right), \end{aligned} \tag{2.23}$$

where $R(t) = \sum_{j=1}^N R_j(t)$.

Proof. The proof consists in writing the energy of u as the sum of the energies of the modulated solitary waves and a quadratic form of ε, which is done using the decomposition of u and the orthogonality conditions satisfied by ε. Indeed, using the decomposition of u (2.1), a straightforward calculation gives :

$$\begin{aligned} E(u(t)) = &\sum_{j=1}^N \int \Big(\frac{1}{2}R_j^2(t) + \frac{1}{p+1}R_j^{p+1}(t)\Big)dx + \frac{1}{2}\int \varepsilon^2(t)dx \\ &+ \frac{p}{2}\int R^{p-1}(t)\varepsilon^2(t)dx + \sum_{j=1}^N \int (R_j + R_j^p)(t)\varepsilon(t)dx + \frac{1}{2}\sum_{j\neq k}\int R_j(t)R_k(t)dx \\ &+ \int\Big\{\frac{(R+\varepsilon)^{p+1}(t)}{p+1} - \sum_{j=1}^N\Big(\frac{R_j^{p+1}}{p+1}(t) + R_j^p(t)\varepsilon(t)\Big) - \frac{p}{2}R^{p-1}(t)\varepsilon^2(t)\Big\}dx \end{aligned}$$

First, thanks to the equation of φ_c (1.5) and the orthogonality condition (2.4) we obtain for all j

$$\int \varepsilon(t)(R_j + R_j^p)(t)dx = c_j \int \varepsilon(t)(1-\partial_x^2)R_j(t)dx = 0.$$

Now, recall that with our choice of σ_0, $|R_j(t,x)| \leqslant Ce^{-\sigma_0|x-x_j(t)|}$, and on the other hand $|x_j(t)-x_k(t)| \geqslant L$ for $j \neq k$. Thus,

$$\left|\int R_j R_k dx\right| \leqslant Ce^{-\sigma_0 L/2}.$$

Finally, for the nonlinear terms : for all $1 \leqslant k \leqslant p-1$,

$$\left|\int \varepsilon^{k+2}R^{p-1-k}\right| \leqslant C\|\varepsilon\|_{L^\infty}^k \int \varepsilon^2 \leqslant K\|\varepsilon(t)\|_{H^1}^3,$$

since $\|\varepsilon(t)\|_{H^1} \leqslant 1$. Hence,

$$\left|E(u(t)) - \sum_{j=1}^N E(R_j(t)) - \frac{1}{2}\int \big(\varepsilon^2(t) + pR^{p-1}(t)\varepsilon^2(t)\big)dx\right|$$
$$\leqslant C(e^{-\sigma_0 L/2} + \|\varepsilon(t)\|_{H^1}^3).$$

To obtain (2.23), it suffices now to use the energy conservation $E(u(t)) = E(u(0))$. Thus Lemma 2.2 is proved.

Finally, we give a generalization of a positivity lemma proved by Weinstein [**21**], Proposition 5.2. The quadratic form $\mathcal{L}_N$ that we consider has a suitable form around each solitary wave, which requires localization arguments.

LEMMA 2.3. *There exists $L_4 > 0$ and $\lambda_0 > 0$ such that the decomposition of u given in Proposition 2.1 satisfies the following : if for all j, $x_j(t) - x_{j-1}(t) \geqslant L$, for some $L \geqslant L_4$, then for all $t \in [0, t_0]$,*

$$(\mathcal{L}_N\varepsilon, \varepsilon) \geqslant \lambda_0 \|\varepsilon(t)\|_{H^1}^2, \tag{2.24}$$

where

$$(\mathcal{L}_N\varepsilon, \varepsilon) = \int \big(c(t,x)\varepsilon_x^2(t,x) + (c(t,x)-1)\varepsilon^2(t,x) - pR^{p-1}(t,x)\varepsilon^2(t,x)\big)\,dx$$

and $c(t,x) = c_1(t) + \sum_{j=2}^N (c_j(t) - c_{j-1}(t))\psi(x - y_j(t))$.

The proof of Lemma 2.3 is given in Appendix A.

3. Stability proof

This section is devoted to the proof of the main result i.e. Theorem 1.1. We follow the strategy described in the Introduction. For $A_0, L, \alpha > 0$, we define

$$\mathcal{V}_{A_0}(\alpha, L) = \Big\{u \in H^1(\mathbb{R}); \inf_{x_j - x_{j-1} \geqslant L} \Big\|u - \sum_{j=1}^N \varphi_{c_j^0}(.-x_j)\Big\|_{H^1} \leqslant A_0\Big(\alpha + e^{-\frac{\sigma_0}{24}L}\Big)\Big\}. \tag{3.1}$$

We claim that there exists $A_0 > 0$, $L_0 > 0$, and $\alpha_0 > 0$ such that, if for some $L > L_0$, $\alpha < \alpha_0$, $\left\|u_0 - \sum_{j=1}^N \varphi_{c_j^0}(.-x_j^0)\right\|_{H^1} \leqslant \alpha$, where $x_j^0 > x_{j-1}^0 + L$, then for all $t \geqslant 0$, $u(t) \in \mathcal{V}_{A_0}(\alpha, L)$, which implies Theorem 1.1. By continuity of $t \mapsto u(t)$ in $H^1(\mathbb{R})$, it is a consequence of the following proposition.

Proposition 3.1. *There exists $A_0 > 0$, $L_0 > 0$, and $\alpha_0 > 0$ such that, if*

$$\Big\| u_0 - \sum_{j=1}^{N} \varphi_{c_j^0}(. - x_j^0) \Big\|_{H^1} \leqslant \alpha, \tag{3.2}$$

for some $L > L_0$, $0 < \alpha < \alpha_0$, $x_j^0 > x_{j-1}^0 + L$, and if for $t^\star > 0$,

$$\forall t \in [0, t^\star], \quad u(t) \in \mathcal{V}_{A_0}(\alpha, L), \tag{3.3}$$

then

$$\forall t \in [0, t^\star], \quad u(t) \in \mathcal{V}_{A_0/2}(\alpha, L). \tag{3.4}$$

Proof of Proposition 3.1. Let $A_0 > 0$ to be fixed later. Since by (3.3), $u(t)$ is close in H^1 to a sum of N sufficiently decoupled solitary waves, we may apply Proposition 2.1 on $[0, t^\star]$. It follows that there exist c_j, x_j as in the statement of the proposition. Since (3.3) involves the constant A_0 to be chosen, we obtain estimates on $\varepsilon(t)$, $|c_j(t) - c_j^0|$, and the quantities in (2.6) all depending on A_0.

However, for the initial data, i.e., at $t = 0$, assumption (3.2) implies directly

$$\|\varepsilon(0)\|_{H^1} + \sum_{j=1}^{N} |c_j(0) - c_j^0| \leqslant K_1 \alpha, \tag{3.5}$$

with no dependency on A_0, using the first part the proof of Proposition 2.1. We choose α_0, L_0 such that we can apply Lemmas 2.1–2.3 on $[0, t^\star]$, in particular $A_0 \alpha_0$ small enough.

Let us define

$$d_j(t) = \sum_{k=j}^{N} m_j(t), \quad \Delta_0^t d_j = d_j(t) - d_j(0). \tag{3.6}$$

The proof proceeds in two steps ((i) and (ii) in the next lemma) : first, we control the variations of the $c_j(t)$. Second, we estimate $\|\varepsilon(t)\|_{H^1}$, which gives the stability result.

Lemma 3.1. *(i) There exists $K_5 > 0$ such that for all $t \in [0, t^\star]$,*

$$\sum_{j=1}^{N} |c_j(t) - c_j(0)| \leqslant K_5 \left(\|\varepsilon(0)\|_{H^1}^2 + \|\varepsilon(t)\|_{H^1}^2 + e^{-\frac{\sigma_0}{12} L} \right). \tag{3.7}$$

(ii) There exists $K_6 > 0$ such that for all $t \in [0, t^\star]$,

$$\|\varepsilon(t)\|_{H^1}^2 \leqslant K_6 \left(\|\varepsilon(0)\|_{H^1}^2 + e^{-\frac{\sigma_0}{12} L} \right). \tag{3.8}$$

Proof of Lemma 3.1. We recall the three estimates that will be used in the proof: (a) The conclusion of Lemma 2.2:

$$\begin{aligned} &\left| \sum_{j=1}^{N} \Delta_0^t E(R_j) + \frac{1}{2} \int \left(\varepsilon^2 + p R^{p-1} \varepsilon^2 \right)(t) \right| \\ &\leqslant K \left(\|\varepsilon(0)\|_{H^1}^2 + \|\varepsilon(t)\|_{H^1}^3 + e^{-\frac{\sigma_0}{12} L} \right), \end{aligned} \tag{3.9}$$

(b) From Lemma 2.1 and the orthogonality conditions on $\varepsilon(t)$:

$$\Delta_0^t d_j + \frac{1}{2} \int (\varepsilon^2(t) + \varepsilon_x^2(t)) \psi_j(t) \leqslant K \left(\|\varepsilon(0)\|_{H^1}^2 + e^{-\frac{\sigma_0}{12} L} \right), \tag{3.10}$$

(c) An estimate

$$\left|\Delta_0^t E(R_j) - c_j(t)\Delta_0^t m(R_j)\right| \leqslant K|c_j(t)-c_j(0)|^2, \tag{3.11}$$

which follows from $\frac{d}{dc}E(\varphi_c) = c\frac{d}{dc}m(\varphi_c)$.

We also write an identity that relates $\Delta_0^t m(R_j)$ to $\Delta_0^t d_j$:

$$\begin{aligned}\sum_{j=1}^{N} c_j(t)\Delta_0^t m(R_j) &= \sum_{j=1}^{N-1} c_j(t)\left[\Delta_0^t d_j - \Delta_0^t d_{j+1}\right] + c_N(t)\Delta_0^t d_N \\ &= \sum_{j=2}^{N}(c_j(t)-c_{j-1}(t))\Delta_0^t d_j + c_1(t)\Delta_0^t d_1. \end{aligned} \tag{3.12}$$

Proof of (i). We combine (3.9)–(3.11) and the above identity to obtain (i). Let

$$\mathcal{Q}(t) = \|\varepsilon(0)\|_{H^1}^2 + \|\varepsilon(t)\|_{H^1}^2 + e^{-\frac{\sigma_0}{12}L} + |c_j(t)-c_j(0)|^2.$$

From (3.9), we have

$$\left|\sum_{j=1}^{N}\Delta_0^t E(R_j)\right| \leqslant K\mathcal{Q}(t),$$

and so using (3.11) and (3.12), we obtain

$$\left|\sum_{j=2}^{N}(c_j(t)-c_{j-1}(t))\Delta_0^t d_j + c_1(t)\Delta_0^t d_1\right| \leqslant K\mathcal{Q}(t). \tag{3.13}$$

Note that directly from (3.10), for all $j\in\{1,\ldots,N\}$, we have

$$\Delta_0^t d_j \leqslant K\left(\|\varepsilon(0)\|_{H^1}^2 + e^{-\frac{\sigma_0}{12}L}\right).$$

Using this estimate for $j\geqslant 2$ in (3.13), we deduce

$$c_1(t)\Delta_0^t d_1 \geqslant -K\mathcal{Q}(t)$$

Since $c_1(t)\geqslant\sigma_0$, we obtain $|\Delta_0^t d_1|\leqslant K\mathcal{Q}(t)$. Similarly, for $j\in\{2,\ldots,N\}$, we have from (3.13),

$$(c_j(t)-c_{j-1}(t))\Delta_0^t d_j \geqslant -K\mathcal{Q}(t),$$

and so, for all $j\in\{1,\ldots,N\}$, we have $|\Delta_0^t d_j|\leqslant K\mathcal{Q}(t)$. It follows that, for all $j\in\{1,\ldots,N\}$, $|\Delta_0^t m(R_j)|\leqslant K\mathcal{Q}(t)$. Since

$$m(R_j)(t) = f(c_j(t))$$

where $f'(c)>0$, for $c=c_j(t)>c^*(p)$ (see (1.7) in the Introduction), we obtain

$$\sum_{j=1}^{N}|c_j(t)-c_j(0)| \leqslant K\mathcal{Q}(t).$$

Thus (i) is proved.

