Global regularity of the two-dimensional regularized MHD equations

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ABSTRACT. In this paper, we consider the Cauchy problem of the two-dimensional regularized incompressible magnetohydrodynamics equations. The main objective of this paper is to establish the global regularity of classical solutions of the magnetohydrodynamics equations with the minimal dissipation. Consequently, our results significantly improve the previous works.

CONTENTS

1. Introduction and main results

Whether or not the smooth solutions of the classical two-dimensional (2D) incompressible magnetohydrodynamics (MHD) equations with only velocity dissipation or with only magnetic diffusion develop finite time singularities is an extremely difficult problem and remains open. The main difficulty arising here is a strong impact of the higher modes to the leading order dynamics through the nonlinearity which a priori may destroy the regularity of a solution and thus lead to the formation of singularities. To bypass this difficulty, considerable models have been proposed in order to capture the leading dynamics of the flow on the one hand and somehow suppress the higher modes on the other hand, see for example

[**7, 13, 15, 14, 27, 24, 45, 46**]. In this paper, we are interested in studying the Cauchy problem of the following 2D regularized MHD equations

$$(1.1) \quad \begin{cases} \partial_t v + (u \cdot \nabla)v + (-\Delta)^\alpha v + \sum_{j=1}^{2} v_j \nabla u_j + \nabla\left(p + \frac{1}{2}|b|^2\right) = (b \cdot \nabla)b, \\ \partial_t b + (u \cdot \nabla)b + (-\Delta)^\beta b = (b \cdot \nabla)u, \\ v = u + (-\Delta)^\gamma u, \\ \nabla \cdot u = \nabla \cdot v = \nabla \cdot b = 0, \\ v(x,0) = v_0(x), \quad b(x,0) = b_0(x), \quad x \in \mathbb{R}^2, \end{cases}$$

where $v = (v_1, v_2)$ denotes the velocity vector, $b = (b_1, b_2)$ the magnetic field, $u = (u_1, u_2)$ the "filtered" velocity and p the scalar "filtered" pressure, respectively, $\alpha \in [0, 2]$, $\beta \in [0, 2]$ and $\gamma \in [0, 2]$ are real parameters. $v_0(x)$ and $b_0(x)$ are the given initial data satisfying $\nabla \cdot v_0 = \nabla \cdot b_0 = 0$. The fractional Laplacian operator $(-\Delta)^\sigma$ is defined through the Fourier transform, namely

$$\widehat{(-\Delta)^\sigma} f(\xi) = |\xi|^{2\sigma} \hat{f}(\xi),$$

where

$$\hat{f}(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) \, dx.$$

For simplicity, we use the notation $\Lambda := (-\Delta)^{\frac{1}{2}}$. We remark the convention that by $\alpha = 0$ we mean that there is no dissipation in $(1.1)_1$, and similarly $\beta = 0$ represents that there is no dissipation in $(1.1)_2$. Moreover, $\gamma = 0$ represents that the system (1.1) without regularizing, namely, $v = 2u$. Recently in the literature great attention has been devoted to the study of fractional Laplace-type problems, not only for pure mathematical generalization, but also for various applications in different fields. In fact, the application background of fractional problems can be founded in fractional quantum mechanics [**21**], probability [**2**], overdriven detonations in gases [**8**], anomalous diffusion in semiconductor growth [**31**], physics and chemistry [**26**] and so on.

Let us briefly discuss the rich history concerning the system (1.1). When $\gamma = 0$, (1.1) reduces to the 2D generalized magnetohydrodynamics (GMHD) system, which describes the motion of electrically conducting fluids and has broad applications in applied sciences such as astrophysics, geophysics and plasma physics (see, e.g., [**10**]). Besides their wide physical applicability, the GMHD equations are also of great interest in mathematics. As a coupled system, the GMHD equations contain much richer structures than the Navier-Stokes-type equations. The GMHD equations are not merely a combination of two parallel Navier-Stokes type equations but an interactive and integrated system. These distinctive features make analytic studies a great challenge but offer new opportunities. Since the work [**32**], the mathematical analysis (specially the global regularity problem) of the 2D GMHD equations has attracted considerable interests in the last few years and progress has been made (see [**6, 4, 5, 11, 16, 22, 23, 28, 30, 34, 33, 41, 37, 29, 43**] with no intention to be complete). We remark that the latest global regularity results of the 2D GMHD equations can be summarized as

(1) $\alpha > 0$, $\beta = 1$;   (2) $\alpha = 0$, $\beta > 1$;   (3) $\alpha = 2$, $\beta = 0$;   (4) $\alpha + \beta = 2$, $\beta < 1$,

see [**6, 11, 16**] for details (one also refers to [**1, 41, 43**] for logarithmic type result).

Now we would like to review some very related works about the system (1.1) with $\gamma = 1$, which is the so-called classical regularized GMHD equations (so-called Leray-alpha model). This system was obtained from variational principles by modifying the Hamiltonian associated with the standard MHD equations (see [**14**]). Linshiz-Titi [**24**] introduced the system (1.1) with $\alpha = \beta = \gamma = 1$ and proved that the the corresponding system admits a unique global smooth solution both in 2D case and 3D case. Later, Fan and Ozawa [**12**] established the global existence of smooth solution for the classical 2D regularized GMHD equations with $\alpha = 0$, $\beta = 1$ or $\alpha = 1$, $\beta = 0$, which was further extended logarithmically by [**18**]. Recently, Zhao and Zhu [**44**] proved the global regularity result for another special case $\alpha = \beta = \frac{1}{2}$ in 2D case. Very recently, Yamazaki [**39**] further examined the general case, namely, $\alpha + \beta = 1$ with α, $\beta \in (0, 1)$ and obtained the global regularity for this general case. The author [**42**] considered the 3D case and proved the global regularity under the assumption $\alpha + \beta \geq \frac{3}{2}$ and $\frac{3}{4} \leq \alpha < \frac{3}{2}$. For the endpoint case $\alpha = \frac{3}{2}$, $\beta = 0$, we refer to [**36, 38**].

The aim of this paper is to gain further understanding of the global regularity problem for the 2D regularized system (1.1) with only velocity dissipation or with only magnetic diffusion. We present two main results in hope that they may shed light on the eventual resolution of the global regularity problem of the classical 2D incompressible MHD equations with only velocity dissipation or with only magnetic diffusion, namely (1.1) with $\alpha = 1$, $\beta = \gamma = 0$ or $\beta = 1$, $\alpha = \gamma = 0$. More precisely, the first result concerns (1.1) with no velocity dissipation, which can be stated as follows.

THEOREM 1.1. *Consider (1.1) with $\alpha = 0$, namely,*

$$
(1.2) \quad
\begin{cases}
\partial_t v + (u \cdot \nabla)v + \sum_{j=1}^{2} v_j \nabla u_j + \nabla\left(p + \frac{1}{2}|b|^2\right) = (b \cdot \nabla)b, \\[2mm]
\partial_t b + (u \cdot \nabla)b + (-\Delta)^\beta b = (b \cdot \nabla)u, \\[2mm]
v = u + (-\Delta)^\gamma u, \\[2mm]
\nabla \cdot u = \nabla \cdot v = \nabla \cdot b = 0, \\[2mm]
v(x,0) = v_0(x), \quad b(x,0) = b_0(x), \quad x \in \mathbb{R}^2.
\end{cases}
$$

Assume that $v_0 \in H^\rho(\mathbb{R}^2)$, $b_0 \in H^{\rho+1-\beta}(\mathbb{R}^2)$ with $\rho > \max\{2, 1+\beta\}$ and $\nabla \cdot v_0 = \nabla \cdot b_0 = 0$. If β and γ satisfy

$$
(1.3) \qquad\qquad \beta > 1 - \frac{\gamma}{2} \quad \text{with } \gamma \in [0, 2],
$$

then the system (1.2) admits a unique global regular solution (v, b) such that for any given $T > 0$,

$v \in L^\infty([0,T]; H^\rho(\mathbb{R}^2))$, $\quad b \in L^\infty([0,T]; H^{\rho+1-\beta}(\mathbb{R}^2)) \cap L^2([0,T]; H^{\rho+1}(\mathbb{R}^2))$.

REMARK 1.1. As stated above, Fan and Ozawa [**12**] established the global regularity for the system (1.1) with $\alpha = 0$, $\beta = \gamma = 1$, which was further extended logarithmically by [**18**]. We remark that under the assumption $\alpha = 0$, $\gamma = 1$, the requirement $\beta > \frac{1}{2}$ actually would ensure the global regularity. Consequently,

Theorem 1.1 significantly improves the previous works [**12, 18, 39, 44**] and answers an interesting problem remarked in [**39**, Remark 1.2].

The following theorem concerns (1.1) with no magnetic diffusion.

THEOREM 1.2. *Consider (1.1) with $\beta = 0$ and the logarithmic dissipation in velocity equation, namely,*

$$
(1.4) \quad
\begin{cases}
\partial_t v + (u \cdot \nabla)v + (-\Delta)^\alpha \mathcal{L}^2 v + \displaystyle\sum_{j=1}^{2} v_j \nabla u_j + \nabla\left(p + \frac{1}{2}|b|^2\right) = (b \cdot \nabla)b, \\[2mm]
\partial_t b + (u \cdot \nabla)b = (b \cdot \nabla)u, \\[1mm]
v = u + (-\Delta)^\gamma \mathcal{L}^2 u, \\[1mm]
\nabla \cdot u = \nabla \cdot v = \nabla \cdot b = 0, \\[1mm]
v(x,0) = v_0(x), \quad b(x,0) = b_0(x), \quad x \in \mathbb{R}^2,
\end{cases}
$$

where the operator $\mathcal{L}$ is defined by

$$
\widehat{\mathcal{L}v}(\xi) = \frac{1}{g(\xi)}\widehat{v}(\xi)
$$

for some non-decreasing symmetric function $g(\tau) \geq 1$ defined on $\tau \geq 0$. Let $v_0 \in H^\rho(\mathbb{R}^2)$, $b_0 \in H^{\rho+1-\alpha'}(\mathbb{R}^2)$ with $\alpha > \alpha'$ and $\rho > \max\{2, 1+\alpha'\}$ as well as $\nabla \cdot v_0 = \nabla \cdot b_0 = 0$. If $\alpha + \gamma = 2$ with $\alpha \in (0, 2]$ and g satisfies the following growth condition

$$
(1.5) \qquad \int_e^\infty \frac{d\tau}{\tau\sqrt{\ln \tau}\, g^2(\tau)} = \infty,
$$

then the system (1.4) admits a unique global regular solution (v, b) such that for any given $T > 0$

$$
v \in L^\infty([0,T]; H^\rho(\mathbb{R}^2)) \cap L^2([0,T]; H^{\rho+\alpha'}(\mathbb{R}^2)), \quad b \in L^\infty([0,T]; H^{\rho+1-\alpha'}(\mathbb{R}^2)).
$$

REMARK 1.2. We point out that the global regularity of the system (1.1) with $\beta = 0$, $\alpha = \gamma = 1$ was proved by Fan and Ozawa [**12**], which was further extended logarithmically by [**18**, Theorem 1.1]. Consequently, Theorem 1.2 improves the results of [**12**, Theorem 1.1] and [**18**, Theorem 1.1].

REMARK 1.3. We should remark that the typical examples satisfying the condition (1.5) are as follows

$$
g(\xi) = \left[\ln(e + |\xi|)\right]^{\frac{1}{4}};
$$
$$
g(\xi) = \left[\ln(e + |\xi|)\right]^{\frac{1}{4}}\left[\ln(e + \ln(e + |\xi|))\right]^{\frac{1}{2}};
$$
$$
g(\xi) = \left[\ln(e + |\xi|)\right]^{\frac{1}{4}}\left[\ln(e + \ln(e + |\xi|))\ln(e + \ln(e + \ln(e + |\xi|)))\right]^{\frac{1}{2}}.
$$

The next main result concerned with the endpoint case, namely, (1.1) with $\alpha = \beta = 0$ and $\gamma = 2$, can be stated as follows.

THEOREM 1.3. *Consider (1.1) with $\alpha = \beta = 0$ and $\gamma = 2$, namely*

(1.6)
$$\begin{cases} \partial_t v + (u \cdot \nabla)v + \sum_{j=1}^{2} v_j \nabla u_j + \nabla\Big(p + \frac{1}{2}|b|^2\Big) = (b \cdot \nabla)b, \\[2mm] \partial_t b + (u \cdot \nabla)b = (b \cdot \nabla)u, \\[1mm] v = u + (-\Delta)^2 \mathcal{L}^2 u, \\[1mm] \nabla \cdot u = \nabla \cdot v = \nabla \cdot b = 0, \\[1mm] v(x,0) = v_0(x), \quad b(x,0) = b_0(x), \quad x \in \mathbb{R}^2, \end{cases}$$

where the operator $\mathcal{L}$ is defined by

$$\widehat{\mathcal{L}v}(\xi) = \frac{1}{g(\xi)}\widehat{v}(\xi)$$

for some non-decreasing symmetric function $g(\tau) \geq 1$ defined on $\tau \geq 0$. Assume that $v_0 \in H^\varrho(\mathbb{R}^2)$, $b_0 \in H^{\rho+1}(\mathbb{R}^2)$ with $\rho > 2$ and $\nabla \cdot v_0 = \nabla \cdot b_0 = 0$. If g satisfies the following growth condition

(1.7)
$$\int_e^\infty \frac{d\tau}{\tau\sqrt{\ln \tau}\, g(\tau)} = \infty,$$

then the corresponding system (1.6) admits a unique global regular solution (v, b) such that for any given $T > 0$,

$$v \in L^\infty([0,T]; H^\rho(\mathbb{R}^2)), \quad b \in L^\infty([0,T]; H^{\rho+1}(\mathbb{R}^2)).$$

REMARK 1.4. We also remark that the typical examples satisfying the condition (1.7) are

$$g(\xi) = \big[\ln(e + |\xi|)\big]^{\frac{1}{2}};$$

$$g(\xi) = \big[\ln(e + |\xi|)\big]^{\frac{1}{2}} \ln(e + \ln(e + |\xi|));$$

$$g(\xi) = \big[\ln(e + |\xi|)\big]^{\frac{1}{2}} \ln(e + \ln(e + |\xi|)) \ln(e + \ln(e + \ln(e + |\xi|))).$$

REMARK 1.5. The requirements $\rho > \max\{2, 1 + \beta\}$ in Theorem 1.1 and $\rho > \max\{2, 1+\alpha\}$ in Theorem 1.2 guarantee the initial data belonging to the lipschitzian class. As a matter of fact, this requirements can be weakened. In order to avoid the tedious computations, we just work on these functional spaces.

REMARK 1.6. Using the techniques and arguments of this paper, it is not hard to derive the global regularity of the system (1.1) with $\alpha + \beta + \gamma = 2$. We would like to leave it to the interested readers. However, it seems to be an interesting problem whether or not the global regularity result of the system (1.1) with $\alpha + \beta + \gamma < 2$ and $\beta \leq 1 - \frac{\gamma}{2}$.

Finally, inspired by the proof of Theorem 1.1, we are able to show the global regularity for the following 2D regularized MHD equations [24]

(1.8)
$$\begin{cases} \partial_t v + (u \cdot \nabla)v + \nabla\Big(p + \frac{1}{2}|b|^2\Big) = (b \cdot \nabla)b, \\[1mm] \partial_t b + (u \cdot \nabla)b + (-\Delta)^\beta b = (b \cdot \nabla)u, \\[1mm] v = u - \Delta u, \\[1mm] \nabla \cdot u = \nabla \cdot v = \nabla \cdot b = 0, \\[1mm] v(x,0) = v_0(x), \quad b(x,0) = b_0(x), \quad x \in \mathbb{R}^2. \end{cases}$$

More precisely, we have the following result.

THEOREM 1.4. *Assume that $v_0 \in H^\rho(\mathbb{R}^2)$ and $b_0 \in H^{\rho+1-\beta}(\mathbb{R}^2)$ with $\rho > \max\{2, 1+\beta\}$ and $\nabla \cdot v_0 = \nabla \cdot b_0 = 0$. If $\beta > \frac{1}{2}$, then the system (1.8) admits a unique global regular solution (v, b) such that for any given $T > 0$,*

$$v \in L^\infty([0,T]; H^\rho(\mathbb{R}^2)), \quad b \in L^\infty([0,T]; H^{\rho+1-\beta}(\mathbb{R}^2)) \cap L^2([0,T]; H^{\rho+1}(\mathbb{R}^2)).$$

REMARK 1.7. The global regularity result of the system (1.8) with $\beta = 1$ was established in [**45**, Theorem 1.2]. Very recently, this global regularity result in the logarithmically supercritical regime was extended by KC and Yamazaki in [**19**, Theorem 1.1]. Clearly, Theorem 1.4 can be regarded as a significant improvement of [**45**, Theorem 1.2] and [**19**, Theorem 1.1]. As the proof of Theorem 1.4 can be performed via the similar arguments adopted in proving Theorem 1.1 with suitable modifications, we only sketch its proof in Appendix A.

