Dynamics of PDE, Vol.16, No.4, 359-382, 2019

Predual forms, harmonic maps &liquid crystals of (BMO-Q) & (BMO-Q)$^{-1}$

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Communicated by Y. Charles Li, received June 25, 2019.

ABSTRACT. Under $(\alpha, n-1) \in [0,1) \times \mathbb{N}$ this paper explores the fractional Sobolev type inclusion and the Fefferman-Stein type decomposition of the predual forms (unifying ones in [**5**] under $\alpha = 0$ & [**2**] under $\alpha \in (0,1)$) of the so-called $(BMO-Q)\,\&\,(BMO-Q)^{-1}$ spaces

$$(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n) \ \ \& \ \ ((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^{-1} = \mathrm{div}((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^n$$

and their natural actions on revealing

$$(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n) \ \ \& \ \ ((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^{-1}$$

analogues of the global results in [**26**] about the heat flow of harmonic maps & the hydrodynamic flow of nematic liquid crystals.

CONTENTS

1. Introduction

1.1. Describing $(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$ and its predual. To begin with, let us recall the following concept.

DEFINITION 1.1. Let $(\alpha, \lambda, p, n) \in (-\infty,\infty) \times (-\infty,\infty) \times [1,\infty) \times \mathbb{N}$.

1991 *Mathematics Subject Classification.* Primary 35Q30, 42B37, 46E35; Secondary 30H25.

Key words and phrases. BMO-Q, heat flow of harmonic maps, hydrodynamic flow of liquid crystals.

JX was supported by NSERC of Canada (#20171864); JZ was supported by the Science Foundation of Hebei Normal University of China (L2019B02) and Hebei Natural Science Foundation of China (A2019205218) and NSERC of Canada (#20171864).

▷ $(-\Delta)^{\frac{\alpha}{2}}$ is the $\frac{\alpha}{2}$-th power of the standard Laplacian operator $-\Delta$ - in other words - the following Fourier transform formula

$$\widehat{(-\Delta)^{\frac{\alpha}{2}} f}(\xi) = |\xi|^{\alpha} \hat{f}(\xi)$$

holds for any function or tempered distribution for which the right-hand-side is meaningful. In particular, $(-\Delta)^0$ is defined as the identity; and if $0 < \alpha < 2$ then there are two constants $c_{1,\alpha}, c_{2,\alpha}$ such that

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = c_{1,\alpha} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+\alpha}}\, dy \ \ \& \ \ (-\Delta)^{-\frac{\alpha}{2}} f(x) = c_{2,\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|y-x|^{n-\alpha}}\, dy$$

hold for any regular enough function f on $\mathbb{R}^n$.

▷ $\mathscr{L}^{p,\lambda}(\mathbb{R}^n)$ and $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ express respectively the (p,λ)- Campanato space and the (p,λ)-Morrey space whose norms are decided respectively by (cf. [**6, 20**])

$$\begin{cases} \|f\|_{\mathscr{L}^{p,\lambda}(\mathbb{R}^n)} = \sup_{(r_0,x_0)\in\mathbb{R}^{1+n}_+} \left(r_0^{\lambda} ⨍_{B(x_0,r_0)} \left| f(y) - ⨍_{B(x_0,r_0)} f(x)\,dx \right|^p dy \right)^{\frac{1}{p}}; \\ \|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} = \sup_{(r_0,x_0)\in\mathbb{R}^{1+n}_+} \left(r_0^{\lambda} ⨍_{B(x_0,r_0)} |f(y)|^p\, dy \right)^{\frac{1}{p}}, \end{cases}$$

where $\mathbb{R}^{1+n}_+ = (0,\infty)\times\mathbb{R}^n$ is the $(1+n)$-dimensional half space based on $\mathbb{R}^n$, $B(x_0,r_0)$ stands for the Euclidean ball with centre $x_0 \in \mathbb{R}^n$ and radius $r_0 \in (0,\infty)$, and $⨍_{B(x_0,r_0)}$ is the integral average over $B(x_0,r_0)$ with respect to the Lebesgue measure element dx.

▷ Below are a group of the well-known inclusions (cf. [**9, 6, 24, 29**])

$$\begin{cases} L^{\infty}(\mathbb{R}^n) = \mathcal{L}^{p,0}(\mathbb{R}^n) \subsetneq \mathscr{L}^{p,0}(\mathbb{R}^n) = BMO(\mathbb{R}^n); \\ L^{\frac{n}{\alpha}}(\mathbb{R}^n) \subsetneq \mathcal{L}^{2,2\alpha}(\mathbb{R}^n) \subsetneq \mathscr{L}^{2,2\alpha}(\mathbb{R}^n) \ \text{ as } \ 0 < 2\alpha < n, \end{cases}$$

and a $(BMO-Q)$ identification (cf. [**30, 31**])

$$\begin{cases} (-\Delta)^{-\frac{\alpha}{2}} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n) = \begin{cases} BMO(\mathbb{R}^n) & \text{as } \ \alpha = 0; \\ Q_{\alpha}(\mathbb{R}^n) & \text{as } \ 0 < \alpha < 1, \end{cases} \\ \|\cdot\|_{(-\Delta)^{-\frac{\alpha}{2}} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n)} = \begin{cases} \|\cdot\|_{BMO(\mathbb{R}^n)} & \text{as } \ \alpha = 0; \\ \|\cdot\|_{Q_{\alpha}(\mathbb{R}^n)} & \text{as } \ 0 < \alpha < 1, \end{cases} \end{cases}$$

where a measurable function f is in $Q_{\alpha}(\mathbb{R}^n)$ if and only if

$$\|f\|_{Q_{\alpha}(\mathbb{R}^n)} = \sup_{(r_0,x_0)\in\mathbb{R}^{1+n}_+} \left(r_0^{n+2\alpha} ⨍_{B(x_0,r_0)} ⨍_{B(x_0,r_0)} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}}\, dxdy \right)^{\frac{1}{2}} < \infty,$$

see also [**4**]. Note that if c is constant then

$$\|c\|_{\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)} = 0 = \|c\|_{Q_{\alpha}(\mathbb{R}^n)}.$$

So, all the constants form an equivalent class, i.e., the zero element in $\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$ (including $BMO(\mathbb{R}^n)$) and $Q_{\alpha}(\mathbb{R}^n)$, and consequently the above identification via $(-\Delta)^{-\frac{\alpha}{2}}$ makes sense for the involved quotient spaces:

$$\begin{cases} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n) & \text{as } \ \alpha \in (0,1); \\ BMO(\mathbb{R}^n) = \mathscr{L}^{2,2\alpha}(\mathbb{R}^n) & \text{as } \ \alpha = 0; \\ Q_{\alpha}(\mathbb{R}^n) & \text{as } \ \alpha \in (0,1). \end{cases}$$

▷ $[0,1) \ni \alpha \mapsto (-\Delta)^{-\frac{\alpha}{2}} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$ is decreasing and

$$\begin{cases} L^{\infty}(\mathbb{R}^n) \subseteq (-\Delta)^{-\frac{\alpha}{2}} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n) \Leftrightarrow \alpha = 0; \\ (-\Delta)^{-\frac{\alpha}{2}} L^{\frac{n}{\alpha}}(\mathbb{R}^n) \subsetneq (-\Delta)^{-\frac{\alpha}{2}} \mathcal{L}^{2,2\alpha}(\mathbb{R}^n) \subsetneq (-\Delta)^{-\frac{\alpha}{2}} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n) \ \text{ as } \ 0 < 2\alpha < n, \end{cases}$$

Moreover, if $n \ge 2$ then (cf. [**29**, Theorem 4.1] & [**31**, Theorem 3.1])

$$\dot{W}^{1,n}(\mathbb{R}^n) \subseteq CIS(\mathbb{R}^n) \subsetneq Q_{0<\alpha<1}(\mathbb{R}^n) \subsetneq BMO(\mathbb{R}^n), \tag{1.1}$$

where $\dot{W}^{1,n}(\mathbb{R}^n)$ and $CIS(\mathbb{R}^n)$ represent the conformally-invariant-Sobolev spaces of all $f \in C^1(\mathbb{R}^n)$ (continuously differentiable functions in $\mathbb{R}^n$) with

$$\|f\|_{\dot{W}^{1,n}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\nabla f(x)|^n \, dx\right)^{\frac{1}{n}} < \infty$$

and

$$\|f\|_{CIS(\mathbb{R}^n)} = \sup_{(r_0,x_0)\in\mathbb{R}^{1+n}_+} \left(r_0^2 \fint_{B(x_0,r_0)} |\nabla f(x)|^2 \, dx\right)^{\frac{1}{2}} < \infty$$

respectively.