Proof of (ii). We use again (3.9)–(3.12) together with Lemma 2.3 to obtain (ii). Recall that in Lemma 2.3, we have defined

$$(\mathcal{L}_N\varepsilon(t),\varepsilon(t)) = \int\left(c(t,x)\varepsilon_x^2(t,x) + (c(t,x)-1)\varepsilon^2(t,x) - pR^{p-1}(t,x)\varepsilon^2(t,x)\right)dx,$$

where $c(t,x) = c_1(t) + \sum_{j=2}^N (c_j(t) - c_{j-1}(t))\psi_j(t,x)$. In the following we set $c_0 \equiv 0$ and $\psi_1 \equiv 1$ for the reader convenience. Inserting (3.9) and (3.12) into (3.11), and then using (i), we have

$$\begin{aligned}
&\left|\frac{1}{2}(\mathcal{L}_N\varepsilon(t),\varepsilon(t)) - \sum_{j=1}^N (c_j(t) - c_{j-1}(t))\left[\Delta_0^t d_j + \frac{1}{2}\int(\varepsilon^2 + \varepsilon_x^2)(t)\psi_j(t)\right]\right| \\
&\leqslant K\Big(\|\varepsilon(0)\|_{H^1}^2 + \|\varepsilon(t)\|_{H^1}^3 + e^{-\frac{\sigma_0}{12}L} + \sum_{j=1}^N |c_j(t) - c_j(0)|^2\Big) \\
&\leqslant K\Big(\|\varepsilon(0)\|_{H^1}^2 + \|\varepsilon(t)\|_{H^1}^3 + e^{-\frac{\sigma_0}{12}L}\Big).
\end{aligned}$$

Thus, by (3.10), and $c_j(t) - c_{j-1}(t) > \sigma_0$, we obtain

$$(\mathcal{L}_N\varepsilon(t),\varepsilon(t)) \leqslant K\Big(\|\varepsilon(0)\|_{H^1}^2 + \|\varepsilon(t)\|_{H^1}^3 + e^{-\frac{\sigma_0}{12}L}\Big).$$

By Lemma 2.3, we have $(\mathcal{L}_N\varepsilon,\varepsilon) \geqslant \lambda_0\|\varepsilon(t)\|_{H^1}^2$, and so we obtain (ii). Thus Lemma 3.1 is proved.

Conclusion of the proof of Proposition 3.1. By (3.5) and Lemma 3.1, we have

$$\begin{aligned}
&\Big\|u(t) - \sum_{j=1}^N \varphi_{c_j^0}(x - x_j(t))\Big\|_{H^1} \\
&\quad\leqslant \Big\|u\varphi(t) - \sum_{j=1}^N R_j(t)\Big\|_{H^1} + \Big\|\sum_{j=1}^N R_j(t) - \sum_{j=1}^N \varphi_{c_j^0}(x - x_j(t))\Big\|_{H^1} \\
&\quad\leqslant \|\varepsilon(t)\|_{H^1} + C\sum_{j=1}^N |c_j(t) - c_j^0| \\
&\quad\leqslant \|\varepsilon(t)\|_{H^1} + C\sum_{j=1}^N |c_j(t) - c_j(0)| + C\sum_{j=1}^N |c_j(0) - c_j^0| \\
&\quad\leqslant \|\varepsilon(t)\|_{H^1} + C(\|\varepsilon(0)\|_{H^1}^2 + e^{-\frac{\sigma_0}{12}L}) + CK_1\alpha \\
&\quad\leqslant K\left(\alpha + e^{-\frac{\sigma_0}{24}L}\right),
\end{aligned}$$

where $K > 0$ is a constant independent of A_0. Thus the proposition is proved with $A_0 = 2K$, and $A_0\alpha_0$ small enough.

4. Proof of the asymptotic stability

4.1. Rigidity property. Together with the monotonicity property described in Lemma 2.1, the second main ingredient of the proof of the asymptotic stability of the family of solitary waves in [**6**], [**17**] is the following rigidity property.

THEOREM 4.1 ([**6**], [**17**]). *Let $p = 2,3$. Let $u_0 \in H^1(\mathbb{R})$. There exists a set $E \subset (1,+\infty)$ without accumulation points (E may be empty) such that, for any $c_0 \in (1,+\infty)\setminus E$, there exists $\alpha_1 > 0$ such that if*

$$\|u_0 - \varphi_{c_0}\|_{H^1} < \alpha_1, \tag{4.1}$$

and if the corresponding solution $u(t)$ of (1.1) satisfies : for all $\delta > 0$, there exists $B_\delta > 0$ such that for all $t \in \mathbb{R}$,

$$\int_{|x|>B_\delta} (u^2 + u_x^2)(t, x + y(t))dx < \delta, \tag{4.2}$$

for some function $y(t)$, then there exists $x_1 \in \mathbb{R}$, and $c_1 > 1$ such that

$$u(t,x) = \varphi_{c_1}(x - x_1 - c_1 t),$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}$.

An analogous result, without the restriction of the set E, was proved in [**12**] for the subcritical gKdV equations. Recall that property (4.2) implies (without assumption (4.1) of closeness to φ_{c_0}) complete smoothness and exponential decay of the solution $u(t)$ of the gBBM equation (see Theorem 1.1 in [**7**]).

Recall also that the proof of Theorem 4.1 is mainly based on a rigidity property of a linear equation:

$$(1 - \partial_x^2)w_t - \partial_x(-c\,\partial_x^2 w + (c-1)w - p\varphi_c^{p-1}w) = 0. \tag{4.3}$$

The proof of this linear property in [**6**] uses a spectral result due to Miller and Weinstein [**16**], and for this reason requires the introduction of the set E. For the gKdV equation, the proof of the linear rigidity property is obtained in a different way, see [**12**].

We turn now to the proof of Theorem 1.2. By Theorem 1.1, the solution $u(t)$ is close to the sum of N solitary waves for all time $t \geqslant 0$, and admits a decomposition as in Proposition 2.1. The proof of Theorem 1.2 then proceeds into two steps. First, using Theorem 4.1 and monotonicity properties, we prove the convergence of $\varepsilon(t)$ to 0 around each solitary wave (§4.2). Second, we prove convergence of $\varepsilon(t)$ in H^1 in the region $x > \frac{1+c_1^0}{2}t$ by monotonicity arguments (§4.3).

4.2. Convergence around the solitary waves. We claim the following convergence result.

Proposition 4.1. *Under the assumptions of Theorem 1.2, for any $j \in \{1, \dots, N\}$, there exists $c_j^{+\infty} \in (1, +\infty) \setminus E$ such that*

$$\varepsilon(t, . + x_j(t)) \to 0 \quad \text{in } H^1_{loc}, \quad c_j(t) \to c_j^{+\infty}, \quad \dot{x}_j(t) \to c_j^{+\infty}, \quad \text{as } t \to +\infty. \tag{4.4}$$

Proof. Proposition 4.1 is a property of the flow of the gBBM equation around the solitary waves, which is a consequence of the rigidity property Theorem 4.1. We sketch the argument, which follows the strategy of section 4 of [**6**], and we refer to [**6**] for more details. Let $j \in \{1, \dots, N\}$.

First, we prove that $\varepsilon(t, . + x_j(t)) \rightharpoonup 0$ in $H^1(\mathbb{R})$. For the sake of contradiction, assume that there exists $\tilde{\varepsilon}_0 \in H^1(\mathbb{R})$, $\tilde{\varepsilon}_0 \not\equiv 0$, and $\tilde{c}_0 > 1$, such that for a sequence $t_n \to +\infty$,

$$\varepsilon(t_n, . + x_j(t_n)) \rightharpoonup \tilde{\varepsilon}_0 \quad \text{in } H^1, \quad c_j(t_n) \to \tilde{c}_0 \quad \text{as } n \to +\infty.$$

Consider the solution $\tilde{u}(t)$ of the gBBM equation with initial data $\tilde{u}_0 \equiv \varphi_{\tilde{c}_0} + \tilde{\varepsilon}_0$. It also admits a decomposition, with parameters $\tilde{c}(t)$, $\tilde{x}(t)$ and $\tilde{\varepsilon}(t)$. By weak convergence and uniqueness of the decomposition of $\tilde{u}_0$, we have $\tilde{\varepsilon}(0) = \tilde{\varepsilon}_0$, $\tilde{c}(0) = \tilde{c}_0$ and $\tilde{x}(0) = 0$.

By the arguments of the proof of Lemma 4.2 of [**6**] (see also Lemma 6 in [**17**]), we have

$$u(t_n+t,.+x(t_n+t))\to\tilde u(t,.+\tilde x(t))\quad\text{in } H^1_{loc} \text{ as } n\to+\infty. \tag{4.5}$$

Here, we obtain convergence in H^1_{loc} from a convergence in H^1 weak, which is a special feature of the gBBM equation, described in the proof of Lemma 4.2 in [**6**]. The main ingredient is that if $u(t_n)$ converges weakly in H^1 then by the equation of u, $u_t(t_n)$ converges weakly in H^2.

This convergence result and the monotonicity property (Lemma 2.1) imply that $\tilde u$ satisfies the assumptions of Theorem 4.1, and thus is equal to a solitary wave solution. We omit the detail of the proof since it is similar to Proposition 4.1 in [**6**]. Since $\tilde u(0)=\varphi_{\tilde c_0}+\tilde\varepsilon_0=\varphi_{c^*}(.-x^*)$, for some $c^*>1$, $x^*\in\mathbb{R}$, by uniqueness of the decomposition of $\tilde u(0)$, we have $c^*=\tilde c_0$ and $\tilde\varepsilon_0\equiv 0$, which is a contradiction.

Second, from the weak convergence to zero, we obtain as in (4.5) a strong convergence result: $\varepsilon(t,.+x_j(t))\to 0$ in $H^1_{loc}(\mathbb{R})$.

Finally, the convergence of $c_j(t)$ to some limit value $c_j^{+\infty}$ is a consequence of another monotonicity property of the gBBM equation, true on quantities related to the energy conservation:

$$\mathcal{J}_j(t)=\int\left(\frac12u^2+\frac{1}{p+1}u^{p+1}\right)(t,x)\psi_j(t,x)dx.$$

We refer to Proposition 4.2 in [**6**] for the proof.

4.3. Asymptotic behavior for $x>\frac{1+c_1^0}{2}t$. Now, we prove the following result, which completes the proof of Theorem 1.2.

Proposition 4.2. *Under the assumptions of Theorem 1.2, the following holds*

$$\|\varepsilon(t)\|_{H^1(x>\frac{1+c_1^0}{2}t)}\to 0\quad \text{as } t\to+\infty. \tag{4.6}$$

The proof of this Proposition is based only on Proposition 4.1 and monotonicity properties. It is very similar to the proof of Proposition 3 in [**14**], but we repeat the proof for the reader's convenience.