Now we give some rough ideas on our proof of Theorem 1.1 and Theorem 1.2. The proof is not straightforward and demands new techniques. We describe the main difficulties and explain the techniques to overcome them. The key to the global regularity is the global *a priori* bounds. We begin with Theorem 1.1.

$$\textbf{Case 1}: \ \beta > 1 - \frac{\gamma}{2} \ \text{ with } \gamma \in (0, 1]$$

The following basic global L^2-energy is immediate due to the special structure of (1.2)

$$(1.9) \qquad \|u(t)\|_{H^\gamma}^2 + \|b(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\beta b(\tau)\|_{L^2}^2 \, d\tau \le C(v_0, b_0),$$

which serves as a preparation for higher regularity estimates. The next step is to derive the global H^1-bound, but direct energy estimates do not appear to easily yield this bound. One of the difficulties comes from the nonlinear term in the velocity equation. Moreover, there is no dissipation in the velocity equation and the dissipation in the magnetic equation is not strong enough (only $\beta \le 1$). To overcome these difficulties, we first make use of the bound (1.9) to show the following optimal regularity for b

$$(1.10) \qquad \sup_{t \in [0, T]} \|\Lambda^{\beta+\gamma-1} b(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{2\beta+\gamma-1} b(\tau)\|_{L^2}^2 \, d\tau \le C(T, v_0, b_0).$$

Next, if one tries to obtain more regularity for b, then the higher regularity for u or v will be needed. To this end, we appeal to the vorticity $\omega := \nabla^\perp \cdot v \equiv \partial_1 v_2 - \partial_2 v_1$ equation

$$(1.11) \qquad \qquad \partial_t \omega + (u \cdot \nabla)\omega = \nabla^\perp \nabla \cdot (b \otimes b).$$

More precisely, combining the estimates for ω and b, we derive by using (1.10) that

$$(1.12) \quad \sup_{t \in [0, T]} (\|v(t)\|_{H^1}^2 + \|\Lambda^{2\gamma+\beta} b(t)\|_{L^2}^2) + \int_0^T \|\Lambda^{2\gamma+2\beta} b(\tau)\|_{L^2}^2 \, d\tau \le C(T, v_0, b_0),$$

$$(1.13) \qquad \qquad \sup_{t \in [0, T]} \|b(t)\|_{L^\infty} + \int_0^T \|\nabla b(\tau)\|_{L^\infty}^2 \, d\tau \le C(T, v_0, b_0).$$

At this moment, the estimate for b in (1.13) is enough to obtain the higher regularity. Consequently, it suffices to derive more regularity estimate for u or v. Due to the nonlinear term in the velocity equation, it is natural to take advantage of the vorticity equation (1.11). But the obstacle is to control the two-order derivatives of b. To achieve this goal, we first rewrite the equation $(1.2)_2$ due to $\nabla \cdot u = \nabla \cdot b = 0$

$$(1.14) \qquad \partial_t b + \Lambda^{2\beta} b = \nabla \cdot (b \otimes u - u \otimes b).$$

Applying the space-time estimate (see Lemma B.2) to (1.14), it allows us to show the following crucial inequality (see (2.23) for details)

$$(1.15) \qquad \|\Lambda^{2\beta+2\gamma-\epsilon} b\|_{L_T^1 L^q} \leq C(T, v_0, b_0) + C(T, v_0, b_0)\|\omega\|_{L_T^1 L^q}.$$

Unfortunately, at present we have no estimate for $\|\omega\|_{L_T^1 L^q}$. In order to overcome this difficulty and to close (1.15), we deduce from the vorticity equation (1.11) that (see (2.26) for details)

$$(1.16) \qquad \|\omega(t)\|_{L_T^\infty L^q} \leq C(T, v_0, b_0) + C(T, v_0, b_0)\|\Lambda^{2\beta+2\gamma-\epsilon} b\|_{L_T^1 L^q}^{\frac{2}{2\beta+2\gamma-\epsilon}},$$

where $\epsilon \in (0, \min\{2\beta - 1, 2\beta + 2\gamma - 2\})$ and $q \in \left(\frac{1}{2\beta-1-\epsilon}, \infty\right)$. Combining (1.15) and (1.16) yields

$$(1.17) \qquad \|\Lambda^{2\beta+2\gamma-\epsilon} b\|_{L_T^1 L^q} \leq C(T, v_0, b_0).$$

As β and γ satisfy (1.3), we can take q large enough to get from (1.17) that

$$\|\nabla^2 b\|_{L_T^1 L^\infty} \leq C(T, v_0, b_0),$$

which along with the vorticity equation (1.11) implies

$$(1.18) \qquad \sup_{t \in [0, T]} \|\omega(t)\|_{L^\infty} \leq C(T, v_0, b_0).$$

Actually, the above estimate (1.18) is a key component to obtain the higher regularity.

$$\textbf{Case 2}: \ \beta > 1 - \frac{\gamma}{2} \ \text{ with } \gamma \in (1, 2]$$

We remark that the proof for **Case 2** can be performed as that of **Case 1**, but some different techniques and observations are required. In this case, basic global L^2-energy (1.9) is still valid. Moreover, we can also show that the optimal regularity for b, namely, (1.10) holds true. Our next step is to derive (1.12), namely,

$$(1.19) \quad \sup_{t \in [0, T]} (\|v(t)\|_{H^1}^2 + \|\Lambda^{2\gamma+\beta} b(t)\|_{L^2}^2) + \int_0^T \|\Lambda^{2\gamma+2\beta} b(\tau)\|_{L^2}^2 \, d\tau \leq C(T, v_0, b_0).$$

With the observation $2\gamma + 2\beta > 3$ due to $\beta > 1 - \frac{\gamma}{2}$ with $\gamma \in (1, 2]$, this implies that the regularity for b of (1.19) is good enough. More precisely, we have at least

$$\|\nabla^2 b\|_{L_T^2 L^\infty} \leq C(T, v_0, b_0),$$

Consequently, this along with the vorticity equation (1.11) leads to

$$(1.20) \qquad \sup_{t \in [0, T]} \|\omega(t)\|_{L^\infty} \leq C(T, v_0, b_0).$$

Finally, the estimates (1.13) and (1.18) or ((1.19) and (1.20)) will allow us to propagate all the higher regularities for (v, b). This ends the proof of Theorem 1.1.

We now explain the proof of Theorem 1.2. We first have the following basic global L^2-energy

$$(1.21) \qquad \|u(t)\|_{H^\gamma}^2 + \|b(t)\|_{L^2}^2 + \int_0^t \left\| \frac{\Lambda^2}{g^2(\Lambda)} u \right\|_{L^2}^2 d\tau \le C(v_0, b_0).$$

The regularity for u in (1.21) seems to be good, which is quite important to derive the higher regularity of the solution. Our next target is to show the estimate for b. This is not trivial as there is no dissipation in b-equation. Actually, by direct computations, we may conclude

$$\frac{d}{dt} V(t) + H(t) \le C(1 + \|\nabla u\|_{L^\infty} + \|\Delta u\|_{L^2}) V(t),$$

where

$$V(t) := \|b(t)\|_{L^\infty}^3 + \|b(t)\|_{L^\infty}^6 + \|\nabla b(t)\|_{L^2}^3 + \|v(t)\|_{L^2}^2, \quad H(t) := \|\mathcal{L}\Lambda^\alpha v(t)\|_{L^2}^2.$$

In order to handle the two terms $\|\nabla u\|_{L^\infty}$ and $\|\Delta u\|_{L^2}$, we take fully exploit of the Littlewood-Paley technique, which allows us to deduce

$$\frac{d}{dt} V(t) + H(t) \le C \left(1 + \left\| \frac{\Lambda^2}{g^2(\Lambda)} u \right\|_{L^2} \right) g^2 \big[(e + V(t))^{\frac{1}{2(r_1 - 2)}} \big] \sqrt{\ln \big(e + V(t) \big)}$$
$$\times \big(e + V(t) \big).$$

Using (1.21) and the condition (1.5), we end up with

$$\sup_{t \in [0, T]} V(t) + \int_0^T H(\tau) \, d\tau \le C(T, v_0, b_0).$$

However, the regularity for b here is still not enough. To gain higher regularity, we go through the process by improving the regularity for v to improve the regularity for b. More precisely, the boundedness of $V(t)$ allows us to show (see Lemma 3.3 for details)

$$(1.22) \qquad \sup_{t \in [0, T]} \|\Lambda^{1+\nu} b(t)\|_{L^2} \le C(T, v_0, b_0), \quad \forall \nu < \gamma.$$

The bound (1.22) plays an important role in deriving the following estimate

$$\sup_{t \in [0, T]} \|\Lambda^\alpha v(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{2\alpha} \mathcal{L} v(\tau)\|_{L^2}^2 \, d\tau \le C(T, v_0, b_0),$$

which further implies (see Lemma 3.4 for details)

$$(1.23) \qquad \sup_{t \in [0, T]} \|\nabla b(t)\|_{L^\infty} \le C(T, v_0, b_0).$$

Consequently, by (1.23), there holds (see Lemma 3.5 for details)

$$\sup_{t \in [0, T]} \|\omega(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha \mathcal{L}\omega(\tau)\|_{L^2}^2 \, d\tau \le C(T, v_0, b_0).$$

Finally, with the above estimates at our disposal, we can propagate all the higher regularities and thus complete the proof of Theorem 1.2.

The rest of the paper is organized as follows. In Section 2 we carry out the proof of Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2 and Theorem 1.3. We sketch the proof of Theorem 1.4 in Appendix A, while in Appendix B, we present some useful lemmas of this paper.

2. The proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Before the proof, we will state a notation. For a quasi-Banach space X and for any $0 < T \leq \infty$, we use standard notation $L^p(0, T; X)$ or $L^p_T(X)$ for the quasi-Banach space of Bochner measurable functions f from $(0, T)$ to X endowed with the norm

$$\|f\|_{L^p_T(X)} := \begin{cases} \left(\int_0^T \|f(.,t)\|_X^p \, dt \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \sup_{0 \leq t \leq T} \|f(.,t)\|_X, & p = \infty. \end{cases}$$

In this paper, we shall use the convention that C denotes a generic constant, whose value may change from line to line. We shall write $C(\lambda_1, \lambda_2, \cdots, \lambda_k)$ as the constant C depends on the quantities $\lambda_1, \lambda_2, \cdots, \lambda_k$. We also denote $\Psi \approx \Upsilon$ if there exist two constants $C_1 \leq C_2$ such that $C_1 \Upsilon \leq \Psi \leq C_2 \Upsilon$.

We state that the existence and uniqueness of local smooth solutions can be performed through the standard approach (see for example [**9, 25**]). Thus, in order to complete the proof of Theorem 1.1, it is sufficient to establish *a priori* estimates that hold for any fixed $T > 0$.

Keeping in mind the fact that when $\alpha = \gamma = 0$ and $\beta > 1$, the corresponding system admits a unique global regular solution [**6, 1, 41, 16**]. As a result, it suffices to consider the case $\beta \leq 1$ (thus $\gamma > 0$) in this section.

$$\textbf{Case 1}: \ \beta > 1 - \frac{\gamma}{2} \ \text{ with } \gamma \in (0, 1]$$

In this case, we should keep in mind that $\beta > \frac{1}{2}$. Now we present the basic L^2-energy estimate as follows.

LEMMA 2.1. *Assume (v_0, b_0) satisfies the assumptions stated in Theorem 1.1. Then the corresponding solution (v, b) of (1.2) admits the following bound for any $t \in [0, T]$*

$$(2.1) \qquad \|u(t)\|_{L^2}^2 + \|\Lambda^\gamma u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\beta b(\tau)\|_{L^2}^2 \, d\tau \leq C(v_0, b_0).$$

PROOF. Taking the inner product of $(1.2)_1$ with u and the inner product of $(1.2)_2$ with b, using the divergence free property and summing them up, we have

$$(2.2) \qquad \frac{1}{2} \frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|\Lambda^\gamma u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + \|\Lambda^\beta b\|_{L^2}^2 = 0,$$

where we have used the following cancelations

$$(2.3) \qquad \int_{\mathbb{R}^2} (b \cdot \nabla b) \cdot u \, dx + \int_{\mathbb{R}^2} (b \cdot \nabla u) \cdot b \, dx = 0,$$

$$(2.4) \qquad \int_{\mathbb{R}^2} (u \cdot \nabla v) \cdot u \, dx + \int_{\mathbb{R}^2} \Big(\sum_{j=1}^{2} v_j \nabla u_j \Big) \cdot u \, dx = 0.$$

Integrating (2.2) in time yields the desired (2.1). This ends the proof of Lemma 2.1. □

Next we will derive the following regularity estimate for b.

LEMMA 2.2. *Assume (v_0, b_0) satisfies the assumptions stated in Theorem 1.1. If $\beta > 1 - \frac{\gamma}{2}$ with $\gamma \in (0, 1]$, then the corresponding solution (v, b) of (1.2) admits the following bound*

$$(2.5) \qquad \sup_{t \in [0, T]} \|\Lambda^{\beta+\gamma-1} b(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{2\beta+\gamma-1} b(\tau)\|_{L^2}^2 \, d\tau \leq C(T, v_0, b_0).$$

In particular, due to $\beta > 1 - \frac{\gamma}{2}$, it holds from (2.5) that

$$(2.6) \qquad \int_0^T \|b(\tau)\|_{L^\infty}^2 \, d\tau \leq C(T, v_0, b_0).$$

PROOF. Applying $\Lambda^{\beta+\gamma-1}$ to the second equation of (1.2) and multiplying it by $\Lambda^{\beta+\gamma-1} b$ yield

$$(2.7) \qquad \frac{1}{2} \frac{d}{dt} \|\Lambda^{\beta+\gamma-1} b(t)\|_{L^2}^2 + \|\Lambda^{2\beta+\gamma-1} b\|_{L^2}^2 = J_1 + J_2,$$

where

$$J_1 = \int_{\mathbb{R}^2} \Lambda^{\beta+\gamma-1} (b \cdot \nabla u) \cdot \Lambda^{\beta+\gamma-1} b \, dx, \quad J_2 = -\int_{\mathbb{R}^2} \Lambda^{\beta+\gamma-1} (u \cdot \nabla b) \cdot \Lambda^{\beta+\gamma-1} b \, dx.$$

By $\nabla \cdot b = 0$, Sobolev embedding inequalities and (B.2), we deduce

$$J_1 = \int_{\mathbb{R}^2} \Lambda^{\beta+\gamma-1} \nabla \cdot (b \otimes u) \cdot \Lambda^{\beta+\gamma-1} b \, dx$$

$$\leq C \|\Lambda^\gamma (ub)\|_{L^2} \|\Lambda^{2\beta+\gamma-1} b\|_{L^2}$$

$$\leq C (\|\Lambda^\gamma u\|_{L^2} \|b\|_{L^\infty} + \|u\|_{L^{p_0}} \|\Lambda^\gamma b\|_{L^{\frac{2p_0}{p_0-2}}}) \|\Lambda^{2\beta+\gamma-1} b\|_{L^2}$$

$$\leq C \|u\|_{H^\gamma} \|b\|_{L^2}^{\frac{2\beta+\gamma-2}{2\beta+\gamma-1}} \|\Lambda^{2\beta+\gamma-1} b\|_{L^2}^{\frac{1}{2\beta+\gamma-1}} \|\Lambda^{2\beta+\gamma-1} b\|_{L^2}$$

$$\quad + \|u\|_{H^\gamma} \|b\|_{L^2}^{\frac{(2\beta-1)p_0-2}{(2\beta+\gamma-1)p_0}} \|\Lambda^{2\beta+\gamma-1} b\|_{L^2}^{\frac{\gamma p_0+2}{(2\beta+\gamma-1)p_0}} \|\Lambda^{2\beta+\gamma-1} b\|_{L^2}$$

$$(2.8) \qquad \leq \frac{1}{4} \|\Lambda^{2\beta+\gamma-1} b\|_{L^2}^2 + C \|u\|_{H^\gamma}^{\frac{2(2\beta+\gamma-1)}{2\beta+\gamma-2}} \|b\|_{L^2}^2 + C \|u\|_{H^\gamma}^{\frac{2(2\beta+\gamma-1)p_0}{(2\beta-1)p_0-2}} \|b\|_{L^2}^2,$$

where $p_0 > 2$ satisfies

$$\frac{1-\gamma}{2} < \frac{1}{p_0} < \frac{2\beta-1}{2}.$$

We remark that as β and $\gamma > 0$ satisfy (1.3), the above p_0 actually would work. Thanks to $\nabla \cdot u = 0$, it also yields

$$J_2 = \int_{\mathbb{R}^2} \Lambda^{\beta+\gamma-1} \nabla \cdot (u \otimes b) \cdot \Lambda^{\beta+\gamma-1} b \, dx$$

$$\leq C \|\Lambda^\gamma (ub)\|_{L^2} \|\Lambda^{2\beta+\gamma-1} b\|_{L^2}$$

$$\leq \frac{1}{4} \|\Lambda^{2\beta+\gamma-1} b\|_{L^2}^2 + C \|u\|_{H^\gamma}^{\frac{2(2\beta+\gamma-1)}{2\beta+\gamma-2}} \|b\|_{L^2}^2 + C \|u\|_{H^\gamma}^{\frac{2(2\beta+\gamma-1)p_0}{(2\beta-1)p_0-2}} \|b\|_{L^2}^2.$$

Inserting the above estimates into (2.7) implies

(2.9)
$$\frac{d}{dt}\|\Lambda^{\beta+\gamma-1}b(t)\|_{L^2}^2 + \|\Lambda^{2\beta+\gamma-1}b\|_{L^2}^2 \le C(\|u\|_{H^\gamma}^{\frac{2(2\beta+\gamma-1)}{2\beta+\gamma-2}} + \|u\|_{H^\gamma}^{\frac{2(2\beta+\gamma-1)p_0}{(2\beta-1)p_0-2}})\|b\|_{L^2}^2.$$

Recalling (2.1), one deduces from (2.9) that

$$\sup_{t\in[0,T]}\|\Lambda^{\beta+\gamma-1}b(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{2\beta+\gamma-1}b(\tau)\|_{L^2}^2\,d\tau \le C(T,v_0,b_0).$$

We thus end the proof of Lemma 2.2. $\square$

With the help of Lemma 2.2, we continue to improve the regularity of b.

LEMMA 2.3. *Assume (v_0, b_0) satisfies the assumptions stated in Theorem 1.1. If $\beta > 1 - \frac{\gamma}{2}$ with $\gamma \in (0, 1]$, then the corresponding solution (v, b) of (1.2) admits the following bound*

(2.10) $$\sup_{t\in[0,T]}(\|v(t)\|_{H^1}^2 + \|\Lambda^{2\gamma+\beta}b(t)\|_{L^2}^2) + \int_0^T \|\Lambda^{2\gamma+2\beta}b(\tau)\|_{L^2}^2\,d\tau \le C(T,v_0,b_0).$$

In particular, we have

(2.11) $$\sup_{t\in[0,T]}\|b(t)\|_{L^\infty} + \int_0^T \|\nabla b(\tau)\|_{L^\infty}^2\,d\tau \le C(T,v_0,b_0).$$

PROOF. Applying $\nabla^\perp := (-\partial_2, \partial_1)^{\mathrm{T}}$ to the first equation of (1.2), we show that the vorticity $\omega := \nabla^\perp \cdot v \equiv \partial_1 v_2 - \partial_2 v_1$ satisfies

(2.12) $$\partial_t \omega + (u \cdot \nabla)\omega = \nabla^\perp \nabla \cdot (b \otimes b).$$

It follows from (2.12) that

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{L^2}^2 \le C\|\nabla^\perp \nabla \cdot (b \otimes b)\|_{L^2}\|\omega\|_{L^2}$$
$$\le C\|b\|_{L^\infty}\|\Delta b\|_{L^2}\|\omega\|_{L^2}$$
$$\le C\|b\|_{L^\infty}\|b\|_{L^2}^{\frac{\gamma+\beta-1}{\gamma+\beta}}\|\Lambda^{2\gamma+2\beta}b\|_{L^2}^{\frac{1}{\gamma+\beta}}\|\omega\|_{L^2}$$

(2.13) $$\le \frac{1}{4}\|\Lambda^{2\gamma+2\beta}b\|_{L^2}^2 + C(1 + \|b\|_{L^\infty}^2)(1 + \|\omega\|_{L^2}^2).$$

Applying $\Lambda^{2\gamma+\beta}$ to the second equation of (1.2) and multiplying it by $\Lambda^{2\gamma+\beta}b$, one has

(2.14) $$\frac{1}{2}\frac{d}{dt}\|\Lambda^{2\gamma+\beta}b(t)\|_{L^2}^2 + \|\Lambda^{2\gamma+2\beta}b\|_{L^2}^2 = J_3 + J_4,$$

where

$$J_3 = \int_{\mathbb{R}^2} \Lambda^{2\gamma+\beta}(b \cdot \nabla u) \cdot \Lambda^{2\gamma+\beta}b\,dx, \qquad J_4 = -\int_{\mathbb{R}^2} \Lambda^{2\gamma+\beta}(u \cdot \nabla b) \cdot \Lambda^{2\gamma+\beta}b\,dx.$$