Clearly, the above structure is an important motivation to explore $(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$ and its applications in many other areas including harmonic analysis and partial differential equations; see also [**2, 3, 8, 12, 27, 28, 33, 32, 35, 37, 34, 14, 15, 36**].

The first purpose of this paper is to present an idea of revealing the predual form of every single space $(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$ and its fractional Sobolev type imbedding of independent interest.

THEOREM 1.2. *Let $(\alpha, n-1) \in [0,1) \times \mathbb{N}$. Then:*

(i) *The space duality*

$$(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n) = [HH^1_{-\alpha}(\mathbb{R}^n)]^*$$

holds in the sense that if $f \in (-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$ then the functional

$$g \mapsto \int_{\mathbb{R}^n} f(x)g(x)\, dx$$

(initially for all $g \in HH^1_{-\alpha}(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n)$) has a bounded linear extension to all $g \in HH^1_{-\alpha}(\mathbb{R}^n)$. Conversely, if L is a bounded linear functional on the α-Hardy-Hausdorff space α $HH^1_{-\alpha}(\mathbb{R}^n)$ (cf. Definition 2.1 below) then there is $f \in (-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$ such that

$$L(g) = \int_{\mathbb{R}^n} f(x)g(x)\, dx \ \ \forall \ g \in HH^1_{-\alpha}(\mathbb{R}^n).$$

(ii) *The fractional Sobolev type inclusion*

$$(-\Delta)^{-\frac{\alpha}{2}} HH^1_{-\alpha}(\mathbb{R}^n) \subseteq \mathcal{L}^{\frac{n}{n-\alpha},n}(\mathbb{R}^n) = L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$$

is valid.

Here it should be noticed that the cases $\alpha = 0$ & $\alpha \in (0,1)$ of Theorem 1.2(i) are essentially due to Fefferman-Stein [**5**] (showing that the dual of the Hardy space $H^1(\mathbb{R}^n)$ is $BMO(\mathbb{R}^n)$) and Dafni-Xiao [**2**] (showing that the dual of $HH^1_{-\alpha}(\mathbb{R}^n)$ is $Q_\alpha(\mathbb{R}^n)$) respectively, and Theorem 1.2(ii) is a somewhat surprising consequence of Theorem 1.2(i), not only strengthening the Stein-Weiss Sobolev type inclusion (cf. [**25**])

$$(-\Delta)^{-\frac{\alpha}{2}} H^1(\mathbb{R}^n) \subseteq L^{\frac{n}{n-\alpha}}(\mathbb{R}^n) \ \text{ due to } \ H^1(\mathbb{R}^n) \subseteq HH^1_{-\alpha}(\mathbb{R}^n).$$

but also corresponding to Schikorra-Spector-Schaftingen's [**23**, Theorem A]

$$\|(-\Delta)^{-\frac{\alpha}{2}} f\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \lesssim \|R(f)\|_{L^1(\mathbb{R}^n)} \lesssim \|f\|_{H^1(\mathbb{R}^n)} \ \ \forall \ f \in H^1(\mathbb{R}^n),$$

where

$$\begin{aligned}R(f) &= (R_1(f), \cdots, R_n(f))\\ &= \left(\Gamma\Big(\frac{n+1}{2}\Big)\pi^{\frac{n+1}{2}}\,\text{p.v.}\int_{\mathbb{R}^n}\frac{(x_j-y_j)f(y)}{|x-y|^{n+1}}\,dy\right)_{j=1}^{n}\\ &= \left(\frac{\Gamma\big(\frac{n+1}{2}\big)}{(1-n)\pi^{\frac{n+1}{2}}c_{2,1}}\right)\nabla(-\Delta)^{-\frac{1}{2}}f.\end{aligned}$$

1.2. Characterizing $((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^{-1}$ and its predual. Below is a natural notion.

DEFINITION 1.3. Let $(\alpha, n) \in [0, 1) \times \mathbb{N}$.

▷ $f \in ((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^{-1}$ provided

$$\exists\ f_1, \cdots, f_n \in (-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$$

such that

$$f(x) = \text{div}\big(f_1(x), \cdots, f_n(x)\big) = \sum_{j=1}^{n}\partial_{x_j}f_j(x) \quad \forall \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n.$$

Moreover,

$$\|f\|_{\big((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\big)^{-1}} = \inf\left\{\sum_{j=1}^{n}\|f_j\|_{(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)}\right\},$$

where the infimum ranges over all

$$f = \text{div}(f_1, \cdots, f_n)$$

enjoying

$$\begin{aligned}(f_1, \cdots, f_n) &\in \underbrace{(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n) \times \ldots \times (-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)}_{n\ copies}\\ &= ((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^n.\end{aligned}$$

▷ As $(BMO - Q)^{-1}$ space chain,

$$[0, 1) \ni \alpha \mapsto ((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^{-1}$$

is decreasing, and if $n \geq 2$ then

$$L^n(\mathbb{R}^n) \subseteq \mathcal{L}^{2,2}(\mathbb{R}^n) \subsetneq ((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^{-1} \subseteq BMO^{-1}(\mathbb{R}^n) = \text{div}(BMO(\mathbb{R}^n))^n.$$

The second purpose of this paper is to use Theorem 1.2 to reveal that $(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$ has a Fefferman-Stein type decomposition (cf. [**5**]) and yet $((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^{-1}$ exists as the dual to the Hardy-Hausdorff-Sobolev space $HHS^{1}_{-\alpha}$ which, under $\alpha = 0$, embedds into the homogeneous Sobolev space $\dot{W}^{1,1}(\mathbb{R}^n)$ (defined as the closure of $C_0^\infty(\mathbb{R}^n)$-functions f with respect to the norm $\int_{\mathbb{R}^n}|\nabla f(x)|\,dx$) whose duality (cf. [**22**, Lemma 4.1]) can be identified with

$$(L^\infty(\mathbb{R}^n))^{-1} = \text{div}(L^\infty(\mathbb{R}^n))^n.$$

THEOREM 1.4. *Let* $(\alpha, n-1) \in [0, 1) \times \mathbb{N}$. *Then:*

(i)

$$(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n) + \sum_{j=1}^n R_j(L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))$$

holds in the sense that

$$f\in(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$$

if and only if

$$\exists\ g_0,\cdots,g_n\in L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$$

such that

$$f = g_0+\sum_{j=1}^n R_jg_j = g_0+\left(\frac{\Gamma\left(\frac{n+1}{2}\right)}{(1-n)\pi^{\frac{n+1}{2}}c_{2,1}}\right)(-\Delta)^{-\frac{1}{2}}\nabla\cdot(g_1,\cdots,g_n).$$