Proof. Set $\gamma_0=\sigma_0/24$. Let $y_0>0$. First, using the arguments of the proof of Lemma 2.1, we have

$$\begin{aligned}&\int(u^2+u_x^2)(t,x)\psi(x-y_0-x_N(t))dx\\&\leqslant\int(u^2+u_x^2)(0,x)\psi(x-y_0-x_N(0)-\tfrac{\sigma_0}{2}t)dx+Ce^{-\gamma_0y_0}.\end{aligned}$$

Therefore, by the decay properties of φ_c, we obtain

$$\int_{x>x_N(t)+y_0}(\varepsilon^2+\varepsilon_x^2)(t)\leqslant 2\int(u^2+u_x^2)(0,x)\psi(x-y_0-x_N(0)-\tfrac{\sigma_0}{2}t)dx+Ce^{-\gamma_0y_0}.$$

Since, for fixed y_0, $\int_{x_N(t)<x<x_N(t)+y_0}(\varepsilon^2+\varepsilon_x^2)(t,x)dx\to 0$ as $t\to+\infty$ (by Proposition 4.1), we obtain

$$\lim_{t\to+\infty}\int_{x>x_N(t)}(\varepsilon^2+\varepsilon_x^2)(t,x)dx=0.$$

Now, we prove that for all j, $\int_{x>x_j(t)}(\varepsilon^2+\varepsilon_x^2)(t)\to 0$ as $t\to+\infty$, by backwards induction on j. Assume that for $j_0\in\{1,\dots,N\}$, we have $\int_{x>x_{j_0}(t)}(\varepsilon^2+\varepsilon_x^2)(t)\to 0$ as $t\to+\infty$. For $t\geqslant 0$ large enough, there exists $0<t'=t'(t)<t$, satisfying

$$x_{j_0}(t')-x_{j_0-1}(t')-\tfrac{\sigma_0}{2}(t-t')=2y_0.$$

Indeed, for t large enough, $x_{j_0}(t)-x_{j_0-1}(t)\geqslant\frac{\sigma_0}{2}t\geqslant 2y_0$, and $x_{j_0}(0)-x_{j_0-1}(0)-\frac{\sigma_0}{2}t<0<2y_0$. Then,

$$\int\psi(.-(x_{j_0-1}(t)+y_0))(u^2+u_x^2)(t) \tag{4.7}$$
$$\leqslant\int\psi(.-(x_{j_0-1}(t')+\tfrac{\sigma_0}{2}(t-t')+y_0))(u^2+u_x^2)(t')+Ce^{-\gamma_0 y_0}$$
$$\leqslant\int\psi(.-(x_{j_0}(t')-y_0))(u^2+u_x^2)(t')+Ce^{-\gamma_0 y_0}. \tag{4.8}$$

Since $t'(t)\to+\infty$ as $t\to+\infty$, by H^1_{loc} convergence of $\varepsilon(t,.+x_{j_0}(t))$ and the induction assumption, we have, for fixed y_0,

$$\lim_{t\to+\infty}\int_{x>x_{j_0}(t')-2y_0}(\varepsilon^2+\varepsilon_x^2)(t')=0.$$

Therefore, by Proposition 4.1,

$$\limsup_{t\to+\infty}\int\psi(.-(x_{j_0}(t')-y_0))(u^2+u_x^2)(t')-\sum_{k=j_0}^N m(\varphi_{c_k^{+\infty}})\leqslant Ce^{-\gamma_0 y_0}. \tag{4.9}$$

Moreover, by the decomposition of $u(t)$,

$$\int\psi(.-(x_{j_0-1}(t)+y_0))(\varepsilon^2+\varepsilon_x^2)(t)$$
$$\leqslant\int\psi(.-(x_{j_0-1}(t)+y_0))(u^2+u_x^2)(t)-\sum_{k=j_0}^N m(\varphi_{c_k(t)})+Ce^{-\gamma_0 y_0},$$

and since $c_k(t)\to c_k^{+\infty}$, we obtain by (4.8):

$$\lim_{t\to+\infty}\int_{x>x_{j_0-1}(t)+y_0}(\varepsilon^2+\varepsilon_x^2)(t,x)dx=0,$$

and so

$$\lim_{t\to+\infty}\int_{x>x_{j_0-1}(t)}(\varepsilon^2+\varepsilon_x^2)(t,x)dx=0.$$

Thus the induction argument yields

$$\lim_{t\to+\infty}\int_{x>x_1(t)}(\varepsilon^2+\varepsilon_x^2)(t,x)dx=0.$$

Finally, we prove $\int_{x>\frac{1+c_1^0}{2}t}(\varepsilon^2+\varepsilon_x^2)(t)\to 0$ as $t\to+\infty$. Indeed, let $0<t'=t'(t)<t$ such that $x_1(t')-t'-\frac{c_1^0-1}{4}(t+t')=y_0$. Then, for $\sup_{t\geqslant 0}\|\varepsilon(t)\|_{H^1}$ small

enough, applying the arguments of Lemma 2.1, we obtain

$$\begin{aligned}
&\int \psi\Big(x - \frac{1+c_1^0}{2}t\Big)(u^2+u_x^2)(t)\\
&\leqslant \int \psi\Big(x - \Big(\frac{1+c_1^0}{2}t' + \frac{c_1^0-1}{4}(t-t')\Big)\Big)(u^2+u_x^2)(t') + Ce^{-\gamma_0 y_0}\\
&\leqslant \int \psi(x-(x_1(t')-y_0))(u^2+u_x^2)(t') + Ce^{-\gamma_0 y_0}.
\end{aligned}$$

The conclusion is obtained as before.

5. Existence and uniqueness of N solitary waves

The aim of this section is to prove Theorem 1.3, i.e. the existence and uniqueness of an asymptotic N solitary wave solution. It follows the strategy of [**11**].

The existence part is done into three steps. First, we consider an increasing sequence $S_n \to +\infty$, and a sequence $(u_n)_{n\in\mathbb{N}}$ of global solutions of (1.1) such that $u_n(S_n)$ is equal to the sum of N solitary waves. The desired solution is obtained as the limit of the sequence $(u_n(t))$, provided we have uniform estimates in H^s. The second step is devoted to the proof of the H^1 uniform estimate, which is an adaptation of the stability result (section 3). Then, for $s \geqslant 2$, H^s-estimates are deduced by computation of the variation in time of norms of $(u_n)_{n\in\mathbb{N}}$.

Finally, the uniqueness part is proved using a refined version of the mass monotonicity.

Let $p \geqslant 2$ be an integer, let $N \in \mathbb{N}$, $c^*(p) < c_1^0 < c_2^0 < \ldots < c_N^0$, and $x_1^0, \ldots, x_N^0 \in \mathbb{R}$. Let σ_0 be as in (2.2). We denote by

$$R(t,x) = \sum_{j=1}^{N} \varphi_{c_j^0}(x - x_j^0 - c_j^0 t),$$

the sum of the N solitary wave solutions of (1.1) associated to c_j^0 and x_j^0.

5.1. Construction of a solution assuming uniform estimates. Let $(S_n)_{n\in\mathbb{N}}$ be an increasing sequence of $\mathbb{R}^+$ such that $S_n \to +\infty$ as $n \to +\infty$. For $n > 0$ integer, we define $u_n(t)$, the solution of

$$u_n(S_n) = R(S_n), \quad (1-\partial_x^2)(u_n)_t + (u_n + u_n^p)_x = 0. \tag{5.1}$$

Note that $u_n \in C(\mathbb{R}, H^s(\mathbb{R}))$, for all $s \geqslant 1$. We claim that this sequence satisfies the following uniform estimates :

PROPOSITION 5.1. *Consider the sequence $(u_n)_{n\in\mathbb{N}}$ of solutions of* (5.1)*. There exist $\gamma_1, T_1 > 0$ such that : for any $s \geqslant 1$ there exists a constant $A_s > 0$, for which u_n satisfies the following estimates for all $n > 0$ and for all $t \in [T_1, S_n]$:*

$$\|u_n(t) - R(t)\|_{H^s(\mathbb{R})} \leqslant A_s e^{-\gamma_1 t}. \tag{5.2}$$

This result is the main step of the proof of the existence in Theorem 1.3, it is proved in sections 5.2, 5.3. We assume that $S_n \geqslant T_1$, by possibly taking a subsequence of $(S_n)_{n\in\mathbb{N}}$ satisfying this property. Note that the constants A_s do not depend on n. Thus, assuming this Proposition we prove that the sequence $(u_n(T_1))_{n\in\mathbb{N}}$ is H^1-localized. This allows us to construct the N-solitary wave solution as the solution of the gBBM equation emanating from the limit of $(u_n(T_1))_{n\in\mathbb{N}}$ when $n \to +\infty$. More precisely, Lemma 5.1. *For all $\delta > 0$, there exists $B_\delta > 0$ such that : for all $n \geqslant 1$,*

$$\int_{|x|>B_\delta} (u_n^2 + u_{nx}^2)(T_1, x)dx < \delta. \tag{5.3}$$

Proof. Fix $\delta > 0$. From (5.2) and the decay of solitary waves, there exists $B_1 > 0$ and $t_0 > T_1$ such that for all $n \in \mathbb{N}$

$$\|u_n(t_0)\|^2_{H^1(|x|>B_1)} < \delta/2. \tag{5.4}$$

We fix such value of B_1 and t_0 and we study the evolution of the $H^1(|x| > B)$-norm of $u_n(t)$ between T_1 and t_0. Let $g : \mathbb{R} \to [0, 1]$, be a C^1 function, such that $g \equiv 0$ on $[-1, 1]$, $g \equiv 1$ on $\mathbb{R}\backslash[-2, 2]$, and $\sup_{x\in\mathbb{R}} |g'(x)| \leqslant 2$. We introduce for $B > 0$, to be fixed later,

$$J_n(t) = \int (u^2(t, x) + u_x^2(t, x))g\Big(\frac{x}{B}\Big)dx.$$

Differentiating $J_n(t)$ with respect to time, using the gBBM equation, and integrating by parts, we find :

$$\begin{aligned} J_n'(t) \quad &= \quad -\frac{1}{B}\int \Big(u_n^2(t) + \frac{2}{p+1}u_n^{p+1}(t)\Big)g'\Big(\frac{x}{B}\Big)dx \\ &\quad +\frac{2}{B}\int g'\Big(\frac{x}{B}\Big)u_n(t)(1-\partial_x^2)^{-1}(u_n(t) + u_n^p(t))dx. \end{aligned}$$

Note that $u_n(t)$ is bounded in H^1 uniformly in t and n. This implies that for all $n \in \mathbb{N}$ and for all $t \in [T_1, t_0]$, $|J_n'(t)| \leqslant C_2/B$, where C_2 is independent of n. Thus, taking $B = \max\Big(1, B_1, \frac{C_2(t_0-T_1)}{\delta}\Big)$ implies $|J_n'(t)| \leqslant \frac{\delta}{2(t_0-T_1)}$, and so

$$J_n(T_1) \leqslant J_n(t_0) + \delta/2 \leqslant \|u_n(t_0)\|^2_{H^1(|x|>B_1)} + \delta/2 \leqslant \delta. \tag{5.5}$$

To achieve the proof of Lemma 5.1, we note that $\|u_n(T_1)\|^2_{H^1(|x|>2B)} \leqslant J_n(T_1)$.

We are now able to construct the N-solitary wave solution. Indeed, from the uniform H^2-estimate, (corresponding to the case $s = 2$ in (5.2)), and the fact that $R(t)$ in uniformly bounded in $H^2(\mathbb{R})$, it follows that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $H^2(\mathbb{R})$. Thus there exists $U_{T_1} \in H^2(\mathbb{R})$, such that $u_n(T_1) \to U_{T_1}$ in $H^1_{loc}(\mathbb{R})$. From Lemma 5.1, it follows that $u_n(T_1) \to U_{T_1}$ in $H^1(\mathbb{R})$.

Note also that by (5.2), $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $H^s(\mathbb{R})$, for all $s \geqslant 1$. Thus by interpolation between $H^1(\mathbb{R})$ and $H^{2s-2}(\mathbb{R})$ for $s \geqslant 2$, we deduce that $U_{T_1} \in H^s(\mathbb{R})$, and

$$u_n(T_1) \to U_{T_1} \text{ in } H^s(\mathbb{R}), \text{ for all } s \geqslant 1. \tag{5.6}$$

Now, we define the N-solitary waves solution U, as the unique solution of

$$\begin{cases} (1-\partial_x^2)U_t + (U + U^p)_x = 0, \ (t, x) \in \mathbb{R}\times\mathbb{R}, \\ U(T_1, x) = U_{T_1}(x), \ x \in \mathbb{R}. \end{cases} \tag{5.7}$$

The Cauchy problem of this equation is globally well posed in $H^s(\mathbb{R})$ for all $s \in \mathbb{R}$, thus $U \in C(\mathbb{R}, H^s(\mathbb{R}))$. Recall that the Cauchy problem is solved via the contraction principle, which ensures the continuity of the flow of the gBBM equation in $H^s(\mathbb{R})$. It follows from (5.6) that for all t, $U_n(t) \to U(t)$ in H^s, and so passing to the limit as $n \to +\infty$ in (5.2), we get, for all $s \geqslant 1$, for any $t \geqslant T_1$,

$$\Big\|U(t) - R(t)\Big\|_{H^s} \leqslant A_s e^{-\gamma_1 t}. \tag{5.8}$$

This estimate also holds for $t \in [0, T_1]$ by possibly taking larger A_s. Thus, the proof of the existence in Theorem 1.3 is reduced to the proof of Proposition 5.1.