By the argument adopted in dealing with (2.8), it leads to

$$J_3 = \int_{\mathbb{R}^2} \Lambda^{2\gamma+\beta}\nabla \cdot (b \otimes u) \cdot \Lambda^{2\gamma+\beta}b\,dx$$
$$\le C\|\Lambda^{2\gamma+1}(ub)\|_{L^2}\|\Lambda^{2\gamma+2\beta}b\|_{L^2}$$
$$\le C(\|\Lambda^{2\gamma+1}u\|_{L^2}\|b\|_{L^\infty} + \|u\|_{L^{p_0}}\|\Lambda^{2\gamma+1}b\|_{L^{\frac{2p_0}{p_0-2}}})\|\Lambda^{2\gamma+2\beta}b\|_{L^2}$$

$$\leq C(\|\nabla v\|_{L^2}\|b\|_{L^\infty} + \|u\|_{H^\gamma}\|b\|_{L^2}^{\frac{(2\beta-1)p_0-2}{2(\gamma+\beta)p_0}}\|\Lambda^{2\gamma+2\beta}b\|_{L^2}^{\frac{(2\gamma+1)p_0+2}{2(\gamma+\beta)p_0}})\|\Lambda^{2\gamma+2\beta}b\|_{L^2}$$

$$(2.15) \quad \leq \frac{1}{8}\|\Lambda^{2\gamma+2\beta}b\|_{L^2}^2 + C\|b\|_{L^\infty}^2\|\omega\|_{L^2}^2 + C\|u\|_{H^\gamma}^{\frac{4(\beta+\gamma)p_0}{(2\beta-1)p_0-2}}\|b\|_{L^2}^2,$$

$$J_4 = -\int_{\mathbb{R}^2}\Lambda^{2\gamma+\beta}\nabla\cdot(u\otimes b)\cdot\Lambda^{2\gamma+\beta}b\,dx$$

$$\leq C\|\Lambda^{2\gamma+1}(ub)\|_{L^2}\|\Lambda^{2\gamma+2\beta}b\|_{L^2}$$

$$(2.16) \quad \leq \frac{1}{8}\|\Lambda^{2\gamma+2\beta}b\|_{L^2}^2 + C\|b\|_{L^\infty}^2\|\omega\|_{L^2}^2 + C\|u\|_{H^\gamma}^{\frac{4(\beta+\gamma)p_0}{(2\beta-1)p_0-2}}\|b\|_{L^2}^2.$$

Putting (2.15) and (2.16) into (2.14) gives

(2.17)
$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{2\gamma+\beta}b(t)\|_{L^2}^2 + \frac{3}{4}\|\Lambda^{2\gamma+2\beta}b\|_{L^2}^2 \leq C\|b\|_{L^\infty}^2\|\omega\|_{L^2}^2 + C\|u\|_{H^\gamma}^{\frac{4(\beta+\gamma)p_0}{(2\beta-1)p_0-2}}\|b\|_{L^2}^2,$$

which along with (2.13) further leads to

$$\frac{d}{dt}(\|\Lambda^{2\gamma+\beta}b(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2) + \|\Lambda^{2\gamma+2\beta}b\|_{L^2}^2 \leq C(1+\|b\|_{L^\infty}^2)(1+\|\omega\|_{L^2}^2)$$

$$(2.18) \qquad\qquad\qquad\qquad\qquad\qquad + C\|u\|_{H^\gamma}^{\frac{4(\beta+\gamma)p_0}{(2\beta-1)p_0-2}}\|b\|_{L^2}^2.$$

Thanks to (2.6), we deduce by applying the Gronwall inequality to (2.18) that

(2.19)
$$\sup_{t\in[0,T]}(\|\nabla v(t)\|_{L^2}^2 + \|\Lambda^{2\gamma+\beta}b(t)\|_{L^2}^2) + \int_0^T\|\Lambda^{2\gamma+2\beta}b(\tau)\|_{L^2}^2\,d\tau \leq C(T,v_0,b_0),$$

where we have used the simple fact $\|\omega\|_{L^2} \approx \|\nabla v\|_{L^2}$. In order to obtain the estimate of $\|v(t)\|_{L^2}$, we take the L^2-inner product of $(1.2)_1$ with v to get

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 = -\int_{\mathbb{R}^2}\Big(\sum_{j=1}^2 v_j\nabla u_j\Big)\cdot v\,dx + \int_{\mathbb{R}^2}(b\cdot\nabla b)\cdot v\,dx$$

$$\leq C\|\nabla u\|_{L^2}\|v\|_{L^4}^2 + C\|b\|_{L^\infty}\|\nabla b\|_{L^2}\|v\|_{L^2}$$

$$\leq C\|\nabla v\|_{L^2}(\|v\|_{L^2}\|\nabla v\|_{L^2}) + C\|b\|_{H^{2\gamma+\beta}}^2\|v\|_{L^2}$$

$$\leq C(\|\nabla v\|_{L^2}^2 + \|b\|_{H^{2\gamma+\beta}}^2)\|v\|_{L^2},$$

which yields

$$(2.20) \qquad\qquad \frac{d}{dt}\|v(t)\|_{L^2} \leq C(\|\nabla v\|_{L^2}^2 + \|b\|_{H^{2\gamma+\beta}}^2).$$

Thanks to (2.19), we obtain

$$(2.21) \qquad\qquad \sup_{t\in[0,T]}\|v(t)\|_{L^2} \leq C(T,v_0,b_0).$$

The desired estimate (2.10) follows by combining (2.19) and (2.21). This completes the proof of Lemma 2.3. □

Now we are in the position to show the following crucial bound which allows us to derive the higher regularity of (v,b).

LEMMA 2.4. *Assume (v_0, b_0) satisfies the assumptions stated in Theorem 1.1. If $\beta > 1 - \frac{\gamma}{2}$ with $\gamma \in (0, 1]$, then the corresponding solution (v, b) of (1.2) admits the following bound*

$$\sup_{t \in [0, T]} \|\omega(t)\|_{L^\infty} \leq C(T, v_0, b_0).$$

PROOF. We rewrite the equation $(1.2)_2$ as follows

$$\partial_t b + \Lambda^{2\beta} b = \nabla \cdot (b \otimes u - u \otimes b).$$

Applying $\Lambda^{2\gamma+1}$ to both sides of the above equation, we have

$$(2.22) \qquad \partial_t \Lambda^{2\gamma+1} b + \Lambda^{2\beta} \Lambda^{2\gamma+1} b = \Lambda^{2\gamma+1} \nabla \cdot (b \otimes u - u \otimes b).$$

Applying Lemma B.2 to (2.22), it yields that for any $\epsilon \in (0, 2\beta - 1)$ and for any $q \in \left(\frac{1}{2\beta - 1 - \epsilon}, \infty \right)$

$$\begin{aligned}
\|\Lambda^{2\beta+2\gamma-\epsilon} b\|_{L^1_T L^q} &= \|\Lambda^{2\beta-1-\epsilon} \Lambda^{2\gamma+1} b\|_{L^1_T L^q} \\
&\leq C(T, v_0, b_0) + C\|\Lambda^{2\gamma+1}(ub)\|_{L^1_T L^q} \\
&\leq C(T, v_0, b_0) + C\|u\|_{L^\infty_T L^{2q}} \|\Lambda^{2\gamma+1} b\|_{L^1_T L^{2q}} \\
&\quad + C\|b\|_{L^\infty_T L^\infty} \|\Lambda^{2\gamma+1} u\|_{L^1_T L^q} \\
&\leq C(T, v_0, b_0) + C\|v\|_{L^\infty_T H^1} \|b\|_{L^1_T L^q}^{\frac{(2\beta-1-\epsilon)q-1}{(2\beta+2\gamma-\epsilon)q}} \|\Lambda^{2\beta+2\gamma-\epsilon} b\|_{L^1_T L^q}^{\frac{(2\gamma+1)q+1}{(2\beta+2\gamma-\epsilon)q}} \\
&\quad + C\|b\|_{L^\infty_T L^\infty} \|\omega\|_{L^1_T L^q} \\
&\leq \frac{1}{2} \|\Lambda^{2\beta+2\gamma-\epsilon} b\|_{L^1_T L^q} + C(T, v_0, b_0) + C(T, v_0, b_0)\|\omega\|_{L^1_T L^q},
\end{aligned}$$

which leads to

$$(2.23) \qquad \|\Lambda^{2\beta+2\gamma-\epsilon} b\|_{L^1_T L^q} \leq C(T, v_0, b_0) + C(T, v_0, b_0)\|\omega\|_{L^1_T L^q}.$$

In order to close the above inequality, we multiply (2.12) by $|\omega|^{q-2}\omega$ and integrate it over whole space to deduce

$$\begin{aligned}
\frac{1}{q} \frac{d}{dt} \|\omega(t)\|_{L^q}^q &\leq C\|\nabla^\perp \nabla \cdot (b \otimes b)\|_{L^q} \|\omega\|_{L^q}^{q-1} \\
&\leq C\|b\|_{L^\infty} \|\Delta b\|_{L^q} \|\omega\|_{L^q}^{q-1} \\
&\leq C\|b\|_{L^\infty} \|b\|_{L^q}^{1 - \frac{2}{2\gamma+2\beta-\epsilon}} \|\Lambda^{2\beta+2\gamma-\epsilon} b\|_{L^q}^{\frac{2}{2\beta+2\gamma-\epsilon}} \|\omega\|_{L^q}^{q-1} \\
(2.24) \qquad &\leq C(T, v_0, b_0)\|\Lambda^{2\beta+2\gamma-\epsilon} b\|_{L^q}^{\frac{2}{2\beta+2\gamma-\epsilon}} \|\omega\|_{L^q}^{q-1},
\end{aligned}$$

where we further restrict ϵ to satisfy $\epsilon \in (0, 2\beta + 2\gamma - 2)$. Therefore, it is easy to deduce from (2.24) that

$$(2.25) \qquad \frac{d}{dt} \|\omega(t)\|_{L^q} \leq C(T, v_0, b_0)\|\Lambda^{2\beta+2\gamma-\epsilon} b\|_{L^q}^{\frac{2}{2\beta+2\gamma-\epsilon}}.$$

Integrating (2.25) in time yields

$$(2.26) \qquad \|\omega(t)\|_{L^\infty_T L^q} \leq C(T, v_0, b_0) + C(T, v_0, b_0)\|\Lambda^{2\beta+2\gamma-\epsilon} b\|_{L^1_T L^q}^{\frac{2}{2\beta+2\gamma-\epsilon}}.$$

Combining (2.23) and (2.26), we derive

$$\begin{aligned}
\|\Lambda^{2\beta+2\gamma-\epsilon} b\|_{L^1_T L^q} &\leq C(T, v_0, b_0) + C(T, v_0, b_0)\|\omega\|_{L^1_T L^q} \\
&\leq C(T, v_0, b_0) + C(T, v_0, b_0)\|\omega\|_{L^\infty_T L^q}
\end{aligned}$$

$$\leq C(T, v_0, b_0) + C(T, v_0, b_0)\|\Lambda^{2\beta+2\gamma-\epsilon}b\|_{L_T^1 L^q}^{\frac{2}{2\beta+2\gamma-\epsilon}}$$

$$\leq \frac{1}{2}\|\Lambda^{2\beta+2\gamma-\epsilon}b\|_{L_T^1 L^q} + C(T, v_0, b_0),$$

which allows us to conclude

$$\|\Lambda^{2\beta+2\gamma-\epsilon}b\|_{L_T^1 L^q} \leq C(T, v_0, b_0).$$

Keeping in mind that $\beta > 1 - \frac{\gamma}{2}$, we may take q suitably large such that

(2.27) $\qquad \|\nabla^2 b\|_{L_T^1 L^\infty} \leq C\|b\|_{L_T^1 L^\infty} + C\|\Lambda^{2\beta+2\gamma-\epsilon}b\|_{L_T^1 L^q} \leq C(T, v_0, b_0).$

Combing back to (2.24), we can deduce

$$\frac{1}{q}\frac{d}{dt}\|\omega(t)\|_{L^q}^q \leq C\|\nabla^\perp\nabla\cdot(b\otimes b)\|_{L^q}\|\omega\|_{L^q}^{q-1}$$

$$\leq C(\|b\|_{L^\infty}\|\nabla^2 b\|_{L^q} + \|\nabla b\|_{L^\infty}\|\nabla b\|_{L^q})\|\omega\|_{L^q}^{q-1},$$

which gives

$$\frac{d}{dt}\|\omega(t)\|_{L^q} \leq C(\|b\|_{L^\infty}\|\nabla^2 b\|_{L^q} + \|\nabla b\|_{L^\infty}\|\nabla b\|_{L^q}).$$

Letting $p \to \infty$, one has

(2.28) $\qquad \frac{d}{dt}\|\omega(t)\|_{L^\infty} \leq C(\|b\|_{L^\infty}\|\nabla^2 b\|_{L^\infty} + \|\nabla b\|_{L^\infty}^2).$

Making use of (2.11) and (2.27), we get by integrating (2.28) in time that

$$\sup_{t\in[0,T]}\|\omega(t)\|_{L^\infty} \leq C(T, v_0, b_0).$$

We therefore complete the proof of Lemma 2.4. $\qquad\qquad\qquad\qquad\square$

Case 2 : $\beta > 1 - \dfrac{\gamma}{2}$ with $\gamma \in (1, 2]$

In this case, we first state that the case $\alpha = \beta = 0$, $\gamma = 2$ will be considered in Theorem 1.3. Therefore, it suffices to consider the case $\beta > 1 - \frac{\gamma}{2}$ with $\gamma \in (1, 2)$. Without loss of generality, we may assume $\beta \leq \frac{1}{2}$ as $\beta > \frac{1}{2}$ can be handled as that of **Case 1**. We remark that in this case, we still have the estimate (2.1). Now we will establish the following result as Lemma 2.2.

LEMMA 2.5. *Assume (v_0, b_0) satisfies the assumptions stated in Theorem 1.1. If $\beta > 1 - \frac{\gamma}{2}$ with $\gamma \in (1, 2)$, then the corresponding solution (v, b) of (1.2) admits the following bound*

(2.29) $\qquad \displaystyle\sup_{t\in[0,T]}\|\Lambda^{\beta+\gamma-1}b(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{2\beta+\gamma-1}b(\tau)\|_{L^2}^2\,d\tau \leq C(T, v_0, b_0).$

In particular, due to $\beta > 1 - \frac{\gamma}{2}$, it holds from (2.29) that

(2.30) $\qquad \displaystyle\int_0^T (\|\nabla b(\tau)\|_{L^2}^2 + \|b(\tau)\|_{L^\infty}^2)\,d\tau \leq C(T, v_0, b_0).$

PROOF. Recalling (2.7), we get

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{\beta+\gamma-1}b(t)\|_{L^2}^2 + \|\Lambda^{2\beta+\gamma-1}b\|_{L^2}^2 = J_1 + J_2,$$

where

$$J_1 = \int_{\mathbb{R}^2} \Lambda^{\beta+\gamma-1}(b\cdot\nabla u)\cdot\Lambda^{\beta+\gamma-1}b\,dx, \quad J_2 = -\int_{\mathbb{R}^2}\Lambda^{\beta+\gamma-1}(u\cdot\nabla b)\cdot\Lambda^{\beta+\gamma-1}b\,dx.$$

It follows from (B.2) that

$$\begin{aligned}
J_1 \leq &C\|\Lambda^{\gamma-1}(b\cdot\nabla u)\|_{L^2}\|\Lambda^{2\beta+\gamma-1}b\|_{L^2}\\
\leq &C(\|\Lambda^\gamma u\|_{L^2}\|b\|_{L^\infty} + \|\nabla u\|_{L^{\frac{2}{2-\gamma}}}\|\Lambda^{\gamma-1}b\|_{L^{\frac{2}{\gamma-1}}})\|\Lambda^{2\beta+\gamma-1}b\|_{L^2}\\
\leq &C\|\Lambda^\gamma u\|_{L^2}\|b\|_{L^2}^{\frac{2\beta+\gamma-2}{2\beta+\gamma-1}}\|\Lambda^{2\beta+\gamma-1}b\|_{L^2}^{\frac{1}{2\beta+\gamma-1}}\|\Lambda^{2\beta+\gamma-1}b\|_{L^2}\\
\leq &\frac{1}{4}\|\Lambda^{2\beta+\gamma-1}b\|_{L^2}^2 + C\|\Lambda^\gamma u\|_{L^2}^{\frac{2(2\beta+\gamma-1)}{2\beta+\gamma-2}}\|b\|_{L^2}^2.
\end{aligned}$$

According to (B.1), it leads to

$$\begin{aligned}
J_4 = -&\int_{\mathbb{R}^2}[\Lambda^{\beta+\gamma-1},\,u\cdot\nabla]b\cdot\Lambda^{\beta+\gamma-1}b\,dx\\
\leq &C\|[\Lambda^{\beta+\gamma-1},\,u\cdot\nabla]b\|_{L^{\frac{2}{1+\beta}}}\|\Lambda^{\beta+\gamma-1}b\|_{L^{\frac{2}{1-\beta}}}\\
\leq &C(\|\nabla u\|_{L^p}\|\Lambda^{\beta+\gamma-1}b\|_{L^{\frac{2p}{(1+\beta)p-2}}} + \|\nabla b\|_{L^2}\|\Lambda^{\beta+\gamma-1}u\|_{L^{\frac{2}{\beta}}})\|\Lambda^{2\beta+\gamma-1}b\|_{L^2}\\
\leq &C(\|u\|_{H^\gamma}\|b\|_{L^2}^{1-\lambda}\|\Lambda^{2\beta+\gamma-1}b\|_{L^2}^\lambda + \|b\|_{L^2}^{\frac{2\beta+\gamma-2}{2\beta+\gamma-1}}\|\Lambda^{2\beta+\gamma-1}b\|_{L^2}^{\frac{1}{2\beta+\gamma-1}}\|\Lambda^\gamma u\|_{L^2})\\
&\times \|\Lambda^{2\beta+\gamma-1}b\|_{L^2}\\
\leq &\frac{1}{8}\|\Lambda^{2\beta+\gamma-1}b\|_{L^2}^2 + C(\|u\|_{H^\gamma}^{\frac{2}{1-\lambda}} + \|\Lambda^\gamma u\|_{L^2}^{\frac{2(2\beta+\gamma-1)}{2\beta+\gamma-2}})\|b\|_{L^2}^2,
\end{aligned}$$

where p and λ satisfy

$$\max\left\{\frac{2-\gamma}{2},\,\frac{\beta}{2}\right\} < \frac{1}{p} < \min\left\{\frac{1}{2},\,\beta\right\}, \qquad \lambda = \frac{(\gamma-1)p+2}{(2\beta+\gamma-1)p}\in(0,\,1).$$

We obtain by combining all the above estimates

$$\frac{d}{dt}\|\Lambda^{\beta+\gamma-1}b(t)\|_{L^2}^2 + \|\Lambda^{2\beta+\gamma-1}b\|_{L^2}^2 \leq C(\|u\|_{H^\gamma}^{\frac{2}{1-\lambda}} + \|\Lambda^\gamma u\|_{L^2}^{\frac{2(2\beta+\gamma-1)}{2\beta+\gamma-2}})\|b\|_{L^2}^2.$$

Using (2.1), we deduce

$$\sup_{t\in[0,\,T]}\|\Lambda^{\beta+\gamma-1}b(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{2\beta+\gamma-1}b(\tau)\|_{L^2}^2\,d\tau \leq C(T,v_0,b_0),$$

which is (2.29). This completes the proof of Lemma 2.5. $\qquad\square$

Next, we will derive the following result as Lemma 2.3.