(ii)

$$((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^{-1} = [HHS^{\,1}_{-\alpha}(\mathbb{R}^n)]^*$$

provided that $HHS^{\,1}_{-\alpha}(\mathbb{R}^n)$ is the Hardy-Hausdorff-Sobolev space of all functions f on $\mathbb{R}^n$ satisfying

$$\nabla f\in\underbrace{HH^1_{-\alpha}(\mathbb{R}^n)\times\cdots\times HH^1_{-\alpha}(\mathbb{R}^n)}_{n\ copies} = (HH^1_{-\alpha}(\mathbb{R}^n))^n.$$

Actually, the cases $\alpha = 0$ and $\alpha\in(0,1)$ of Theorem 1.4(i) are due to Fefferman-Stein [**5**] and Yang-Qian-Li [**35**] (extending Nicolau-Xiao's one-dimensional-result [**21**]) respectively - moreover - according to the inclusion chain (1.1), Theorem 1.4(i) which especially implies

$$(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\supseteq\sum_{j=1}^n R_j(L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)),$$

and the following Bourgain-Brezis decomposition under $n\geq 2$ (cf. [**1**, p.305])

$$\dot W^{1,n}(\mathbb{R}^n) = \sum_{j=1}^n R_j(L^\infty(\mathbb{R}^n)\cap\dot W^{1,n}(\mathbb{R}^n))\ \text{ with }\ \dot W^{1,2}(\mathbb{R}^2) = CIS(\mathbb{R}^2)$$

we conjecture that any

$$X\in\left\{CIS(\mathbb{R}^{n>2}),(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^{n\geq 2})\right\}$$

is an answer of the Bourgain-Brezis question (cf. [**1**, p.396]):

What are the function spaces X, $\dot W^{1,n}(\mathbb{R}^n)\subseteq X\subseteq BMO(\mathbb{R}^n)$, such that every $F\in X$ has a decomposition $F = \sum_{j=1}^n R_jY_j$ where $Y_j\in L^\infty(\mathbb{R}^n)\cap X$ and $n\geq 2$?

But nevertheless Theorem 1.4(ii) is new and useful, at least thanks to some applications of both $BMO^{-1}(\mathbb{R}^n)$ and $Q^{-1}_\alpha(\mathbb{R}^n)$ in studying the Navier-Stokes systems (see e.g. [**11, 30, 31, 13, 8, 33, 32**]).

1.3. Applied to heat flow of harmonic maps. Being inspired by [**19, 26**], we recall the following concept.

DEFINITION 1.5. Let $(n,k) \in \mathbb{N}\times\mathbb{N}$, $\mathcal{N}$ be a k-dimensional compact Riemannian manifold without boundary, isometrically embedded in some Euclidean space $\mathbb{R}^l$, T_x be the tangent plane at $x\in\mathcal{N}$, and

$$\mathrm{II}(x):\ T_x\mathcal{N}\times T_x\mathcal{N}\to (T_x\mathcal{N})^{\perp}$$

be the second fundamental form of $\mathcal{N}\subseteq\mathbb{R}^l$ at $x\in\mathcal{N}$. Then the system of heat flow of harmonic maps u from $\mathbb{R}^{1+n}_+ = (0,\infty)\times\mathbb{R}^n$ to $\mathcal{N}$ is:

$$\begin{cases} u_t - \Delta u = \mathrm{II}(u)(\nabla u,\nabla u) & \text{in } \mathbb{R}^{1+n}_+;\\ u|_{t=0} = u_0 & \text{in } \mathbb{R}^n. \end{cases} \tag{1.2}$$

The third purpose of this paper is to utilize $(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$ and some ideas developed in [**33, 32**] to improve the well-posedness for the heat flow of harmonic maps within $BMO(\mathbb{R}^n)$.

THEOREM 1.6. *Let $(\alpha,k,n-1)\in[0,1)\times\mathbb{N}\times\mathbb{N}$ and $\mathcal{N}$ be a k-dimensional compact Riemannian manifold without boundary, isometrically embedded in some Euclidean space $\mathbb{R}^l$. There exists $\varepsilon>0$ such that if $u_0:\mathbb{R}^n\to\mathcal{N}$ obeys*

$$\|u_0\|_{\left((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\right)^l}\le\varepsilon$$

then (1.2) has a unique mild solution

$$u=(u_1,\cdots,u_l)\in X^l_\alpha(\mathbb{R}^{1+n}_+)$$

with

$$|||u|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)}=\sum_{j=1}^{l}\|u_j\|_{L^\infty(\mathbb{R}^{1+n}_+)}+\sum_{j=1}^{l}\|u_j\|_{X_\alpha(\mathbb{R}^{1+n}_+)}\lesssim\varepsilon$$

and

$$\|u_j\|_{X_\alpha(\mathbb{R}^{1+n}_+)}=\sup_{(r,x)\in\mathbb{R}^{1+n}_+}\sqrt{r}|\nabla u_j(r,x)|+\sup_{(r,x)\in\mathbb{R}^{1+n}_+}\left(\int_0^{r^2}\int_{B(x,r)}|\nabla u_j(t,y)|^2\left(\frac{r^2}{t}\right)^{\alpha}\frac{dydt}{r^n}\right)^{\frac12}.$$

The case $\alpha=0$ of Theorem 1.6 is due to Wang [**26**] (cf. Koch-Lamm's survey [**10**]). However, it is worth mentioning that $Q_{0<\alpha<1}(\mathbb{R}^n)$ is more regular than $BMO(\mathbb{R}^n)$ and so $X^l_{0<\alpha<1}(\mathbb{R}^{1+n}_+)$ is more regular than $X^l_0(\mathbb{R}^{1+n}_+)$ - in other words - $\alpha=0$ of Theorem 1.6 cannot directly derive $\alpha\in(0,1)$ of Theorem 1.6.

1.4. Applied to flow of liquid crystals. Stemming from a understanding of the well-posedness for the liquid crystal flow (coupling the incompressible Navier-Stokes system and the transported heat flow of harmonic maps into $\mathbb{S}^2$) investigated initially by Lin-Liu [**17, 18**], Lin-Lin-Wang [**16**] and Wang [**26**], we review the following notion.

DEFINITION 1.7. Let $n-1\in\mathbb{N}$ and $\mathrm{p}:\mathbb{R}^n\to\mathbb{R}$ be a pressure and $\mathrm{d}:\mathbb{R}^n\to\mathbb{S}^2$ (the unit sphere of $\mathbb{R}^3$) be a unit-vector field representing the macroscopic molecular orientation of the nematic liquid crystal material. Then the system generated by an incompressible hydrodynamic flow of nematic liquid crystals is

$$\begin{cases} u_t + u\cdot\nabla u - \Delta u + \nabla\mathrm{p} = -\nabla\cdot(\nabla\mathrm{d}\otimes\nabla\mathrm{d}) & \text{in } \mathbb{R}^{1+n}_+;\\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^{1+n}_+;\\ \mathrm{d}_t + u\cdot\nabla\mathrm{d} - \Delta\mathrm{d} = |\nabla\mathrm{d}|^2\mathrm{d} & \text{in } \mathbb{R}^{1+n}_+;\\ (u|_{t=0},\mathrm{d}|_{t=0}) = (u_0,\mathrm{d}_0) & \text{in } \mathbb{R}^n. \end{cases} \tag{1.3}$$

The fourth purpose of this paper is to employ Theorem 1.6 to discover the existence of a unique mild solution to (1.3) within $(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$.

