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# Asymptotic autonomy of kernel sections for Newton-Boussinesq equations on unbounded zonary domains

Renhai Wang and Yangrong Li

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ABSTRACT. We study asymptotic autonomy of the kernel sections of an evolution process, which has a forward limiting semigroup. We show that the forward compactness of the kernel sections for the process is a necessary and sufficient condition such that the kernel sections are attracted by the global attractor for the semigroup. The criterion of forward compactness is also established by using the forward-pullback asymptotic compactness of the process. As applications, we obtain nonempty, uniformly bounded and forward compact kernel sections for the non-autonomous Newton-Boussinesq equation defined on an unbounded zonary domain and perturbed by longtime convergent forces. More importantly, the kernel sections are attracted by the global attractor of the autonomous equation.

CONTENTS



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## 1. Introduction

A non-autonomous dynamics system (evolution process) $S$ uniquely determines a family of **kernel sections** given by

$$\mathcal{K}(t) := \{u(t) : u(\cdot) \text{ is a bounded complete trajectory for } S\} \quad \forall t \in \mathbb{R}.$$

This classical concept was first introduced by Chepyzhov and Vishik [**5**] under the framework of uniform attractors (see [**7, 8, 3, 12**]). However, even if there does not exist a uniform attractor, the kernel sections can be defined and investigated (see [**2, 31, 39, 42**]).

Consider another autonomous system (semigroup) $T$ with a global attractor $A$ in a Banach space $(X, \|\cdot\|)$. In fact, $A$ is the kernel section of $T$ at zero (see [**4**]).

We concern a new subject on **asymptotic autonomy** of kernel sections. More precisely, we consider the following problem.

Question 1. What is the condition, in terms of $S$ and $T$, to ensure that the kernel section $\mathcal{K}(t)$ is **attracted** by the global attractor $A$? that is,

$$\text{dist}(\mathcal{K}(t), A) := \sup_{x\in\mathcal{K}(t)} \inf_{a\in A} d(x, a) \to 0 \text{ as } t \to +\infty? \tag{1.1}$$

Such asymptotic autonomy for a pullback attractor had been investigated by Kloeden et al.[**14, 15, 11**] and Li et al.[**18**].

The kernel section is different from a pullback attractor, although some relations between kernel section and attracting set can be established in Section 2.

In particular, both kernel section and pullback attractor are invariant sets. So, we try to consider a more general problem.

Question 2. What is the condition, in terms of $S$ and $T$, to ensure that an invariant set $\mathcal{P}(\cdot)$ of $S$ is **attracted** by $A$ in the sense of (1.1)?

A basic condition is the convergence from $S$ to $T$ in the following sense.

Definition 1.1. We say that a semigroup $T$ is a **forward limit** of an evolution process $S$ if for any compact set $C \subset X$,

$$\lim_{\tau\to+\infty} \sup_{x\in C} \|S(t+\tau, \tau)x - T(t)x\|_X = 0, \ \forall t \geq 0. \tag{1.2}$$

We also need some time-dependent compactness of a family of sets in $X$.

Definition 1.2. A family $\{\mathcal{P}(t) : t \in \mathbb{R}\}$ of sets in $X$ is called

(a) **compact (closed, bounded)** if so is each component $\mathcal{P}(t)$,

(b) **eventually compact** if any sequence $x_n \in \mathcal{P}(t_n)$ with $t_n \uparrow +\infty$ has a convergent subsequence,

(c) **forward compact** if both $\mathcal{P}(t)$ and $\overline{\cup_{s\geq t}\mathcal{P}(s)}$ are compact for all $t \in \mathbb{R}$,

(d) **forward bounded** if $\cup_{s\geq s_0}\mathcal{P}(s)$ is bounded for some $s_0 \in \mathbb{R}$.

Theorem 1.3. *Suppose a semigroup $T$ is a forward limit of a process $S$ with a global attractor $A$ and the process $S$ has an invariant set $\mathcal{P}(\cdot)$. Then, $\mathcal{P}(\cdot)$ is attracted by $A$ if and only if $\mathcal{P}(\cdot)$ is eventually compact and forward bounded.*

The above theorem deduces a satisfactory result for compact kernel sections.

Theorem 1.4. *Suppose a semigroup $T$ is a forward limit of a process $S$ with a global attractor $A$ and the process $S$ has a family $\mathcal{K}(\cdot)$ of compact kernel sections. Then, $\mathcal{K}(\cdot)$ is attracted by $A$ if and only if $\mathcal{K}(\cdot)$ is forward compact.*

Another issue is to look for some conditions (depending on the system itself) such that the family of kernel sections is forward compact. One condition is the uniformly bounded absorption, which ensures that the kernel section is nonempty. Another important condition is that the process is **forward-pullback asymptotically compact**, that is, the usual pullback asymptotic compactness ([**17, 40**]) is uniform in the future, see Theorem 3.4.

We consider two models to illustrate the abstract results. One is the simple reaction-diffusion equation on a bounded domain. The main assumption is the convergence condition from the time-dependent force to the fixed force. On one hand, this convergence condition ensures the convergence (like (1.2)) of the systems, on the other hand, both convergence and tempered conditions can imply the **forward tempered** condition (see Lemma 4.1). The latter condition ensures that the evolution process is forward-pullback asymptotically compact.

Another model is the Newton-Boussinesq equation defined on an unbounded zonary domain $\Omega = (0, L) \times \mathbb{R}$:

$$(1.3) \quad \begin{cases} \xi_t + u\xi_x + v\xi_y = \Delta\xi - \dfrac{R}{P}\theta_x + f(t, x, y), \quad \xi = \Delta\Psi, \ u = \Psi_y, \ v = -\Psi_x, \\ \theta_t + u\theta_x + v\theta_y = \dfrac{1}{P}\Delta\theta + g(t, x, y), \ (x, y) \in \Omega, \ t \in \mathbb{R}, \end{cases}$$

where $R, P > 0$, $\xi$ is the vortex, $\theta$ is the flow temperature, $\Psi$ is the flow function, $\overrightarrow{u} = (u, v)$ is the velocity field of the fluid. The Newton-Boussinesq equation is widely used to model the dynamics of Bénard flow [**9, 24**].

The previous equation can help us to look for some suitable conditions for the present equation on the unbounded domain.

The main assumption is still the convergence condition that $f$ and $g$ converge to time-independent functions $\hat{f}$ and $\hat{g}$ respectively (see Hypothesis **C**). Since the domain is unbounded, we need to deduce the **forward tail-smallness** of $f, g$ from the condition **C**, see Lemma 6.2. Then, the forward-uniform tail-estimate is available and thus the forward-pullback asymptotic compactness is obtained.

## 2. Kernel sections and invariant sets of a process

**2.1. Basic properties of kernel sections.** An **evolution process** $S$ on a Banach space $X$ means that it is a family of mappings $S(t, s) : X \to X$ such that $S(s, s) = I$, $S(t, s) = S(t, r)S(r, s)$ for all $t \geq r \geq s$. We assume that $S(t, s)x$ is continuous in $t$ and in $x$.

DEFINITION 2.1. A function $u : \mathbb{R} \to X$ is called a complete trajectory for an evolution process $S$ if $S(t, s)u(s) = u(t)$ for $t \geq s$. A complete trajectory is bounded if $\{u(t) : t \in \mathbb{R}\}$ is a bounded set in $X$.

The continuity of a complete trajectory was assumed in the literature (see [**38, 41, 43**]). However, this continuity is automatic if the process is continuous.

LEMMA 2.2. *A complete trajectory $u : \mathbb{R} \to X$ is continuous.*

PROOF. For any $t_0 \in \mathbb{R}$, by the continuity of $t \mapsto S(t, t_0 - 1)u(t_0 - 1)$, we have

$$\lim_{t \to t_0} u(t) = \lim_{t \to t_0} S(t, t_0 - 1)u(t_0 - 1) = S(t_0, t_0 - 1)u(t_0 - 1) = u(t_0),$$

which means that $u(\cdot)$ is continuous from $\mathbb{R}$ to $X$. □

DEFINITION 2.3. The class $\mathbf{K}$ of all bounded complete trajectories is called a **kernel**. The set $\mathcal{K}(t) := \{u(t) \in X : u(\cdot) \in \mathbf{K}\}$ is called a **kernel section** at $t$.

The kernel section $\mathcal{K}(t)$ may be empty. For example, the ODE $\dot{x} + x = e^t$ has a single-trajectory $\{e^t/2\}$ [**19**], which is unbounded and thus the kernel section $\mathcal{K}(t) = \emptyset$. Another extreme case is $\mathcal{K}(t) \equiv X$ for the process $S(t,s) \equiv I$.

The following result concerns the relation between the kernel section and an attracting set.

LEMMA 2.4. *Let $\mathcal{K}(\cdot)$ be the family of all kernel sections for a process $S$. Then,*

*(i) $\mathcal{K}(\cdot)$ is* **invariant**, *that is, $S(t,s)\mathcal{K}(s) = \mathcal{K}(t)$ for $t \geq s$.*

*(ii) $\mathcal{K}(t) \subset \mathcal{P}(t)$, if $\mathcal{P}(\cdot)$ is closed and* **pullback attracting**, *the latter means*

$$\lim_{s \to -\infty} dist(S(t,s)B, \mathcal{P}(t)) = 0, \ \forall t \in \mathbb{R}, \ B \in \mathfrak{B}(X),$$

*where $\mathfrak{B}(X)$ denotes the class of all bounded sets.*

*(iii) $dist(\mathcal{K}(t), \mathcal{F}(t)) \to 0$ as $t \to +\infty$, if $\mathcal{K}(\cdot)$ is bounded, and $\mathcal{F}(\cdot)$ is* **forward attracting** [**16**], *that is,*

$$\lim_{t \to +\infty} dist(S(t,s)B, \mathcal{F}(t)) = 0, \ \forall s \in \mathbb{R}, \ B \in \mathfrak{B}(X).$$

PROOF. (i) The assertion is well known (see [**5**]).