LEMMA 2.6. *Assume (v_0, b_0) satisfies the assumptions stated in Theorem 1.1. If $\beta > 1 - \frac{\gamma}{2}$ with $\gamma\in(1,\,2)$, then the corresponding solution (v,b) of (1.2) admits the following bound*

$$(2.31)\quad \sup_{t\in[0,\,T]}(\|v(t)\|_{H^1}^2 + \|\Lambda^{2\gamma+\beta}b(t)\|_{L^2}^2) + \int_0^T \|\Lambda^{2\gamma+2\beta}b(\tau)\|_{L^2}^2\,d\tau \leq C(T,v_0,b_0).$$

In particular, there holds

$$(2.32) \qquad \sup_{t \in [0, T]} \|b(t)\|_{L^\infty} + \int_0^T \|\nabla b(\tau)\|_{L^\infty}^2 \, d\tau \leq C(T, v_0, b_0).$$

PROOF. Recalling (2.13), we have

$$(2.33) \qquad \frac{1}{2} \frac{d}{dt} \|\omega(t)\|_{L^2}^2 \leq \frac{1}{4} \|\Lambda^{2\gamma+2\beta} b\|_{L^2}^2 + C(1 + \|b\|_{L^\infty}^2)(1 + \|\omega\|_{L^2}^2).$$

We have from (2.14) that

$$(2.34) \qquad \frac{1}{2} \frac{d}{dt} \|\Lambda^{2\gamma+\beta} b(t)\|_{L^2}^2 + \|\Lambda^{2\gamma+2\beta} b\|_{L^2}^2 = J_3 + J_4,$$

where

$$J_3 = \int_{\mathbb{R}^2} \Lambda^{2\gamma+\beta}(b \cdot \nabla u) \cdot \Lambda^{2\gamma+\beta} b \, dx, \qquad J_4 = -\int_{\mathbb{R}^2} \Lambda^{2\gamma+\beta}(u \cdot \nabla b) \cdot \Lambda^{2\gamma+\beta} b \, dx.$$

Making use of (B.2), one deduces

$$\begin{aligned}
J_3 &\leq C \|\Lambda^{2\gamma}(b \nabla u)\|_{L^2} \|\Lambda^{2\gamma+2\beta} b\|_{L^2} \\
&\leq C(\|b\|_{L^\infty} \|\Lambda^{2\gamma+1} u\|_{L^2} + \|\nabla u\|_{L^{\frac{2}{2-\gamma}}} \|\Lambda^{2\gamma} b\|_{L^{\frac{2}{\gamma-1}}}) \|\Lambda^{2\gamma+2\beta} b\|_{L^2} \\
&\leq C(\|\nabla v\|_{L^2} \|b\|_{L^\infty} + \|\Lambda^\gamma u\|_{L^2} \|b\|_{L^2}^{\frac{2\beta+\gamma-2}{2(\beta+\gamma)}} \|\Lambda^{2\gamma+2\beta} b\|_{L^2}^{\frac{2+\gamma}{2(\beta+\gamma)}}) \|\Lambda^{2\gamma+2\beta} b\|_{L^2} \\
&\leq \frac{1}{8} \|\Lambda^{2\gamma+2\beta} b\|_{L^2}^2 + C \|b\|_{L^\infty}^2 \|\omega\|_{L^2}^2 + C \|\Lambda^\gamma u\|_{L^2}^{\frac{4(\beta+\gamma)}{2\beta+\gamma-2}} \|b\|_{L^2}^2.
\end{aligned}$$

For J_4, we have by (B.1) that

$$\begin{aligned}
J_4 &= -\int_{\mathbb{R}^2} [\Lambda^{2\gamma+\beta}, \, u \cdot \nabla] b \cdot \Lambda^{2\gamma+\beta} b \, dx \\
&\leq C \|[\Lambda^{2\gamma+\beta}, \, u \cdot \nabla] b\|_{L^{\frac{2}{1+\beta}}} \|\Lambda^{2\gamma+\beta} b\|_{L^{\frac{2}{1-\beta}}} \\
&\leq C(\|\nabla u\|_{L^p} \|\Lambda^{2\gamma+\beta} b\|_{L^{\frac{2p}{(1+\beta)p-2}}} + \|\nabla b\|_{L^2} \|\Lambda^{2\gamma+\beta} u\|_{L^{\frac{2}{\beta}}}) \|\Lambda^{2\gamma+2\beta} b\|_{L^2} \\
&\leq C(\|u\|_{H^\gamma} \|b\|_{L^2}^{1-\theta} \|\Lambda^{2\gamma+2\beta} b\|_{L^2}^\theta + \|\nabla b\|_{L^2} \|\Lambda^{2\gamma+1} u\|_{L^2}) \|\Lambda^{2\gamma+2\beta} b\|_{L^2} \\
&\leq \frac{1}{8} \|\Lambda^{2\gamma+2\beta} b\|_{L^2}^2 + C \|\nabla b\|_{L^2}^2 \|\omega\|_{L^2}^2 + C \|u\|_{H^\gamma}^{\frac{2}{1-\theta}} \|b\|_{L^2}^2,
\end{aligned}$$

where p and θ satisfy

$$\max \left\{ \frac{2-\gamma}{2}, \frac{\beta}{2} \right\} < \frac{1}{p} < \min \left\{ \frac{1}{2}, \beta \right\}, \qquad \theta = \frac{\gamma p + 1}{(\beta + \gamma)p} \in (0, 1).$$

Substituting the above two estimates back into (2.34) gives

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\Lambda^{2\gamma+\beta} b(t)\|_{L^2}^2 + \frac{3}{4} \|\Lambda^{2\gamma+2\beta} b\|_{L^2}^2 &\leq C(\|b\|_{L^\infty}^2 + \|\nabla b\|_{L^2}^2) \|\omega\|_{L^2}^2 \\
&\quad + C(\|\Lambda^\gamma u\|_{L^2}^{\frac{4(\beta+\gamma)}{2\beta+\gamma-2}} + \|u\|_{H^\gamma}^{\frac{2}{1-\theta}}) \|b\|_{L^2}^2,
\end{aligned}$$

which along with (2.33) gives

$$\begin{aligned}
&\frac{d}{dt}(\|\Lambda^{2\gamma+\beta} b(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2) + \|\Lambda^{2\gamma+2\beta} b\|_{L^2}^2 \\
&\leq C(1 + \|b\|_{L^\infty}^2 + \|\nabla b\|_{L^2}^2)(1 + \|\omega\|_{L^2}^2) + C(\|\Lambda^\gamma u\|_{L^2}^{\frac{4(\beta+\gamma)}{2\beta+\gamma-2}} + \|u\|_{H^\gamma}^{\frac{2}{1-\theta}}) \|b\|_{L^2}^2.
\end{aligned}$$

Thanks to (2.1), (2.30) and the Gronwall inequality, we have that

(2.35)

$$\sup_{t\in[0,T]} (\|\nabla v(t)\|_{L^2}^2 + \|\Lambda^{2\gamma+\beta} b(t)\|_{L^2}^2) + \int_0^T \|\Lambda^{2\gamma+2\beta} b(\tau)\|_{L^2}^2 \, d\tau \le C(T, v_0, b_0).$$

Coming back to (2.20), we get

$$\frac{d}{dt} \|v(t)\|_{L^2} \le C(\|\nabla v\|_{L^2}^2 + \|b\|_{H^{2\gamma+\beta}}^2).$$

Using (2.35), it leads to

$$\sup_{t\in[0,T]} \|v(t)\|_{L^2} \le C(T, v_0, b_0).$$

This completes the proof of Lemma 2.6. $\qquad\square$

Now let us to show the following crucial estimate.

LEMMA 2.7. *Assume (v_0, b_0) satisfies the assumptions stated in Theorem 1.1. If $\beta > 1 - \frac{\gamma}{2}$ with $\gamma \in (1, 2)$, then the corresponding solution (v, b) of (1.2) admits the following bound*

$$\sup_{t\in[0,T]} \|\omega(t)\|_{L^\infty} \le C(T, v_0, b_0).$$

PROOF. The proof is different from Lemma 2.4 due to the fact that we only consider the case $\beta \le \frac{1}{2}$. Fortunately, the bound (2.31) allows us to conclude the desired estimate. It follows from (2.12) that

$$\frac{d}{dt} \|\omega(t)\|_{L^\infty} \le \|\nabla^2(bb)\|_{L^\infty}.$$

Direct computations yield due to $\beta > 1 - \frac{\gamma}{2}$ with $\gamma \in (1, 2)$

$$\|\nabla^2(bb)\|_{L^\infty} \le C(\|bb\|_{L^2} + \|\Lambda^{2\gamma+2\beta}(bb)\|_{L^2})$$
$$\le C(\|b\|_{L^\infty} \|b\|_{L^2} + \|b\|_{L^\infty} \|\Lambda^{2\gamma+2\beta} b\|_{L^2}).$$

Thanks to (2.31) and (2.32), it implies

$$\int_0^T \|\nabla^2(bb)(\tau)\|_{L^\infty} \, d\tau \le C(T, v_0, b_0).$$

Consequently, we have

$$\sup_{t\in[0,T]} \|\omega(t)\|_{L^\infty} \le C(T, v_0, b_0).$$

We thus complete the proof of Lemma 2.7. $\qquad\square$

With the above estimates of both **Case 1** and **Case 2** at our disposal, we are now ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Applying Λ^s to (2.12) and multiplying it by $\Lambda^s \omega$ yield

(2.36) $$\frac{1}{2} \frac{d}{dt} \|\Lambda^s \omega(t)\|_{L^2}^2 = - \int_{\mathbb{R}^2} [\Lambda^s, u \cdot \nabla] \omega \Lambda^s \omega \, dx + \int_{\mathbb{R}^2} \Lambda^s \nabla^\perp \nabla \cdot (b \otimes b) \Lambda^s \omega.$$

Applying $\Lambda^{s+2-\beta}$ to $(1.2)_2$ and multiplying it by $\Lambda^{s+2-\beta}b$, we get

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{s+2-\beta}b(t)\|_{L^2}^2 + \|\Lambda^{s+2}b\|_{L^2}^2 = \int_{\mathbb{R}^2}\Lambda^{s+2-\beta}(b\cdot\nabla u)\cdot\Lambda^{s+2-\beta}b\,dx$$

$$(2.37) \qquad\qquad\qquad\qquad - \int_{\mathbb{R}^2}\Lambda^{s+2-\beta}(u\cdot\nabla b)\cdot\Lambda^{s+2-\beta}b\,dx.$$

Due to $\nabla\cdot u = 0$, we have by (B.3) that

$$\int_{\mathbb{R}^2}[\Lambda^s,\,u\cdot\nabla]\omega\Lambda^s\omega\,dx = \int_{\mathbb{R}^2}[\Lambda^s\partial_i,\,u_i]\omega\Lambda^s\omega\,dx$$

$$\le C\|[\Lambda^s\partial_i,\,u_i]\omega\|_{L^2}\|\Lambda^s\omega\|_{L^2}$$

$$\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s\omega\|_{L^2} + \|\omega\|_{L^\infty}\|\Lambda^{s+1}u\|_{L^2})\|\Lambda^s\omega\|_{L^2}$$

$$\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s\omega\|_{L^2} + \|\omega\|_{L^\infty}\|\Lambda^s\omega\|_{L^2})\|\Lambda^s\omega\|_{L^2}$$

$$(2.38) \qquad\qquad \le C(\|\omega\|_{L^2} + \|\omega\|_{L^\infty})\|\Lambda^s\omega\|_{L^2}^2,$$

where we have applied the facts due to $v = u + (-\Delta)^\gamma u$ with $\gamma > 0$

$$(2.39)\quad \|\nabla u\|_{L^\infty} \le C(\|\omega\|_{L^2} + \|\omega\|_{L^\infty}), \qquad \|\Lambda^{s+\sigma}u\|_{L^2} \le \|\Lambda^s\omega\|_{L^2}, \quad \sigma \le 2\gamma + 1.$$

The simple proof of (2.39) will be given at the end of Appendix B. In view of (B.2), it entails

$$\int_{\mathbb{R}^2}\Lambda^s\nabla^\perp\nabla\cdot(b\otimes b)\Lambda^s\omega \le C\|\Lambda^s\nabla^\perp\nabla\cdot(b\otimes b)\|_{L^2}\|\Lambda^s\omega\|_{L^2}$$

$$\le C\|b\|_{L^\infty}\|\Lambda^{s+2}b\|_{L^2}\|\Lambda^s\omega\|_{L^2}$$

$$(2.40) \qquad\qquad \le \frac{1}{8}\|\Lambda^{s+2}b\|_{L^2}^2 + C\|b\|_{L^\infty}^2\|\Lambda^s\omega\|_{L^2}^2.$$

Noticing $\nabla\cdot b = 0$, it follows from (B.2) and (2.39) that

$$\int_{\mathbb{R}^2}\Lambda^{s+2-\beta}(b\cdot\nabla u)\cdot\Lambda^{s+2-\beta}b\,dx$$

$$\le C\|\Lambda^{s+2-2\beta}(b\cdot\nabla u)\|_{L^2}\|\Lambda^{s+2}b\|_{L^2}$$

$$\le C(\|\nabla u\|_{L^\infty}\|\Lambda^{s+2-2\beta}b\|_{L^2} + \|b\|_{L^\infty}\|\Lambda^{s+3-2\beta}u\|_{L^2})\|\Lambda^{s+2}b\|_{L^2}$$

$$\le C(\|\nabla u\|_{L^\infty}\|b\|_{L^2}^{\frac{\beta}{s+2-\beta}}\|\Lambda^{s+2-\beta}b\|_{L^2}^{\frac{s+2-2\beta}{s+2-\beta}} + \|b\|_{L^\infty}\|\Lambda^s\omega\|_{L^2})\|\Lambda^{s+2}b\|_{L^2}$$

$$\le \frac{1}{8}\|\Lambda^{s+2}b\|_{L^2}^2 + C\|b\|_{L^\infty}^2\|\Lambda^s\omega\|_{L^2}^2$$

$$(2.41) \qquad + C(\|\omega\|_{L^2} + \|\omega\|_{L^\infty})^2\|b\|_{L^2}^{\frac{2\beta}{s+2-\beta}}(1 + \|\Lambda^{s+2-\beta}b\|_{L^2}^2).$$

Since $\nabla\cdot u = 0$, we also conclude

$$\int_{\mathbb{R}^2}\Lambda^{s+2-\beta}(u\cdot\nabla b)\cdot\Lambda^{s+2-\beta}b\,dx$$

$$= \int_{\mathbb{R}^2}[\Lambda^{s+2-\beta},\,u\cdot\nabla]b\cdot\Lambda^{s+2-\beta}b\,dx$$

$$\le C\|[\Lambda^{s+2-\beta},\,u\cdot\nabla]b\|_{L^2}\|\Lambda^{s+2-\beta}b\|_{L^2}$$

$$\le C(\|\nabla u\|_{L^\infty}\|\Lambda^{s+2-\beta}b\|_{L^2} + \|\nabla b\|_{L^\infty}\|\Lambda^{s+2-\beta}u\|_{L^2})\|\Lambda^{s+2-\beta}b\|_{L^2}$$

$$(2.42) \quad \le C(\|\omega\|_{L^2} + \|\omega\|_{L^\infty})\|\Lambda^{s+2-\beta}b\|_{L^2}^2 + C\|\nabla b\|_{L^\infty}\|\Lambda^s\omega\|_{L^2}\|\Lambda^{s+2-\beta}b\|_{L^2}.$$

Summing up (2.36), (2.37), (2.38), (2.40), (2.41) and (2.42), it directly gives

$$\frac{d}{dt}(\|\Lambda^s\omega(t)\|_{L^2}^2 + \|\Lambda^{s+2-\beta}b(t)\|_{L^2}^2) + \|\Lambda^{s+2}b\|_{L^2}^2$$

(2.43) $$\leq C(1 + \|\omega\|_{L^2}^2 + \|\omega\|_{L^\infty}^2 + \|\nabla b\|_{L^\infty} + \|b\|_{L^\infty}^2)(\|\Lambda^s\omega\|_{L^2}^2 + \|\Lambda^{s+2-\beta}b\|_{L^2}^2).$$

The estimates of Lemma 2.1-Lemma 2.4 (or Lemma 2.5-Lemma 2.7) allow us to show

$$\int_0^T (1 + \|\omega\|_{L^2}^2 + \|\omega\|_{L^\infty}^2 + \|\nabla b\|_{L^\infty} + \|b\|_{L^\infty}^2)(t)\, dt \leq C(T, v_0, b_0).$$

This together with the Gronwall inequality, we deduce from (2.43) that

$$\sup_{t\in[0,T]}(\|\Lambda^s\omega(t)\|_{L^2}^2 + \|\Lambda^{s+2-\beta}b(t)\|_{L^2}^2) + \int_0^T \|\Lambda^{s+2}b(t)\|_{L^2}^2\, dt \leq C(T, v_0, b_0).$$

Consequently, this completes the proof of Theorem 1.1.   □

3. The proof of Theorem 1.2 and Theorem 1.3

We remark that when $\gamma = 0$ and $\alpha = 2$, the corresponding system admits a unique global regular solution, see [**29, 37**] for details. Consequently, it is sufficient to consider the case $\alpha + \gamma = 2$ with $\alpha \in [0, 2)$.

The proof of Theorem 1.2

In this case, we should keep in mind that $\alpha + \gamma = 2$ with $\alpha \in (0, 2)$. Now we begin with the basic energy estimate.

LEMMA 3.1. *Assume (v_0, b_0) satisfies the assumptions stated in Theorem 1.2. Then the corresponding solution (v, b) of (1.4) admits the following bound*

$$\|u(t)\|_{L^2}^2 + \|\Lambda^\gamma \mathcal{L}u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^t (\|\Lambda^\alpha \mathcal{L}u(\tau)\|_{L^2}^2 + \|\Lambda^{\alpha+\gamma}\mathcal{L}^2 u(\tau)\|_{L^2}^2)\, d\tau$$
$$\leq C(v_0, b_0).$$

In particular, due to $\alpha + \gamma = 2$, there holds true

(3.1) $$\|u(t)\|_{L^2}^2 + \|\Lambda^\gamma \mathcal{L}u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^t \left\|\frac{\Lambda^2}{g^2(\Lambda)}u(\tau)\right\|_{L^2}^2 d\tau \leq C(v_0, b_0).$$

PROOF. The proof can be performed as that of Lemma 2.1. We thus omit the details.   □

Our next goal is to derive the following key estimate.