5.2. H^1 **estimate.** We claim the following result.

PROPOSITION 5.2. *There exists $T_1 > 0$, $A_1 > 0$, $\alpha_1 > 0$ with $A_1 e^{-\sigma_0^2 T_1/24} \leqslant \alpha_1/2$ such that, for all $n \geqslant 0$, if for some $t^* \in [T_1, S_n]$, for all $t \in [t^*, S_n]$,*

$$\|u_n(t) - R(t)\|_{H^1} \leqslant \alpha_1, \tag{5.9}$$

then, for all $t \in [t^*, S_n]$,

$$\|u_n(t) - R(t)\|_{H^1} \leqslant A_1 e^{-\frac{\sigma_0^2}{24} t}. \tag{5.10}$$

This implies (5.2) for $s = 1$. Indeed, if we assume Proposition 5.2, since $u_n(S_n) = R(S_n)$, by continuity of $u_n(t)$ and $R(t)$ in time in $H^1(\mathbb{R})$, there exists $\tau_0 = \tau_0(n) > 0$ such that (5.9) is true on the interval $[S_n - \tau_0, S_n]$. Let

$$t^* = t^*(n) = \inf\{T_1 \leqslant t \leqslant S_n,\ \|u_n(t') - R(t')\|_{H^1} \leqslant \alpha_1,\ \forall t' \in [t, S_n]\}.$$

Looking for contradiction, we assume that $t^* > T_1$, then by Proposition 5.2, we have for all $t \in [t^*, S_n]$, $\|u_n(t) - R(t)\|_{H^1} \leqslant A_1 e^{-\sigma_0^2 t/24} \leqslant A_1 e^{-\sigma_0^2 T_1/24} \leqslant \alpha_1/2$. Thus, by continuity in $H^1(\mathbb{R})$, there exists $\tau_1 = \tau_1(n) > 0$ such that $\|u_n(t) - R(t)\|_{H^1} \leqslant \frac{3}{4}\alpha_1$ for all $t \in [t^* - \tau_1, S_n]$, which is a contradiction with the definition of t^*. Therefore, $t^* = T_1$ and (5.10) holds on $[T_1, S_n]$.

Proof of Proposition 5.2. The proof follows the same lines as the proof of the H^1 stability result (Proposition 3.1). However, we point out two main differences : first, we consider here stability of solution emanating exactly from the sum of N-solitary waves $u_n(S_n) = R(S_n)$, i.e. $\varepsilon(S_n) \equiv 0$. Second, the H^1 estimate is proved backwards in time on $[t^*, S_n]$. Since the gBBM equation is invariant under the transformation $x \to -x$, $t \to t$, if we simply reverse time, the solitary waves are sorted by decreasing sizes, and we cannot apply directly the proof of Section 3.

In what follows, u_n will be denoted by u for the sake of simplicity. We assume that (5.9) holds. Then, assuming α_1 small enough and T_1 large enough, we use the results of Section 2 concerning the modulation of the solution. We obtain:

- There exist unique C^1 functions

$$c_j \ :\ [t^*, S_n] \to (c^\star(p), +\infty), \quad x_j \ :\ [t^*, S_n] \to \mathbb{R},$$

such that if we define ε by

$$\varepsilon(t) = u(t) - \sum_{j=1}^{N} R_j(t), \text{ where } R_j(t) = \varphi_{c_j(t)}(\cdot - x_j(t)),$$

then the following properties are satisfied for all $j \in \{1, \cdots, N\}$, for all $t \in [t^*, S_n]$:

$$\int \varepsilon(t)(1 - \partial_x^2) R_j(t) dx = \int \varepsilon(t)(1 - \partial_x^2)\partial_x R_j(t) dx = 0, \tag{5.11}$$

$$\|\varepsilon(t)\|_{H^1} + \sum_{j=1}^{N} |c_j(t) - c_j^0| \leqslant K\alpha_1, \tag{5.12}$$

$$|\dot{c}_j(t)| + |\dot{x}_j(t) - c_j(t)| \leqslant K \left(\int e^{-\sigma_0 |x - x_j(t)|} \varepsilon^2(t) dx \right)^{\frac{1}{2}} + K e^{-\frac{\sigma_0 t}{4}}, \tag{5.13}$$

for some constant $K > 0$.

Following the strategy of section 3, we first control the variation of the velocities: there exist $C_1 > 0$, such that for all $t \in [t^*, S_n]$,

$$\sum_{j=1}^{N} |c_j(S_n) - c_j(t)| \leqslant C_1 \left(\|\varepsilon(t)\|_{H^1}^2 + e^{-\frac{\sigma_0}{12} t} \right). \tag{5.14}$$

To conclude the proof, it remains to estimate $\|\varepsilon(t)\|_{H^1}$, this is the main difficulty in the proof of Proposition 5.2. We need to introduce a new monotonicity property.

First, we introduce some notations. For $j = 2, \cdots, N$, and ψ_j defined as in section 2.2, we set

$$m_j^R(t) = \frac{1}{2} \int \left(u^2(t,x) + u_x^2(t,x) \right) \psi_j(t,x) dx,$$

and

$$E_j^R(t) = \frac{1}{2} \int \left(u^2(t,x) + \frac{2}{p+1} u^{p+1}(t,x) \right) \psi_j(t,x) dx. \tag{5.15}$$

In the same way we introduce for $j = 1, \cdots, N-1$

$$m_j^L(t) = \frac{1}{2} \int \left(u^2(t,x) + u_x^2(t,x) \right) (1 - \psi_{j+1}(t,x)) dx,$$

and

$$E_j^L(t) = \frac{1}{2} \int \left(u^2(t,x) + \frac{2}{p+1} u^{p+1}(t,x) \right) (1 - \psi_{j+1}(t,x)) dx.$$

Remark that

$$m(u(t)) = m_j^L(t) + m_{j+1}^R(t), \text{ and } E(u(t)) = E_j^L(t) + E_{j+1}^R(t). \tag{5.16}$$

In order to write a new monotonicity property, we introduce

$$\rho(t,x) = \frac{c_N^0}{c_N^0 - 1} + \sum_{j=1}^{N-1} \Big(\frac{c_j^0}{c_j^0 - 1} - \frac{c_{j+1}^0}{c_{j+1}^0 - 1} \Big) (1 - \psi_{j+1})(t,x),$$

and we define

$$\begin{aligned} H(t) &= \frac{1}{2} \int \big\{ \rho(t,x)(u^2(t,x) + u_x^2(t,x)) - (\rho(t,x) - 1)(u^2(t,x) \\ &\quad + \frac{2}{p+1} u^{p+1}(t,x)) \big\} dx. \end{aligned} \tag{5.17}$$

Observe that locally around $x_j(t)$, ρ is close to $\frac{c_j^0}{c_j^0 - 1}$, and thus ρ is close to $c_j^0(\rho - 1)$. This is crucial in the definition of $H(t)$. Note also that $\frac{c_j^0}{c_j^0 - 1}$ is decreasing with respect to j, and thus ρ is decreasing in x.

We can write H in terms of m_j^L and E_j^L:

$$\begin{aligned} H(t) &= \sum_{j=1}^{N} \Big[\Big(\frac{c_j^0}{c_j^0 - 1} - \frac{c_{j+1}^0}{c_{j+1}^0 - 1} \Big) m_j^L(t) - \Big(\frac{1}{c_j^0 - 1} - \frac{1}{c_{j+1}^0 - 1} \Big) E_j^L(t) \Big] \\ &= \sum_{j=1}^{N} r_j \Big(m_j^L(t) - E_j^L(t) \Big), \end{aligned} \tag{5.18}$$

where $c_{N+1}=0$, $r_j=\left(\frac{c_j^0}{c_j^0-1}-\frac{c_{j+1}^0}{c_{j+1}^0-1}\right)=\left(\frac{1}{c_j^0-1}-\frac{1}{c_{j+1}^0-1}\right)$, and $m_N^L(t)=m(u(t))$, $E_N^L(t)=E(u(t))$. We also introduce

$$(\tilde{\mathcal{L}}\varepsilon,\varepsilon)=\frac{1}{2}\int\Big(\rho(t,x)(\varepsilon^2+\varepsilon_x^2)(t,x)-(\rho(t,x)-1)(\varepsilon^2+pR^{p-1}\varepsilon^2)(t,x)\Big)dx.$$

It is easy to check that $\tilde{\mathcal{L}}$ satisfies the same positivity property as $\mathcal{L}_N$ under the orthogonality conditions on $\varepsilon(t)$ (see Lemma 2.3), thus there exists $\lambda_1>0$ such that

$$(\tilde{\mathcal{L}}\varepsilon,\varepsilon)\geqslant\lambda_1\|\varepsilon(t)\|_{H^1}^2. \tag{5.19}$$

Now we define, for $\gamma>0$:

$$H^\gamma(t)=H(t)+\frac{\gamma}{2}\int\rho(t,x)(u^2(t,x)+u_x^2(t,x))dx.$$

In order to fix γ, we need to remark that using (5.14), there exists $C_3>0$, such that for all $t\in[t^*,S_n]$,

$$\sum_{j=1}^N\frac{c_j^0}{c_j^0-1}\Delta_t^{S_n}m(R_j)\leqslant C_3\left(\|\varepsilon(t)\|_{H^1}^2+e^{-\frac{\sigma_0}{12}t}\right). \tag{5.20}$$

We now fix $\gamma\leqslant\min(1,\frac{\lambda_1}{2C_3})$. We claim that

LEMMA 5.2. *Under the above assumptions, there exists $C_4>0$, such that for all $t\in[t^*,S_n]$,*

$$\Delta_t^{S_n}H^\gamma\geqslant-C_4e^{-\frac{\sigma_0}{12}t}. \tag{5.21}$$

Let us assume for the moment this lemma and prove the control of the H^1-norm of ε. Therefore, by calculations similar to the ones in the proof of Lemma 2.2 we obtain from (5.21)

$$\begin{aligned}\sum_{j=1}^N\frac{1}{c_j^0-1}\Delta_t^{S_n}(c_j^0m(R_j)-E(R_j))+\Delta_t^{S_n}(\tilde{\mathcal{L}}\varepsilon,\varepsilon)\\+\frac{\gamma}{2}\Delta_t^{S_n}\int\rho(\varepsilon^2+\varepsilon_x^2)dx+\gamma\sum_{j=1}^N\frac{c_j^0}{c_j^0-1}\Delta_t^{S_n}m(R_j)\\\geqslant-K\Big(\|\varepsilon(t)\|_{H^1}^3+e^{-\sigma_0^2t/12}\Big),\end{aligned} \tag{5.22}$$

This implies, thanks to (3.11) and (5.14), that

$$\begin{aligned}\Delta_t^{S_n}(\tilde{\mathcal{L}}\varepsilon,\varepsilon)+\frac{\gamma}{2}\Delta_t^{S_n}\int\rho(\varepsilon^2+\varepsilon_x^2)dx+\gamma\sum_{j=1}^N\frac{c_j^0}{c_j^0-1}\Delta_t^{S_n}m(R_j)\\\geqslant-K\Big(\|\varepsilon(t)\|_{H^1}^3+e^{-\sigma_0^2t/12}\Big).\end{aligned} \tag{5.23}$$

Using (5.20), the fact that $\varepsilon(S_n)=0$, and (5.19) we find that

$$\lambda_1\|\varepsilon(t)\|_{H^1}^2\leqslant(\tilde{\mathcal{L}}\varepsilon(t),\varepsilon(t))\leqslant K\Big(\|\varepsilon(t)\|_{H^1}^3+e^{-\sigma_0^2t/12}\Big)+\gamma C_3\left(\|\varepsilon(t)\|_{H^1}^2+e^{-\frac{\sigma_0}{12}t}\right).$$

Now, using our choice of γ, we deduce that

$$\|\varepsilon(t)\|_{H^1}^2\leqslant Ce^{-\sigma_0^2t/12}, \tag{5.24}$$

for $\alpha_1 > 0$ sufficiently small.

Proof of Lemma 5.2. Thanks to identities (5.18), (5.16) and the conservation of mass and energy, we find that

$$\frac{d}{dt}H^\gamma(t) = -\sum_{j=1}^{N-1} r_j\Big((1+\gamma)\frac{d}{dt}m^R_{j+1}(t) - \frac{d}{dt}E^R_{j+1}(t)\Big). \tag{5.25}$$

Remark that from the monotonicity results related to mass (Lemma 2.1) and energy (Proposition 4.2 in [**6**]), we know that m_j^R is almost decreasing with respect to time and $-E_j^R$ is almost increasing with respect to time. This prevents us to conclude directly using H.