(ii) Given $u_t \in \mathcal{K}(t)$, we take a bounded complete trajectory $u(\cdot)$ such that $u(t) = u_t$. Hence, $B := \{u(s) : s \in \mathbb{R}\}$ is bounded and thus attracted by $\mathcal{P}(\cdot)$,

$$\text{dist}(u_t, \mathcal{P}(t)) = \text{dist}(u(t), \mathcal{P}(t)) = \text{dist}(S(t,s)u(s), \mathcal{P}(t)) \leq \text{dist}(S(t,s)B, \mathcal{P}(t)),$$

which converges to zero as $s \to -\infty$. So, $u_t \in \overline{\mathcal{P}(t)} = \mathcal{P}(t)$ and thus $\mathcal{K}(t) \subset \mathcal{P}(t)$.

(iii) Let $t \geq 0$. By the invariance of $\mathcal{K}(\cdot)$ and boundedness of $\mathcal{K}(0)$, we have

$$\text{dist}(\mathcal{K}(t), \mathcal{F}(t)) = \text{dist}(S(t,0)\mathcal{K}(0), \mathcal{F}(t)) \to 0 \ \text{ as } t \to +\infty,$$

which proves the convergence in (iii). □

The following result concerns closedness of a kernel section, which is useful later and has its own meaning.

PROPOSITION 2.5. *Each kernel section $\mathcal{K}(t)$ of an evolution process is compact if $\mathcal{K}(\cdot)$ is pre-compact and uniformly bounded.*

PROOF. It suffices to prove that $\mathcal{K}(\tau)$ is closed for each $\tau \in \mathbb{R}$. Let $u_{\tau,n} \in \mathcal{K}(\tau)$ such that $u_{\tau,n} \to u(\tau) \in X$ as $n \to \infty$. We can find a sequence $\{u_n(\cdot)\}$ of complete trajectories such that $u_n(\tau) = u_{\tau,n} \to u(\tau)$. By the uniformly bounded assumption,

$$\sup_{n \in \mathbb{N}} \sup_{s \in \mathbb{R}} \|u_n(s)\| \leq \sup_{s \in \mathbb{R}} \|\mathcal{K}(s)\| =: c < +\infty.$$

By the pre-compactness of $\mathcal{K}(\tau - 1)$, we know that there are a subsequence $\{u_n^1(\cdot)\}$ of trajectories $\{u_n(\cdot)\}$ and $u(\tau-1) \in X$ such that $u_n^1(\tau-1) \to u(\tau-1)$. For each $k = 2, 3, \cdots$, we sequentially choose a subsequence $\{u_n^k(\cdot)\}$ of $\{u_n^{k-1}(\cdot)\}$ and $u(\tau-k) \in X$ such that $u_n^k(\tau-k) \to u(\tau-k)$.

Thereby, the diagonal subsequence $\{u_n^n(\cdot)\}$ satisfies the following convergence:

$$u_n^n(\tau-k) \to u(\tau-k) \text{ as } n \to \infty \ \text{ for each } k \in \mathbb{N} \cup \{0\}.$$

Note that we have defined all values $u(\tau-k)$ ($k\in\mathbb{N}\cup\{0\}$) of a mapping $u:\mathbb{R}\to X$. We define other values by

$$u(s)=\begin{cases}S(s,\tau)u(\tau), & s\geq\tau,\\ S(s,\tau-k)u(\tau-k), & \tau-k\leq s<\tau-k+1,\end{cases}\tag{2.1}$$

We claim that $\{u(s):\ s\in\mathbb{R}\}$ is bounded. Indeed, the continuity of $S:X\to X$ implies that for $s\geq\tau$,

$$\|u(s)\|=\|S(s,\tau)u(\tau)\|=\|S(s,\tau)\lim_{n\to\infty}u_n^n(\tau)\|=\|\lim_{n\to\infty}u_n^n(s)\|\leq c.$$

If $\tau-k\leq s\leq\tau-k+1$, then

$$\|u(s)\|=\|S(s,\tau-k)\lim_{n\to\infty}u_n^n(\tau-k)\|=\|\lim_{n\to\infty}u_n^n(s)\|\leq c.$$

We then prove that $u$ is a complete trajectory. Let $s_1\geq s_2$. If $s_2\geq\tau$, then

$$u(s_1)=S(s_1,\tau)u(\tau)=S(s_1,s_2)S(s_2,\tau)u(\tau)=S(s_1,s_2)u(s_2).$$

If $s_1\geq\tau$ and $\tau-k\leq s_2<\tau-k+1$ for some $k\in\mathbb{N}$, then

$$\begin{aligned}u(s_1)&=S(s_1,\tau)u(\tau)=S(s_1,\tau)\lim_{n\to\infty}u_n^n(\tau)\\&=S(s_1,\tau)\lim_{n\to\infty}S(\tau,\tau-k)u_n^n(\tau-k)=S(s_1,\tau)S(\tau,\tau-k)u(\tau-k)\\&=S(s_1,\tau)S(\tau,s_2)S(s_2,\tau-k)u(\tau-k)=S(s_1,s_2)u(s_2).\end{aligned}$$

It is similar to prove the cases that $\tau-k_1\leq s_1<\tau-k_1+1$ and $\tau-k_2\leq s_2<\tau-k_2+1$ for $k_1\leq k_2$. Therefore, $u$ is a bounded complete trajectory and thus $u(\tau)\in\mathcal{K}(\tau)$ as desired. □

**2.2. Asymptotic autonomy of an invariant set.** We first concern the condition that $T$ is the limiting semigroup of an evolution process $S$ in the sense of Definition 1.1. Notice that the continuity of $T$ has been assumed as a special process.

LEMMA 2.6. *Suppose a semigroup $T$ is a* **forward limit** *of a process $S$ in the sense of Definition 1.1. Then, we have*

$$\lim_{\tau\to+\infty}\|S(t+\tau,\tau)x_\tau-T(t)x_0\|_X=0,\ \forall t\geq0,\tag{2.2}$$

*whenever $\|x_\tau-x_0\|\to0$ as $\tau\to+\infty$.*

PROOF. Suppose $\tau_n\to+\infty$ and $\|x_{\tau_n}-x_0\|\to0$ as $n\to+\infty$. Then, $C=\{x_{\tau_n}\}\cup\{x_0\}$ is a compact set. By (1.2) and the continuity of $T(\cdot)$, we have

$$\begin{aligned}&\|S(t+\tau_n,\tau_n)x_{\tau_n}-T(t)x_0\|_X\\&\leq\sup_{x\in C}\|S(t+\tau_n,\tau_n)x-T(t)x\|_X+\|T(t)x_{\tau_n}-T(t)x_0\|_X\to0\end{aligned}$$

as $n\to\infty$. So (2.2) holds true. □

The pointwise convergence in (2.2) is obviously weaken than the uniform convergence (w.r.t. $t\geq0$), the latter was used by Kloeden et al.[**14, 15, 11**] to deduce the asymptotic autonomy of a pullback attractor.

Next, we prove Theorem 1.3, which is rewritten as follows.

THEOREM 2.7. *Let a semigroup $T$ be a forward limit of a process $S$ with a global attractor $A$ and $\mathcal{P}(\cdot)$ be an invariant set for $S$. Then,*

*(I) $\mathcal{P}(\cdot)$ is attracted by $A$, i.e.*

$$dist(\mathcal{P}(s), A) \to 0 \quad as\ s \to +\infty \tag{2.3}$$

*if and only if $\mathcal{P}$ is eventually compact and forward bounded.*

*(II) If $\mathcal{P}(\cdot)$ is also compact then it is attracted by $A$ if and only if $\mathcal{P}(\cdot)$ is forward compact.*

PROOF. (I) *Sufficiency.* If (2.3) is not true, then there are $\delta > 0$, $s_n \uparrow +\infty$ and $x_n \in \mathcal{P}(s_n)$ such that

$$\operatorname{dist}(x_n, A) \geq \operatorname{dist}(\mathcal{P}(s_n), A) - \delta \geq 2\delta, \quad \forall n \in \mathbb{N}. \tag{2.4}$$

By the forward boundedness (Definition 1.2) of $\mathcal{P}(\cdot)$, there is an $\hat{s} \in \mathbb{R}$ such that the set $B := \overline{\cup_{s \geq \hat{s}} \mathcal{P}(s)}$ is bounded in $X$. By the attraction of $A$ under $T$, there is an $n_0 \in \mathbb{N}$ such that

$$\operatorname{dist}(T(s_{n_0})B, A) < \delta. \tag{2.5}$$

By the invariance of $\mathcal{P}(\cdot)$, for any $n \geq n_0$,

$$x_n = S(s_n, s_n - s_{n_0})\hat{x}_n, \quad \text{for some } \hat{x}_n \in \mathcal{P}(s_n - s_{n_0}).$$

By the eventual compactness (Definition 1.2), the sequence $\{\hat{x}_n\}$ has a convergent subsequence $\hat{x}_{n_k} \to \hat{x}$. Hence, it follows from (2.2) in Lemma 2.6 that

$$d(x_{n_k}, T(s_{n_0})\hat{x}) = d(S(s_{n_0} + s_{n_k} - s_{n_0}, s_{n_k} - s_{n_0})\hat{x}_{n_k}, T(s_{n_0})\hat{x}) < \delta$$

provided $k$ is large enough. It is easy to show $\hat{x} \in B$, then, by (2.5),

$$\operatorname{dist}(x_{n_k}, A) \leq d(x_{n_k}, T(s_{n_0})\hat{x}) + \operatorname{dist}(T(s_{n_0})B, A) < \delta.$$

This contradicts with (2.4).

*Necessity.* Suppose $\operatorname{dist}(\mathcal{P}(s), A) \to 0$ as $s \uparrow +\infty$. Then, we can find an $s_0$ such that

$$\bigcup_{s \geq s_0} \mathcal{P}(s) \subset N(A, 1) := \{x \in X : d(x, A) \leq 1\},$$

which is bounded. So, $\mathcal{P}(\cdot)$ is forward bounded.

On the other hand, let $x_n \in \mathcal{P}(s_n)$ with $s_n \uparrow +\infty$. We have

$$\operatorname{dist}(x_n, A) \leq \operatorname{dist}(\mathcal{P}(s_n), A) \to 0.$$

Since $A$ is compact, it follows from [**4**, Theorem 2.3] that $\{x_n\}$ has a convergent subsequence. So, $\mathcal{P}$ is eventually compact.