LEMMA 3.2. *Assume (v_0, b_0) satisfies the assumptions stated in Theorem 1.2. Then the corresponding solution (v, b) of (1.4) admits the following bound*

(3.2)

$$\sup_{t\in[0,T]}(\|b(t)\|_{L^\infty} + \|\nabla b(t)\|_{L^2} + \|v(t)\|_{L^2}) + \int_0^T \|\Lambda^\alpha \mathcal{L}v(\tau)\|_{L^2}^2\, d\tau \leq C(T, v_0, b_0).$$

In particular, it holds

$$(3.3) \qquad \int_0^T \left(\|\nabla u(\tau)\|_{L^\infty}^2 + \|\Delta u(\tau)\|_{L^2}^2 \right) d\tau \leq C(T, v_0, b_0).$$

PROOF. We recall the second equation of (1.4), namely,

$$(3.4) \qquad \partial_t b + (u \cdot \nabla)b = (b \cdot \nabla)u.$$

Multiplying (3.4) by $|b|^{q-2}b$ and integrating it over whole space, it implies

$$\frac{1}{q} \frac{d}{dt} \|b(t)\|_{L^q}^q \leq \|\nabla u\|_{L^\infty} \|b\|_{L^q}^q,$$

which leads to

$$\frac{d}{dt} \|b(t)\|_{L^q} \leq \|\nabla u\|_{L^\infty} \|b\|_{L^q}.$$

By letting $q \to \infty$, we deduce

$$\frac{d}{dt} \|b(t)\|_{L^\infty} \leq C \|\nabla u\|_{L^\infty} \|b\|_{L^\infty},$$

which implies

$$(3.5) \qquad \frac{d}{dt} \|b(t)\|_{L^\infty}^m \leq C(m) \|\nabla u\|_{L^\infty} \|b\|_{L^\infty}^m, \quad \forall m \in [2, \infty).$$

Applying ∇ to (3.4), we get

$$(3.6) \qquad \partial_t \nabla b + (u \cdot \nabla)\nabla b = \nabla b \cdot \nabla u + b \cdot \nabla \nabla u - \nabla u \cdot \nabla b.$$

Taking the L^2-inner product of (3.6) with ∇b, one arrives at

$$\frac{d}{dt} \|\nabla b(t)\|_{L^2}^2 \leq C\|\nabla u\|_{L^\infty} \|\nabla b\|_{L^2}^2 + C\|b\|_{L^\infty} \|\Delta u\|_{L^2} \|\nabla b\|_{L^2}$$
$$\leq C(\|\nabla u\|_{L^\infty} + \|\Delta u\|_{L^2})(\|\nabla b\|_{L^2}^2 + \|b\|_{L^\infty}^2),$$

which allows us to show

$$(3.7) \qquad \frac{d}{dt} \|\nabla b(t)\|_{L^2}^3 \leq C(\|\nabla u\|_{L^\infty} + \|\Delta u\|_{L^2})(\|\nabla b\|_{L^2}^3 + \|b\|_{L^\infty}^3).$$

Taking the L^2-inner product of $(1.4)_1$ with v, it gives that

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_{L^2}^2 + \|\Lambda^\alpha \mathcal{L}v\|_{L^2}^2 = -\int_{\mathbb{R}^2} \left(\sum_{j=1}^2 v_j \nabla u_j \right) \cdot v \, dx + \int_{\mathbb{R}^2} (b \cdot \nabla b) \cdot v \, dx$$
$$\leq C\|\nabla u\|_{L^\infty} \|v\|_{L^2}^2 + C\|b\|_{L^\infty} \|\nabla b\|_{L^2} \|v\|_{L^2}$$
$$\leq C\|\nabla u\|_{L^\infty} \|v\|_{L^2}^2 + C\|v\|_{L^2}^2 + C\|\nabla b\|_{L^2}^3 + C\|b\|_{L^\infty}^6.$$

This allows us to deduce

$$(3.8) \qquad \frac{d}{dt} \|v(t)\|_{L^2}^2 + \|\Lambda^\alpha \mathcal{L}v\|_{L^2}^2 \leq C(1 + \|\nabla u\|_{L^\infty})\|v\|_{L^2}^2 + C\|\nabla b\|_{L^2}^3 + C\|b\|_{L^\infty}^6.$$

Taking $m = 3$ and $m = 6$ in (3.5), we then get by summing up (3.5), (3.7) and (3.8)

$$(3.9) \qquad \frac{d}{dt} V(t) + H(t) \leq C(1 + \|\nabla u\|_{L^\infty} + \|\Delta u\|_{L^2})V(t),$$

where

$$V(t) := \|b(t)\|_{L^\infty}^3 + \|b(t)\|_{L^\infty}^6 + \|\nabla b(t)\|_{L^2}^3 + \|v(t)\|_{L^2}^2, \quad H(t) := \|\mathcal{L}\Lambda^\alpha v(t)\|_{L^2}^2.$$

According to the assumptions on g (more precisely, g grows logarithmically), one may conclude that for any fixed $\delta > 0$, there exists $N = N(\delta)$ satisfying

$$g(r) \leq \widetilde{C} r^\delta, \quad \forall\, r \geq N$$

with the constant $\widetilde{C} = \widetilde{C}(\delta)$. With this observation, one has for any $\delta > 0$

$$\|\Lambda^\kappa \mathcal{L} u\|_{L^2}^2 = \int_{|\xi|<N(\delta)} \frac{|\xi|^{2\kappa}}{g^2(|\xi|)} |\widehat{u}(\xi)|^2 \, d\xi + \int_{|\xi|\geq N(\delta)} \frac{|\xi|^{2\kappa}}{g^2(|\xi|)} |\widehat{u}(\xi)|^2 \, d\xi$$

$$\geq \int_{|\xi|\geq N(\delta)} \frac{|\xi|^{2\kappa}}{\left[\widetilde{C}|\xi|^\delta\right]^2} |\widehat{u}(\xi)|^2 \, d\xi$$

$$= \int_{\mathbb{R}^2} \frac{|\xi|^{2\kappa}}{\left[\widetilde{C}|\xi|^\delta\right]^2} |\widehat{u}(\xi)|^2 \, d\xi - \int_{|\xi|<N(\delta)} \frac{|\xi|^{2\kappa}}{\left[\widetilde{C}|\xi|^\delta\right]^2} |\widehat{u}(\xi)|^2 \, d\xi$$

$$(3.10) \qquad \geq C_0 \|\Lambda^{\kappa-\delta} u\|_{L^2}^2 - \widetilde{C_0} \|u\|_{L^2}^2,$$

where C_0 and $\widetilde{C_0}$ depend only on δ. Due to $v = u + (-\Delta)^\gamma \mathcal{L}^2 u$, we may deduce by the argument used in proving (3.10) that

$$(3.11) \qquad \|\Lambda^{r_1} u\|_{L^2} \leq C(\|u\|_{L^2} + \|\mathcal{L}\Lambda^\alpha v\|_{L^2})$$

for any $r_1 < \alpha + 2\gamma \equiv 2 + \gamma > 2$. Invoking the high-low frequency technique gives

$$\|\nabla u\|_{L^\infty} \leq \|\Delta_{-1}\nabla u\|_{L^\infty} + \sum_{l=0}^{N-1} \|\Delta_l \nabla u\|_{L^\infty} + \sum_{l=N}^{\infty} \|\Delta_l \nabla u\|_{L^\infty},$$

where Δ_l ($l = -1, 0, 1, \cdots$) denote the frequency operator (see Appendix for details). One obtains that by using the Bernstein inequality (see Lemma B.1)

$$\|\Delta_{-1}\nabla u\|_{L^\infty} \leq C\|u\|_{L^2},$$

$$\sum_{l=N}^{\infty} \|\Delta_l \nabla u\|_{L^\infty} \leq C \sum_{l=N}^{\infty} 2^{2l} \|\Delta_l u\|_{L^2}$$

$$= C \sum_{l=N}^{\infty} 2^{l(2-r_1)} \|\Delta_l \Lambda^{r_1} u\|_{L^2}$$

$$\leq C 2^{N(2-r_1)} \|\Lambda^{r_1} u\|_{L^2},$$

where $r_1 > 2$. By the Bernstein inequality again and the Plancherel theorem, one gets

$$\sum_{l=0}^{N-1} \|\Delta_l \nabla u\|_{L^\infty} \leq C \sum_{l=0}^{N-1} 2^{2l} \|\Delta_l u\|_{L^2} \leq C \sum_{l=0}^{N-1} \|\Delta_l \Lambda^2 u\|_{L^2}$$

$$\leq C \sum_{l=0}^{N-1} \|\varphi(2^{-l}\xi)|\xi|^2 \widehat{u}(\xi)\|_{L^2}$$

$$= C \sum_{l=0}^{N-1} \left\| \varphi(2^{-l}\xi) g^2(|\xi|) \frac{|\xi|^2}{g^2(|\xi|)} \widehat{u}(\xi) \right\|_{L^2}$$

$$\leq C \sum_{l=0}^{N-1} g^2(2^l) \left\| \frac{|\xi|^2}{g^2(|\xi|)} \widehat{\Delta_l u}(\xi) \right\|_{L^2}$$

$$\leq C\Big(\sum_{l=0}^{N-1} g^4(2^l)\Big)^{\frac{1}{2}} \Big(\sum_{l=0}^{N-1}\Big\|\frac{|\xi|^2}{g^2(|\xi|)}\widehat{\Delta_l u}(\xi)\Big\|_{L^2}^2\Big)^{\frac{1}{2}}$$

$$\leq C g^2(2^N)\Big(\sum_{l=1}^{N-1} 1\Big)^{\frac{1}{2}}\Big\|\frac{\Lambda^2}{g^2(\Lambda)}u\Big\|_{L^2}$$

$$\leq C g^2(2^N)\sqrt{N}\Big\|\frac{\Lambda^2}{g^2(\Lambda)}u\Big\|_{L^2},$$

where we have used the fact that g is a non-decreasing function. Taking $2 < r_1 < 2+\gamma$, we therefore obtain by summarizing the above three estimates

$$\|\nabla u\|_{L^\infty} \leq C\|u\|_{L^2} + C g^2(2^N)\sqrt{N}\Big\|\frac{\Lambda^2}{g^2(\Lambda)}u\Big\|_{L^2} + C2^{N(2-r_1)}\|\mathcal{L}\Lambda^\alpha v\|_{L^2}.$$

By the same argument, we have

$$\|\Delta u\|_{L^2} \leq C\|u\|_{L^2} + C g^2(2^N)\sqrt{N}\Big\|\frac{\Lambda^2}{g^2(\Lambda)}u\Big\|_{L^2} + C2^{N(2-r_1)}\|\mathcal{L}\Lambda^\alpha v\|_{L^2}.$$

It follows from (3.9) and (3.11) that

$$\frac{d}{dt}V(t) + H(t) \leq CV(t) + C g^2(2^N)\sqrt{N}\Big\|\frac{\Lambda^2}{g^2(\Lambda)}u\Big\|_{L^2}V(t) + C2^{N(2-r_1)}H^{\frac{1}{2}}(t)V(t).$$

Taking N such

$$2^N \approx (e+V(t))^{\frac{1}{2(r_1-2)}},$$

it yields

$$\frac{d}{dt}V(t) + H(t)$$
$$\leq C\Big(1 + \Big\|\frac{\Lambda^2}{g^2(\Lambda)}u\Big\|_{L^2}\Big) g^2\big[(e+V(t))^{\frac{1}{2(r_1-2)}}\big]\sqrt{\ln\big(e+V(t)\big)}\big(e+V(t)\big)$$
$$+ CH^{\frac{1}{2}}(t)\big(e+V(t)\big)^{\frac{1}{2}}$$
$$\leq C\Big(1 + \Big\|\frac{\Lambda^2}{g^2(\Lambda)}u\Big\|_{L^2}\Big) g^2\big[(e+V(t))^{\frac{1}{2(r_1-2)}}\big]\sqrt{\ln\big(e+V(t)\big)}\big(e+V(t)\big)$$
$$+ \frac{1}{2}H(t) + C\big(e+V(t)\big).$$

Using the following fact

$$g^2\big[(e+V(t))^{\frac{1}{2(r_1-2)}}\big]\sqrt{\ln\big(e+V(t)\big)}\big(e+V(t)\big) \geq 1,$$

we thus have

$$\frac{d}{dt}V(t) + H(t) \leq C\Big(1 + \Big\|\frac{\Lambda^2}{g^2(\Lambda)}u\Big\|_{L^2}\Big) g^2\big[(e+V(t))^{\frac{1}{2(r_1-2)}}\big]\sqrt{\ln\big(e+V(t)\big)}$$
$$\times \big(e+V(t)\big).$$

As a result, one deduces from the above inequality that

$$\int_{e+V(0)}^{e+V(t)} \frac{d\tau}{\tau\sqrt{\ln\tau}\,g^2(\tau^{\frac{1}{2(r_1-2)}})} \leq C\int_0^t \Big(1 + \Big\|\frac{\Lambda^2}{g^2(\Lambda)}u(\tau)\Big\|_{L^2}\Big)\,d\tau.$$

Making use of the following facts due to (1.5) and (3.1), respectively

$$\int_e^\infty \frac{d\tau}{\tau\sqrt{\ln\tau}g^2(\tau^{\frac{1}{2(r_1-2)}})} = \sqrt{2(r_1-2)}\int_{e^{\frac{1}{2(r_1-2)}}}^\infty \frac{d\tau}{\tau\sqrt{\ln\tau}g^2(\tau)} = \infty$$

and

$$\int_0^T \left(1 + \left\|\frac{\Lambda^2}{g^2(\Lambda)}u(\tau)\right\|_{L^2}\right)d\tau \le C(T, u_0, w_0),$$

we have

$$\sup_{t\in[0,T]} V(t) + \int_0^T H(\tau)\,d\tau \le C(T, v_0, b_0),$$

which is nothing but (3.2). The desired (3.3) follows from (3.11). We therefore finish the proof of Lemma 3.2. $\qquad\square$

Now we are going to improve the regularity of b, which will be used to derive (3.13).

LEMMA 3.3. *Assume (v_0, b_0) satisfies the assumptions stated in Theorem 1.2. Then the corresponding solution (v, b) of (1.4) admits the following bound for any $0 < \nu < \gamma$*

$$(3.12) \qquad\qquad \sup_{t\in[0,T]} \|\Lambda^{1+\nu}b(t)\|_{L^2} \le C(T, v_0, b_0).$$

PROOF. Applying $\Lambda^{1+\nu}$ to (3.4) and multiplying it by $\Lambda^{1+\nu}b$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{1+\nu}b(t)\|_{L^2}^2 = -\int_{\mathbb{R}^2}[\Lambda^{1+\nu}, u\cdot\nabla]b\cdot\Lambda^{1+\nu}b\,dx + \int_{\mathbb{R}^2}\Lambda^{1+\nu}(b\cdot\nabla u)\cdot\Lambda^{1+\nu}b\,dx.$$

Thanks to $\nabla\cdot u = 0$, we get by (B.3) that

$$\begin{aligned}
\int_{\mathbb{R}^2}[\Lambda^{1+\nu}, u\cdot\nabla]b\cdot\Lambda^{1+\nu}b\,dx &= \int_{\mathbb{R}^2}[\Lambda^{1+\nu}\partial_i, u_i]b_j\Lambda^{1+\nu}b_j\,dx \\
&\le C\|[\Lambda^{1+\nu}\partial_i, u_i]b_j\|_{L^2}\|\Lambda^{1+\nu}b\|_{L^2} \\
&\le C(\|\nabla u\|_{L^\infty}\|\Lambda^{1+\nu}b\|_{L^2} + \|b\|_{L^\infty}\|\Lambda^{2+\nu}u\|_{L^2}) \\
&\quad \times \|\Lambda^{1+\nu}b\|_{L^2} \\
&\le C\|\nabla u\|_{L^\infty}\|\Lambda^{1+\nu}b\|_{L^2}^2 \\
&\quad + C\|b\|_{L^\infty}(\|u\|_{L^2} + \|\mathcal{L}\Lambda^\alpha v\|_{L^2})\|\Lambda^{1+\nu}b\|_{L^2},
\end{aligned}$$

where we have used the following fact for any $0 < \nu < \gamma$ (see (3.11) for details)

$$\|\Lambda^{2+\nu}u\|_{L^2} \le C(\|u\|_{L^2} + \|\mathcal{L}\Lambda^\alpha v\|_{L^2}).$$

In view of (B.2) and the above fact, it yields

$$\begin{aligned}
\int_{\mathbb{R}^2}\Lambda^{1+\nu}(b\cdot\nabla u)\cdot\Lambda^{1+\nu}b\,dx &\le C\|\Lambda^{1+\nu}(b\cdot\nabla u)\|_{L^2}\|\Lambda^{1+\nu}b\|_{L^2} \\
&\le C(\|\nabla u\|_{L^\infty}\|\Lambda^{1+\nu}b\|_{L^2} + \|b\|_{L^\infty}\|\Lambda^{2+\nu}u\|_{L^2}) \\
&\quad \times \|\Lambda^{1+\nu}b\|_{L^2} \\
&\le C\|\nabla u\|_{L^\infty}\|\Lambda^{1+\nu}b\|_{L^2}^2 \\
&\quad + C\|b\|_{L^\infty}(\|u\|_{L^2} + \|\mathcal{L}\Lambda^\alpha v\|_{L^2})\|\Lambda^{1+\nu}b\|_{L^2}.
\end{aligned}$$

Consequently, we deduce

$$\frac{d}{dt}\|\Lambda^{1+\nu}b(t)\|_{L^2}^2 \leq C\|\nabla u\|_{L^\infty}\|\Lambda^{1+\nu}b\|_{L^2}^2 + C\|b\|_{L^\infty}(\|u\|_{L^2} + \|\mathcal{L}\Lambda^\alpha v\|_{L^2})\|\Lambda^{1+\nu}b\|_{L^2}.$$

By the Gronwall inequality, we conclude by using (3.3) and (3.2) that

$$\sup_{t\in[0,T]} \|\Lambda^{1+\nu}b(t)\|_{L^2} \leq C(T, v_0, b_0).$$

This ends the proof of Lemma 3.3. $\square$

The following lemma concerns the higher regularity of v.