However, with $\gamma > 0$ we can prove that $H^\gamma(t)$ is almost increasing. Indeed, using identities (2.14) – (2.15), and setting $h = (1-\partial_x^2)^{-1}(u+u^p)$, we find as in the proof of Proposition 4.2 in [**6**]:

$$\frac{d}{dt}\Big((1+\gamma)m_j^R(t) - E_j^R(t)\Big) = -\frac{1+\gamma(1+y_j')}{2}\int u^2\psi_j' dx$$
$$-\frac{1-y_j'-\gamma}{p+1}\int u^{p+1}\psi_j' dx - (1+\gamma)\frac{y_j'}{2}\int (u_x^2)\psi_j' dx + \frac{1+2\gamma}{2}\int h^2\psi_j' dx$$
$$+\frac{3+2\gamma}{2}\int h_x^2\psi_j' dx - \frac{1+\gamma}{2}\int h^2\psi_j''' dx - (1+\gamma)\int u^p h\psi_j' dx.$$

As in Lemma 2.1, we set $I = [x_{j-1}(t)+L/4, x_j(t)-L/4]$ and $I^C = \mathbb{R}\setminus I$. For $x \in I$, taking T_1 sufficiently large, and $\alpha_1 > 0$ sufficiently small, we write $|u| = \frac{|h-h_{xx}|}{|1+u^{p-1}|}$. This allows us to compare $\int_I u^2\psi_j' dx$ with $\int_I h^2\psi_j' dx$. Note that the inequality

$$\frac{1+\gamma(1+y_j')}{2} > \frac{1+2\gamma}{2},$$

is crucial to treat these terms. We refer to Proposition 4.2 in [**6**] for more details on this calculation. This proves Lemma 5.2.

End of the proof of Proposition 5.2. To complete the proof of (5.10), we have to compare $c_j(t)$ with c_j^0 and $x_j(t)$ with $x_j^0 + c_j^0 t$. First, using (5.13) then (5.24), we find

$$|c_j(t) - c_j^0| = \Big|\int_t^{S_n} \dot{c}_j(s)ds\Big| \leqslant C\int_t^{S_n} e^{-\sigma_0^2 s/24} ds \leqslant Ce^{-\sigma_0^2 t/24}. \tag{5.26}$$

Similarly, using also (5.13) and (5.24), we find

$$|x_j(t) - x_j^0 - c_j^0 t| \leqslant Ce^{-\sigma_0^2 t/24}. \tag{5.27}$$

The last two identities imply that

$$\Big\|\sum_{j=1}^N R_j(t) - R(t)\Big\|_{H^1} \leqslant Ce^{-\sigma_0^2 t/24}. \tag{5.28}$$

Now remark that (5.24) with (5.28) imply (5.10). This achieves the proof of Proposition 5.2.

A consequence of Proposition 5.2 and its proof, which is also an important tool in the proof of the uniqueness, is the following :

Proposition 5.3. *Let $u(t)$ be an H^1 solution of (1.1) such that*

$$\lim_{t\to+\infty} \|u(t)-U(t)\|_{H^1}=0, \tag{5.29}$$

then there exists $C>0$ such that for all $t>0$,

$$\|u(t)-U(t)\|_{H^1}\leqslant Ce^{-\sigma_0^2 t/24}. \tag{5.30}$$

Proof. Let $T>0$ be such that

$$\|u(t)-U(t)\|_{H^1}\leqslant \alpha_1, \text{ for all } t\in[T,+\infty[,$$

where α_1 is as in Proposition 5.2. Remark that it suffices to prove estimate (5.30) for $t\in[T,+\infty[$. Consider a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n\in[T,+\infty[$ for all $n\in\mathbb{N}$, and $t_n\to+\infty$ when $n\to+\infty$. Following the proof of Proposition 5.2, we can prove that for all $t\in[T,t_n]$,

$$\|\varepsilon(t)\|_{H^1}\leqslant Ce^{-\sigma_0^2 t/24}+C\|u(t_n)-U(t_n)\|_{H^1}.$$

For $t\geqslant T$, taking the limit as $n\to+\infty$, we obtain $\|\varepsilon(t)\|_{H^1}\leqslant Ce^{-\sigma_0^2 t/24}$. We conclude as before.

5.3. Estimates of higher Sobolev norms.

Proposition 5.4. *For any $s\geqslant 1$, there exists $A_s>0$ such that for all $t\in[0,S_n]$,*

$$\|u_n(t)-R(t)\|_{H^s}\leqslant A_s e^{-2\gamma_1 t}, \tag{5.31}$$

where $\gamma_1=\sigma_0^2/96$.

Proof. We follow the strategy of [**11**], Proposition 5. We have already established in the previous section:

$$\|u_n(t)-R(t)\|_{H^1}\leqslant A_1 e^{-4\gamma_1 t}. \tag{5.32}$$

In what follows, u_n will be denoted by u. To treat the case $s=2$, we consider

$$G_2(t)=\int(u_x^2(t)+u_{xx}^2(t))dx.$$

Differentiating $G_2(t)$, using (1.1), and integrating by parts, we find

$$\begin{aligned} G_2'(t) &= 2\int(u_x u_{xt}+u_{xx}u_{xxt})dx=-2\int u_{xx}(1-\partial_x^2)u_t dx\\ &= 2\int u_{xx}(u+u^p)_x dx = p\int u^{p-1}(u_x^2)_x dx=-p(p-1)\int u^{p-2}u_x^3 dx. \end{aligned}$$

Replacing $u=v+R$ in the above expression of $G_2'(t)$, we find

$$G_2'(t)=-p(p-1)\int(v+R)^{p-2}(v_x+R_x)^3dx.$$

Now remark that $\int R_j^{p-2}R_{jx}^3dx=0$, and recall that the N solitary waves R_j are sufficiently decoupled, thus we deduce that

$$\left|\int R^{p-2}R_x^3dx\right|\leqslant Ce^{-8\gamma_1 t}.$$

We decompose

$$\int u^{p-2}u_x^3 dx = \int \big((v_x+R_x)^3 - R_x^3\big)(v+R)^{p-2}dx + \int R_x^3\big((v+R)^{p-2} - R^{p-2}\big) + \int R^{p-2}R_x^3 dx.$$

Recall that we have

$$\|v(t)\|_{H^1} \leqslant A_1 e^{-4\gamma_1 t}. \tag{5.33}$$

Thus, using the Gagliardo-Nirenberg inequality :

$$\int |v_x|^3 dx \leqslant C\Big(\int v_{xx}^2 dx\Big)^{1/4}\Big(\int v_x^2\Big)^{5/4},$$

we find that for all $t \in [0, S_n]$,

$$|G_2'(t)| \leqslant C(1+G_2(t))e^{-4\gamma_1 t}.$$

By direct integration, since

$$G_2(S_n) = \int (R_x^2(S_n) + R_{xx}^2(S_n))dx \tag{5.34}$$

is uniformly bounded, we find that G_2 is uniformly bounded on $[0, S_n]$. The function G_2 being bounded, integrating between t and S_n, we find, for all $t \in [0, S_n]$,

$$|G_2(t) - G_2(S_n)| \leqslant Ce^{-4\gamma_1 t}. \tag{5.35}$$

On the other hand, replacing $u = v + R$ in the expression of G_2, then integrating by parts, we find

$$G_2(t) = \int (v_x^2(t) + v_{xx}^2(t))dx + \int (R_x^2(t) + R_{xx}^2(t))dx + 2\int (-R_{xx}(t) + R_{xxxx}(t))v(t)dx.$$

Using this identity, (5.34) and $v(S_n) \equiv 0$, we compute

$$\int v_{xx}^2(t)dx \leqslant |G_2(t) - G_2(S_n)| + 2\|v(t)\|_{L^\infty}\int |R_{xx}(t)| + |R_{xxxx}(t)|dx + \Big|\int (R_x^2(t) + R_{xx}^2(t))dx - \int (R_x^2(S_n) + R_{xx}^2(S_n))dx\Big|.$$

Using estimates (5.33), (5.35), and the exponential decay of the N decoupled solitary waves R_j, we find that there exists a constant $A_2 > 0$ such that for all $t \in [0, S_n]$,

$$\|v(t)\|_{H^2} \leqslant A_1 e^{-2\gamma_1 t}. \tag{5.36}$$

This proves the case $s = 2$.

We prove by induction on s that the following holds for all $s \geqslant 3$:

$$\forall n \geqslant 0, \forall t \in [T_1, S_n], \quad \|v(t)\|_{H^s} \leqslant A_s e^{-2\gamma_1 t}. \tag{5.37}$$

We have already proved that (5.37) is true for $s = 2$. Now, we assume that it is true for $s - 1$, for $s \geqslant 3$, and we prove that it also holds for s.

We write the equation of $v = u - R$:

$$(1-\partial_x^2)v_t = -\Big(v + (R+v)^p - \sum_{j=1}^N \tilde{R}_j^p\Big)_x,$$

where $\tilde{R}_j(t,x) = \varphi_{c_j^0}(x - x_j^0 - c_j^0 t)$. We define

$$F_s(t) = \int (\partial_x^{s-1} v(t))^2 + (\partial_x^s v(t))^2.$$

We compute $F_s'(t)$:

$$\begin{aligned} F_s'(t) &= 2\int \Big(\partial_x^{s-1} v\, \partial_x^{s-1} v_t + \partial_x^s v\, \partial_x^s v_t\Big) dx \\ &= -2\int \partial_x^s v (1-\partial_x^2)\, \partial_x^{s-2} v_t\, dx \\ &= 2\int \partial_x^s v\, \partial_x^{s-1}\Big(v + (R+v)^p - \sum_{j=1}^N \tilde{R}_j^p\Big) dx. \end{aligned}$$

After integrations by parts, we observe that derivatives of v of order s disappear, and that the second-hand term is controlled by $K(F_{s-1}(t) + e^{-4\gamma_1 t})$, and thus, using the induction assumption, is controlled by $Ke^{-4\gamma_1 t}$. Integrating between t and $+\infty$, we obtain (5.37) for s. Thus, the proof of Proposition 5.4 is complete.

5.4. Uniqueness. We denote by $U(t)$ the solution of (1.1) constructed in Sections 5.1 and 5.2. Recall that it satisfies, for $\gamma_1 = \sigma_0^2/96$, and $A_s > 0$, for all $s \geqslant 0$, for all $t \geqslant 0$,

$$\|U(t) - R(t)\|_{H^s} \leqslant A_s e^{-\gamma_1 t}, \tag{5.38}$$

where $R(t) = \sum_{j=1}^N \tilde{R}_j(t)$, and $\tilde{R}_j(t,x) = \varphi_{c_j^0}(x - x_j^0 - c_j^0 t)$. In this section, we prove the following result, which implies the uniqueness part of Theorem 1.3.

Proposition 5.5. *Let $u(t)$ be an H^1 solution of (1.1) on $\mathbb{R}$. Assume that $u(t)$ satisfies*

$$\lim_{t\to+\infty} \|u(t) - U(t)\|_{H^1} = 0, \tag{5.39}$$

then $u(t) \equiv U(t)$.

Proof. Assume (5.39). By Proposition 5.3, for all $t > 0$,

$$\|u(t) - U(t)\|_{H^1} \leqslant Ce^{-\gamma_1 t}.$$

Let

$$z(t) = u(t) - U(t) \quad \text{so that for all } t \geqslant T_0, \quad \|z(t)\|_{H^1} \leqslant Ce^{-\gamma_1 t}. \tag{5.40}$$

We write the equation of $z(t)$:

$$(1-\partial_x^2)z_t = -\big((u-U) + u^p - U^p\big)_x = -\big(z + (z+U)^p - U^p\big)_x. \tag{5.41}$$

Step 1. Monotonicity property of the energy. The function ψ being defined in section 2.2, just before Lemma 2.1, we set

$$\tilde{\rho}(t,x) = \frac{c_N^0}{c_N^0 - 1} + \sum_{j=1}^{N-1}\Big(\frac{c_j^0}{c_j^0 - 1} - \frac{c_{j+1}^0}{c_{j+1}^0 - 1}\Big)(1-\psi)(x - \tilde{y}_j(t)),$$

where
$$\tilde{y}_j(t) = \frac{c_j^0 + c_{j+1}^0}{2} t + \frac{x_j^0 + x_{j+1}^0}{2}.$$
Observe that the function $\tilde{\rho}$ takes values close to $\frac{c_j^0}{c_j^0 - 1}$ for x close to $c_j^0 t + x_j^0$, and has large variations only in regions far away from the solitary waves (for instance we have for all j, for all $t \geqslant T_0$, $\|R_j(t)\tilde{\rho}_x(t)\|_{L^\infty} \leqslant Ce^{-\gamma_1 t}$). We also define a quantity related to the energy for z:
$$\tilde{H}(t) = \int \big\{\tilde{\rho}(t,x) z_x^2(t,x) - (\tilde{\rho}(t,x) - 1) F(t, z(t,x)) + z^2(t,x)\big\} dx$$
where
$$F(t,z) = 2\left[\frac{(z + U(t))^{p+1}}{p+1} - U^p(t) z - \frac{U^{p+1}(t)}{p+1}\right].$$
Note that
$$\tilde{H}(t) = \int \big\{\tilde{\rho}(t,x)(z_x^2(t,x) + z^2(t,x)) - (\tilde{\rho}(t,x) - 1)(F(t, z(t,x)) + z^2(t,x))\big\} dx.$$
We have the following property.