(II) *Sufficiency.* Since $\mathcal{P}(\cdot)$ is forward compact, it is easy to show that $\mathcal{P}(\cdot)$ is eventually compact and forward bounded. Hence, the needed asymptotic autonomy follows from the assertion (I) immediately.

*Necessity.* Suppose (2.3) holds true. We need to show that $\mathcal{P}(\cdot)$ is forward compact. By the compactness assumption of $\mathcal{P}(\cdot)$. It suffices to show that $\cup_{s \geq t} \mathcal{P}(s)$ is pre-compact for any fixed $t \in \mathbb{R}$. For this end, we take $x_n \in \mathcal{P}(s_n)$ with $s_n \geq t$, and then show that $\{x_n\}$ has a convergent subsequence in two cases.

If $\sup_{n \in \mathbb{N}} s_n = +\infty$, then there is a subsequence $s_{n_k} \uparrow +\infty$. By (I), $\mathcal{P}(\cdot)$ is eventually compact, which implies that the subsequence $\{x_{n_k}\}$ has a convergent subsequence.

If $s_0 := \sup_{n\in\mathbb{N}} s_n < +\infty$, we have $\{x_n\} \subset \bigcup_{t\le s\le s_0} \mathcal{P}(s)$. By the invariance of $\mathcal{P}(\cdot)$, we have

$$\bigcup_{t\le s\le s_0} \mathcal{P}(s) = \Lambda_t([t, s_0] \times \mathcal{P}(t)),\ \text{ where } \Lambda_t : (s, x) \to S(s, t)x.$$

By the continuity of $\Lambda_t$ and the compactness of $\mathcal{P}(t)$, we know that $\cup_{t\le s\le s_0}\mathcal{P}(s)$ is compact and thus $\{x_n\}$ has a convergent subsequence. □

## 3. Asymptotic autonomy and forward compactness of kernel sections

**3.1. General case.** We come back to investigate the asymptotic autonomy of kernel sections. We will establish some criteria depending on the process itself.

Definition 3.1. A family $\mathcal{P}(\cdot)$ of sets in $X$ is a *pullback absorbing set* for the process $S$ if for each $(t, B) \in \mathbb{R} \times \mathfrak{B}(X)$, there is a $\tau_0 = \tau_0(t, B) > 0$ such that $S(t, t-\tau)B \subset \mathcal{P}(t)$ for all $\tau \ge \tau_0$.

Definition 3.2. The process $S$ is **forward-pullback asymptotically compact** if for each $t \in \mathbb{R}$, the sequence $\{S(s_n, s_n-\tau_n)x_n\}_{n=1}^{\infty}$ is pre-compact whenever $s_n \ge t$, $\tau_n \to +\infty$ and $\{x_n\}$ is bounded in $X$.

The forward-pullback asymptotic compactness is contrary with the concept of *backward-pullback asymptotic compactness* [**20, 21, 30, 35, 36**] (or *strong pullback asymptotic compactness* [**4**]), the latter means the asymptotic compactness is uniform in the past.

We introduce the concept of a **forward-pullback omega-limit set**:

$$\Omega(B, t) := \bigcap_{T>0} \overline{\bigcup_{\tau\ge T} \bigcup_{s\ge t} S(s, s-\tau)B}, \quad \forall (B, t) \in \mathfrak{B}(X) \times \mathbb{R},$$

which contains the pullback omega-limit set $\omega(B, t) := \cap_{T>0}\overline{\cup_{\tau\ge T} S(t, t-\tau)B}$.

Lemma 3.3. *(i) $x \in \Omega(B, t)$ if and only if there are $s_n \ge t$, $\tau_n \uparrow +\infty$ and $x_n \in B$ such that*

$$x = \lim_{n\to\infty} S(s_n, s_n-\tau_n)x_n \quad \text{in } X.$$

*(ii) $\Omega(B, \cdot)$ is decreasing, that is, $\Omega(B, t_1) \supset \Omega(B, t_2)$ if $t_1 \le t_2$.*

*(iii) If $S$ is forward-pullback asymptotically compact, then, $\Omega(B, \cdot)$ is forward compact.*

Proof. The proof of (i) is similar to the case of usual omega-limit set (see [**4**]). The assertion (ii) follows from the definition immediately.

It suffices to prove (iii). Indeed, by (ii), $\Omega(B, \cdot)$ is decreasing, then $\cup_{s\ge t}\Omega(B, s) = \Omega(B, t)$. Hence, it suffices to show $\Omega(B, t)$ is compact. Let $\{y_n\}_{n=1}^{\infty} \subset \Omega(B, t)$. By (i), there are $s_n \ge t$, $\tau_n \uparrow +\infty$ and $x_n \in B$ such that

$$\|S(s_n, s_n-\tau_n)x_n - y_n\| \le \frac{1}{n}.$$

By the forward-pullback asymptotic compactness of $S$, passing to a subsequence,

$$S(s_{n_k}, s_{n_k}-\tau_{n_k})x_{n_k} \to y_0 \ \text{ in } X.$$

So, $y_{n_k} \to y_0$ in $X$. Therefore, $\Omega(B, t)$ is pre-compact and thus compact. □

THEOREM 3.4. *Let an evolution process $S$ be forward-pullback asymptotically compact with a uniformly bounded absorbing set $\mathcal{P}(\cdot)$. Then, the kernel sections $\mathcal{K}(\cdot)$ are nonempty, forward compact, uniformly bounded and pullback attracting.*

*Furthermore, if a semigroup $T$ is a forward limit of $S$ with a global attractor $A$, then, the kernel section $\mathcal{K}(\tau)$ is attracted by $A$:*

$$dist(\mathcal{K}(\tau), A) \to 0 \quad as\ \tau \to +\infty. \tag{3.1}$$

PROOF. We show the conclusions in three steps.

*Step 1.* We show that $\mathcal{K}(\cdot)$ is uniformly bounded. Indeed, by Lemma 2.4, the kernel section is included into the closed attracting set. So,

$$\bigcup_{s\in\mathbb{R}} \mathcal{K}(s) \subset \bigcup_{s\in\mathbb{R}} \overline{\mathcal{P}(s)} \subset \overline{\bigcup_{s\in\mathbb{R}} \mathcal{P}(s)} =: P,$$

where $P$ is bounded, closed and pullback absorbing.

*Step 2.* We show that $\mathcal{K}(\cdot)$ is forward compact. If this is verified, then, by Theorem 2.7 (II), the kernel section $\mathcal{K}(\tau)$ (regarded as an invariant set) is attracted by $A$, i.e. (3.1) holds true.

For this end, we show that $\Omega(P,\cdot)$ is pullback attracting. Indeed, if it is not true, then, there are $\delta > 0$, $t \in \mathbb{R}$, $\tau_n \to +\infty$ and a bounded sequence $\{x_n\}$ such that

$$\mathrm{dist}(S(t, t-\tau_n)x_n, \Omega(P,t)) \geq \delta, \quad \forall n \in \mathbb{N}. \tag{3.2}$$

By the asymptotic compactness, passing to a subsequence, $S(t, t-\tau_n)x_n \to x$. For each $k \in \mathbb{N}$, we choose an $\tau_{n_k}$ such that $\tau_{n_k} - k$ is large enough and so

$$y_k := S(t-k, t-\tau_{n_k})x_{n_k} \in \mathcal{P}(t-k) \subset P.$$

Note that $S(t, t-k)y_k = S(t, t-\tau_{n_k})x_{n_k} \to x$. By Lemma 3.3 (i), we know $x \in \Omega(P,t)$, which contradicts with (3.2).

By Lemma 2.4 (ii), we have $\mathcal{K}(t) \subset \Omega(P,t)$ for each $t \in \mathbb{R}$. Since $S$ is forward-uniformly compact, by Lemma 3.3 (iii), $\Omega(P,t)$ is compact and so $\mathcal{K}(t)$ is pre-compact. By Step 1, $\mathcal{K}(\cdot)$ is uniformly bounded. Therefore, by Proposition 2.5, the kernel section $\mathcal{K}(t)$ is compact. On the the hand, by Lemma 3.3 (iii) again, $\Omega(P,\cdot)$ is forward compact and so $\mathcal{K}(\cdot)$ is forward compact as required.

*Step 3.* We show that the kernel section $\mathcal{K}(t)$ is nonempty and pullback attracting at any time $t \in \mathbb{R}$. Since $\omega(P,t)$ is nonempty, pullback attracting and invariant (see [**4**]), it suffices to prove $\mathcal{K}(t) \supset \omega(P,t)$. For this end, let $u_t \in \omega(P,t)$, we show $S$ has a bounded complete trajectory $u : \mathbb{R} \mapsto X$ such that $u(t) = u_t$. Let $u(s) := S(s,t)u_t$ for all $s \geq t$, then

$$u(s) = S(s,t)u_t \in S(s,t)\omega(P,t) = \omega(P,s) \subset \overline{\mathcal{P}(s)} \subset \overline{P} = P, \quad \forall s \geq t, \tag{3.3}$$

in view of the invariance of $\omega(P,t)$. Then $\{u(s) : s \geq t\}$ is bounded.

Let us focuss on the construction for $s \leq t$. By $u_t \in S(t,t-1)\omega(P,t-1)$, we can take a $u_{t-1} \in \omega(P,t-1)$ such that $u_t = S(t,t-1)u_{t-1}$, and then set

$$u(s) := S(s,t-1)u_{t-1} \quad \text{for all } s \in [t-1,t].$$

Repeating this procedure, we take $u_{t-n} \in \omega(P,t-n)$ such that $u_{t-n+1} = S(t-n+1, t-n)u_{t-n}$ and then set

$$u(s) := S(s,t-n)u_{t-n} \quad \text{for all } s \in [t-n, t-n+1].$$

So, $\{u(s) : s \leq t\}$ is well-defined. In addition, for all $n \in \mathbb{N}$,

$$u(s) = S(s, t-n)u_{t-n} \subset S(s, t-n)\omega(P, t-n) = \omega(P, s) \subset P, \quad t-n \leq s \leq t.$$

So, $\{u(s) : s \in \mathbb{R}\}$ is bounded in $X$. It is similar as the proof of Proposition 2.5 to prove that $u$ is a complete trajectory. Therefore, $u_t = u(t) \in \mathcal{K}(t)$ as required. □

**3.2. Special case: a uniform attractor exists.** We discuss a special case that a uniform attractor exists. We consider the uniform attractor with respect to $\mathbb{R}$ rather than other symbol space $\Sigma$, see [**4, 6**].