LEMMA 3.4. *Assume (v_0, b_0) satisfies the assumptions stated in Theorem 1.2. Then the corresponding solution (v, b) of (1.4) admits the following bound*

$$(3.13) \qquad \sup_{t\in[0,T]} \|\Lambda^\alpha v(t)\|_{L^2} + \int_0^T \|\Lambda^{2\alpha}\mathcal{L}v(\tau)\|_{L^2}^2 \, d\tau \leq C(T, v_0, b_0).$$

Moreover, we have

$$(3.14) \qquad \sup_{t\in[0,T]} \|\nabla b(t)\|_{L^\infty} \leq C(T, v_0, b_0).$$

PROOF. Multiplying $(1.1)_1$ by $\Lambda^{2\alpha}v$ and integrating it over $\mathbb{R}^2$, we get

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^\alpha v(t)\|_{L^2}^2 + \|\Lambda^{2\alpha}\mathcal{L}v\|_{L^2}^2 = -\int_{\mathbb{R}^2}\Big(\sum_{j=1}^2 v_j\nabla u_j\Big)\cdot\Lambda^{2\alpha}v\,dx$$

$$+ \int_{\mathbb{R}^2}(b\cdot\nabla b)\cdot\Lambda^{2\alpha}v\,dx$$

$$- \int_{\mathbb{R}^2}[\Lambda^\alpha, u\cdot\nabla]v\cdot\Lambda^\alpha v\,dx.$$

Thanks to (3.10), we have for any $r_3 < 2\alpha$

$$(3.15) \qquad \|\Lambda^{r_3}v\|_{L^2} \leq C(\|v\|_{L^2} + \|\Lambda^{2\alpha}\mathcal{L}v\|_{L^2}).$$

Taking $0 < \delta < \min\{1, \alpha\}$ and using (3.15), the first term can be bounded by

$$\int_{\mathbb{R}^2}\Big(\sum_{j=1}^2 v_j\nabla u_j\Big)\cdot\Lambda^{2\alpha}v\,dx \leq C\|\Lambda^\delta(v\nabla u)\|_{L^2}\|\Lambda^{2\alpha-\delta}v\|_{L^2}$$

$$\leq C(\|\nabla u\|_{L^\infty}\|\Lambda^\delta v\|_{L^2} + \|\Lambda^\delta\nabla u\|_{L^{\frac{2}{\delta}}}\|v\|_{L^{\frac{2}{1-\delta}}})$$

$$\times \|\Lambda^{2\alpha-\delta}v\|_{L^2}$$

$$\leq C(\|\nabla u\|_{L^\infty} + \|\Delta u\|_{L^2})\|v\|_{L^2}^{\frac{\alpha-\delta}{\alpha}}\|\Lambda^\alpha v\|_{L^2}^{\frac{\delta}{\alpha}}$$

$$\times (\|v\|_{L^2} + \|\Lambda^{2\alpha}\mathcal{L}v\|_{L^2})$$

$$\leq \frac{1}{8}\|\Lambda^{2\alpha}\mathcal{L}v\|_{L^2}^2 + C(1 + \|\nabla u\|_{L^\infty}^2 + \|\Delta u\|_{L^2}^2)$$

$$\times (1 + \|\Lambda^\alpha v\|_{L^2}^2).$$

On the other hand, one may obtain

$$\int_{\mathbb{R}^2}(b\cdot\nabla b)\cdot\Lambda^{2\alpha}v\,dx \leq C\|\Lambda^{1+\nu}(bb)\|_{L^2}\|\Lambda^{2\alpha-\nu}v\|_{L^2}$$

$$\leq C\|b\|_{L^\infty}\|\Lambda^{1+\nu}b\|_{L^2}(\|v\|_{L^2} + \|\Lambda^{2\alpha}\mathcal{L}v\|_{L^2})$$

$$\leq \frac{1}{8}\|\Lambda^{2\alpha}\mathcal{L}v\|_{L^2}^2 + C\|b\|_{L^\infty}^2\|\Lambda^{1+\nu}b\|_{L^2}^2$$
$$+ C\|b\|_{L^\infty}\|v\|_{L^2}\|\Lambda^{1+\nu}b\|_{L^2},$$

where ν is given by (3.12). For the last term, due to $\nabla \cdot u = 0$, we have by (B.3) that

$$-\int_{\mathbb{R}^2} [\Lambda^\alpha,\ u \cdot \nabla]v \cdot \Lambda^\alpha v\, dx = -\int_{\mathbb{R}^2} [\Lambda^\alpha \partial_i,\ u_i]v_j \Lambda^\alpha v_j\, dx$$
$$\leq C\|[\Lambda^\alpha \partial_i,\ u_i]v_j\|_{L^2}\|\Lambda^\alpha v\|_{L^2}$$
$$\leq C(\|\nabla u\|_{L^\infty}\|\Lambda^\alpha v\|_{L^2} + \|v\|_{L^{p_1}}\|\Lambda^{\alpha+1}u\|_{L^{\frac{2p_1}{p_1-2}}})\|\Lambda^\alpha v\|_{L^2}$$
$$\leq C(\|v\|_{L^2} + \|\Lambda^\alpha \mathcal{L}v\|_{L^2})\|\Lambda^{2\alpha}\mathcal{L}v\|_{L^2}\|\Lambda^\alpha v\|_{L^2}$$
$$+ C\|\nabla u\|_{L^\infty}\|\Lambda^\alpha v\|_{L^2}^2$$
$$\leq \frac{1}{8}\|\Lambda^{2\alpha}\mathcal{L}v\|_{L^2}^2 + C(\|\nabla u\|_{L^\infty} + \|v\|_{L^2}^2 + \|\Lambda^\alpha \mathcal{L}v\|_{L^2}^2)$$
$$\times \|\Lambda^\alpha v\|_{L^2}^2,$$

where $p_1 > 2$ satisfies

$$\frac{1-\alpha}{2} < \frac{1}{p_1} < \frac{\alpha + 2\gamma - 1}{2}.$$

Combining all the above estimates implies that

$$\frac{d}{dt}\|\Lambda^\alpha v(t)\|_{L^2}^2 + \|\Lambda^{2\alpha}\mathcal{L}v\|_{L^2}^2$$
$$\leq C(1 + \|\nabla u\|_{L^\infty}^2 + \|\Delta u\|_{L^2}^2 + \|\Lambda^\alpha \mathcal{L}v\|_{L^2}^2)(1 + \|\Lambda^\alpha v\|_{L^2}^2) + C\|b\|_{L^\infty}^2\|\Lambda^{1+\nu}b\|_{L^2}^2$$
$$+ C\|b\|_{L^\infty}\|v\|_{L^2}\|\Lambda^{1+\nu}b\|_{L^2}.$$

Thanks to (3.2), (3.3) and (3.12), we have by using the Gronwall inequality

$$\sup_{t\in[0,T]}\|\Lambda^\alpha v(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{2\alpha}\mathcal{L}v(\tau)\|_{L^2}^2\, d\tau \leq C(T, v_0, b_0).$$

It follows from (3.6) and (3.15) that

$$\frac{d}{dt}\|\nabla b(t)\|_{L^\infty} \leq C\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^\infty} + C\|b\|_{L^\infty}\|\nabla^2 u\|_{L^\infty}\|\nabla b\|_{L^\infty}$$
$$(3.16) \qquad \leq C\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^\infty} + C\|b\|_{L^\infty}(\|u\|_{L^2} + \|\Lambda^{2\alpha}\mathcal{L}v\|_{L^2})\|\nabla b\|_{L^\infty}.$$

Recalling (3.2) and (3.3), one deduces by the Gronwall inequality

$$\sup_{t\in[0,T]}\|\nabla b(t)\|_{L^\infty} \leq C(T, v_0, b_0).$$

Consequently, the proof of Lemma 3.4 is completed. $\qquad\square$

The following lemma concerns the L^2-estimate of ω.

LEMMA 3.5. *Assume (v_0, b_0) satisfies the assumptions stated in Theorem 1.2. Then the corresponding solution (v, b) of (1.4) admits the following bound*

$$(3.17) \qquad \sup_{t\in[0,T]}\|\omega(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha \mathcal{L}\omega(\tau)\|_{L^2}^2\, d\tau \leq C(T, v_0, b_0).$$

PROOF. We first claim

$$\sup_{t \in [0, T]} \|\Delta b(t)\|_{L^2} \le C(T, v_0, b_0). \tag{3.18}$$

To this end, applying Δ to (3.4) and multiplying the resultant by Δb, it leads to

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta b(t)\|_{L^2}^2 &= -\int_{\mathbb{R}^2} [\Delta, u \cdot \nabla] b \cdot \Delta b\, dx + \int_{\mathbb{R}^2} \Delta(b \cdot \nabla u) \cdot \Delta b\, dx \\
&\le C\|[\Delta, u \cdot \nabla]b\|_{L^2}\|\Delta b\|_{L^2} + C\|\Delta(b \cdot \nabla u)\|_{L^2}\|\Delta b\|_{L^2} \\
&\le C(\|\nabla u\|_{L^\infty}\|\Delta b\|_{L^2} + \|\nabla b\|_{L^\infty}\|\Delta u\|_{L^2})\|\Delta b\|_{L^2} \\
&\quad + C(\|\nabla u\|_{L^\infty}\|\Delta b\|_{L^2} + \|b\|_{L^\infty}\|\Lambda^3 u\|_{L^2})\|\Delta b\|_{L^2},
\end{aligned}
$$

which implies

$$\frac{d}{dt}\|\Delta b(t)\|_{L^2} \le C\|\nabla u\|_{L^\infty}\|\Delta b\|_{L^2} + C(\|\nabla b\|_{L^\infty} + \|b\|_{L^\infty})(\|u\|_{L^2} + \|\Lambda^{2\alpha}\mathcal{L}v\|_{L^2}).$$

By means of (3.2), (3.3), (3.14) and the Gronwall inequality, we thus obtain

$$\sup_{t \in [0, T]} \|\Delta b(t)\|_{L^2} \le C(T, v_0, b_0),$$

which is (3.18). We remark that in this case, the vorticity $\omega := \nabla^\perp \cdot v$ reads

$$\partial_t \omega + (u \cdot \nabla)\omega + \Lambda^{2\alpha}\mathcal{L}^2 \omega = \nabla^\perp \nabla \cdot (b \otimes b). \tag{3.19}$$

Taking the L^2 inner product of (3.19) with ω yields

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{L^2}^2 + \|\Lambda^\alpha \mathcal{L}\omega\|_{L^2}^2 &\le C\|\nabla^\perp \nabla \cdot (b \otimes b)\|_{L^2}\|\omega\|_{L^2} \\
&\le C\|b\|_{L^\infty}\|\Delta b\|_{L^2}\|\omega\|_{L^2}.
\end{aligned}
$$

The Gronwall inequality and (3.18) allow us to get

$$\sup_{t \in [0, T]} \|\omega(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha \mathcal{L}\omega(\tau)\|_{L^2}^2\, d\tau \le C(T, v_0, b_0).$$

Consequently, we complete the proof of Lemma 3.5. $\square$

With the above estimates at our disposal, we will prove Theorem 1.2.

PROOF OF THEOREM 1.2. Applying Λ^s to (3.19) and multiplying it by $\Lambda^s \omega$ yield

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Lambda^s \omega(t)\|_{L^2}^2 + \|\Lambda^{s+\alpha}\mathcal{L}\omega\|_{L^2}^2 &= \int_{\mathbb{R}^2} [\Lambda^s,\, u \cdot \nabla]\omega \Lambda^s \omega\, dx \\
&\quad + \int_{\mathbb{R}^2} \Lambda^s \nabla^\perp \nabla \cdot (b \otimes b)\Lambda^s \omega.
\end{aligned}
$$

Applying $\Lambda^{s+2-\alpha'}$ to $(1.4)_2$ and multiplying it by $\Lambda^{s+2-\alpha'}b$ with $\alpha' < \alpha$, one gets

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Lambda^{s+2-\alpha'}b(t)\|_{L^2}^2 &= \int_{\mathbb{R}^2} \Lambda^{s+2-\alpha'}(b \cdot \nabla u) \cdot \Lambda^{s+2-\alpha'}b\, dx \\
&\quad - \int_{\mathbb{R}^2} [\Lambda^{s+2-\alpha'},\, u \cdot \nabla]b \cdot \Lambda^{s+2-\alpha'}b\, dx.
\end{aligned}
$$

Thanks to (3.10), we get that

$$\|\Lambda^{s+\alpha'}\omega\|_{L^2} \le C(\|\omega\|_{L^2} + \|\Lambda^{s+\alpha}\mathcal{L}\omega\|_{L^2}), \quad \text{for } 0 < \alpha' < \alpha. \tag{3.20}$$

Due to $\nabla \cdot u = 0$ and (B.3), it implies

$$\int_{\mathbb{R}^2} [\Lambda^s, \, u \cdot \nabla]\omega \Lambda^s \omega \, dx$$

$$= \int_{\mathbb{R}^2} [\Lambda^s \partial_i, \, u_i]\omega \Lambda^s \omega \, dx$$

$$\leq C \|[\Lambda^s \partial_i, \, u_i]\omega\|_{L^2} \|\Lambda^s \omega\|_{L^2}$$

$$\leq C(\|\nabla u\|_{L^\infty} \|\Lambda^s \omega\|_{L^2} + \|\omega\|_{L^{p_2}} \|\Lambda^{s+1} u\|_{L^{\frac{2p_2}{p_2-2}}}) \|\Lambda^s \omega\|_{L^2}$$

$$\leq C \left(\|\nabla u\|_{L^\infty} \|\Lambda^s \omega\|_{L^2} + (\|\omega\|_{L^2} + \|\Lambda^\alpha \mathcal{L}\omega\|_{L^2}) \|\Lambda^{s+\alpha} \mathcal{L}\omega\|_{L^2} \right) \|\Lambda^s \omega\|_{L^2}$$

$$\leq \frac{1}{8} \|\Lambda^{s+\alpha} \mathcal{L}\omega\|_{L^2}^2 + C(\|\nabla u\|_{L^\infty} + \|\omega\|_{L^2}^2 + \|\Lambda^\alpha \mathcal{L}\omega\|_{L^2}^2) \|\Lambda^s \omega\|_{L^2}^2,$$

where $p_2 > 2$ satisfies

$$\frac{1-\alpha}{2} < \frac{1}{p_2} < \frac{\alpha + 2\gamma}{2}.$$

By means of (B.2) and (3.20), we get

$$\int_{\mathbb{R}^2} \Lambda^s \nabla^\perp \nabla \cdot (b \otimes b) \Lambda^s \omega \leq C \|\Lambda^{s-\alpha'} \nabla^\perp \nabla \cdot (b \otimes b)\|_{L^2} \|\Lambda^{s+\alpha'} \omega\|_{L^2}$$

$$\leq C \|b\|_{L^\infty} \|\Lambda^{s+2-\alpha'} b\|_{L^2} \|\Lambda^{s+\alpha'} \omega\|_{L^2}$$

$$\leq C \|b\|_{L^\infty} \|\Lambda^{s+2-\alpha'} b\|_{L^2} (\|\omega\|_{L^2} + \|\Lambda^{s+\alpha} \mathcal{L}\omega\|_{L^2})$$

$$\leq \frac{1}{8} \|\Lambda^{s+\alpha} \mathcal{L}\omega\|_{L^2}^2 + C \|b\|_{L^\infty}^2 \|\Lambda^{s+2-\alpha'} b\|_{L^2}^2$$

$$+ C \|b\|_{L^\infty} \|\omega\|_{L^2} \|\Lambda^{s+2-\alpha'} b\|_{L^2}.$$

Similarly, we can verify

$$\int_{\mathbb{R}^2} \Lambda^{s+2-\alpha'} (b \cdot \nabla u) \cdot \Lambda^{s+2-\alpha'} b \, dx$$

$$\leq C(\|b\|_{L^\infty} \|\Lambda^{s+3-\alpha'} u\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Lambda^{s+2-\alpha'} b\|_{L^2}) \|\Lambda^{s+2-\alpha'} b\|_{L^2}$$

$$\leq C(\|b\|_{L^\infty} \|\Lambda^{s+\alpha'} \omega\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Lambda^{s+2-\alpha'} b\|_{L^2}) \|\Lambda^{s+2-\alpha'} b\|_{L^2}$$

$$\leq C(\|b\|_{L^\infty} \|\omega\|_{L^2} + \|b\|_{L^\infty} \|\Lambda^{s+\alpha} \mathcal{L}\omega\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Lambda^{s+2-\alpha'} b\|_{L^2}) \|\Lambda^{s+2-\alpha'} b\|_{L^2}$$

$$\leq \frac{1}{8} \|\Lambda^{s+\alpha} \mathcal{L}\omega\|_{L^2}^2 + C(\|\nabla u\|_{L^\infty} + \|b\|_{L^\infty}^2) \|\Lambda^{s+2-\alpha'} b\|_{L^2}^2$$

$$+ C \|b\|_{L^\infty} \|\omega\|_{L^2} \|\Lambda^{s+2-\alpha'} b\|_{L^2}.$$

Using $\nabla \cdot u = 0$ and (B.3), it yields

$$\int_{\mathbb{R}^2} \Lambda^{s+2-\alpha'} (u \cdot \nabla b) \cdot \Lambda^{s+2-\alpha'} b \, dx$$

$$= \int_{\mathbb{R}^2} [\Lambda^{s+2-\alpha'} \partial_{x_i}, \, u_i] b_j \Lambda^{s+2-\alpha'} b_j \, dx$$

$$\leq C \|[\Lambda^{s+2-\alpha'} \partial_{x_i}, \, u_i] b_j\|_{L^2} \|\Lambda^{s+2-\alpha'} b\|_{L^2}$$

$$\leq C(\|b\|_{L^\infty} \|\Lambda^{s+3-\alpha'} u\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Lambda^{s+2-\alpha'} b\|_{L^2}) \|\Lambda^{s+2-\alpha'} b\|_{L^2}$$

$$\leq C(\|b\|_{L^\infty} \|\Lambda^{s+\alpha'} \omega\|_{L^2} + \|\nabla u\|_{L^\infty} \|\Lambda^{s+2-\alpha'} b\|_{L^2}) \|\Lambda^{s+2-\alpha'} b\|_{L^2}$$

$$\leq \frac{1}{8} \|\Lambda^{s+\alpha} \mathcal{L}\omega\|_{L^2}^2 + C(\|\nabla u\|_{L^\infty} + \|b\|_{L^\infty}^2) \|\Lambda^{s+2-\alpha'} b\|_{L^2}^2$$

$$+ C\|b\|_{L^\infty}\|\omega\|_{L^2}\|\Lambda^{s+2-\alpha'}b\|_{L^2}.$$

Collecting all the above estimates, one derives

$$\frac{d}{dt}(\|\Lambda^s\omega(t)\|_{L^2}^2 + \|\Lambda^{s+2-\alpha'}b(t)\|_{L^2}^2) + \|\Lambda^{s+\alpha}\mathcal{L}\omega\|_{L^2}^2$$

$$\leq C(\|\nabla u\|_{L^\infty} + \|\omega\|_{L^2}^2 + \|\Lambda^\alpha\mathcal{L}\omega\|_{L^2}^2 + \|b\|_{L^\infty}^2)(\|\Lambda^s\omega\|_{L^2}^2 + \|\Lambda^{s+2-\alpha'}b\|_{L^2}^2)$$

$$+ C\|b\|_{L^\infty}\|\omega\|_{L^2}\|\Lambda^{s+2-\alpha'}b\|_{L^2}.$$

By (3.2), (3.3) and (3.17), we have

$$\int_0^T (\|\nabla u\|_{L^\infty} + \|\omega\|_{L^2}^2 + \|\Lambda^\alpha\mathcal{L}\omega\|_{L^2}^2 + \|b\|_{L^\infty}^2 + \|b\|_{L^\infty}\|\omega\|_{L^2})(t)\,dt \leq C(T, v_0, b_0),$$

which together with the Gronwall inequality entails

$$\sup_{t\in[0,T]}(\|\Lambda^s\omega(t)\|_{L^2}^2 + \|\Lambda^{s+2-\alpha'}b(t)\|_{L^2}^2) + \int_0^T \|\Lambda^{s+\alpha}\mathcal{L}\omega(\tau)\|_{L^2}^2\,d\tau \leq C(T, v_0, b_0).$$

Consequently, this completes the proof of Theorem 1.2. $\qquad\square$

The proof of Theorem 1.3

The proof of Theorem 1.3 can be performed as that of Theorem 1.2. For the sake of convenience, we sketch its proof as follows. First, we have

LEMMA 3.6. *Assume* (v_0, b_0) *satisfies the assumptions stated in Theorem 1.3. Then the corresponding solution* (v, b) *of (1.6) admits the following bound*

$$(3.21) \qquad \|u(t)\|_{L^2}^2 + \left\|\frac{\Lambda^2}{g(\Lambda)}u(t)\right\|_{L^2}^2 + \|b(t)\|_{L^2}^2 \leq C(v_0, b_0).$$

Modifying the proof of Lemma 3.2, we will derive the following crucial estimate.