LEMMA 5.3. *There exists $K > 0$ such that for all $t \geqslant 0$,*
$$\tilde{H}(t) \leqslant K\, e^{-\gamma_1 t} \sup_{t' \geqslant t} \|z(t')\|_{H^1}^2. \tag{5.42}$$

Note that such a result is possible because we estimate the difference of two solutions, and not the difference of a solution with a sum of solitary waves as in Lemma 5.2.

Proof of Lemma 5.3. By direct calculations (we present formal calculations that can be justified by a regularization argument on $u(t)$, using well-posedness and continuous dependence of solutions of (1.1) in the Sobolev spaces H^s, $s \geqslant 1$),
$$\begin{aligned} \frac{d\tilde{H}}{dt} &= \int \big(z_x^2 - F(z)\big)\tilde{\rho}_t - 2\int z_t (z_x \tilde{\rho})_x - 2\int z_t\big((z+U)^p - U^p\big)(\tilde{\rho} - 1) \\ &\quad + 2\int z_t z - 2\int U_t\big((z+U)^p - pU^{p-1}z - U^p\big)(\tilde{\rho} - 1), \end{aligned}$$
where we have used $\frac{dF}{dz} = 2\big[(z+U)^p - U^p\big]$ and for any function $f(x)$ of class C^1,
$$\frac{d}{dt}\int z_x^2 f = 2\int z_{tx} z_x f = -2\int z_t (z_x f)_x.$$
Using the equation of z, we obtain the following expression for $d\tilde{H}/dt$:
$$\begin{aligned} \frac{d\tilde{H}}{dt} &= \int \big(z_x^2 - F(z)\big)\tilde{\rho}_t + 2\int (1 - \partial_x^2)^{-1}\big(z + (z+U)^p - U^p\big)_x (z_x \tilde{\rho})_x \\ &\quad + 2\int (1 - \partial_x^2)^{-1}\big(z + (z+U)^p - U^p\big)_x \big((z+U)^p - U^p\big)(\tilde{\rho} - 1) \\ &\quad - 2\int (1 - \partial_x^2)^{-1}\big(z + (z+U)^p - U^p\big)_x z \\ &\quad + 2\int U_x\big((z+U)^p - pU^{p-1}z - U^p\big)\tilde{\rho} \\ &\quad - 2\int \big(U_t(\tilde{\rho} - 1) + U_x\tilde{\rho}\big)\big((z+U)^p - pU^{p-1}z - U^p\big). \end{aligned}$$

Since $\lim_{t\to+\infty}\tilde{H}(t)=0$, in view of (5.42), our objective is to find a lower bound on $d\tilde{H}/dt$.

- First, we consider the term: $\int\big(z_x^2-F(z)\big)\tilde{\rho}_t$. Note that

$$\tilde{\rho}_t(t,x)=\sum_{j=1}^{N-1}\Big(\frac{c_j^0}{c_j^0-1}-\frac{c_{j+1}^0}{c_{j+1}^0-1}\Big)\frac{c_j^0+c_{j+1}^0}{2}\psi'\Big(x-\tilde{y}_j(t)\Big).$$

and

$$\tilde{\rho}_x(t,x)=-\sum_{j=1}^{N-1}\Big(\frac{c_j^0}{c_j^0-1}-\frac{c_{j+1}^0}{c_{j+1}^0-1}\Big)\psi'\Big(x-\tilde{y}_j(t)\Big).$$

Since $\psi'>0$ and $c_j^0>1+2\sigma_0$, we have

$$\forall t,x\in\mathbb{R}^+\times\mathbb{R},\quad \tilde{\rho}_x(t,x)<0,\quad (1+2\sigma_0)|\tilde{\rho}_x(t,x)|<\tilde{\rho}_t(t,x)<K|\tilde{\rho}_x(t,x)|. \tag{5.43}$$

We also note that

$$|F(z)|\leqslant C|z|^{p+1}+Cz^2|U|^{p-1}\leqslant C|z|^{p+1}+Cz^2|U-R|^{p-1}+Cz^2|R|^{p-1}, \tag{5.44}$$

and so $|F(z)|\leqslant C|z|^2\big(e^{-\gamma_1 t}+R\big)$, by (5.38) and (5.40). Moreover, $\|R\tilde{\rho}_x\|_{L^\infty}\leqslant Ce^{-\gamma_1 t}$, so that we obtain

$$\int\big(z_x^2-F(z)\big)\tilde{\rho}_t\geqslant(1+2\sigma_0)\int z_x^2|\tilde{\rho}_x|-Ce^{-\gamma_1 t}\int z^2. \tag{5.45}$$

- Second, we take care of the quadratic terms, i.e. $2\int(1-\partial_x^2)^{-1}z_x(z_x\tilde{\rho})_x$ and $-2\int(1-\partial_x^2)^{-1}z_xz$. Setting $a=(1-\partial_x^2)^{-1}z$, we see immediately that the second term is zero. For the first term, we have

$$2\int(1-\partial_x^2)^{-1}z_x(z_x\tilde{\rho})_x=-2\int a_{xx}((a_x-a_{xxx})\tilde{\rho})=\int(a_x^2-a_{xx}^2)\tilde{\rho}_x\geqslant-\int a_x^2|\tilde{\rho}_x|.$$

Note that

$$\begin{aligned}&-\int z_x^2\tilde{\rho}_x=-\int(-a_{xxx}+a_x)^2\tilde{\rho}_x\\&=-\int a_{xxx}^2\tilde{\rho}_x-2\int a_{xx}^2\tilde{\rho}_x-\int a_x^2\tilde{\rho}_x+\int a_x^2\tilde{\rho}_{xxx}\geqslant\int a_x^2|\tilde{\rho}_x|-\int a_x^2|\tilde{\rho}_{xxx}|.\end{aligned}$$

Since $|\tilde{\rho}_{xxx}|\leqslant\frac{\sigma_0^2}{9}|\tilde{\rho}_x|$, by the properties of ψ, we obtain $\int a_x^2|\tilde{\rho}_x|\leqslant\frac{1}{1-\frac{\sigma_0^2}{9}}\int z_x^2|\tilde{\rho}_x|\leqslant\Big(1+2\frac{\sigma_0^2}{9}\Big)\int z_x^2|\tilde{\rho}_x|$, and so

$$2\int(1-\partial_x^2)^{-1}z_x(z_x\tilde{\rho})_x\geqslant-\Big(1+2\frac{\sigma_0^2}{9}\Big)\int z_x^2|\tilde{\rho}_x|.$$

This term can then be controlled by (5.45).

- Third, we consider the following terms in the expression of $\frac{d\tilde{H}}{dt}$:

$$
\begin{aligned}
&2\int(1-\partial_x^2)^{-1}\big((z+U)^p-U^p\big)_x[(z_x\tilde{\rho})_x-z]\\
&+2\int\big[(1-\partial_x^2)^{-1}z_x\big]\big((z+U)^p-U^p\big)(\tilde{\rho}-1)\\
&=2\int(1-\partial_x^2)^{-1}\big((z+U)^p-U^p\big)[-(z_x\tilde{\rho})_{xx}+z_x\tilde{\rho}]\\
&+2\int\Big\{\big[(1-\partial_x^2)^{-1}z_x\big](\tilde{\rho}-1)-(1-\partial_x^2)^{-1}[z_x(\tilde{\rho}-1)]\Big\}\big((z+U)^p-U^p\big).
\end{aligned}
$$

The first term becomes:

$$
2\int\big((z+U)^p-U^p\big)z_x\tilde{\rho}, \tag{5.46}
$$

it will be combined with some other term later on. The second term can be controlled completely. Indeed, we have $-a_{xx}+a=z$ and so $-(a_x(\tilde{\rho}-1))_{xx}+a_x(\tilde{\rho}-1)=z_x(\tilde{\rho}-1)-2a_{xx}\tilde{\rho}_x-a_x\tilde{\rho}_{xx}$. Thus

$$
\big[(1-\partial_x^2)^{-1}z_x\big](\tilde{\rho}-1)-(1-\partial_x^2)^{-1}[z_x(\tilde{\rho}-1)]=-(1-\partial_x^2)^{-1}(2a_{xx}\tilde{\rho}_x+a_x\tilde{\rho}_{xx}).
$$

Therefore, this second term is $-2\int(1-\partial_x^2)^{-1}(2a_{xx}\tilde{\rho}_x+a_x\tilde{\rho}_{xx})\big((z+U)^p-U^p\big)$. It is known that $f\leqslant g$ implies $(1-\partial_x^2)^{-1}f\leqslant(1-\partial_x^2)^{-1}g$; moreover, for $K>2$, $(1-\partial_x^2)^{-1}e^{-\frac{|x-x_0|}{K}}\leqslant 2e^{-\frac{|x-x_0|}{K}}$. Since $|\tilde{\rho}_{xx}(t,x)|+|\tilde{\rho}_x(t,x)|\leqslant C\sum_{j=1}^N e^{-\frac{\sigma_0}{3}|x-\tilde{y}_j(t)|}$, we obtain

$$
|(1-\partial_x^2)^{-1}(2a_{xx}\tilde{\rho}_x+a_x\tilde{\rho}_{xx})|\leqslant C(\|a_{xx}\|_{L^\infty}+\|a_x\|_{L^\infty})\sum_{j=1}^N e^{-\frac{\sigma_0}{3}|x-\tilde{y}_j(t)|}.
$$

Moreover, $\|a_{xx}\|_{L^\infty}+\|a_x\|_{L^\infty}\leqslant C\|z\|_{H^1}$, and $|(z+U)^p-U^p|\leqslant C|z||U|+C|z|^p$, so that

$$
\begin{aligned}
&\left|2\int(1-\partial_x^2)^{-1}(2a_{xx}\tilde{\rho}_x+a_x\tilde{\rho}_{xx})\big((z+U)^p-U^p\big)\right|\\
&\leqslant C\|z\|_{H^1}^2\int\sum_{j=1}^N e^{-\frac{\sigma_0}{3}|x-\tilde{y}_j(t)|}|U(t,x)|dx+C\|z\|_{H^1}^{p+1}\leqslant Ce^{-\gamma_1 t}\|z\|_{H^1}^2.
\end{aligned}
$$

- Fourth, we consider the term $-2\int(U_t(\tilde{\rho}-1)+U_x\tilde{\rho})\big((z+U)^p-pU^{p-1}z-U^p\big)$. We have

$$
\begin{aligned}
&\|U_t(\tilde{\rho}-1)+U_x\tilde{\rho}\|_{L^\infty}\leqslant\|(1-\partial_x^2)^{-1}\big[(U+U^p)-(R+R^p)\big]_x(\tilde{\rho}-1)\|_{L^\infty}\\
&+\|(1-\partial_x^2)^{-1}(R+R^p)_x(\tilde{\rho}-1)-R_x\tilde{\rho}\|_{L^\infty}+\|R_x\tilde{\rho}-U_x\tilde{\rho}\|_{L^\infty}.
\end{aligned}
$$

The first and the third terms are controlled by $\|U-R\|_{H^2}\leqslant Ce^{-\gamma_1 t}$. For the second term, we have $(1-\partial_x^2)^{-1}(\tilde{R}_j+\tilde{R}_j^p)=c_j\tilde{R}_j$, so that

$$
\begin{aligned}
&\|(1-\partial_x^2)^{-1}(R+R^p)_x(\tilde{\rho}-1)-R_x\tilde{\rho}\|_{L^\infty}\\
&\leqslant C\|(1-\partial_x^2)^{-1}\big(R^p-\sum_{j=1}^N\tilde{R}_j^p\big)_x\|_{L^\infty}+\sum_{j=1}^N\|c_j(\tilde{\rho}-1)\tilde{R}_{jx}-\tilde{\rho}\tilde{R}_{jx}\|_{L^\infty}\leqslant Ce^{-\gamma_1 t}.
\end{aligned}
$$