Definition 3.5. A set $E$ is called a $\mathbb{R}$-**uniform attractor** if it is a minimal compact $\mathbb{R}$-uniformly attracting set, where the uniform attraction means that for each $B \in \mathfrak{B}(X)$,

$$\lim_{t\to+\infty} \sup_{s\in\mathbb{R}} \operatorname{dist}(S(s+t, s)B, E) = 0. \tag{3.4}$$

Proposition 3.6. *Suppose a process $S$ has a $\mathbb{R}$-uniform attractor $E$ and a semigroup $T$ has a global attractor $A$ such that $T$ is a forward limit of $S$. Then,*

*(a) the kernel section $\mathcal{K}(t)$ for $S$ is attracted by $A$,*

*(b) we have the following inclusion:*

$$\lim_{t\to+\infty} \mathcal{K}(t) := \bigcap_{T>0} \overline{\bigcup_{t\geq T} \mathcal{K}(t)} = \bigcap_{T>0} \overline{\bigcup_{t\geq T} S(t, 0)\mathcal{K}(0)} = \omega(\mathcal{K}(0)) \subset A. \tag{3.5}$$

Proof. (a) Since there is a $\mathbb{R}$-uniform attractor $E$, by [**4, 5**], the kernel section $\mathcal{K}(t)$ is nonempty compact. Moreover, we have

$$\bigcup_{t\in\mathbb{R}} \mathcal{K}(t) \subset E. \tag{3.6}$$

Since $E$ is compact, we know that the kernel brochette $\mathcal{K}$ is uniformly compact and (particularly) forward compact. So, by Theorem 2.7(II), $\mathcal{K}$ is attracted by $A$.

(b) Since there is a $\mathbb{R}$-uniform attractor $E$, it is easy to prove that $\mathcal{K}(t) = \mathcal{A}(t)$, where $\mathcal{A}(\cdot)$ is the pullback attractor for $S$ and, by (3.6), $\mathcal{A}(\cdot)$ is uniformly compact. Therefore, the needed conclusion follows from [**18**, Prop.2.6]. □

In general, we cannot prove $\mathcal{K}(t) \subset A$ except for $E = A$.

Proposition 3.7. *If a process $S$ has a $\mathbb{R}$-uniform attractor $E$ and a semigroup $T$ has a global attractor $A$ such that $A = E$, then the kernel section $\mathcal{K}(t)$ satisfies $\mathcal{K}(t) \subset A$ for all $t \in \mathbb{R}$.*

Proof. The assertion follows from (3.6) immediately. □

However, it is possible that $E \neq A$ in an asymptotically autonomous system. Here, we consider a simple example as given in [**4**, p.389].

Example 3.8. We consider a non-autonomous ODE on $\mathbb{R}$:

$$\dot{x} = x - x^3 \text{ if } t < 0 \text{ and } \dot{x} = -x \text{ if } t \geq 0.$$

The uniform attractor is $E = [-1, 1]$ and the kernel section is given by

$$\mathcal{K}(t) = [-1, 1] \text{ if } t < 0, \ \mathcal{K}(t) = [-e^{-t}, e^{-t}] \text{ if } t \geq 0.$$

The autonomous ODE is $\dot{x}(t) = -x(t)$ for $t \geq 0$, which has a global attractor $A = \{0\}$. We have $A \neq E$ and $\mathcal{K}(t)$ is not included into $A$.

For the above example, both inclusions (3.5) and (3.6) become equalities. However, consider another example

$$\dot{x} = -x \text{ if } t < 0 \text{ and } \dot{x} = x - x^3 \text{ if } t \geq 0.$$

Then, the uniform attractor $E = [-1, 1]$, the kernel section $\mathcal{K}(t) \equiv \{0\}$ and the global attractor $A = [-1, 1]$. So, both inclusions (3.5) and (3.6) are strict.

REMARK 3.9. In [**4**, p385], the *asymptotically autonomous system* is defined by

$$\omega(K_\Sigma) := \bigcap_{t>0} \overline{\bigcup_{s\geq t} \theta_s K_\Sigma} \text{ is a point,} \tag{3.7}$$

where $K_\Sigma$ is a compact attracting set for the flow $\theta_t$ on the symbol space $\Sigma$. This concept is completely different from the one as given in Definition 1.1.

In fact, if we take $\Sigma = \mathbb{R}$, then there does not exist any compact attracting set for the flow $\theta_t s = t + s$.

## 4. A simple reaction-diffusion model

We consider a simple non-autonomous reaction-diffusion equation on a bounded domain $D \subset \mathbb{R}^n$:

$$u_t - \Delta u + u^3 - u = f(t, x),\ t > \tau,\ u|_{\partial D} = 0,\ u(\tau) = u_\tau. \tag{4.1}$$

As usual, one can obtain uniformly bounded and compact kernel sections $\mathcal{K}(\cdot)$ in $L^2(D)$ if $f \in L^2_{loc}(\mathbb{R}, L^2(D))$ satisfies some suitable conditions, which at least contain the tempered condition:

$$\int_{-\infty}^{t} e^{\lambda r}\|f(r)\|^2 dr < \infty, \quad \forall t \in \mathbb{R}, \tag{4.2}$$

where $\lambda > 0$ is the Poincaré constant such that $\lambda\|u\|^2 \leq \|\nabla u\|^2$. We then consider the autonomous equation:

$$v_t - \Delta v + v^3 - v = f_\infty(x),\ t \geq 0,\ v|_{\partial D} = 0,\ v(0) = v_0. \tag{4.3}$$

We assume that $f_\infty \in L^2(D)$ and it is the limit of $f(t)$ in the integral sense:

$$\lim_{t\to+\infty} \int_t^\infty \|f(r) - f_\infty\|^2 dr = 0. \tag{4.4}$$

Both conditions (4.2) and (4.4) can deduce that $f(t)$ is **forward tempered**.

LEMMA 4.1. *If* (4.2) *and* (4.4) *hold true, then*

$$I(t) := \sup_{s\geq t} \int_{-\infty}^{s} e^{\lambda(r-s)}\|f(r)\|^2 dr < +\infty, \quad \forall t \in \mathbb{R}. \tag{4.5}$$

PROOF. Let $t \in \mathbb{R}$ be fixed. By (4.2) and $\sup_{s\geq t} e^{-\lambda s} = e^{-\lambda t}$,

$$I_1(t) := \sup_{s\geq t} \int_{-\infty}^{t} e^{\lambda(r-s)}\|f(r)\|^2 dr = \int_{-\infty}^{t} e^{\lambda(r-t)}\|f(r)\|^2 dr < \infty.$$

By (4.4), there is $T > t$ such that $\int_T^\infty \|f(r) - f_\infty\|^2 dr \leq 1$ and thus

$$\int_t^\infty \|f(r) - f_\infty\|^2 \leq \int_T^\infty \|f(r) - f_\infty\|^2 + 2\int_t^T \|f(r)\|^2 + 2(T - t)\|f_\infty\|^2,$$

which is finite and further implies that

$$\begin{aligned}I_2(t) &:= \sup_{s\geq t}\int_t^s e^{\lambda(r-s)}\|f(r)\|^2dr\\ &\leq 2\sup_{s\geq t}\int_t^s e^{\lambda(r-s)}\|f(r)-f_\infty\|^2dr + 2\|f_\infty\|^2\sup_{s\geq t}\int_t^s e^{\lambda(r-s)}dr\\ &\leq 2\int_t^\infty \|f(r)-f_\infty\|^2dr + \frac{2}{\lambda}\|f_\infty\|^2 < \infty.\end{aligned}$$

Therefore, $I(t) \leq I_1(t) + I_2(t) < \infty$ as desired. □

LEMMA 4.2. *For each bound set $B \subset L^2(D)$ and $t \in \mathbb{R}$, there is a $\tau_0 = \tau_0(B,t) > 1$ such that for all $\tau \geq \tau_0$ and $u_{s-\tau} \in B$,*

$$\sup_{s\geq t}\|u(s,s-\tau,u_{s-\tau})\|^2 \leq c(1+I(t)) < \infty, \tag{4.6}$$

$$\sup_{s\geq t}\|\nabla u(s,s-\tau,u_{s-\tau})\|^2 \leq c(1+I(t))e^{c(1+I(t))} < \infty. \tag{4.7}$$

PROOF. By multiplying Eq.(4.1) with $u$ and using Young inequality and Poincaré inequality, we obtain

$$\frac{d}{dr}\|u\|^2 + \lambda\|u\|^2 + \|\nabla u\|^2 + \|u\|_4^4 \leq c(1+\|f(r)\|^2)$$

By the Gronwall lemma w.r.t. $r \in [s-\tau, s]$,

$$\begin{aligned}\|u(s,s-\tau,u_{s-\tau})\|^2 &\leq e^{-\lambda\tau}\|u_{s-\tau}\|^2 + c\int_{s-\tau}^s e^{\lambda(r-s)}(1+\|f(r)\|^2)dr\\ &\leq e^{-\lambda\tau}\|B\|^2 + \frac{c}{\lambda} + \int_{-\infty}^s e^{\lambda(r-s)}\|f(r)\|^2dr.\end{aligned}$$

By taking the maximum w.r.t. $s \in [t,\infty)$ and letting $\tau \to \infty$, we obtain (4.6).

By multiplying Eq.(4.1) with $-\Delta u$ and using the uniform Gronwall lemma, we can prove (4.7) as desired. □

By the compactness of the Sobolev embedding, it follows from (4.7) that the evolution process is forward-pullback asymptotically compact.

LEMMA 4.3. *Assume* (4.4)*, we have*

$$\lim_{\tau\to+\infty}\|u(t+\tau,\tau,u_\tau) - v(t,v_0)\| = 0, \quad \forall t \geq 0,$$

*whenever $\|u_\tau - v_0\| \to 0$ as $\tau \to +\infty$.*

Lemma 4.3 tells us the convergence of the solution operator. For the details of the proof, one can refer Proposition 5.1 later.

THEOREM 4.4. *The kernel section $\mathcal{K}(t)$ is forward compact and attracted by the attractor $A$ of the autonomous equation* (4.3).