THEOREM 1.8. *Let* $(\alpha, n-1) \in [0,1) \times \mathbb{N}$. *There exists* $\varepsilon > 0$ *such that if the initial pair*

$$(u_0, \mathrm{d}_0) \in \left(((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^{-1}\right)^n \times ((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^3$$

satisfies

$$div\ u_0 = 0 \quad \& \quad \|u_0\|_{\left(((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^{-1}\right)^n} + \|\mathrm{d}_0\|_{((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^3} \le \varepsilon$$

then (1.3) has a unique mild solution

$$(u, \mathrm{d}) \in Z^n_\alpha(\mathbb{R}^{1+n}_+) \times X^3_\alpha(\mathbb{R}^{1+n}_+)$$

with

$$\|u\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} + |||\mathrm{d}|||_{X^3_\alpha(\mathbb{R}^{1+n}_+)} = \sum_{j=1}^n \|u_j\|_{Z_\alpha(\mathbb{R}^{1+n}_+)} + |||\mathrm{d}|||_{X^3_\alpha(\mathbb{R}^{1+n}_+)} \lesssim \varepsilon$$

and

$$\|u_j\|_{Z_\alpha(\mathbb{R}^{1+n}_+)} = \sup_{(r,x)\in\mathbb{R}^{1+n}_+} \sqrt{r}|u_j(r,x)| + \sup_{(r,x)\in\mathbb{R}^{1+n}_+} \left(\int_0^{r^2}\int_{B(x,r)} |u_j(t,y)|^2 \left(\frac{r^2}{t}\right)^\alpha \frac{dydt}{r^n}\right)^{\frac{1}{2}}.$$

Here it should be pointed out that the case $\alpha = 0$ of Theorem 1.8 is due to Wang [**26**], but nevertheless this situation cannot be used to establish the others $\alpha \in (0,1)$. Since $Q_{0<\alpha<1}(\mathbb{R}^n)$ and $Q^{-1}_{0<\alpha<1}(\mathbb{R}^n)$ are more regular than $BMO(\mathbb{R}^n)$ and $BMO^{-1}(\mathbb{R}^n)$ respectively, the solution space

$$Z^n_{0<\alpha<1}(\mathbb{R}^{1+n}_+) \times X^3_{0<\alpha<1}(\mathbb{R}^{1+n}_+)$$

is more regular than

$$Z^n_0(\mathbb{R}^{1+n}_+) \times X^3_0(\mathbb{R}^{1+n}_+).$$

The rest of the paper is organized as follows:

§2 is utilized to give an intrinsic argument for Theorem 1.2;
§3 is devoted to validating Theorem 1.4 via utilizing Theorem 1.2;
§4 is designed to verify Theorem 1.6 via the fixed-point-principle;
§5 is emplyed to demonstrate Theorem 1.8 via the Carleson measure nature of $((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^{-1}$ and the idea of validating Theorem 1.6.

Notation. In the above and below $U \lesssim V$ means $U \le CV$ for a positive constant C; $U \approx V$ means both $U \lesssim V$ and $V \lesssim U$.

2. Predual form of $(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$

2.1. Proof of Theorem 1.2(i). The forward implication of Theorem 1.2(i) comes from the first part of [**2**, Theorem 7.1].

Firstly, let us recall the following concept.

DEFINITION 2.1. Let $(\alpha, n-1) \in (-\infty, \infty) \times \mathbb{N}$.

▷ $\dot{L}^2_\alpha(\mathbb{R}^n)$ is the homogeneous Sobolev α-space with norm

$$\|a\|_{\dot{L}^2_\alpha(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\hat{a}(\xi)|^2|\xi|^{2\alpha}\,d\xi\right)^{2^{-1}}.$$

In particular, $\dot{L}^2_0(\mathbb{R}^n)$ is identified with $L^2(\mathbb{R}^n)$.

▷ $T^1_{n-2\alpha}$ with $0 \le \alpha < 1$ is the class of all measurable functions f on $\mathbb{R}^{1+n}_+$ satisfying

$$\|f\|_{T^1_{n-2\alpha}} = \inf_\omega \left(\int_{\mathbb{R}^{1+n}_+} \left(\frac{|f(t,y)|^2}{\omega(t,y)} \right) t^{2\alpha-1}\, dydt \right)^{\frac{1}{2}} < \infty,$$

where the infimum is taken over all nonnegative measurable functions ω on $\mathbb{R}^{1+n}_+$ with its nontangential maximal function $N(\omega)$ obeying

$$\int_{\mathbb{R}^n} N(\omega)\, d\Lambda^{(\infty)}_{n-2\alpha} \le 1$$

for which $\Lambda^{(\infty)}_{n-2\alpha}(\cdot)$ represents $n-2\alpha$ dimensional Hausdorff capacity in $\mathbb{R}^n$, and the restriction that ω is allowed to vanish only where f vanishes.

▷ $HH^1_{-\alpha}(\mathbb{R}^n)$ with $0 \le \alpha < 1$ is the so-called Hady-Hausdorff class of all distributions $f \in \dot{L}^2_{-\frac{n}{2}}(\mathbb{R}^n)$ obeying

$$\|f\|_{HH^1_{-\alpha}(\mathbb{R}^n)} = \|f * \phi_\cdot(\cdot)\|_{T^1_{n-2\alpha}} < \infty,$$

where ϕ is a radial $C^\infty(\mathbb{R}^n)$-function satisfying

$$\begin{cases} \operatorname{supp}(\phi) \subseteq B(0,1); \\ \phi_t(x) = t^{-n}\phi(t^{-1}x) \ \text{ provided } (t,x) \in \mathbb{R}^{1+n}_+; \\ \int_{\mathbb{R}^n} x^\vartheta \phi(x)\, dx = 0 \ \text{ provided } \vartheta \in \mathbb{N}^n,\ x^\vartheta = x_1^{\vartheta_1} \cdots x_n^{\vartheta_n},\ |\vartheta| = \sum_{i=1}^n \vartheta_i; \\ \int_0^\infty (\hat{\phi}(t\xi))^2 t^{-1}\, dt = 1 \ \text{ provided } \xi \in \mathbb{R}^n \backslash \{0\}. \end{cases}$$

▷ A tempered distribution a is called an $HH^1_{-\alpha}$-atom if a is supported in a ball $B(x_0, r_0)$ and satisfies the following two conditions for the Schwarz class $\mathscr{S}(\mathbb{R}^n)$:

▷ A local α-Sobolev inequality

$$|\langle a, \psi \rangle| \le r_0^{\alpha - \frac{n}{2}} \|\psi\|_{\dot{L}^2_\alpha(\mathbb{R}^n)} \ \ \forall\ \ \psi \in \mathscr{S}(\mathbb{R}^n);$$

▷ A cancellation

$$\langle a, \psi \rangle = 0 \ \ \forall\ \ \psi \in \mathscr{S}(\mathbb{R}^n)$$

whose restriction to a neighborhood of $B(x_0, r_0)$ is a polynomial of degree $\le 1 + n/2$.