-Fifth, we consider $2\int(1-\partial_x^2)^{-1}\big((z+U)^p-U^p\big)_x\big((z+U)^p-U^p\big)(\tilde\rho-1)$. Setting $b=(1-\partial_x^2)^{-1}\big((z+U)^p-U^p\big)$, this term is equal to

$$2\int b_x(-b_{xx}+b)(\tilde\rho-1)=\int(b_x^2-b^2)\tilde\rho_x\geqslant-\int b_x^2|\tilde\rho_x|.$$

Moreover, as before, we have

$$\int b_x^2|\tilde\rho_x|\leqslant 2\int\big((z+U)^p-U^p\big)_x^2|\tilde\rho_x|\leqslant Ce^{-\gamma_1 t}\|z(t)\|_{H^1}^2.$$

- Finally, there remains only the following term:

$$2\int U_x\big((z+U)^p-pU^{p-1}z-U^p\big)\tilde\rho. \tag{5.47}$$

We have to combine it with (5.46). Indeed, if we sum them, terms which are at least cubic in z are controlled by $C\|z\|_{L^\infty}\|z\|_{H^1}^2\leqslant Ce^{-\gamma_1 t}\|z(t)\|_{H^1}^2$. On the other hand, the quadratic terms in z are

$$2\int pzU^{p-1}z_x\tilde\rho+2\int U_x\frac{p(p-1)}{2}U^{p-2}z^2\tilde\rho=p\int(z^2U^{p-1})_x\tilde\rho=-p\int z^2U^{p-1}\tilde\rho_x,$$

and this is controlled by $Ce^{-\gamma_1 t}\|z(t)\|_{H^1}^2$.

In conclusion of these estimates, we have

$$\frac{d\tilde H}{dt}\geqslant-Ce^{-\gamma_1 t}\|z(t)\|_{H^1}^2\geqslant-Ce^{-\gamma_1 t}\sup_{t'>t}\|z(t)\|_{H^1}^2,$$

which proves Lemma 5.3 by integration between t_1 and $+\infty$, and using the fact that $\lim_{t\to+\infty}\tilde H(t)=0$.

Step 2. Control of the $(1-\partial_x^2)\tilde R_j$ directions. We claim the following estimate: for all $t\geqslant T_0$,

$$\sum_{j=1}^N\Big|\int z(t)(1-\partial_x^2)\tilde R_j(t)\Big|\leqslant Ce^{-\gamma_1 t}\sup_{t'\geqslant t}\|z(t')\|_{L^2}. \tag{5.48}$$

We prove (5.48) by using the equation of $z(t)$. By $\frac{d}{dt}\tilde R_j(t)=-c_j^0\tilde R_{jx}(t)$, and the equation of $z(t)$, we have

$$\begin{aligned}\frac{d}{dt}\int z(t)(1-\partial_x^2)\tilde R_j(t)&=\int(1-\partial_x^2)z_t\tilde R_j+\int(1-\partial_x^2)zR_{jt}\\&=\int\big(c_j^0z_{xx}-(c_j^0-1)z+\big[(z+U)^p-U^p\big]\big)\tilde R_{jx}.\end{aligned}$$

Next, note that by differentiating equation (1.5), we have $-c(\varphi_{cx})_{xx}+(c-1)\varphi_{cx}-p\varphi_c^{p-1}\varphi_{cx}=0$. Thus,

$$\begin{aligned}&\Big|\int\big(-c_j^0z_{xx}+(c_j^0-1)z-\big[(z+U)^p-U^p\big]\big)\tilde R_{jx}\Big|\\&\leqslant\Big|\int z\big(-c_j^0(\tilde R_{jx})_{xx}+(c_j^0-1)\tilde R_{jx}-pR_j^{p-1}\tilde R_{jx}\big)\Big|+p\Big|\int z\big(\tilde R_j^{p-1}-R^{p-1}\big)\tilde R_{jx}\Big|\\&\qquad+p\Big|\int z\big(R^{p-1}-U^{p-1}\big)\tilde R_{jx}\Big|+\Big|\int\big((z+U)^p-U^p-pU^{p-1}z\big)\tilde R_{jx}\Big|\\&\leqslant Ce^{-\gamma_1 t}\|z(t)\|_{L^2}+\|z(t)\|_{L^2}^2\leqslant Ce^{-\gamma_1 t}\|z(t)\|_{L^2}.\end{aligned}$$

Therefore, $\left|\frac{d}{dt}\int z(t)(1-\partial_x^2)\tilde{R}_j(t)\right| \leqslant Ce^{-\gamma_1 t}\|z(t)\|_{L^2}$. By integration in time between t and $+\infty$, since $\lim_{t\to+\infty}\int z(t)(1-\partial_x^2)\tilde{R}_j(t)=0$, we obtain (5.48).

Step 3. Control of the $(1-\partial_x^2)\tilde{R}_{jx}$ directions and conclusion. Now, we define

$$\tilde{z}(t) = z(t) + \sum_{j=1}^N a_j(t)\tilde{R}_{jx}(t), \quad \text{where} \quad a_j(t) = -\frac{\int z(t)(1-\partial_x^2)\tilde{R}_{jx}(t)}{\int \tilde{R}_{jxx}^2(t)+\tilde{R}_{jx}^2(t)},$$

so that $\int \tilde{z}(t)(1-\partial_x^2)\tilde{R}_{jx}(t)=0$. Note that for some $C_1, C_2>0$,

$$C_1\|z\|_{H^1} \leqslant \|\tilde{z}\|_{H^1} + \sum_{j=1}^N |a_j(t)| \leqslant C_2\|z\|_{H^1}.$$

We claim the following lemma.

LEMMA 5.4. *For all $t \geqslant 0$,*

(5.49) $$\|\tilde{z}(t)\|_{H^1} + \sum_{j=1}^N |a_j(t)| \leqslant Ce^{-\gamma_1 t}\sup_{t'>t}\|z(t')\|_{H^1}.$$

Assuming this claim, we have $\|z(t)\|_{H^1} \leqslant Ce^{-\frac{\gamma_1}{2}t}\sup_{t'>t}\|z(t')\|_{H^1}$ for all t large enough, which implies $z\equiv 0$ and thus $u(t)\equiv U(t)$.

Proof of Lemma 5.4. This proof proceeds in two steps. First, we prove the estimate of $\|\tilde{z}(t)\|$, and second we consider $|a_j(t)|$.

Let

$$\mathcal{L}_{\tilde{\rho}} z = -\tilde{\rho} z_{xx} - (\tilde{\rho}-1)pR^{p-1}z + z,$$

where $\tilde{\rho}$ is defined in Step 1. We have, by direct calculations

$$\begin{aligned}\int \tilde{\rho} z_x^2 - p(\tilde{\rho}-1)R^{p-1}z^2 + z^2 &= \int (\mathcal{L}_{\tilde{\rho}} z)z + \frac{1}{2}\int z^2\tilde{\rho}_{xx} = \int (\mathcal{L}_{\tilde{\rho}}\tilde{z})\tilde{z} \\ &- \sum_{j=1}^N a_j \int (\mathcal{L}_{\tilde{\rho}}\tilde{z})\tilde{R}_{jx} - \sum_{j=1}^N a_j \int \tilde{z}(\mathcal{L}_{\tilde{\rho}}\tilde{R}_{jx}) + \sum_{j,k=1}^N a_j a_k \int (\mathcal{L}_{\tilde{\rho}}\tilde{R}_{jx})\tilde{R}_{kx} \\ &+ \frac{1}{2}\int \tilde{z}^2\tilde{\rho}_{xx} - \sum_{j=1}^N a_j \int \tilde{z}\tilde{R}_{jx}\tilde{\rho}_{xx} + \frac{1}{2}\int \Big(\sum_{j=1}^N a_j\tilde{R}_{jx}\Big)^2\tilde{\rho}_{xx}.\end{aligned}$$

Now, we have

$$\begin{aligned}\mathcal{L}_{\tilde{\rho}}\tilde{R}_{jx} &= -\frac{c_j^0}{c_j^0-1}(\tilde{R}_{jx})_{xx} - \frac{1}{c_j^0-1}p\tilde{R}_j^{p-1}\tilde{R}_{jx} + \tilde{R}_{jx} \\ &\quad -\Big(\tilde{\rho} - \frac{c_j^0}{c_j^0-1}\Big)(\tilde{R}_{jx})_{xx} - \Big((\tilde{\rho}-1) - \frac{1}{c_j^0-1}\Big)p\tilde{R}_j^{p-1}\tilde{R}_{jx} \\ &\quad -p\big(R^{p-1} - R_j^{p-1}\big)\tilde{R}_{jx}(\tilde{\rho}-1).\end{aligned}$$

Since $-c_j^0(\tilde{R}_{jx})_{xx} + (c_j^0-1)\tilde{R}_{jx} - p\tilde{R}_j^{p-1}\tilde{R}_{jx} = 0$, and $|c_j^0\tilde{\rho} - (c_j^0-1)|e^{-\sqrt{\sigma_0}|x-x_j^0-c_j^0t|} \leqslant Ce^{-\gamma_1 t}e^{-\frac{\sqrt{\sigma_0}}{2}|x-x_j^0-c_j^0t|}$, we have

(5.50) $$|\mathcal{L}_{\tilde{\rho}}(\tilde{R}_{jx})| \leqslant Ce^{-\gamma_1 t}e^{-\frac{\sqrt{\sigma_0}}{2}|x-x_j^0-c_j^0t|}.$$

Note also that

$$\int (\mathcal{L}_{\tilde\rho}\tilde z)\tilde R_{jx} = \int \tilde z(\mathcal{L}_{\tilde\rho}\tilde R_{jx}) - \int \tilde z(\tilde\rho_{xx}\tilde R_{jx} + 2\tilde\rho_x \tilde R_{jxx}),$$

so that by the properties of $\tilde\rho_x$ and $\tilde\rho_{xx}$, and (5.50),

$$\Big|\int (\mathcal{L}_{\tilde\rho}\tilde z)\tilde R_{jx}\Big| + \Big|\int \tilde z(\mathcal{L}_{\tilde\rho}\tilde R_{jx})\Big| \leqslant Ce^{-\gamma_1 t}\|\tilde z\|_{L^2}.$$

Finally, we obtain

$$\begin{aligned}&\int \tilde\rho \tilde z_x^2 - p(\tilde\rho-1)R^{p-1}\tilde z^2 + \tilde z^2\\ &\leqslant \int \big[\tilde\rho z_x^2 - p(\tilde\rho-1)R^{p-1}z^2 + z^2\big] + Ce^{-\gamma_1 t}\sum_{j=1}^N a_j^2 + Ce^{-\gamma_1 t}\|\tilde z\|_{L^2}^2\\ &\leqslant \tilde H(z) + Ce^{-\gamma_1 t}\sum_{j=1}^N a_j^2 + Ce^{-\gamma_1 t}\|\tilde z\|_{L^2}^2,\end{aligned}$$

and so by step 1,

$$\int \tilde\rho \tilde z_x^2 - p(\tilde\rho-1)R^{p-1}\tilde z^2 + \tilde z^2 \leqslant Ce^{-\gamma_1 t}\sup_{t'\geqslant t}\|z(t')\|_{H^1}^2. \tag{5.51}$$

Since $\int \tilde R_j \tilde R_{jx} = 0$ by parity properties, we have

$$\int \tilde z(t)\tilde R_j(t) = \int z(t)\tilde R_j(t) + \sum_{k=1,\dots,N;k\neq j} a_k(t)\int \tilde R_j(t)\tilde R_{kx}(t),$$

and so $\sum_{j=1}^N\big|\int \tilde z(t)\tilde R_j(t)\big| \leqslant Ce^{-\gamma_1 t}\sup_{t'\geqslant t}\|z(t')\|_{H^1}$. By a property similar to Lemma 2.3, we have, for $\lambda_2>0$,

$$\begin{aligned}\int \tilde\rho \tilde z_x^2 - p(\tilde\rho-1)R^{p-1}\tilde z^2 + \tilde z^2 &\geqslant \lambda_2\|\tilde z\|_{H^1}^2\\ &-\frac{1}{\lambda_2}\sum_{j=1}^N\left(\Big|\int \tilde z(1-\partial_x^2)\tilde R_j\Big|^2 + \Big|\int \tilde z(1-\partial_x^2)\tilde R_{jx}\Big|^2\right).\end{aligned}$$