## 5. Convergence of solutions for Newton-Boussinesq equations

We rewrite the non-autonomous Newton-Boussinesq equations (1.3) as follows.

$$
\begin{cases} \xi_t - \Delta\xi + J(\Psi,\xi) + \dfrac{R}{P}\theta_x = f(t,x,y),\ \Delta\Psi = \xi, \quad t \geq \tau, \\ \theta_t - \dfrac{1}{P}\Delta\theta + J(\Psi,\theta) = g(t,x,y),\ (x,y) \in \Omega := (0,L)\times\mathbb{R}, \\ \xi(x,y,\tau) = \xi_\tau, \quad \theta(x,y,\tau) = \theta_\tau,\ \text{and } \xi,\theta,\Psi = 0 \text{ on } \partial\Omega, \end{cases} \tag{5.1}
$$

where $J$ is given by $J(u,v) = u_y v_x - u_x v_y$. It is easy to check that

$$
J(u_1 - u_2, v) = J(u_1,v) - J(u_2,v), \tag{5.2}
$$

$$
J(u, v_1 - v_2) = J(u,v_1) - J(u,v_2), \quad (J(u,v),v) = 0. \tag{5.3}
$$

If $f,g \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$, then the same method as given in [**1, 23, 27, 33, 34**] can show that, for $(\xi_\tau,\theta_\tau) \in L^2(\Omega)^2$, the system (5.1) has a unique solution

$$
(\xi(\cdot,\tau,\xi_\tau),\theta(\cdot,\tau,\theta_\tau)) \in C([\tau,+\infty), L^2(\Omega)^2) \cap L^2(\tau,\tau+T; H_0^1(\Omega)^2).
$$

The solution continuously depends on the initial data and thus it defines a continuous process $S(\cdot,\cdot) : L^2(\Omega)^2 \to L^2(\Omega)^2$ by

$$
S(t,\tau)(\xi_\tau,\theta_\tau) = (\xi(t,\tau,\xi_\tau),\theta(t,\tau,\theta_\tau)), \quad \forall t \geq s. \tag{5.4}
$$

We further assume a convergence condition.

**Hypothesis C.** There are $\hat{f},\hat{g} \in L^2(\Omega)$ such that

$$
\lim_{t\to+\infty} \int_t^\infty (\|f(r) - \hat{f}\|^2 + \|g(r) - \hat{g}\|^2) dr = 0.
$$

The autonomous equation for the time-independent forces $\hat{f},\hat{g}$ can be read as:

$$
\begin{cases} \hat{\xi}_t - \Delta\hat{\xi} + J(\hat{\Psi},\hat{\xi}) + \dfrac{R}{P}\hat{\theta}_x = \hat{f}(x,y), \quad \Delta\hat{\Psi} = \hat{\xi}, \\ \hat{\theta}_t - \dfrac{1}{P}\Delta\hat{\theta} + J(\hat{\Psi},\hat{\theta}) = \hat{g}(x,y),\ (x,y) \in \Omega, \end{cases} \tag{5.5}
$$

such that $\hat{\xi}(0) = \hat{\xi}_0$, $\hat{\theta}(0) = \hat{\theta}_0$. It is well known (see [**10, 13, 23, 25, 26, 28, 32**]) that there is a global attractor $A$ for the semigroup $T(\cdot) : L^2(\Omega)^2 \to L^2(\Omega)^2$ given by

$$
T(t)(\hat{\xi}_0,\hat{\theta}_0) = (\hat{\xi}(t,\hat{\xi}_0),\hat{\theta}(t,\hat{\theta}_0)), \quad t \in \mathbb{R}^+. \tag{5.6}
$$

Proposition 5.1. *Let the hypothesis* **C** *be satisfied. Then, the semigroup $T$ is a forward limit of the process $S$ in the following sense:*

$$
\lim_{\tau\to+\infty} \|S(t+\tau,\tau)(\xi_\tau,\theta_\tau) - T(t)(\hat{\xi}_0,\hat{\theta}_0)\| = 0, \quad \forall t \in \mathbb{R}^+,
$$

*whenever $\|(\xi_\tau,\theta_\tau) - (\hat{\xi}_0,\hat{\theta}_0)\| \to 0$ as $\tau \to +\infty$.*

Proof. For each $\tau \geq 0$, we consider the following functions:

$$
\tilde{\xi}^\tau(t) := \xi(t+\tau,\tau,\xi_\tau) - \hat{\xi}(t,\hat{\xi}_0),\ \tilde{\theta}^\tau(t) := \theta(t+\tau) - \hat{\theta}(t),\ \tilde{\Psi}^\tau := \Psi - \hat{\Psi}
$$

for $t \geq 0$. By the difference of both equations (5.1) and (5.5), we obtain that

$$
\begin{aligned}
(5.7)\qquad & \frac{1}{2}\frac{d}{dt}(\|\tilde{\xi}^\tau\|^2 + \|\tilde{\theta}^\tau\|^2) + \|\nabla\tilde{\xi}^\tau\|^2 + \frac{1}{P}\|\nabla\tilde{\theta}^\tau\|^2 \\
&= -(J(\Psi,\xi) - J(\hat{\Psi},\hat{\xi}),\tilde{\xi}^\tau) - (J(\Psi,\theta) - J(\hat{\Psi},\hat{\theta}),\tilde{\theta}^\tau) \\
&\quad + (f(t+\tau) - \hat{f},\tilde{\xi}^\tau) + (g(t+\tau) - \hat{g},\tilde{\theta}^\tau) - \frac{R}{P}(\tilde{\theta}^\tau_x,\tilde{\xi}^\tau)
\end{aligned}
$$

By (5.2)-(5.3), we have

$$
\begin{aligned}
(J(\Psi,\xi) - J(\hat{\Psi},\hat{\xi}),\tilde{\xi}^\tau) &= (J(\Psi,\xi) - J(\Psi,\hat{\xi}),\tilde{\xi}^\tau) + (J(\Psi,\hat{\xi}) - J(\hat{\Psi},\hat{\xi}),\tilde{\xi}^\tau) \\
&= (J(\Psi,\tilde{\xi}^\tau),\tilde{\xi}^\tau) + (J(\tilde{\Psi}^\tau,\hat{\xi}),\tilde{\xi}^\tau) = (J(\tilde{\Psi}^\tau,\hat{\xi}),\tilde{\xi}^\tau).
\end{aligned}
$$

Hence, by the Hölder inequality and the embedding $H^1 \hookrightarrow L^4$,

$$
\begin{aligned}
(5.8)\qquad & |(J(\Psi,\xi) - J(\hat{\Psi},\hat{\xi}),\tilde{\xi}^\tau)| = |(J(\tilde{\Psi}^\tau,\hat{\xi}),\tilde{\xi}^\tau)| \\
&\leq |(\tilde{\Psi}^\tau_y\hat{\xi}_x,\tilde{\xi}^\tau)| + |(\tilde{\Psi}^\tau_x\hat{\xi}_y,\tilde{\xi}^\tau)| \leq \|\hat{\xi}_x\|_4\|\tilde{\Psi}^\tau_y\|_4\|\tilde{\xi}^\tau\| + \|\hat{\xi}_y\|_4\|\tilde{\Psi}^\tau_x\|_4\|\tilde{\xi}^\tau\| \\
&\leq c\|\hat{\xi}_x\|_{H^1}\|\tilde{\Psi}^\tau_y\|_{H^1}\|\tilde{\xi}^\tau\| + c\|\hat{\xi}_y\|_{H^1}\|\tilde{\Psi}^\tau_x\|_{H^1}\|\tilde{\xi}^\tau\| \\
&\leq c\|\hat{\xi}\|_{H^2}\|\tilde{\xi}^\tau\|^2 \leq c\|\tilde{\xi}^\tau\|^2 + c\|\hat{\xi}\|^2_{H^2}\|\tilde{\xi}^\tau\|^2.
\end{aligned}
$$

Similarly, $|(J(\Psi,\theta) - J(\hat{\Psi},\hat{\theta}),\tilde{\theta}^\tau)| \leq c\|\tilde{\theta}^\tau\|^2 + \|\hat{\theta}\|^2_{H^2}\|\tilde{\xi}^\tau\|^2$. The Young inequality gives

$$
\begin{aligned}
(5.9)\qquad & (f(t+\tau) - \hat{f},\tilde{\xi}^\tau) + (g(t+\tau) - \hat{g},\tilde{\theta}^\tau) - \frac{R}{P}(\tilde{\theta}^\tau_x,\tilde{\xi}^\tau) \\
&\leq \frac{1}{P}\|\nabla\tilde{\theta}^\tau\|^2 + c(\|\tilde{\xi}^\tau\|^2 + \|\tilde{\theta}^\tau\|^2) + c\|f(t+\tau) - \hat{f}\|^2 + c\|g(t+\tau) - \hat{g}\|^2.
\end{aligned}
$$

We substitute (5.8)-(5.9) into (5.7) to find that

$$
\begin{aligned}
\frac{d}{dt}(\|\tilde{\xi}^\tau\|^2 + \|\tilde{\theta}^\tau\|^2) \leq & c(1 + \|\hat{\xi}\|^2_{H^2} + \|\hat{\theta}\|^2_{H^2})(\|\tilde{\xi}^\tau\|^2 + \|\tilde{\theta}^\tau\|^2) \\
& + c\|f(t+\tau) - \hat{f}\|^2 + c\|g(t+\tau) - \hat{g}\|^2.
\end{aligned}
$$

Applying the Gronwall inequality over $(0,t)$, we have

$$
\begin{aligned}
& \|\tilde{\xi}^\tau(t)\|^2 + \|\tilde{\theta}^\tau(t)\|^2 \\
\leq & e^{I_5(t)}\Big(\|\tilde{\xi}^\tau(0)\|^2 + \|\tilde{\theta}^\tau(0)\|^2 + \int_0^\infty (\|f(r+\tau) - \hat{f}\|^2 + \|g(r+\tau) - \hat{g}\|^2)dr\Big)
\end{aligned}
$$

where, by the condition **C**, the last integral tends to zero as $\tau \to +\infty$. By using [**10**, Lemma 4.2] for the autonomous equation, we know

$$
I_5(t) := c\int_0^t (1 + \|\hat{\xi}(r)\|^2_{H^2} + \|\hat{\theta}(r)\|^2_{H^2})dr < \infty,\ \forall t \geq 0.
$$

Note that $\|\tilde{\xi}^\tau(0)\|^2 + \|\tilde{\theta}^\tau(0)\|^2 \to 0$ as $\tau \to +\infty$. Therefore, $\|\tilde{\xi}^\tau(t)\|^2 + \|\tilde{\theta}^\tau(t)\| \to 0$ as $\tau \to +\infty$ for each $t \geq 0$. □

## 6. Forward-uniform estimates for the non-autonomous equation

**6.1. Further hypotheses.** In order to give longtime estimates, we assume the usual tempered condition, see [**22, 27, 29, 37**].