Secondly, in accordance with [**2**, Theorem 6.3] and its proof, a tempered distribution f on $\mathbb{R}^n$ belongs to $HH^1_{-\alpha}(\mathbb{R}^n)$ if and only if there is a sequence of $HH^1_{-\alpha}$-atoms a_j and an l^1-sequence $\{\lambda_j\}$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ holds distributionally with

$$\|f\|_{HH^1_{-\alpha}(\mathbb{R}^n)} \approx \inf \left\{ \sum_{j=1}^\infty |\lambda_j| : f = \sum_{j=1}^\infty \lambda_j a_j \right\}.$$

Meanwhile,

$$g \in (-\Delta)^{-\frac{\alpha}{2}} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$$

amounts to

$$\begin{cases} \int_{\mathbb{R}^n} |g(x)|(1+|x|)^{-1-n}\, dx < \infty; \\ \|g\|_{(-\Delta)^{-\frac{\alpha}{2}} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n)} = \sup_{(r_0,x_0)\in\mathbb{R}^{1+n}_+} \left(r_0^{2\alpha} \int_0^{r_0} ⨍_{B(x_0,r_0)} |g * \phi_t(x)|^2 \frac{dxdt}{t^{1+2\alpha}} \right)^{\frac{1}{2}} < \infty. \end{cases}$$

If

$$f \in HH^1_{-\alpha}(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n) \ \ \&\ \ g \in (-\Delta)^{-\frac{\alpha}{2}} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n),$$

then the Calderón reproducing formula

$$f(x) = \int_0^\infty f * \phi_t * \phi_t(x)\, t^{-1} dt$$

is used to deduce

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| &\approx \left|\int_{\mathbb{R}^{1+n}_+} (f*\phi_t)(t,y)\cdot(g*\phi_t)(t,y)t^{-1}dxdt\right| \\ &\lesssim \|f\|_{HH^1_{-\alpha}(\mathbb{R}^n)}\|g\|_{(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)},\end{aligned} \tag{2.1}$$

and hence

$$g \in (-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$$

induces a bounded linear functional on

$$HH^1_{-\alpha}(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n).$$

Since this last function class is dense on $HH^1_{-\alpha}(\mathbb{R}^n)$, we utilize (2.1) to obtain that

$$f \mapsto \int_{\mathbb{R}^n} f(x)g(x)\,dx$$

can be extended to a bounded linear functional L on $HH^1_{-\alpha}(\mathbb{R}^n)$ with its norm

$$\|L\| \lesssim \|g\|_{(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)}.$$

The backward implication of Theorem 1.2(i) just follows from the second part of the argument for [**2**, Theorem 7.1]. Briefly, if L is a bounded linear functional on $HH^1_{-\alpha}(\mathbb{R}^n)$, then an application of the Hahn-Banach theorem yields an extension $\tilde{L}$ which is bounded linear functional on $T^1_{n-2\alpha}$ and induces $G(\cdot,\cdot) \in T^\infty_{n-2\alpha}$ such that

$$\begin{cases}\|G\|_{T^\infty_{n-2\alpha}} = \sup_{(r_0,x_0)\in\mathbb{R}^{1+n}_+}\left(r_0^{2\alpha}\int_0^{r_0} \fint_{B(x_0,r_0)} |G(t,x)|^2\,\frac{dxdt}{t^{1+2\alpha}}\right)^{\frac{1}{2}} < \infty; \\ \tilde{L}(F) = \int_{\mathbb{R}^{1+n}_+} F(t,x)G(t,x)\,t^{-1}dxdt \quad \forall \quad F \in T^1_{n-2\alpha}.\end{cases} \tag{2.2}$$

Accordingly, if

$$f \in HH^1_{-\alpha}(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n) \ \ \& \ \ g(x) = \int_0^\infty G(t,\cdot) * \phi_t(x)\,t^{-1}dt,$$

then

$$\begin{aligned}L(f) &= \int_{\mathbb{R}^{1+n}_+} f * \phi_t(y)\,G(t,y)\,t^{-1}dydt \\ &= \int_{\mathbb{R}^{1+n}_+} f(x)G(t,\cdot) * \phi_t(x)\,t^{-1}dxdt \\ &= \int_{\mathbb{R}^n} f(x)g(x)\,dx.\end{aligned}$$

Thanks to the fact that (2.2) implies

$$\|g\|_{(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)} \lesssim \|G\|_{T^\infty_{n-2\alpha}} \lesssim \|L\|,$$

we arrive at the desired result.

2.2. Proof of Theorem 1.2(ii). If

$$f \in (-\Delta)^{-\frac{\alpha}{2}} HH^1_{-\alpha}(\mathbb{R}^n) \ \ \& \ \ h \in L^{\frac{n}{\alpha}}(\mathbb{R}^n)$$

then

$$\exists\, g \in HH^1_{-\alpha}(\mathbb{R}^n) \ \text{ such that } \ f = (-\Delta)^{-\frac{\alpha}{2}} g \ \text{ with } \ \|f\|_{(-\Delta)^{-\frac{\alpha}{2}} HH^1_{-\alpha}(\mathbb{R}^n)} \approx \|g\|_{HH^1_{-\alpha}(\mathbb{R}^n)}$$

and hence

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)h(x)\,dx\right| &= \left|\int_{\mathbb{R}^n} g(x)(-\Delta)^{-\frac{\alpha}{2}} h(x)\,dx\right| \\ &\le \|g\|_{HH^1_{-\alpha}(\mathbb{R}^n)} \|(-\Delta)^{-\frac{\alpha}{2}} h\|_{(-\Delta)^{-\frac{\alpha}{2}} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n)} \\ &\lesssim \|g\|_{HH^1_{-\alpha}(\mathbb{R}^n)} \|h\|_{L^{\frac{n}{\alpha}}(\mathbb{R}^n)}.\end{aligned}$$

This, along with the duality

$$L^{\frac{n}{n-\alpha}}(\mathbb{R}^n) = [L^{\frac{n}{\alpha}}(\mathbb{R}^n)]^*,$$

implies

$$\|f\|_{L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \lesssim \|f\|_{(-\Delta)^{-\frac{\alpha}{2}} HH^1_{-\alpha}(\mathbb{R}^n)}.$$

3. Predual form of $((-\Delta)^{-\frac{\alpha}{2}} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^{-1}$

3.1. Proof of Theorem 1.4(i). On the one hand, let us prove

$$\begin{aligned}(-\Delta)^{-\frac{\alpha}{2}} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n) \supseteq\ & (-\Delta)^{-\frac{\alpha}{2}} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \\ & + \sum_{j=1}^n R_j((-\Delta)^{-\frac{\alpha}{2}} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)).\end{aligned} \tag{3.1}$$

To do so, let

$$\begin{cases} f = g_0 + \sum_{j=1}^n R_j g_j; \\ g_j \in L^\infty(\mathbb{R}^n) \cap (-\Delta)^{-\frac{\alpha}{2}} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n); \\ g_j(t,x) = e^{-t\sqrt{-\Delta}} g_j(x) = g_j * P_t(x); \\ P_t(x) = \Gamma\left(\frac{n+1}{2}\right) \pi^{-\frac{n+1}{2}} t(t^2+|x|^2)^{-\frac{n+1}{2}}. \end{cases}$$

According to [**7**, p.263, 4.1.8], we have

$$(R_j P_t)(x) \approx x_j (t^2+|x|^2)^{-\frac{n+1}{2}} \quad \forall \quad j \in \{1, \cdots, n\} \,\&\, (t,x) \in \mathbb{R}^{1+n}_+,$$

thereby obtaining (cf. [**15**, Theorem 3.6 & Corollary 3.7])

$$\begin{aligned}\|R_j g_j\|_{(-\Delta)^{-\frac{\alpha}{2}} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n)} &\approx \sup_{(r_0,x_0)\in\mathbb{R}^{1+n}_+} \left(\int_0^{r_0} \int_{B(x_0,r_0)} \left|\nabla \left(R_j g_j(t,x)\right)\right|^2 \left(\frac{t}{r_0}\right)^{n-2\alpha} \frac{dxdt}{t^{n-1}}\right)^{\frac{1}{2}} \\ &= \sup_{(r_0,x_0)\in\mathbb{R}^{1+n}_+} \left(\int_0^{r_0} \int_{B(x_0,r_0)} \left|\nabla \left((R_j g_j) * P_t(x)\right)\right|^2 \left(\frac{t}{r_0}\right)^{n-2\alpha} \frac{dxdt}{t^{n-1}}\right)^{\frac{1}{2}} \\ &= \sup_{(r_0,x_0)\in\mathbb{R}^{1+n}_+} \left(\int_0^{r_0} \int_{B(x_0,r_0)} \left|\nabla \left(g_j * (R_j P_t)(x)\right)\right|^2 \left(\frac{t}{r_0}\right)^{n-2\alpha} \frac{dxdt}{t^{n-1}}\right)^{\frac{1}{2}} \\ &\lesssim \sup_{(r_0,x_0)\in\mathbb{R}^{1+n}_+} \left(\int_0^{r_0} \int_{B(x_0,r_0)} \left|\nabla \left(g_j * P_t(x)\right)\right|^2 \left(\frac{t}{r_0}\right)^{n-2\alpha} \frac{dxdt}{t^{n-1}}\right)^{\frac{1}{2}} \\ &\lesssim \|g_j\|_{L^\infty(\mathbb{R}^n) \cap (-\Delta)^{-\frac{\alpha}{2}} \mathscr{L}^{2,2\alpha}(\mathbb{R}^n)}.\end{aligned}$$