Therefore by (5.51), the orthogonality $\int \tilde z(1-\partial_x^2)\tilde R_{jx} = 0$, and the control on $\int \tilde z(1-\partial_x^2)\tilde R_j$, we obtain

$$\|\tilde z(t)\|_{H^1}^2 \leqslant Ce^{-\gamma_1 t}\sup_{t'\geqslant t}\|z(t')\|_{H^1}^2.$$

Second, we prove (5.49) for $a_j(t)$, using the equation of $\tilde z$ and integration in time. Note that $\tilde z$ satisfies

$$\begin{aligned}(1-\partial_x^2)\tilde z_t + (\tilde z + pU^{p-1}\tilde z)_x &= \sum_{k=1}^N a_k'(t)(1-\partial_x^2)\tilde R_{kx}\\ &+\sum_{k=1}^N a_k(t)\big(c_k^0(\tilde R_{kx})_{xx} - (c_k^0-1)\tilde R_{kx} + pU^{p-1}\tilde R_{kx}\big)_x\\ &-\big((z+U)^p - pU^{p-1}z - U^p\big)_x.\end{aligned}$$

Since $\int \tilde{z}(1-\partial_x^2)\tilde{R}_{jx}=0$, we have

$$0=\frac{d}{dt}\int \tilde{z}(1-\partial_x^2)\tilde{R}_{jx}=\int(1-\partial_x^2)\tilde{z}_t\tilde{R}_{jx}-c_j^0\int\tilde{z}(1-\partial_x^2)\tilde{R}_{jxx},$$

thus, by integration by parts, and using $\left|c_j^0(\tilde{R}_{kx})_{xx}-(c_j^0-1)\tilde{R}_{jx}+pU^{p-1}\tilde{R}_{jx}\right|\leqslant Ce^{-\gamma_1 t}$, as before, we have

$$\left|a_j'(t)\int(\tilde{R}_{jxx}^2+\tilde{R}_{jx}^2)+\sum_{k=1,\dots,N;k\neq j}a_k'(t)\int\tilde{R}_{jx}(1-\partial_x^2)\tilde{R}_{kx}\right|$$
$$\leqslant C\|\tilde{z}(t)\|_{H^1}+Ce^{-\gamma_1 t}\sum_{k=1}^N|a_k|+C\|z(t)\|_{H^1}^2\leqslant Ce^{-\gamma_1 t}\|z(t)\|_{H^1}.$$

Thus, $|a_j'(t)|\leqslant Ce^{-\gamma_1 t}\|z(t)\|_{H^1}$, for any $j\in\{1,\dots,N\}$, and by integration between t and $+\infty$, since $\lim_{t\to+\infty}a_j(t)=0$, we obtain

$$|a_j(t)|\leqslant Ce^{-\gamma_1 t}\sup_{t'\geqslant t}\|z(t')\|_{H^1},$$

which completes the proof of (5.49).

Appendix A. Proof of Lemma 2.3

Lemma 2.3 is a generalization of the positivity of the "orbital stability operator"

$$\mathcal{L}_c=(-E''+cm'')(\varphi_c)=-c\partial_x^2+(c-1)-p\varphi_c^{p-1}, \tag{A.1}$$

under suitable orthogonality conditions ; the proof uses the "continuity" of this family of self-adjoint operators with respect to perturbations of their potentials and coefficients, the arguments are very similar to those used in [**14**]. We present here a complete proof for the reader's convenience.

Let $c>c^\star(p)$, from Proposition 5.2 in [**21**] there exists $C>0$ such that if $v\in H^1(\mathbb{R})$ satisfies

$$(v,(1-\partial_x^2)\varphi_c)=(v,(1-\partial_x^2)\partial_x\varphi_c)=0, \tag{A.2}$$

then

$$(\mathcal{L}_c v,v)\geqslant C\|v\|_{H^1}^2. \tag{A.3}$$

First we give a local version of $(A.3)$. Let $\Phi\in C^2(\mathbb{R})$, $\Phi(x)=\Phi(-x)$, $\Phi'\leqslant 0$ on $\mathbb{R}^+$, be such that $e^{-x}\leqslant\Phi(x)\leqslant 3e^{-x}$ for all $x\in\mathbb{R}^+$ and

$$\Phi(x)=1 \text{ on } [0,1], \Phi(x)=e^{-x} \text{ on } [2,+\infty].$$

Let $\Phi_B(x)=\Phi(\frac{x}{B})$. There exists $B_0>0$, such that for all $B>B_0$, if $v\in H^1(\mathbb{R})$ satisfies $(A.2)$ then

$$\int\Phi_B(x)(cv_x^2+(c-1)v^2-pU_c^{p-1}v^2)dx\geqslant C/2\int\Phi_B(x)(v_x^2+v^2)dx. \tag{A.4}$$

In order to prove $(A.4)$, we need a perturbed version of $(A.3)$. We set $\mathcal{V}_1(c)=(1-\partial_x^2)\varphi_c$ and $\mathcal{V}_2(c)=(1-\partial_x^2)\partial_x\varphi_c$, note that $(\mathcal{V}_1(c),\mathcal{V}_2(c))=0$. One can easily prove, using the orthogonal projection on $span(\mathcal{V}_1(c),\mathcal{V}_2(c))^\perp$, that there exists $\delta>0$ such that if

$$|(v,\mathcal{V}_1(c))|+|(v,\mathcal{V}_2(c))|\leqslant\delta\|v\|_{H^1} \tag{A.5}$$

then

$$(\mathcal{L}_c v, v) \geqslant \frac{3C}{4}\|v\|_{H^1}^2. \tag{A.6}$$

Now, we write

$$\int \Phi_B(x)(cv_x^2 + (c-1)v^2 - p\varphi_c^{p-1}v^2)dx$$

$$= (\mathcal{L}_c(\sqrt{\Phi_B}v), \sqrt{\Phi_B}v) - \frac{c}{4}\int \frac{(\Phi_B')^2}{\Phi_B}v^2dx + \frac{c}{2}\int \Phi_B^{''}v^2dx, \tag{A.7}$$

and

$$\|\sqrt{\Phi_B}v\|_{H^1}^2 = \int \Phi_B(x)(v_x^2+v^2)dx + \frac{1}{4}\int \frac{(\Phi_B')^2}{\Phi_B}v^2dx - \frac{1}{2}\int \Phi_B^{''}v^2dx. \tag{A.8}$$

Note that for $B>0$ large enough $\sqrt{\Phi_B}v$ satisfies $(A.5)$, thus $(A.6)$ and the identities $(A.7)$, $(A.8)$ imply

$$\int \Phi_B(x)(cv_x^2 + (c-1)v^2 - p\varphi_c^{p-1}v^2)dx$$

$$\geqslant \frac{3C}{4}\int \Phi_B(x)(v_x^2+v^2)dx + \frac{3C/4-c}{4}\int \frac{(\Phi_B'')^2}{\Phi_B}v^2dx - \frac{3C/4-c}{2}\int \Phi_B^{''}v^2dx, \tag{A.9}$$

Now, note that for $B \geqslant 1$, $|\Phi_B{}'(x)| + |\Phi_B^{''}(x)| \leqslant \frac{C_1}{B}\Phi_B(x)$, where $C_1>0$ does not depend on B. Hence, taking $B>0$ sufficiently large, $(A.9)$ implies $(A.4)$.

Now, remark that $c(t,x) = \sum_{j=1}^N c_j(t)\zeta_j(t,x)$, where $\zeta_1(t,x) = 1-\psi(x-y_2(t))$, for $j \in [[2,N]]$ $\zeta_1(t,x) = \psi(x-y_j(t)) - \psi(x-y_{j+1}(t))$, $\zeta_N(t,x) = \psi(x-y_N(t))$, hence

$$\begin{aligned}(\mathcal{L}_N\varepsilon,\varepsilon) &= \\ &= \int \left(c(t,x)\varepsilon_x^2(t,x) + (c(t,x)-1)\varepsilon^2(t,x) - pR^{p-1}(t)\varepsilon^2(t)\right)dx \\ &= \sum_{j=1}^N \int \Big(\zeta_j(t)c_j(t)\varepsilon_x^2(t) + (\zeta_j(t)c_j(t)-1)\varepsilon^2(t) - pR_j^{p-1}(t)\varepsilon^2(t)\Big)dx.\end{aligned} \tag{A.10}$$

Thus it is clear that $\mathcal{L}_N$ is the sum of N local operators similar to $(A.4)$, which leads us to the following decomposition :

$$\begin{aligned}(\mathcal{L}_N\varepsilon,\varepsilon) \quad = \quad & \sum_{j=1}^N \int \Phi_B(x-x_j(t))\Big(c_j(t)\varepsilon_x^2 + (c_j(t)-1)\varepsilon^2 - pR_j^{p-1}\varepsilon^2\Big)dx \\ & -p\int\left(R^{p-1} - \sum_{j=1}^N \Phi_B(x-x_j(t))R_j^{p-1}\right)\varepsilon^2(t)dx \\ & +\sum_{j=1}^N \int \Phi_B(x-x_j(t))(c(t,x)-c_j(t))\left(\varepsilon_x^2(t)+\varepsilon^2(t)\right)dx \\ & +\int\left(1-\sum_{j=1}^N \Phi_B(x-x_j(t))\right)\left(c(t,x)\varepsilon_x^2 + (c(t,x)-1)\varepsilon^2\right)dx.\end{aligned}$$

Recall that $c_j(t) \geqslant c^\star(p)+\sigma_0$, hence there exists $C_0>0$ depending only on σ_0 such that $(A.4)$ holds with $c_j(t)$ for all j, this and the orthogonality conditions satisfied by ε imply that for all j

$$\begin{aligned}&\int \Phi_B(x-x_j)\left(c_j\varepsilon_x^2+(c_j-1)\varepsilon^2-pR_j^{p-1}\varepsilon^2\right)dx\\ &\geqslant \frac{C_0}{2}\int \Phi_B(x-x_j)(\varepsilon_x^2+\varepsilon^2)dx.\end{aligned}$$

Let $B>B_0>0$ and $L_4=4kB$, where B_0 and the integer k are to be chosen later. Recall that $\Phi_B(x)=1$ for $|x|\leqslant B$, hence the exponential decay of R_j implies that

$$\begin{aligned}0 &\leqslant R^{p-1}-\sum_{j=1}^N \Phi_B(x-x_j(t))R_j^{p-1}\\ &\leqslant \|R\|^{p-1}_{L^\infty(|x-x_j(t)|>B)}+C\sum_{i\neq j}R_iR_j\leqslant Ce^{-\sigma_0 B}.\end{aligned}$$

Let us estimate the third term in the above decomposition. Remark that $|x-x_j(t)|\leqslant kB$ implies that for all i $|x-y_i(t)|\geqslant kB$ since $|x_j-y_j(t)|\geqslant L_4/2\geqslant 2kB$; hence the decay of ψ and $1-\psi$ imply that for $|x-x_j(t)|\leqslant kB$

$$|\zeta_j(t,x)-1|+\sum_{i\neq j}|\zeta_i(t,x)|\leqslant Ce^{-\sigma_0 kB}.$$

Finally, this inequality and the decay of $\Phi_B(x)$ imply

$$\begin{aligned}|\Phi_B(x-x_j(t))(c(t,x)-c_j(t))| &\leqslant \|c(t,x)-c_j(t)\|_{L^\infty(|x-x_j(t)|\leqslant kB)}+Ce^{-k}\\ &\leqslant Ce^{-\sigma_0 kB}+Ce^{-k}.\end{aligned}$$

Let $\lambda_0=1/2\min(\sigma_0,C_0/2)$, gathering the above estimates and taking B and k large enough imply

$$\begin{aligned}(\mathcal{L}_N\varepsilon,\varepsilon) &\geqslant 2\lambda_0\int(\varepsilon_x^2+\varepsilon^2)dx-C(e^{-\sigma_0 B}+Ce^{-k})\int(\varepsilon_x^2+\varepsilon^2)dx\\ &\geqslant \lambda_0\int(\varepsilon_x^2+\varepsilon^2)dx.\end{aligned}$$

This achieves the proof of Lemma 2.3.

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