**Hypothesis T.** $f, g \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$ and they are **tempered**:

$$\int_{-\infty}^{t} e^{\kappa(r-t)}(\|f(r)\|^2 + \|g(r)\|^2)dr < \infty, \quad \forall t \in \mathbb{R},$$

where $\kappa := \min(\frac{\lambda}{2P}, \frac{\lambda}{2})$ and $\lambda > 0$ is the Poincaré constant such that $\lambda\|u\|^2 \leq \|\nabla u\|^2$ for all $u \in H_0^1(\Omega)$.

We can say more from both hypotheses **C** and **T**.

LEMMA 6.1. *Under the hypotheses* **C, T**, *both* $f, g$ *are* **forward tempered***:*

$$I(t) := 1 + \sup_{s \geq t} \int_{-\infty}^{s} e^{\kappa(r-s)}(\|f(r)\|^2 + \|g(r)\|^2)dr < \infty, \quad \forall t \in \mathbb{R}. \tag{6.1}$$

PROOF. The proof is completely similar to Lemma 4.1. □

LEMMA 6.2. *Under the hypotheses* **C, T**, *both* $f, g$ *are* **forward tail-small***: for all* $t \in \mathbb{R}$,

$$\lim_{k\to\infty} \sup_{s\geq t} \int_{-\infty}^{s} e^{\kappa(r-s)} \int_{\Omega(|y|\geq k)} (|f(r,x,y)|^2 + |g(r,x,y)|^2)dxdydr = 0.$$

PROOF. By the condition **T** and the Lebesgue controlled convergence theorem,

$$\begin{aligned} I_3(k,t) : &= \sup_{s\geq t} \int_{-\infty}^{t} e^{\kappa(r-s)} \int_{\Omega(|y|\geq k)} |f(r,x,y)|^2 dxdydr \\ &= \int_{-\infty}^{t} e^{\kappa(r-t)} \int_{\Omega(|y|\geq k)} |f(r,x,y)|^2 dxdydr \to 0 \end{aligned}$$

as $k \to \infty$ for each $t \in \mathbb{R}$. On the other hand,

$$\begin{aligned} I_4(k,t) : &= \sup_{s\geq t} \int_{t}^{s} e^{\kappa(r-s)} \int_{\Omega(|y|\geq k)} |f(r,x,y)|^2 dxdydr \\ &\leq 2\int_{t}^{\infty} \int_{\Omega(|y|\geq k)} |f(r) - \hat{f}|^2 dxdydr + \frac{2}{\kappa} \int_{\Omega(|y|\geq k)} |\hat{f}|^2 dxdy. \end{aligned}$$

The Lebesgue theorem gives $I_4(k,t) \to 0$ as $k \to \infty$. Therefore,

$$\sup_{s\geq t} \int_{-\infty}^{s} e^{\kappa(r-s)} \int_{\Omega(|y|\geq k)} |f(x,y,r)|^2 dxdydr \leq I_3(k,t) + I_4(k,t) \to 0$$

as $k \to \infty$. Similarly, $g$ is forward tail-small. □

**6.2. Forward-uniform absorption.** We start at the Lebesgue space.

LEMMA 6.3. *Let the hypotheses* **C, T** *be satisfied. Then, for each* $B \in \mathfrak{B}(L^2(\Omega)^2)$ *and* $t \in \mathbb{R}$, *there is a* $\tau_1 = \tau_1(B,t) > 1$ *such that*

$$\sup_{s\geq t}(\|\xi(s, s-\tau, \xi_0)\|^2 + \|\theta(s, s-\tau, \theta_0)\|^2) \leq cI(t), \tag{6.2}$$

$$\sup_{s\geq t} \int_{s-\tau}^{s} e^{\kappa(r-s)}(\|\nabla\xi(r, s-\tau, \xi_0)\|^2 + \|\nabla\theta(r, s-\tau, \theta_0)\|^2)dr \leq cI(t), \tag{6.3}$$

$$\sup_{s\geq t} \int_{s-\tau}^{s} e^{\kappa(r-s)}(\|\xi(r, s-\tau, \xi_0)\|^{2m} + \|\theta(r, s-\tau, \theta_0)\|^{2m})dr \leq cI^m(t) \tag{6.4}$$

*for all* $\tau \geq \tau_1$, $(\xi_0, \theta_0) \in B$ *and* $m \geq 1$. *The function* $I(t)$ *is given in* (6.1).

PROOF. By the inner product of the third equation in (5.1) with $\theta$,

$$\frac{1}{2}\frac{d}{dr}\|\theta\|^2 + \frac{1}{P}\|\nabla\theta\|^2 + (J(\Psi,\theta),\theta) = (g(r),\theta).$$

By (5.3), $(J(\Psi,\theta),\theta) = 0$. So, by the Poincaré inequality, we obtain

$$\frac{d}{dr}\|\theta\|^2 + \kappa\|\theta\|^2 + \frac{1}{P}\|\nabla\theta\|^2 \leq c\|g(r)\|^2. \tag{6.5}$$

where we recall that $\kappa := \min(\frac{\lambda}{2P}, \frac{\lambda}{2})$. For each $s \leq t$ with $t$ fixed, we apply the Gronwall lemma over $(s-\tau, \sigma)$ to see

$$\begin{aligned} &\|\theta(\sigma, s-\tau)\|^2 + \int_{s-\tau}^{\sigma} e^{\kappa(r-\sigma)}\|\nabla\theta(r, s-\tau)\|^2 dr \\ \leq & ce^{-\kappa(\sigma-s+\tau)}\|\theta_0\|^2 + c\int_{s-\tau}^{\sigma} e^{\kappa(r-\sigma)}\|g(r)\|^2 dr, \ \forall \sigma \geq s-\tau. \end{aligned} \tag{6.6}$$

On the other hand, by multiplying the first equation in (5.1) with $\xi$ and by the same method as in (6.5), we obtain

$$\frac{d}{dr}\|\xi\|^2 + \kappa\|\xi\|^2 + \|\nabla\xi\|^2 \leq c\|\nabla\theta\|^2 + c\|f(r)\|^2.$$

The Gronwall inequality gives

$$\begin{aligned} \|\xi(\sigma, s-\tau)\|^2 &+ \int_{s-\tau}^{\sigma} e^{\kappa(r-\sigma)}\|\nabla\xi(r, s-\tau)\|^2 dr \leq ce^{-\kappa(\sigma-s+\tau)}\|\xi_0\|^2 \\ &+ c\int_{s-\tau}^{\sigma} e^{\kappa(r-\sigma)}\|\nabla\theta(r, s-\tau)\|^2 dr + c\int_{s-\tau}^{\sigma} e^{\kappa(r-\sigma)}\|f(r)\|^2 dr. \end{aligned} \tag{6.7}$$

Now, both (6.6) and (6.7) imply that for all $s \geq t$ and $\tau \geq 0$, $\sigma \geq s-\tau$,

$$\begin{aligned} &\|\xi(\sigma, s-\tau)\|^2 + \|\theta(\sigma)\|^2 + \int_{s-\tau}^{\sigma} e^{\kappa(r-\sigma)}(\|\nabla\xi(r)\|^2 + \|\nabla\theta(r)\|^2) dr \\ \leq & ce^{-\kappa(\sigma-s+\tau)}(\|\xi_0\|^2 + \|\theta_0\|^2) + c\int_{s-\tau}^{\sigma} e^{\kappa(r-\sigma)}(\|f(r)\|^2 + \|g(r)\|^2) dr. \end{aligned} \tag{6.8}$$

In particular, letting $\sigma = s$ in (6.8), we can take $\tau_0 > 1$ such that for all $\tau \geq \tau_0$, $s \geq t$ and $(\xi_0, \theta_0) \in B$,

$$\begin{aligned} &\|\xi(s, s-\tau)\|^2 + \|\theta(s)\|^2 + \int_{s-\tau}^{s} e^{\kappa(r-s)}(\|\nabla\xi(r)\|^2 + \|\nabla\theta(r)\|^2) dr \\ \leq & ce^{-\kappa\tau}(\|\xi_0\|^2 + \|\theta_0\|^2) + c\sup_{s\geq t}\int_{-\infty}^{s} e^{\kappa(r-s)}(\|f(r)\|^2 + \|g(r)\|^2) dr \leq cI(t), \end{aligned}$$

where $I(t)$ is finite as proved in Lemma 6.1. We have proved (6.2)-(6.3).