Hence

$$\|f\|_{(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)} \le \|g_0\|_{(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)} + \sum_{j=1}^{n}\|R_jg_j\|_{(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)}$$
$$\lesssim \sum_{j=0}^{n}\|g_j\|_{L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)},$$

which gives (3.1).

On the other hand, let us verify

$$(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$$
$$+\sum_{j=1}^{n}R_j(L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)). \tag{3.2}$$

In doing so, we define $\mathscr{L}_\alpha$ as the space of all functions f on $\mathbb{R}^n$ with its norm

$$\|f\|_{\mathscr{L}_\alpha} = \sup_{\{h\}}\left\{\left|\int_{\mathbb{R}^n} f(x)h(x)\,dx\right|\right\} < \infty,$$

where the supremum is taken over all functions h obeying

$$\|h\|_{L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)} = \|h\|_{L^\infty(\mathbb{R}^n)} + \|h\|_{(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)} \le 1.$$

Let

$$\begin{cases}\mathscr{U} = \Big\{(g_0,g_1,\cdots,g_n):\ \ g_0,g_1,\cdots,g_n\in\mathscr{L}_\alpha\Big\} = \underbrace{\mathscr{L}_\alpha\oplus\cdots\oplus\mathscr{L}_\alpha}_{n+1\ copies};\\ \|(g_0,g_1,\cdots,g_n)\|_{\mathscr{U}} = \sum_{j=0}^{n}\|g_j\|_{\mathscr{L}_\alpha}.\end{cases}$$

If

$$\mathscr{V} = \Big\{(g_0,g_1,\cdots,g_n)\in\mathscr{U}:\ \ g_j = R_jg_0\ \forall\ j\in\{1,\cdots,n\}\Big\},$$

then $\mathscr{V}$ is a closed subset of $\mathscr{U}$, and hence the mapping

$$g_0 \mapsto (g_0,g_1,\cdots,g_n)$$

defines a norm-preserving map from $HH^1_{-\alpha}(\mathbb{R}^n)$ to $\mathscr{V}$. In order to see the last statement, we are only required to prove that if $g_0\in HH^1_{-\alpha}(\mathbb{R}^n)$ then

$$R_jg_0\in\mathscr{L}_\alpha\ \ \forall\ \ j\in\{1,\cdots,n\}.$$

In fact, an application of Theorem 1.2 derives

$$\begin{aligned}\|R_jg_0\|_{\mathscr{L}_\alpha} &= \sup_{\|h\|_{L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)}\le 1}\left|\int_{\mathbb{R}^n}(R_jg_0)(x)h(x)\,dx\right|\\ &= \sup_{\|h\|_{L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)}\le 1}\left|\int_{\mathbb{R}^n}g_0(x)(R_jh)(x)\,dx\right|\\ &\le \sup_{\|h\|_{L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)}\le 1}\Big(\|g_0\|_{HH^1_{-\alpha}(\mathbb{R}^n)}\|R_jh\|_{(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)}\Big)\\ &\lesssim \sup_{\|h\|_{L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)}\le 1}\Big(\|g_0\|_{HH^1_{-\alpha}(\mathbb{R}^n)}\|h\|_{L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)}\Big)\\ &\lesssim \|g_0\|_{HH^1_{-\alpha}(\mathbb{R}^n)},\end{aligned}$$

where we have used the above-established inequality:

$$\|R_jh\|_{(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)} \lesssim \|h\|_{L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)}.$$

Now, any bounded linear functional on $HH^1_{-\alpha}(\mathbb{R}^n)$ can be identified with a corresponding functional on $\mathscr{V}$, and so from the Hahn-Banach theorem it follows that this functional is extendable to a bounded linear functional on $\mathscr{U}$. Note that the dual space of $\mathscr{U}$ is

$$\underbrace{\Big(L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\Big)\oplus\cdots\oplus\Big(L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\Big)}_{n+1\ copies}$$

Thus, upon restricting to $\mathscr{V}$ (and so $HH^1_{-\alpha}(\mathbb{R}^n)$), we have the forthcoming demonstration.

Suppose that

$$f\in(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$$

and L is a bounded linear functional on $HH^1_{-\alpha}(\mathbb{R}^n)$. Then, for each $j=0,\cdots,n$ there is

$$\tilde{f}_j\in L^\infty(\mathbb{R}^n)\cap(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$$

such that for any $g_0\in HH^1_{-\alpha}(\mathbb{R}^n)$ one has

$$\begin{aligned}\mathrm{L}(f)&=\int_{\mathbb{R}^n}f(x)g_0(x)\,dx\\&=\int_{\mathbb{R}^n}\tilde{f}_0(x)g_0(x)\,dx+\sum_{j=1}^n\int_{\mathbb{R}^n}\tilde{f}_j(x)(R_jg_0)(x)\,dx\\&=\int_{\mathbb{R}^n}\tilde{f}_0(x)g_0(x)\,dx-\sum_{j=1}^n\int_{\mathbb{R}^n}R_j\tilde{f}_j(x)g_0(x)\,dx\\&=\int_{\mathbb{R}^n}\Big(\tilde{f}_0(x)-\sum_{j=1}^n(R_j\tilde{f}_j)(x)\Big)g_0(x)\,dx.\end{aligned}$$

Accordingly, we gain

$$f(x)=\tilde{f}_0(x)-\sum_{j=1}^n(R_j\tilde{f}_j)(x),$$

thereby reaching (3.2).

Consequently, putting (3.1) and (3.2) together validates the set equality in Theorem 1.4(i).

3.2. Proof of Theorem 1.4(ii). Note that $HHS^1_{-\alpha}(\mathbb{R}^n)$ can be also identitied with the closure of $C_0^\infty(\mathbb{R}^n)$ in the norm $\big\||\nabla f|\big\|_{HH^1_{-\alpha}(\mathbb{R}^n)}$. Thus, the coming-up-next argument is two-fold.