LEMMA 3.7. *Assume* (v_0, b_0) *satisfies the assumptions stated in Theorem 1.3. Then the corresponding solution* (v, b) *of (1.6) admits the following bound*

$$(3.22) \qquad \sup_{t\in[0,T]}(\|b(t)\|_{L^\infty} + \|\nabla b(t)\|_{L^2} + \|v(t)\|_{L^2} + \|\nabla b(t)\|_{L^\infty}) \leq C(T, v_0, b_0).$$

Due to $v = u + (-\Delta)^2\mathcal{L}^2 u$, *it holds true*

$$(3.23) \qquad \sup_{t\in[0,T]}\|\Lambda^4\mathcal{L}^2 u(t)\|_{L^2} \leq C(T, v_0, b_0).$$

PROOF. Following the arguments in deriving (3.9), it is not difficult to check that

$$(3.24) \qquad \frac{d}{dt}V(t) \leq C(1 + \|\nabla u\|_{L^\infty} + \|\Delta u\|_{L^2})V(t),$$

where

$$V(t) := \|b(t)\|_{L^\infty}^3 + \|b(t)\|_{L^\infty}^6 + \|\nabla b(t)\|_{L^2}^3 + \|v(t)\|_{L^2}^2.$$

Moreover, one may also conclude that for some $r_1 > 2$

$$\|\nabla u\|_{L^\infty} \leq C\|u\|_{L^2} + Cg(2^N)\sqrt{N}\left\|\frac{\Lambda^2}{g(\Lambda)}u\right\|_{L^2} + C2^{N(2-r_1)}\|\Lambda^{r_1}u\|_{L^2},$$

$$\|\Delta u\|_{L^2} \leq C\|u\|_{L^2} + Cg(2^N)\sqrt{N}\Big\|\frac{\Lambda^2}{g(\Lambda)}u\Big\|_{L^2} + C2^{N(2-r_1)}\|\Lambda^{r_1}u\|_{L^2}.$$

Making use of the following fact due to $v = u + (-\Delta)^2\mathcal{L}^2 u$

$$C2^{N(2-r_1)}\|\Lambda^{r_1}u\|_{L^2} \leq C(\|u\|_{L^2} + 2^{N(2-r_1)}\|v\|_{L^2}),$$

it yields that

$$\frac{d}{dt}V(t) \leq CV(t) + Cg(2^N)\sqrt{N}\Big\|\frac{\Lambda^2}{g(\Lambda)}u\Big\|_{L^2}V(t) + C2^{N(2-r_1)}V^{\frac{1}{2}}(t)V(t).$$

Taking N such

$$2^N \approx (e + V(t))^{\frac{1}{2(r_1-2)}},$$

we have

$$\frac{d}{dt}V(t) \leq C\left(1 + \Big\|\frac{\Lambda^2}{g(\Lambda)}u\Big\|_{L^2}\right)g\big[(e+V(t))^{\frac{1}{2(r_1-2)}}\big]\sqrt{\ln\big(e+A(t)\big)}\big(e+V(t)\big).$$

It thus follows from the above inequality that

$$\int_{e+V(0)}^{e+V(t)} \frac{d\tau}{\tau\sqrt{\ln\tau}g(\tau^{\frac{1}{2(r_1-2)}})} \leq C\int_0^t\left(1 + \Big\|\frac{\Lambda^2}{g(\Lambda)}u(\tau)\Big\|_{L^2}\right)d\tau.$$

By (1.7), we have

$$\int_e^{\infty} \frac{d\tau}{\tau\sqrt{\ln\tau}g(\tau^{\frac{1}{2(r_1-2)}})} = \sqrt{2(r_1-2)}\int_{e^{\frac{1}{2(r_1-2)}}}^{\infty} \frac{d\tau}{\tau\sqrt{\ln\tau}g(\tau)} = \infty.$$

Therefore, noticing the following fact due to (3.21)

$$\int_0^T\left(1 + \Big\|\frac{\Lambda^2}{g(\Lambda)}u(\tau)\Big\|_{L^2}\right)d\tau \leq C(T, u_0, w_0),$$

we can verify that

(3.25)
$$\sup_{t\in[0,T]} V(t) \leq C(T, v_0, b_0).$$

Coming back to (3.16), we can show

$$\frac{d}{dt}\|\nabla b(t)\|_{L^\infty} \leq C\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^\infty} + C\|b\|_{L^\infty}\|\nabla^2 u\|_{L^\infty}\|\nabla b\|_{L^\infty}$$
$$\leq C(\|u\|_{L^2} + \|\Lambda^4\mathcal{L}^2 u(t)\|_{L^2})\|\nabla b\|_{L^\infty}$$
$$+ C\|b\|_{L^\infty}(\|u\|_{L^2} + \|\Lambda^4\mathcal{L}^2 u(t)\|_{L^2})\|\nabla b\|_{L^\infty},$$

which along with the Gronwall inequality and (3.25) gives

$$\sup_{t\in[0,T]} \|\nabla b(t)\|_{L^\infty} \leq C(T, v_0, b_0).$$

This concludes the proof of Lemma 3.7. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following lemma concerns the L^q-estimate of ω for some $q > 2$.

LEMMA 3.8. *Assume (v_0, b_0) satisfies the assumptions stated in Theorem 1.3. Then the corresponding solution (v, b) of (1.6) admits the following bound for some $q > 2$*

(3.26)
$$\sup_{t\in[0,T]} \|\omega(t)\|_{L^q} \leq C(T, v_0, b_0).$$

PROOF. For some $0 < \theta < 1$, we claim

(3.27)
$$\sup_{t \in [0, T]} \|\Lambda^{2+\theta} b(t)\|_{L^2} \le C(T, v_0, b_0).$$

Applying $\Lambda^{2+\theta}$ to $(1.6)_2$ and multiplying it by $\Lambda^{2+\theta} b$, we infer

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{2+\theta} b(t)\|_{L^2}^2 = -\int_{\mathbb{R}^2} [\Lambda^{2+\theta}, u \cdot \nabla] b \cdot \Lambda^{2+\theta} b \, dx + \int_{\mathbb{R}^2} \Lambda^{2+\theta}(b \cdot \nabla u) \cdot \Lambda^{2+\theta} b \, dx$$

$$\le C\|[\Lambda^{2+\theta}, u \cdot \nabla] b\|_{L^2}\|\Lambda^{2+\theta} b\|_{L^2} + C\|\Lambda^{2+\theta}(b \cdot \nabla u)\|_{L^2}\|\Lambda^{2+\theta} b\|_{L^2}$$

$$\le C(\|\nabla u\|_{L^\infty}\|\Lambda^{2+\theta} b\|_{L^2} + \|\nabla b\|_{L^\infty}\|\Lambda^{2+\theta} u\|_{L^2})\|\Lambda^{2+\theta} b\|_{L^2}$$

$$+ C(\|\nabla u\|_{L^\infty}\|\Lambda^{2+\theta} b\|_{L^2} + \|b\|_{L^\infty}\|\Lambda^{3+\theta} u\|_{L^2})\|\Lambda^{2+\theta} b\|_{L^2},$$

which allows us to show

$$\frac{d}{dt}\|\Lambda^{2+\theta} b(t)\|_{L^2} \le C(\|u\|_{L^2} + \|\Lambda^4 \mathcal{L}^2 u(t)\|_{L^2})\|\Lambda^{2+\theta} b\|_{L^2}$$

$$+ C(\|\nabla b\|_{L^\infty} + \|b\|_{L^\infty})(\|u\|_{L^2} + \|\Lambda^4 \mathcal{L}^2 u(t)\|_{L^2}).$$

Thanks to (3.22) and (3.23), we thus obtain the desired estimate (3.27). Let us recall the vorticity $\omega := \nabla^\perp \cdot v$, which satisfies

(3.28)
$$\partial_t \omega + (u \cdot \nabla)\omega = \nabla^\perp \nabla \cdot (b \otimes b).$$

Taking the L^2 inner product of (3.28) with $|\omega|^{q-2}\omega$, it implies

$$\frac{1}{q}\frac{d}{dt}\|\omega(t)\|_{L^q}^q \le C\|\nabla^\perp \nabla \cdot (b \otimes b)\|_{L^q}\|\omega\|_{L^q}^{q-1}$$

$$\le C\|b\|_{L^\infty}\|\Delta b\|_{L^q}\|\omega\|_{L^q}^{q-1}$$

$$\le C\|b\|_{L^\infty}\|b\|_{H^{2+\theta}}\|\omega\|_{L^q}^{q-1},$$

which leads to

$$\frac{d}{dt}\|\omega(t)\|_{L^q} \le C\|b\|_{L^\infty}\|b\|_{H^{2+\theta}}.$$

Integrating it in time and using (3.27), we have for some $q > 2$

$$\sup_{t \in [0, T]} \|\omega(t)\|_{L^q} \le C(T, v_0, b_0).$$

Consequently, Lemma 3.8 is proved. $\square$

With the above estimates at our disposal, we will prove Theorem 1.3.

PROOF OF THEOREM 1.3. Applying Λ^s to (3.28) and multiplying it by $\Lambda^s \omega$ yield

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s \omega(t)\|_{L^2}^2 = \int_{\mathbb{R}^2} [\Lambda^s, u \cdot \nabla]\omega \Lambda^s \omega \, dx + \int_{\mathbb{R}^2} \Lambda^s \nabla^\perp \nabla \cdot (b \otimes b)\Lambda^s \omega.$$

Applying Λ^{s+2} to $(1.6)_2$ and multiplying it by $\Lambda^{s+2} b$, we infer

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{s+2} b(t)\|_{L^2}^2 = \int_{\mathbb{R}^2} \Lambda^{s+2}(b \cdot \nabla u) \cdot \Lambda^{s+2} b \, dx - \int_{\mathbb{R}^2} [\Lambda^{s+2}, u \cdot \nabla] b \cdot \Lambda^{s+2} b \, dx.$$

Using the same arguments used in proving Theorem 1.2, we may check

$$\int_{\mathbb{R}^2} [\Lambda^s, u \cdot \nabla]\omega \Lambda^s \omega \, dx \le C\|[\Lambda^s \partial_i, u_i]\omega\|_{L^2}\|\Lambda^s \omega\|_{L^2}$$

$$\le C(\|\nabla u\|_{L^\infty}\|\Lambda^s \omega\|_{L^2} + \|\omega\|_{L^p}\|\Lambda^{s+1} u\|_{L^{\frac{2p}{p-2}}})\|\Lambda^s \omega\|_{L^2}$$

$$\leq C\|\nabla u\|_{L^\infty}\|\Lambda^s\omega\|_{L^2}^2 + C\|\omega\|_{L^p}\|\Lambda^s\omega\|_{L^2}^2,$$

$$\int_{\mathbb{R}^2}\Lambda^s\nabla^\perp\nabla\cdot(b\otimes b)\Lambda^s\omega \leq C\|\Lambda^s\nabla^\perp\nabla\cdot(b\otimes b)\|_{L^2}\|\Lambda^s\omega\|_{L^2}$$

$$\leq C\|b\|_{L^\infty}\|\Lambda^{s+2}b\|_{L^2}\|\Lambda^s\omega\|_{L^2},$$

$$\int_{\mathbb{R}^2}\Lambda^{s+2}(b\cdot\nabla u)\cdot\Lambda^{s+2}b\,dx \leq C(\|b\|_{L^\infty}\|\Lambda^{s+3}u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\Lambda^{s+2}b\|_{L^2})\|\Lambda^{s+2}b\|_{L^2}$$

$$\leq C(\|b\|_{L^\infty}\|\Lambda^s\omega\|_{L^2} + \|\nabla u\|_{L^\infty}\|\Lambda^{s+2}b\|_{L^2})\|\Lambda^{s+2}b\|_{L^2},$$

$$\int_{\mathbb{R}^2}\Lambda^{s+2}(u\cdot\nabla b)\cdot\Lambda^{s+2}b\,dx \leq C\|[\Lambda^{s+2}\partial_i, u_i]b_j\|_{L^2}\|\Lambda^{s+2}b\|_{L^2}$$

$$\leq C(\|b\|_{L^\infty}\|\Lambda^{s+3}u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\Lambda^{s+2}b\|_{L^2})\|\Lambda^{s+2}b\|_{L^2}$$

$$\leq C(\|b\|_{L^\infty}\|\Lambda^s\omega\|_{L^2} + \|\nabla u\|_{L^\infty}\|\Lambda^{s+2}b\|_{L^2})\|\Lambda^{s+2}b\|_{L^2}.$$

Summing up all the above estimates, it holds that

$$\frac{d}{dt}(\|\Lambda^s\omega(t)\|_{L^2}^2 + \|\Lambda^{s+2}b(t)\|_{L^2}^2)$$
$$\leq C(\|\omega\|_{L^p} + \|\nabla u\|_{L^\infty} + \|b\|_{L^\infty})(\|\Lambda^s\omega\|_{L^2}^2 + \|\Lambda^{s+2}b\|_{L^2}^2)$$
$$\leq C(\|\omega\|_{L^p} + \|u\|_{L^2} + \|\Lambda^4\mathcal{L}^2 u\|_{L^2} + \|b\|_{L^\infty})(\|\Lambda^s\omega\|_{L^2}^2 + \|\Lambda^{s+2}b\|_{L^2}^2).$$

In view of the Gronwall inequality, we obtain by using (3.23) and (3.26) that

$$\sup_{t\in[0,T]}(\|\Lambda^s\omega(t)\|_{L^2}^2 + \|\Lambda^{s+2}b(t)\|_{L^2}^2) \leq C(T, v_0, b_0).$$

As a result, we finish the proof of Theorem 1.3. $\qquad\square$

Appendix A. The proof of Theorem 1.4

To begin with, the basic energy estimate reads as follows.

LEMMA A.1. *Assume* (v_0, b_0) *satisfies the assumptions stated in Theorem 1.4. Then the corresponding solution* (v, b) *of (1.8) admits the following bound for any* $t \in [0, T]$

(A.1) $$\|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^t\|\Lambda^\beta b(\tau)\|_{L^2}^2\,d\tau \leq C(v_0, b_0).$$

PROOF. Taking the inner product of $(1.8)_1$ with u and the inner product of $(1.8)_2$ with b, one gets

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + \|\Lambda^\beta b\|_{L^2}^2 = 0,$$

where we have used the cancelations (2.3)-(2.4) and the following crucial identity

$$\int_{\mathbb{R}^2}(u\cdot\nabla v)\cdot u\,dx = 0.$$

Actually, the above equality can be deduced as

$$\int_{\mathbb{R}^2}(u\cdot\nabla v)\cdot u\,dx = \int_{\mathbb{R}^2}(u\cdot\nabla u)\cdot u\,dx - \int_{\mathbb{R}^2}(u\cdot\nabla\Delta u)\cdot u\,dx$$

$$= -\int_{\mathbb{R}^2} (u \cdot \nabla \Delta u) \cdot u \, dx$$

$$= -\int_{\mathbb{R}^2} u_i \partial_i \partial_l^2 u_j u_j \, dx$$

$$= \int_{\mathbb{R}^2} u_i \partial_l^2 u_j \partial_i u_j \, dx$$

$$= \int_{\mathbb{R}^2} (u \cdot \nabla u) \cdot \Delta u \, dx$$

$$= 0,$$

where in the last line we have used the following crucial fact due to $\nabla \cdot u = 0$ (see [**35**, (3.3)] for details)

$$\int_{\mathbb{R}^2} (u \cdot \nabla u) \cdot \Delta u \, dx = 0.$$

Integrating in time yields the desired (A.1). $\qquad\square$

According to the proof of Lemma 2.2 (letting $\gamma = 1$), the following lemma holds immediately.

LEMMA A.2. *Assume (v_0, b_0) satisfies the assumptions stated in Theorem 1.4. If $\beta > \frac{1}{2}$, then the corresponding solution (v, b) of (1.8) admits the following bound*

$$(A.2) \qquad \sup_{t \in [0, T]} \|\Lambda^\beta b(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{2\beta} b(\tau)\|_{L^2}^2 \, d\tau \leq C(T, v_0, b_0).$$

In this case, the vorticity $\omega := \nabla^\perp \cdot v$ satisfies

$$(A.3) \qquad \partial_t \omega + (u \cdot \nabla)\omega = \nabla^\perp \nabla \cdot (b \otimes b) + \nabla^\perp u_i \partial_i v.$$

which is different from (2.12). Due to the presence of $\nabla^\perp u_i \partial_i v$ in the vorticity equation (A.3), the following lemma cannot be obtained as that of Lemma 2.3.