By (6.3) and the Poincaré inequality, it is easy to show (6.4) for $m = 1$. It suffices to show (6.4) for $m > 1$. We use (6.8) to obtain

$$\begin{aligned}
&\int_{s-\tau}^{s} e^{\kappa(\sigma-s)}(\|\xi(\sigma, s-\tau)\|^{2m} + \|\theta(\sigma, s-\tau)\|^{2m})d\sigma \\
\le & c(\|\xi_0\|^{2m} + \|\theta_0\|^{2m}) \sup_{s\ge t} \int_{s-\tau}^{s} e^{\kappa(\sigma-s)} e^{-m\kappa(\sigma-s+\tau)} d\sigma \\
&+ c \sup_{s\ge t} \int_{s-\tau}^{s} e^{\kappa(\sigma-s)} \Big( \int_{-\infty}^{\sigma} e^{\kappa(r-\sigma)}(\|f(r)\|^2 + \|g(r)\|^2)dr \Big)^m d\sigma \\
\le & \frac{ce^{-m\kappa\tau}}{\kappa(m-1)}(\|\xi_0\|^{2m} + \|\theta_0\|^{2m}) + \frac{c}{\kappa} I^m(t)
\end{aligned}$$

for all $s \ge t$. Letting $\tau \to +\infty$, we obtain (6.4) as desired. □

**6.3. Forward-uniform estimates in regular space.** We need a uniform Gronwall inequality (see [**19**]): If non-negative locally integrable functions $y, h_1$, $h_2$ satisfy $\frac{dy}{dr} \le h_1(r)y + h_2(r)$ for $r \in [s-\tau, s]$, then, for all $\sigma \in (0, \tau)$,

$$y(s) \le e^{\int_{s-\sigma}^{s} h_1(r)dr} \Big( \frac{1}{\sigma} \int_{s-\sigma}^{s} y(r)dr + \int_{s-\sigma}^{s} h_2(r)dr \Big).$$

Lemma 6.4. *Let $t, B$ and $\tau_1$ be the same as given in Lemma 6.3. We have*

$$\sup_{\tau\ge\tau_1} \sup_{s\ge t} \sup_{(\xi_0,\theta_0)\in B} (\|\nabla\xi(s, s-\tau, \xi_0)\|^2 + \|\nabla\theta(s, s-\tau, \theta_0)\|^2) \le cI(t)e^{I^2(t)}.$$

Proof. It follows from (5.1) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dr}(\|\nabla\xi\|^2 + \|\nabla\theta\|^2) + \|\Delta\xi\|^2 + \frac{1}{P}\|\Delta\theta\|^2 \\
=&(J(\Psi, \xi), \Delta\xi) + (J(\Psi, \theta), \Delta\theta) + \frac{R}{P}(\theta_x, \Delta\xi) - (f(r), \Delta\xi) - (g(r), \Delta\theta).
\end{aligned}$$

By the Nirenberg-Gagliardo inequality: $\|u\|_4^2 < c\|u\|_{H^1}\|u\|$,

$$\begin{aligned}
&(J(\Psi, \xi), \Delta\xi) = \int_\Omega \Psi_y \xi_x \Delta\xi dxdy - \int_\Omega \Psi_x \xi_y \Delta\xi dxdy \\
&\le \|\Psi_y\|_4 \|\xi_x\|_4 \|\Delta\xi\| + \|\Psi_x\|_4 \|\xi_y\|_4 \|\Delta\xi\| \\
&\le c\|\Psi_y\|^{\frac{1}{2}} \|\Psi_y\|_{H^1}^{\frac{1}{2}} \|\xi_x\|^{\frac{1}{2}} \|\xi_x\|_{H^1}^{\frac{1}{2}} \|\Delta\xi\| + c\|\Psi_x\|^{\frac{1}{2}} \|\Psi_x\|_{H^1}^{\frac{1}{2}} \|\xi_y\|^{\frac{1}{2}} \|\xi_y\|_{H^1}^{\frac{1}{2}} \|\Delta\xi\|, \\
&\le c\|\Psi\|_{H^2} \|\nabla\xi\|^{\frac{1}{2}} \|\Delta\xi\|^{\frac{3}{2}} \le c\|\xi\| \|\nabla\xi\|^{\frac{1}{2}} \|\Delta\xi\|^{\frac{3}{2}} \le \frac{1}{4}\|\Delta\xi\|^2 + c\|\xi\|^4 \|\nabla\xi\|^2.
\end{aligned}$$

Similarly, $(J(\Psi, \theta), \Delta\theta) \le \frac{1}{4P}\|\Delta\theta\|^2 + c\|\xi\|^4\|\nabla\theta\|^2$. By the Young inequality,

$$\begin{aligned}
&\frac{R}{P}(\theta_x, \Delta\xi) - (f(r), \Delta\xi) - (g(r), \Delta\theta) \\
\le & \frac{1}{4}\|\Delta\xi\|^2 + \frac{1}{4P}\|\Delta\theta\|^2 + c\|\nabla\theta\|^2 + c\|f(r)\|^2 + c\|g(r)\|^2.
\end{aligned}$$

From all above estimates, we obtain

$$\begin{aligned}
&\frac{d}{dr}(\|\nabla\xi\|^2 + \|\nabla\theta\|^2) \\
\le & c(1 + \|\xi\|^4)(\|\nabla\xi\|^2 + \|\nabla\theta\|^2) + c(\|f(r)\|^2 + \|g(r)\|^2).
\end{aligned} \tag{6.9}$$

For each $t \in \mathbb{R}$, applying the uniform Gronwall inequality to (6.9) over $(s-1, s)$ with $s \geq t$ to see

$$\|\nabla\xi(s, s-\tau, \xi_0)\|^2 + \|\nabla\theta(s, s-\tau, \theta_0)\|^2) \leq ce^{I_6(s)}(I_7(s) + I_8(s)),$$

where, by (6.3)-(6.4), for all $\tau \geq \tau_1 > 1$ and all $s \geq t$,

$$I_6(s) := c + c\int_{s-1}^{s} \|\xi(r, s-\tau, \xi_0)\|^4 dr \leq c(1 + I^2(t)),$$

$$I_7(s) := \int_{s-1}^{s} (\|\nabla\xi(r, s-\tau, \xi_0)\|^2 + \|\nabla\theta(r, s-\tau, \theta_0)\|^2)dr \leq cI(t),$$

$$I_8(s) := \int_{s-1}^{s} (\|f(r)\|^2 + \|g(r)\|^2) \leq cI(t).$$

The proof is complete. □

**6.4. Forward-uniform tail-estimates.** We use the square of a cut-off function to obtain forward uniform tail-estimates.

LEMMA 6.5. *Let the hypotheses* **C, T** *be satisfied. Then, for each bounded* $B \subset L^2(\Omega)^2$ *and* $t \in \mathbb{R}$, *we have*

$$\lim_{\tau,k\to+\infty} \sup_{s\geq t} \sup_{(\xi_0,\theta_0)\in B} \int_{\Omega(|y|\geq k)} (|\xi(s, s-\tau, \xi_0)|^2 + |\theta(s, s-\tau, \theta_0)|^2)dxdy = 0. \tag{6.10}$$

PROOF. Let $\rho_k(y) := \rho(\frac{y^2}{k^2})$ for $y \in \mathbb{R}$ and $k \geq 1$, where $\rho : [0, \infty) \to [0, 1]$ is smooth such that $\rho \equiv 0$ on $[0, 1]$ and $\rho \equiv 1$ on $[4, \infty)$. It is easy to show

$$\|\rho_{ky}\|_\infty \leq \frac{c}{k}, \ \|(\rho_k^2)_y\|_\infty \leq \frac{c}{k}, \quad \text{where } \rho_{ky} = \frac{d}{dy}\rho_k(y).$$

Multiplying the third equation in (5.1) with $\rho_k^2\theta$, we obtain

$$\begin{aligned} &\frac{d}{dr}\int_\Omega \rho_k^2|\theta|^2 dxdy + \frac{2}{P}\int_\Omega \rho_k^2|\nabla\theta|^2 dxdy \\ &= -2\int_\Omega \rho_k^2\theta J(\Psi, \theta)dxdy + 2\int_\Omega \rho_k^2\theta g(r)dxdy - \frac{2}{P}\int_\Omega (\rho_k^2)_y\theta_y\theta dxdy. \end{aligned} \tag{6.11}$$

We apply the Poincaré inequality on the function $\rho_k\theta$ to obtain

$$\frac{2}{P}\int_\Omega \rho_k^2|\nabla\theta|^2 \geq \frac{\lambda}{P}\int_\Omega \rho_k^2|\theta|^2 - \frac{c}{k}\|\theta\|^2. \tag{6.12}$$

On the other hand,

$$2\rho_k^2\theta J(\Psi, \theta) = 2\rho_k^2\theta(\Psi_y\theta_x - \Psi_x\theta_y) = \rho_k^2\Psi_y(\theta^2)_x - \rho_k^2\Psi_x(\theta^2)_y.$$

Then, the integral by part implies that

$$\begin{aligned} &-2\int_\Omega \rho_k^2\theta J(\Psi, \theta) = \int_\Omega \rho_k^2\Psi_{yx}\theta^2 - \int_\Omega \rho_k^2\Psi_{xy}\theta^2 - \int_\Omega (\rho_k^2)_y\Psi_x\theta^2 \\ &= -\int_\Omega (\rho_k^2)_y\Psi_x\theta^2 \leq \frac{c}{k}\|\Psi_x\|_4\|\theta\|_4\|\theta\| \leq \frac{c}{k}\|\Psi_x\|_{H^1}\|\theta\|_{H^1}\|\theta\| \\ &\leq \frac{c}{k}\|\xi\|(\|\theta\| + \|\nabla\theta\|)\|\theta\| \leq \frac{c}{k}\|\xi\|\|\nabla\theta\|\|\theta\| \leq \frac{c}{k}(\|\xi\|^4 + \|\theta\|^4 + \|\nabla\theta\|^2). \end{aligned} \tag{6.13}$$

The Young inequality deduces that

$$2\int_\Omega \rho_k^2\theta g(r) - \frac{2}{P}\int_\Omega (\rho_k^2)_y\theta_y\theta \leq \frac{\lambda}{4P}\int_\Omega \rho_k^2|\theta|^2 + c\int_\Omega \rho_k^2|g(r)|^2 + \frac{c}{k}\|\nabla\theta\|^2. \tag{6.14}$$

We substitute (6.12)-(6.14) into (6.11) to obtain

$$\begin{aligned}\frac{d}{dr}\int_\Omega \rho_k^2|\theta|^2+\kappa\int_\Omega \rho_k^2|\theta|^2+\frac{\lambda}{4P}\int_\Omega \rho_k^2|\theta|^2 \\ \le \frac{c}{k}(1+\|\xi\|^4+\|\theta\|^4+\|\nabla\theta\|^2)+c\int_\Omega \rho_k^2|g(r)|^2.\end{aligned} \tag{6.15}$$

Let $s\ge t$ with $t$ fixed. Then, by Lemma 6.3, the Gronwall inequality over $(s-\tau,s)$ (omitting the third term in (6.15)) yields

$$\int_\Omega \rho_k^2|\theta(s,s-\tau,\theta_0)|^2\le I_9(\tau,k)+I_{10}(s,\tau,k)+\frac{c}{k}I_{11}(s,\tau),$$

where, for all $k\ge 1$,

$$I_9(\tau,k):=e^{-\kappa\tau}\int_\Omega \rho_k^2|\theta_0|^2\le e^{-\kappa\tau}\|\theta_0\|^2\to 0 \quad \text{as } \tau\to\infty.$$