On the one hand, if

$$f\in((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^{-1}$$

then

$$\exists\ (f_1,\cdots,f_n)\in((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^n\ \text{ such that }\ f(x)=\nabla\cdot(f_1,\cdots,f_n)(x)=\sum_{j=1}^n\partial_{x_j}f_j(x)$$

and hence f induces a bounded linear functional on $HHS^1_{-\alpha}(\mathbb{R}^n)$ thanks to

$$C_0^\infty(\mathbb{R}^n)\ni g\mapsto\int_{\mathbb{R}^n}f(x)g(x)\,dx=-\int_{\mathbb{R}^n}(f_1(x),\cdots,f_n(x))\cdot\nabla g(x)\,dx.$$

This in turn implies

$$((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^{-1}\subseteq[HHS^1_{-\alpha}(\mathbb{R}^n)]^*.$$

On the other hand, suppose $L\in[HHS^1_{-\alpha}(\mathbb{R}^n)]^*$. Since ∇ exists as a bounded linear operator from $HHS^1_{-\alpha}(\mathbb{R}^n)$ to $(HH^1_{-\alpha}(\mathbb{R}^n))^n$, the completeness of $HHS^1_{-\alpha}(\mathbb{R}^n)$ ensures that the range $\mathscr{R}(\nabla\, HHS^{\,1}_{-\alpha}(\mathbb{R}^n))$ of $HHS^{\,1}_{-\alpha}(\mathbb{R}^n)$ under ∇ is a closed subspace of $(HH^1_{-\alpha}(\mathbb{R}^n))^n$. Upon defining

$$L_1(\nabla f) = L(f) \quad \forall \quad \nabla f \in \mathscr{R}(\nabla\, HHS^{\,1}_{-\alpha}(\mathbb{R}^n)),$$

we get

$$\|L_1\|_{\left[\mathscr{R}\left(\nabla\, HHS^{\,1}_{-\alpha}(\mathbb{R}^n)\right)\right]^*} = \|L\|_{[HHS^{\,1}_{-\alpha}(\mathbb{R}^n)]^*}.$$

This last equation, plus the Hanh-Banach Theorem, yields a norm-preserving extension L_2 of L_1 to $(HH^1_{-\alpha}(\mathbb{R}^n))^n$. Meanwhile, the vector-valued form of Theorem 1.2 is used to produce a vector

$$(f_1, \cdots, f_n) \in ((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^n$$

such that

$$L_2((g_1, \cdots, g_n)) = \sum_{j=1}^{n} \int_{\mathbb{R}^n} f_j(x) g_j(x)\, dx \quad \forall \quad (g_1, \cdots, g_n) \in (HH^1_{-\alpha}(\mathbb{R}^n))^n$$

and

$$\begin{aligned}\|(f_1, \cdots, f_n)\|_{((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^n} &= \|L_2\|_{(HH^1_{-\alpha}(\mathbb{R}^n))^n}\\ &= \|L_1\|_{\left[\mathscr{R}\left(\nabla\, HHS^{\,1}_{-\alpha}(\mathbb{R}^n)\right)\right]^*}\\ &= \|L\|_{[HHS^{\,1}_{-\alpha}(\mathbb{R}^n)]^*}.\end{aligned}$$

Consequently, for each $\phi \in C_0^\infty(\mathbb{R}^n)$ we obtain

$$L(\phi) = L_1(\nabla\phi) = L_2(\nabla\phi) = \int_{\mathbb{R}^n} (f_1(x), \cdots, f_n(x)) \cdot \nabla\phi(x)\, dx,$$

thereby reaching

$$\begin{cases} L = -\mathrm{div}(f_1, \cdots, f_n);\\ \|(f_1, \cdots, f_n)\|_{((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^n} = \|L\|_{[HHS^{\,1}_{-\alpha}(\mathbb{R}^n)]^*},\end{cases}$$

and so

$$((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^{-1} \supseteq [HHS^{\,1}_{-\alpha}(\mathbb{R}^n)]^*.$$

Therefore, the above-established set-inclusions derive the desired duality.

4. Application to harmonic map flow

4.1. A caloric estimation. Given the Gauss kernel

$$\mathcal{G}(t, x) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4t}\right) \quad \forall \ (t, x) \in \mathbb{R}^{1+n}_+$$

and a vector-valued function $f : \mathbb{R}^{1+n}_+ \to \mathbb{R}^l$, we define the caloric operator

$$(\mathrm{S}f)(t, x) = \int_0^t \int_{\mathbb{R}^n} \mathcal{G}(t - s, x - y) f(s, y)\, dyds \quad \forall \quad (t, x) \in \mathbb{R}^{1+n}_+,$$

whence establishing its boundedness from $Y^l_\alpha(\mathbb{R}^{1+n}_+)$ to $X^l_\alpha(\mathbb{R}^{1+n}_+)$:

LEMMA 4.1. *For $(\alpha, l, n-1) \in [0, 1) \times \mathbb{N} \times \mathbb{N}$ let $Y^l_\alpha(\mathbb{R}^{1+n}_+)$ be the space of all vector-valued functions*

$$f = (f_1, \cdots, f_l) : \ \mathbb{R}^{1+n}_+ \to \mathbb{R}^l$$

satisfying

$$\begin{cases} \|f\|_{Y^l_\alpha(\mathbb{R}^{1+n}_+)} = \sum_{j=1}^l \|f_j\|_{Y_\alpha(\mathbb{R}^{1+n}_+)} < \infty;\\ \|f_j\|_{Y_\alpha(\mathbb{R}^{1+n}_+)} = \sup_{(r,x)\in\mathbb{R}^{1+n}_+} r|f_j(r, x)| + \sup_{(r,x)\in\mathbb{R}^{1+n}_+} \int_0^{r^2} \int_{B(x,r)} |f_j(t, y)| \left(\frac{r^2}{t}\right)^\alpha \frac{dydt}{r^n}.\end{cases}$$

Then

$$|||\mathrm{S}f|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)} \lesssim \|f\|_{Y^l_\alpha(\mathbb{R}^{1+n}_+)}. \tag{4.1}$$

PROOF. Since $|||\cdot|||_{X^l_\alpha(\mathbb{R}^n)}$ is invariant under both scaling and translation, (4.1) follows from

$$|(\mathrm{S}f)(1,0)| + |\nabla(\mathrm{S}f)(1,0)| + \left(\int_0^1 \int_{B(0,1)} |\nabla(\mathrm{S}f)(t,y)|^2 dydt\right)^{\frac{1}{2}} \lesssim \|f\|_{Y^l_\alpha(\mathbb{R}^{1+n}_+)}.$$

To estimate $|(\mathrm{S}f)(1,0)|$, we write

$$\begin{aligned}(\mathrm{S}f)(1,0) &= \int_0^1 \int_{\mathbb{R}^n} \mathcal{G}(1-s,y) f(s,y)\, dyds \\ &= \left(\int_{2^{-1}}^1 \int_{\mathbb{R}^n} + \int_0^{2^{-1}} \int_{B(0,1)} + \int_0^{2^{-1}} \int_{\mathbb{R}^n\setminus B(0,1)}\right)(\cdots)\, dyds.\end{aligned}$$

By the integrability of $\mathcal{G}(\cdot,\cdot)$ at the origin we have

$$\begin{aligned}\left|\int_{2^{-1}}^1 \int_{\mathbb{R}^n} (\cdots)\, dyds\right| &\le \left(\sup_{2^{-1}\le s\le 1} \|f(s,\cdot)\|_{L^\infty(\mathbb{R}^n)}\right)\left(\int_{2^{-1}}^1 \int_{\mathbb{R}^n} \mathcal{G}(1-s,y)\, dyds\right) \\ &\lesssim \|f\|_{Y^l_\alpha(\mathbb{R}^{1+n}_+)}.\end{aligned}$$