LEMMA A.3. *Assume (v_0, b_0) satisfies the assumptions stated in Theorem 1.4. If $\beta > \frac{1}{2}$, then the corresponding solution (v, b) of (1.8) admits the following bound*

$$(A.4) \qquad \sup_{t \in [0, T]} (\|v(t)\|_{H^1}^2 + \|\Lambda^{2+\beta} b(t)\|_{L^2}^2) + \int_0^T \|\Lambda^{2+2\beta} b(\tau)\|_{L^2}^2 \, d\tau \leq C(T, v_0, b_0).$$

PROOF. First, taking the L^2-inner product of $(1.8)_1$ with v, it gives

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_{L^2}^2 = \int_{\mathbb{R}^2} (b \cdot \nabla b) \cdot v \, dx$$

$$\leq C\|b\|_{L^\infty} \|\nabla b\|_{L^2} \|v\|_{L^2}$$

$$(A.5) \qquad \leq C\|b\|_{H^{2\beta}}^2 \|v\|_{L^2},$$

where here and in what follows we have used

$$\|b\|_{L^\infty} \leq C\|b\|_{H^{2\beta}}, \quad \beta > \frac{1}{2}.$$

Thanks to (A.2), we get from (A.5) that

$$(A.6) \qquad \sup_{t \in [0, T]} \|v(t)\|_{L^2} \leq C(T, v_0, b_0).$$

Keeping in mind the relation $v = u - \Delta u$, we obtain

$$\text{(A.7)} \qquad \sup_{t \in [0, T]} \|u(t)\|_{H^2} \leq C(T, v_0, b_0).$$

Taking the L^2-inner product of (A.3) with ω yields

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{L^2}^2 = \int_{\mathbb{R}^2} \nabla^\perp \nabla \cdot (b \otimes b) \,\omega \, dx + \int_{\mathbb{R}^2} \nabla^\perp u_i \partial_i v \,\omega \, dx.$$

According to (2.13), one has

$$\int_{\mathbb{R}^2} \nabla^\perp \nabla \cdot (b \otimes b) \,\omega \, dx \leq \frac{1}{4}\|\Lambda^{2+2\beta}b\|_{L^2}^2 + C(1 + \|b\|_{L^\infty}^2)(1 + \|\omega\|_{L^2}^2).$$

Moreover, it is obvious that

$$\int_{\mathbb{R}^2} \nabla^\perp u_i \partial_i v \,\omega \, dx \leq C\|\nabla u\|_{L^\infty}\|\nabla v\|_{L^2}\|\omega\|_{L^2} \leq C\|\nabla u\|_{L^\infty}\|\omega\|_{L^2}^2.$$

Whence, we derive

$$\text{(A.8)} \qquad \frac{1}{2}\frac{d}{dt}\|\omega(t)\|_{L^2}^2 \leq \frac{1}{4}\|\Lambda^{2+2\beta}b\|_{L^2}^2 + C(1 + \|\nabla u\|_{L^\infty} + \|b\|_{L^\infty}^2)(1 + \|\omega\|_{L^2}^2).$$

According to (2.17), we deduce

$$\text{(A.9)} \quad \frac{1}{2}\frac{d}{dt}\|\Lambda^{2+\beta}b(t)\|_{L^2}^2 + \frac{3}{4}\|\Lambda^{2+2\beta}b\|_{L^2}^2 \leq C\|b\|_{L^\infty}^2\|\omega\|_{L^2}^2 + C\|u\|_{H^1}^{\frac{4(\beta+1)p_0}{(2\beta-1)p_0-2}}\|b\|_{L^2}^2,$$

where $p_0 > \frac{2}{2\beta-1}$. Summing up (A.8) and (A.9) yields

$$\frac{d}{dt}(\|\omega(t)\|_{L^2}^2 + \|\Lambda^{2+\beta}b(t)\|_{L^2}^2) + \|\Lambda^{2+2\beta}b\|_{L^2}^2 \leq C(1 + \|\nabla u\|_{L^\infty}$$
$$+ \|b\|_{H^{2\beta}}^2)(1 + \|\omega\|_{L^2}^2).$$

Recalling the logarithmic Sobolev inequality (see also [**20**])

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \leq C\Big(1 + \|u\|_{L^2(\mathbb{R}^2)} + \|\Delta u\|_{L^2(\mathbb{R}^2)} \ln\big(e + \|\Lambda^\varrho u\|_{L^2(\mathbb{R}^2)}\big)\Big), \quad \varrho > 2,$$

we have by taking $\varrho = 3$ that

$$\frac{d}{dt}(\|\omega(t)\|_{L^2}^2 + \|\Lambda^{2+\beta}b(t)\|_{L^2}^2) + \|\Lambda^{2+2\beta}b\|_{L^2}^2$$
$$\leq C(1 + \|\Delta u\|_{L^2} + \|b\|_{H^{2\beta}}^2) \ln\big(e + \|\Lambda^3 u\|_{L^2}^2\big)(1 + \|\omega\|_{L^2}^2)$$
$$\leq C(1 + \|\Delta u\|_{L^2} + \|b\|_{H^{2\beta}}^2) \ln\big(e + \|\omega\|_{L^2}^2 + \|\Lambda^{2+\beta}b\|_{L^2}^2\big)(1 + \|\omega\|_{L^2}^2 + \|\Lambda^{2+\beta}b\|_{L^2}^2).$$

Making use of (A.2), (A.7) and the Gronwall type inequality, we deduce that

$$\text{(A.10)} \quad \sup_{t \in [0, T]} (\|\omega(t)\|_{L^2}^2 + \|\Lambda^{2+\beta}b(t)\|_{L^2}^2) + \int_0^T \|\Lambda^{2+2\beta}b(\tau)\|_{L^2}^2 \, d\tau \leq C(T, v_0, b_0).$$

Combining (A.6) and (A.10) leads to (A.4). Therefore, we conclude the proof of Lemma A.3. $\qquad \square$

With the above estimates in hand, we are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Applying $(\Lambda^\rho, \Lambda^{\rho+1-\beta})$ to $((1.8)_1, (1.8)_2)$ and multiplying them by $(\Lambda^\rho v, \Lambda^{\rho+1-\beta}b)$, respectively, we infer

$$(A.11) \qquad \frac{1}{2}\frac{d}{dt}(\|\Lambda^\rho v(t)\|_{L^2}^2 + \|\Lambda^{\rho+1-\beta}b(t)\|_{L^2}^2) + \|\Lambda^{\rho+1}b\|_{L^2}^2 = \sum_{k=1}^4 L_k,$$

where

$$L_1 := -\int_{\mathbb{R}^2} [\Lambda^\rho,\, u\cdot\nabla]v\cdot\Lambda^\rho v\, dx, \qquad L_2 := \int_{\mathbb{R}^2}\Lambda^\rho\nabla\cdot(b\otimes b)\Lambda^s\cdot\Lambda^\rho v,$$

$$L_3 := \int_{\mathbb{R}^2}\Lambda^{\rho+1-\beta}(b\cdot\nabla u)\cdot\Lambda^{\rho+1-\beta}b\, dx, \qquad L_4 := -\int_{\mathbb{R}^2}\Lambda^{\rho+1-\beta}(u\cdot\nabla b)\cdot\Lambda^{\rho+1-\beta}b\, dx.$$

According to $\nabla\cdot u = 0$ and (B.3), one obtains

$$L_1 = \int_{\mathbb{R}^2}[\Lambda^\rho\partial_i,\, u_i]v_j\Lambda^\rho v_j\, dx$$

$$\le C\|[\Lambda^\rho\partial_i,\, u_i]v_j\|_{L^2}\|\Lambda^\rho v_j\|_{L^2}$$

$$\le C(\|\nabla u\|_{L^\infty}\|\Lambda^\rho v\|_{L^2} + \|v\|_{L^4}\|\Lambda^{\rho+1}u\|_{L^4})\|\Lambda^\rho v\|_{L^2}$$

$$\le C(\|v\|_{H^1}\|\Lambda^\rho v\|_{L^2} + \|v\|_{H^1}\|\Lambda^{\rho+\frac{3}{2}}u\|_{L^2})\|\Lambda^\rho v\|_{L^2}$$

$$(A.12) \qquad \le C\|v\|_{H^1}\|\Lambda^\rho v\|_{L^2}^2,$$

where we have used the following facts due to $v = u - \Delta u$ (see (2.39) for similar proof)

$$\|\nabla u\|_{L^\infty} \le C\|v\|_{H^1}, \qquad \|\Lambda^{\rho+\frac{3}{2}}u\|_{L^2} \le C\|\Lambda^\rho v\|_{L^2}.$$

Using (B.2), it holds that

$$L_2 \le C\|\Lambda^\rho\nabla\cdot(b\otimes b)\|_{L^2}\|\Lambda^\rho v\|_{L^2}$$

$$\le C\|b\|_{L^\infty}\|\Lambda^{\rho+1}b\|_{L^2}\|\Lambda^\rho v\|_{L^2}$$

$$\le C\|b\|_{H^{2+\beta}}\|\Lambda^{\rho+1}b\|_{L^2}\|\Lambda^\rho v\|_{L^2}$$

$$(A.13) \qquad \le \frac{1}{8}\|\Lambda^{\rho+1}b\|_{L^2}^2 + C\|b\|_{H^{2+\beta}}^2\|\Lambda^\rho v\|_{L^2}^2.$$

Due to $\nabla\cdot b = 0$, we have by (B.2) that

$$L_3 = \int_{\mathbb{R}^2}\Lambda^{\rho+1-\beta}\nabla\cdot(b\otimes u)\cdot\Lambda^{\rho+1-\beta}b\, dx$$

$$\le C\|\Lambda^{\rho+2-2\beta}(ub)\|_{L^2}\|\Lambda^{\rho+1}b\|_{L^2}$$

$$\le C(\|u\|_{L^\infty}\|\Lambda^{\rho+2-2\beta}b\|_{L^2} + \|b\|_{L^\infty}\|\Lambda^{\rho+2-2\beta}u\|_{L^2})\|\Lambda^{\rho+1}b\|_{L^2}$$

$$\le C(\|u\|_{L^\infty}\|b\|_{L^2}^{\frac{2\beta-1}{\rho+1}}\|\Lambda^{\rho+1}b\|_{L^2}^{\frac{\rho+2-2\beta}{\rho+1}} + \|b\|_{L^\infty}\|\Lambda^\rho v\|_{L^2})\|\Lambda^{\rho+1}b\|_{L^2}$$

$$\le \frac{1}{8}\|\Lambda^{\rho+1}b\|_{L^2}^2 + C\|b\|_{L^\infty}^2\|\Lambda^\rho v\|_{L^2}^2 + C\|u\|_{L^\infty}^{\frac{2(\rho+1)}{2\beta-1}}\|b\|_{L^2}^2$$

$$(A.14) \qquad \le \frac{1}{8}\|\Lambda^{\rho+1}b\|_{L^2}^2 + C\|b\|_{H^{2+\beta}}^2\|\Lambda^\rho v\|_{L^2}^2 + C\|v\|_{H^1}^{\frac{2(\rho+1)}{2\beta-1}}\|b\|_{L^2}^2.$$

Similarly, one concludes

$$L_4 = \int_{\mathbb{R}^2}\Lambda^{\rho+1-\beta}\nabla\cdot(u\otimes b)\cdot\Lambda^{\rho+1-\beta}b\, dx$$

$$\le C\|\Lambda^{\rho+2-2\beta}(ub)\|_{L^2}\|\Lambda^{\rho+1}b\|_{L^2}$$

$$(A.15) \qquad \leq \frac{1}{8}\|\Lambda^{\rho+1}b\|_{L^2}^2 + C\|b\|_{H^{2+\beta}}^2\|\Lambda^\rho v\|_{L^2}^2 + C\|v\|_{H^1}^{\frac{2(\rho+1)}{2\beta-1}}\|b\|_{L^2}^2.$$

Inserting (A.12), (A.13), (A.14) and (A.15) into (A.11) yields

$$\frac{d}{dt}(\|\Lambda^\rho v(t)\|_{L^2}^2 + \|\Lambda^{\rho+1-\beta}b(t)\|_{L^2}^2) + \|\Lambda^{\rho+1}b\|_{L^2}^2$$

$$(A.16) \qquad \leq C(\|v\|_{H^1} + \|v\|_{H^1}^{\frac{2(\rho+1)}{2\beta-1}} + \|b\|_{H^{2+\beta}}^2)(\|\Lambda^\rho v\|_{L^2}^2 + \|\Lambda^{\rho+1-\beta}b\|_{L^2}^2).$$

Recalling (A.3), we find

$$\int_0^T (\|v(t)\|_{H^1} + \|v(t)\|_{H^1}^{\frac{2(\rho+1)}{2\beta-1}} + \|b(t)\|_{H^{2+\beta}}^2)\,dt \leq C(T, v_0, b_0).$$

Applying the Gronwall inequality to (A.16), we finally obtain

$$\sup_{t\in[0,T]}(\|\Lambda^\rho v(t)\|_{L^2}^2 + \|\Lambda^{\rho+1-\beta}b(t)\|_{L^2}^2) + \int_0^T \|\Lambda^{\rho+1}b(t)\|_{L^2}^2\,dt \leq C(T, v_0, b_0).$$

Consequently, this ends the proof of Theorem 1.4. $\qquad\qquad\square$

Appendix B. Several useful facts

We start with the Littlewood-Paley theory. We choose some smooth radial non increasing function χ with values in $[0,1]$ such that $\chi \in C_0^\infty(\mathbb{R}^n)$ is supported in the ball $\mathcal{B} := \{\xi \in \mathbb{R}^n, |\xi| \leq \frac{4}{3}\}$ and and with value 1 on $\{\xi \in \mathbb{R}^n, |\xi| \leq \frac{3}{4}\}$, then we set $\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$. One easily verifies that $\varphi \in C_0^\infty(\mathbb{R}^n)$ is supported in the annulus $\mathcal{C} := \{\xi \in \mathbb{R}^n, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and satisfies

$$\chi(\xi) + \sum_{j\geq 0}\varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n.$$

Let $h = \mathcal{F}^{-1}(\varphi)$ and $\widetilde{h} = \mathcal{F}^{-1}(\chi)$, then we introduce the dyadic blocks Δ_j of our decomposition by setting

$$\Delta_j u = 0, \ \ j \leq -2; \qquad \Delta_{-1}u = \chi(D)u = \int_{\mathbb{R}^n}\widetilde{h}(y)u(x-y)\,dy;$$

$$\Delta_j u = \varphi(2^{-j}D)u = 2^{jn}\int_{\mathbb{R}^n}h(2^j y)u(x-y)\,dy, \ \ \forall j \in \mathbb{N}.$$

We shall also use the following low-frequency cut-off:

$$S_j u = \chi(2^{-j}D)u = \sum_{-1\leq k\leq j-1}\Delta_k u = 2^{jn}\int_{\mathbb{R}^n}\widetilde{h}(2^j y)u(x-y)\,dy, \ \ \forall j \in \mathbb{N}.$$

The nonhomogeneous Besov spaces are defined through the dyadic decomposition.

DEFINITION B.1. Let $s \in \mathbb{R}, (p,r) \in [1,+\infty]^2$. The nonhomogeneous Besov space $B_{p,r}^s$ is defined as a space of $f \in S'(\mathbb{R}^n)$ such that

$$B_{p,r}^s = \{f \in S'(\mathbb{R}^n); \|f\|_{B_{p,r}^s} < \infty\},$$

where

$$\|f\|_{B_{p,r}^s} = \begin{cases} \left(\sum_{j \geq -1} 2^{jrs} \|\Delta_j f\|_{L^p}^r \right)^{\frac{1}{r}}, & \forall\, r < \infty, \\ \sup_{j \geq -1} 2^{js} \|\Delta_j f\|_{L^p}, & \forall\, r = \infty. \end{cases}$$

We now introduce the Bernstein's inequalities, which are useful tools in dealing with Fourier localized functions and these inequalities trade integrability for derivatives. The following lemma provides Bernstein type inequalities for fractional derivatives

LEMMA B.1 (see [**3**]). *Assume $1 \leq a \leq b \leq \infty$. If the integer $j \geq -1$, then it holds*

$$\|\Lambda^k \Delta_j f\|_{L^b} \leq C_1\, 2^{jk + jn(\frac{1}{a} - \frac{1}{b})} \|\Delta_j f\|_{L^a}, \quad k \geq 0.$$

If the integer $j \geq 0$, then we have

$$C_2\, 2^{jk} \|\Delta_j f\|_{L^b} \leq \|\Lambda^k \Delta_j f\|_{L^b} \leq C_3\, 2^{jk + jn(\frac{1}{a} - \frac{1}{b})} \|\Delta_j f\|_{L^a}, \quad k \in \mathbb{R},$$

where n is the space dimension, and C_1, C_2 and C_3 are constants depending on k, a and b only.

Let us recall the following space-time estimate, which plays a crucial role in proving Theorem 1.1 (see [**40**, Lemma 3.1]).

LEMMA B.2. *Consider the following transport-diffusion equation with $\alpha > 0$*

$$\partial_t f + \Lambda^{2\alpha} f = g, \qquad f(x,0) = f_0(x).$$

For any $0 < \widetilde{\varepsilon} \leq 2\alpha$ and for any $1 \leq p, q \leq \infty$, let both g and $e^{-\Lambda^{2\alpha} t} \Lambda^{2\alpha - \widetilde{\varepsilon}} f_0$ belong to $L^q(0, t; L^p(\mathbb{R}^n))$, then we have

$$\|\Lambda^{2\alpha - \widetilde{\varepsilon}} f\|_{L_t^q L_x^p} \leq C(t, f_0) + C(t) \|g\|_{L_t^q L_x^p}.$$

where $C(t, f_0) = \|e^{-\Lambda^{2\alpha} t} \Lambda^{2\alpha - \widetilde{\varepsilon}} f_0\|_{L_t^q L_x^p}$ and $C(t)$ depends on t only.

The following Kato-Ponce type commutator estimate and the product type estimate can be stated as follows (see [**17**] for example).

LEMMA B.3. *Let $s > 0$. Assume that $p, p_1, p_3 \in (1, \infty)$ and $p_2, p_4 \in [1, \infty]$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Then there exists some constants C such that

(B.1) $\|[\Lambda^s, f]g\|_{L^p} \leq C \left(\|\Lambda^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|\Lambda^{s-1} g\|_{L^{p_3}} \|\nabla f\|_{L^{p_4}} \right),$

(B.2) $\|\Lambda^s(fg)\|_{L^p} \leq C \left(\|\Lambda^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|\Lambda^s g\|_{L^{p_3}} \|\nabla f\|_{L^{p_4}} \right).$

In some context, we also need the following variant version of (B.1), whose proof is the same one as for (B.1)

(B.3) $\|[\Lambda^{s-1} \partial_i, f]g\|_{L^r} \leq C \left(\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}} \|g\|_{L^{q_2}} \right).$

We end up this appendix with the proof of (2.39).

The proof of (2.39). According to the Lemma B.1, we get that

$$
\begin{aligned}
\|\nabla u\|_{L^\infty} &= \left\| \frac{\nabla v}{\mathbb{I} + \Lambda^{2\gamma}} \right\|_{L^\infty} \\
&\leq \left\| \Delta_{-1} \frac{\nabla v}{\mathbb{I} + \Lambda^{2\gamma}} \right\|_{L^\infty} + \sum_{j \geq 0} \left\| \Delta_j \frac{\nabla v}{\mathbb{I} + \Lambda^{2\gamma}} \right\|_{L^\infty} \\
&\leq C \|\Delta_{-1} \nabla v\|_{L^\infty} + C \sum_{j \geq 0} \frac{1}{1 + 2^{2\gamma j}} \|\Delta_j \nabla v\|_{L^\infty} \\
&\leq C \|\Delta_{-1} \nabla v\|_{L^2} + C \sum_{j \geq 0} \frac{1}{1 + 2^{2\gamma j}} \|\Delta_j \omega\|_{L^\infty} \\
&\leq C \|\omega\|_{L^2} + C \sum_{j \geq 0} \frac{1}{1 + 2^{2\gamma j}} \|\omega\|_{L^\infty} \\
&\leq C(\|\omega\|_{L^2} + \|\omega\|_{L^\infty}),
\end{aligned}
$$

where we have used $\gamma > 0$ in the last line. This yields the first inequality of (2.39). The second one is the direct consequence of Plancherel theorem

$$
\begin{aligned}
\|\Lambda^{s+\sigma} u\|_{L^2} &= \left\| \frac{\Lambda^{\sigma-1}}{\mathbb{I} + \Lambda^{2\gamma}} \Lambda^{s+1} v \right\|_{L^2} \\
&= \left\| \frac{|\xi|^{\sigma-1}}{1 + |\xi|^{2\gamma}} \widehat{\Lambda^{s+1} v}(\xi) \right\|_{L^2} \\
&\leq C \left\| \widehat{\Lambda^{s+1} v}(\xi) \right\|_{L^2} \\
&= C \left\| \Lambda^{s+1} v \right\|_{L^2} \\
&\leq C \left\| \Lambda^s \omega \right\|_{L^2},
\end{aligned}
$$

where we have used the fact $\sigma \leq 1 + 2\gamma$ in the third line.

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