By Lemma 6.2, $g$ is forward tail-small and so

$$\begin{aligned}\sup_{s\ge t} I_{10}(s,\tau,k):=c\sup_{s\ge t}\int_{s-\tau}^s e^{\kappa(r-s)}\int_\Omega \rho_k^2|g(r)|^2dxdydr \\ \le c\sup_{s\ge t}\int_{-\infty}^s e^{\kappa(r-s)}\int_{\Omega(|y|\ge k)}|g(r)|^2dxdydr\to 0\end{aligned}$$

as $k\to\infty$ for all $\tau\ge 0$. By Lemma 6.3,

$$\begin{aligned}\sup_{s\ge t} I_{11}(s,\tau):=\sup_{s\ge t}\int_{s-\tau}^s e^{\kappa(r-s)}(1+\|\xi(r)\|^4+\|\theta(r)\|^4+\|\nabla\theta(r)\|^2) \\ \le c(I(t)+I^2(t))<+\infty.\end{aligned}$$

Therefore, as $\tau,k\to\infty$,

$$\sup_{s\ge t}\int_{\Omega(|y|\ge 2k)}|\theta(s,s-\tau,\theta_0)|^2\le \sup_{s\ge t}\int_\Omega \rho_k^2|\theta(s,s-\tau,\theta_0)|^2\to 0.$$

In addition, by the Gronwall inequality again (using the third term in (6.15)), it is easy to show that

$$\lim_{\tau,k\to+\infty}\sup_{s\ge t}\int_{s-\tau}^s e^{\kappa(r-s)}\int_\Omega \rho_k^2|\theta(r,s-\tau,\theta_0)|^2dxdydr=0. \tag{6.16}$$

We then take the inner product of the first equation of (5.1) with $\rho_k^2\xi$ to obtain

$$\begin{aligned}\frac{d}{dr}\int_\Omega \rho_k^2|\xi|^2dxdy+2\int_\Omega \rho_k^2|\nabla\xi|^2dxdy \\ =-2\int_\Omega \rho_k^2\xi J(\Psi,\xi)-\frac{2R}{P}\int_\Omega \rho_k^2\xi\theta_x+2\int_\Omega \rho_k^2\xi f(r)-2\int_\Omega (\rho_k^2)_y\xi_y\xi.\end{aligned} \tag{6.17}$$

By the Poincaré inequality on $\rho_k\xi$, it is similar as (6.12) to obtain that

$$2\int_\Omega \rho_k^2|\nabla\xi|^2dxdy\ge\int_\Omega \rho_k^2|\nabla\xi|^2+\frac{1}{2\lambda}\int_\Omega \rho_k^2|\xi|^2dxdy-\frac{c}{k}\|\xi\|^2. \tag{6.18}$$

By the same method as given in (6.13), we have

$$-2\int_\Omega \rho_k^2\xi J(\Psi,\xi)dxdy=-\int_\Omega (\rho_k^2)_y\Psi_x\xi^2dxdy\le\frac{c}{k}(1+\|\nabla\xi\|^2+\|\xi\|^4). \tag{6.19}$$

By the Young inequality,

$$-\frac{2R}{P}\int_\Omega \rho_k^2\xi\theta_x + 2\int_\Omega \rho_k^2\xi f(r) - 2\int_\Omega (\rho_k^2)_y\xi_y\xi \le \int_\Omega \rho_k^2|\nabla\xi|^2dxdy + c\int_{\Omega(|y|\ge k)}(|\theta|^2+|f(t)|^2)dxdy + \frac{c}{k}(\|\xi\|^2+\|\nabla\xi\|^2). \tag{6.20}$$

Substituting (6.18)-(6.20) into (6.17), we have

$$\frac{d}{dr}\int_\Omega \rho_k^2|\xi|^2 + \kappa\int_\Omega \rho_k^2|\xi|^2 \le \frac{c}{k}(1+\|\xi\|^4+\|\nabla\xi\|^2) + c\int_\Omega \rho_k^2(|\theta|^2+|f(r)|^2). \tag{6.21}$$

The Gronwall lemma gives, for all $s \ge t$,

$$\int_\Omega \rho_k^2|\xi(s,s-\tau,\xi_0)|^2 \le e^{-\lambda\tau}\|\xi_0\|^2 + \frac{c}{k}(I(t)+I^2(t)) + c\sup_{s\ge t}\int_{s-\tau}^s e^{\kappa(r-s)}\int_\Omega \rho_k^2|\theta(r)|^2 + c\sup_{s\ge t}\int_{-\infty}^s e^{\kappa(r-s)}\int_{\Omega(|y|\ge k)}|f(r)|^2,$$

which tends to zero as $\tau, k \to \infty$ in view of (6.16) and Lemma 6.2. So, (6.10) holds true for $\xi$. □

## 7. Forward compactness and convergence of kernel sections

In order to ensure that the kernel section is *nonempty*, we need a stronger condition than the tempered condition **T**.

**Hypothesis B.** $f, g \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$ are **backward tempered**:

$$\sup_{s\le s_0}\int_{-\infty}^s e^{\kappa(r-s)}(\|f(r)\|^2+\|g(r)\|^2)dr < \infty, \text{ for some } s_0 \in \mathbb{R}.$$

Now, we state and prove the main application results of this paper as follows.

THEOREM 7.1. *Let the hypotheses* **B** *and* **C** *be satisfied. Then, the kernel sections* $\mathcal{K}(\tau)$ *of non-autonomous Newton-Boussinesq system are nonempty, forward compact, uniformly bounded and pullback attracting in* $L^2(\Omega)^2$.

*Furthermore,* $\mathcal{K}$ *is attracted by the global attractor* $A$ *of the autonomous system:*

$$dist_{L^2(\Omega)^2}(\mathcal{K}(\tau), A) \to 0 \quad as\ \tau \to +\infty. \tag{7.1}$$

PROOF. By Proposition 5.1, the semigroup $T$ in (5.6) is a forward limit of the process $S$ in (5.4).

Let $I(t)$ be the function as given in (6.1) and

$$\mathcal{P}(t) := \{(u,v) \in L^2(\Omega)^2 : \|u\|^2+\|v\|^2 \le cI(t)\},\ \forall t \in \mathbb{R}.$$

By Lemma 6.3, $\mathcal{P}(\cdot)$ is an absorbing set. By Lemma 6.1, $\mathcal{P}(\cdot)$ is forward bounded. So, by the hypothesis **B**,

$$\sup_{t\in\mathbb{R}} I(t) = 1 + \sup_{t\in\mathbb{R}}\sup_{s\ge t}\int_{-\infty}^s e^{\kappa(r-s)}(\|f(r)\|^2+\|g(r)\|^2)dr \le I(s_0) + \sup_{s\le s_0}\int_{-\infty}^s e^{\kappa(r-s)}(\|f(r)\|^2+\|g(r)\|^2)dr < \infty,$$

which means that $\mathcal{P}(\cdot)$ is uniformly bounded.

It suffices to show that $S$ is forward asymptotically compact. Let $s_n \geq t$ with $t$ fixed, $\tau_n \to +\infty$ and $\{(\xi_{0,n}, \theta_{0,n})\}$ is a bounded sequence in $L^2(\Omega)^2$. We consider the sequence given by

$$(u_n, v_n) := S(s_n, s_n - \tau_n)(\xi_{0,n}, \theta_{0,n}) = (\xi(s_n, s_n - \tau_n, \xi_{0,n}),\ \theta(s_n, s_n - \tau_n, \theta_{0,n})).$$

By Lemma 6.3, $\{(u_n, v_n)\}$ is bounded in $L^2(\Omega)^2$, and thus there is $(u, v) \in L^2(\Omega)^2$ such that, up to a subsequence, $(u_n, v_n) \to (u, v)$ weakly in $L^2(\Omega)^2$. By Lemma 6.5, for each $\varepsilon > 0$, there are $k \in \mathbb{N}$ and $N_1 \in \mathbb{N}$ such that

$$\|(u_n, v_n)\|_{L^2(Q_k^c)^2} \leq \frac{\varepsilon}{3}, \quad \forall n \geq N_1 \quad \text{and} \quad \|(u, v)\|_{L^2(\Omega_k^c)^2} \leq \frac{\varepsilon}{3}, \tag{7.2}$$

where $\Omega_k^c = \Omega \setminus \Omega_k$ and $\Omega_k := \{(x, y) \in \Omega, |y| < k\}$. By Lemma 6.4, $\{(u_n, v_n)\}$ is bounded in $H^1(\Omega)^2$. So, by the Sobolev compact embedding, the subsequence $\{(u_n, v_n)\}$ is pre-compact in $L^2(\Omega_k)^2$. Passing to a subsequence again, there is a $N_2 \geq N_1$ such that

$$\|(u_n, v_n) - (u, v)\|_{L^2(\Omega_k)^2} \leq \frac{\varepsilon}{3}, \quad \forall n \geq N_2, \tag{7.3}$$

where the strong-limit must equal to the weak-limit. Both (7.2) and (7.3) imply that for all $n \geq N_2$,

$$\begin{aligned} \|(u_n, v_n) - (u, v)\|_{L^2(\Omega)^2} &\leq \|(u_n, v_n) - (u, v)\|_{L^2(\Omega_k)^2} + \|(u_n, v_n) - (u, v)\|_{L^2(\Omega_k^c)^2} \\ &\leq \frac{\varepsilon}{3} + \|(u_n, v_n)\|_{L^2(Q_k^c)^2} + \|(u, v)\|_{(L^2(\Omega_k^c))^2} \leq \varepsilon. \end{aligned}$$

Therefore, $S$ is forward-pullback asymptotically compact, and so Theorem 3.4 can be applied to obtain all desired conclusions. □

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School of Mathematics and Statistics, Southwest University, Chongqing 400715, China

*Email address*: `a624716519@email.swu.edu.cn (R.H. Wang)`

School of Mathematics and Statistics, Southwest University, Chongqing 400715, China

*Email address*: `liyr@swu.edu.cn (Y.R. Li)`