Also, by the boundedness of $\mathcal{G}(\cdot,\cdot)$ away from the origin we have

$$\begin{aligned}\left|\int_0^{2^{-1}} \int_{B(0,1)} (\cdots)\, dyds\right| &\lesssim \sup_{x\in\mathbb{R}^n} \int_0^1 \int_{B(x,1)} |f(s,y)|\, dyds \\ &\lesssim \sup_{x\in\mathbb{R}^n} \int_0^1 \int_{B(x,1)} |f(s,y)| \frac{dyds}{s^\alpha} \\ &\lesssim \|f\|_{Y^l_\alpha(\mathbb{R}^{1+n}_+)}\end{aligned}$$

and

$$\begin{aligned}\left|\int_0^{2^{-1}} \int_{\mathbb{R}^n\setminus B(0,1)} (\cdots)\, dyds\right| &\lesssim \int_0^{2^{-1}} \int_{\mathbb{R}^n\setminus B(0,1)} \exp\left(-\frac{|y|^2}{2}\right) |f(s,y)|\, dyds \\ &\lesssim \left(\sum_{k=1}^\infty \frac{k^{n-1}}{\exp\left(\frac{k^2}{2}\right)}\right)\left(\sup_{x\in\mathbb{R}^n} \int_0^1 \int_{B(x,1)} |f(s,y)|\, dyds\right) \\ &\lesssim \|f\|_{Y^l_\alpha(\mathbb{R}^{1+n}_+)}.\end{aligned}$$

These estimates show

$$|(\mathrm{S}f)(1,0)| \lesssim \|f\|_{Y^l_\alpha(\mathbb{R}^{1+n}_+)}.$$

In order to estimate $|\nabla(\mathrm{S}f)(1,0)|$, we use

$$|\nabla(\mathrm{S}f)(1,0)| \le \int_0^1 \int_{\mathbb{R}^n} |\mathcal{H}(1-s,-y)|\, |f(s,y)|\, dyds,$$

where

$$\mathcal{H}(t,x) = -(2t)^{-1} x\, \mathcal{G}(t,x)$$

satisfies

$$\begin{cases} \int_0^{\frac{1}{2}} \int_{\mathbb{R}^n} |\mathcal{H}(t,x)|\, dxdt \lesssim 1; \\ \sup_{(t,x)\in[2^{-1},1]\times\mathbb{R}^n} |\mathcal{H}(t,x)| \lesssim 1, \end{cases}$$

and then replace $\mathcal{G}(1-s,y)$ (in dominating $|(\mathrm{S}f)(1,0)|$) with $\mathcal{H}(1-s,-y)$ to obtain

$$|\nabla(\mathrm{S}f)(1,0)| \lesssim \|f\|_{Y^l_\alpha(\mathbb{R}^{1+n}_+)}.$$

In order to control

$$\int_0^1\int_{B(0,1)}|\nabla(\mathrm{S}f)(s,y)|^2\,dyds,$$

we recall that $\mathrm{S}f$ solves

$$\begin{cases}(\partial_t-\Delta)(\mathrm{S}f)(t,x)=f(t,x) & \forall\quad (t,x)\in[0,1]\times\mathbb{R}^n;\\ (\mathrm{S}f)(0,x)=0 & \forall\quad x\in\mathbb{R}^n.\end{cases}\tag{4.2}$$

Suppose that η is a cut-off function in $C_0^\infty(\mathbb{R}^n)$ and satisfies

$$\begin{cases}\mathrm{supp}(\eta)\subseteq B(0,2);\\ \eta(x)=1 & \forall\quad x\in B(0,1);\\ |\nabla\eta(x)|\lesssim 1 & \forall\quad x\in B(0,2).\end{cases}$$

Now, the first equation of (4.2) derives

$$(\mathrm{S}f)(t,x)\eta^2(x)((\partial_t-\Delta)(\mathrm{S}f)(t,x))=(\mathrm{S}f)(t,x)\eta^2(x)f(t,x)\ \ \forall\ \ (t,x)\in[0,1]\times\mathbb{R}^n.$$

Furthermore, an integration of the last equation over $[0,1]\times\mathbb{R}^n$ yields

$$\begin{aligned}&\int_0^1\int_{B(0,2)}(\mathrm{S}f)(t,x)\eta^2(x)f(t,x)\,dxdt\\ &=\int_0^1\int_{\mathbb{R}^n}(\mathrm{S}f)(t,x)\eta^2(x)f(t,x)\,dxdt\\ &=2^{-1}\int_{\mathbb{R}^n}\eta^2(x)\int_0^1\partial_t((\mathrm{S}f)(t,x))^2\,dt-\int_0^1\int_{\mathbb{R}^n}(\mathrm{S}f)(t,x)\eta^2(x)\Delta(\mathrm{S}f)(t,x)\,dxdt\\ &=2^{-1}\int_{B(0,2)}\eta^2(x)|(\mathrm{S}f)(1,x)|^2\,dx+\int_0^1\int_{B(0,2)}\nabla((\mathrm{S}f)(t,x))\cdot\nabla((\mathrm{S}f)(t,x)\eta^2(x))\,dxdt.\end{aligned}$$

Upon observing

$$\begin{aligned}&\nabla((\mathrm{S}f)(t,x))\cdot\nabla((\mathrm{S}f)(t,x)\eta^2(x))\\ &\quad=\eta^2(x)|\nabla(\mathrm{S}f)(t,x)|^2+2(\eta(x)\nabla(\mathrm{S}f)(t,x))\cdot((\mathrm{S}f)(t,x)\nabla\eta(x))\ \ \forall\ \ (t,x)\in[0,1]\times\mathbb{R}^n\end{aligned}$$

and

$$\begin{cases}2ab\le\epsilon a^2+\epsilon^{-1}b^2 & \forall\quad (a,b,\epsilon)\in(0,\infty)\times(0,\infty)\times(0,\infty);\\ (\mathrm{S}f)(0,x)=0 & \forall\quad x\in\mathbb{R}^n,\end{cases}$$

we can make the following estimation

$$\begin{aligned}&\int_0^1\int_{B(0,2)}\eta^2(x)|\nabla(\mathrm{S}f)(t,x)|^2\,dxdt\\ &\quad\le 2^{-1}\int_{B(0,2)}\eta^2(x)|(\mathrm{S}f)(1,x)|^2\,dx+\int_0^1\int_{B(0,2)}\eta^2(x)|(\mathrm{S}f)(t,x)|\,|f(t,x)|\,dxdt\\ &\quad\quad+2\int_0^1\int_{B(0,2)}(\eta(x)\nabla(\mathrm{S}f)(t,x))\cdot((\mathrm{S}f)(t,x)\nabla\eta(x))\,dxdt\\ &\quad\le 2^{-1}\int_{B(0,2)}\eta^2(x)|(\mathrm{S}f)(1,x)|^2\,dx+\int_0^1\int_{B(0,2)}\eta^2(x)|(\mathrm{S}f)(t,x)|\,|f(t,x)|\,dxdt\\ &\quad\quad+\epsilon\int_0^1\int_{B(0,2)}\eta^2(x)|\nabla(\mathrm{S}f)(t,x)|^2\,dxdt+\epsilon^{-1}\int_0^1\int_{B(0,2)}|\nabla\eta(x)|^2|(\mathrm{S}f)(t,x)|^2\,dxdt.\end{aligned}$$

Via choosing $\epsilon = 2^{-1}$ and using the properties of η we find

$$\begin{aligned}
&\int_0^1 \int_{B(0,1)} \left|\nabla(\mathrm{S}f)(t,x)\right|^2 dxdt \\
&\le \int_0^1 \int_{B(0,2)} \eta^2(x)\left|\nabla(\mathrm{S}f)(t,x)\right|^2 dxdt \\
&\le \int_{B(0,2)} \eta^2(x)|(\mathrm{S}f)(1,x)|^2\,dx + 2\int_0^1 \int_{B(0,2)} \eta^2(x)|(\mathrm{S}f)(t,x)|\,|f(t,x)|\,dxdt \\
&\quad + 2^2 \int_0^1 \int_{B(0,2)} |\nabla\eta(x)|^2|(\mathrm{S}f)(t,x)|^2\,dxdt \\
&\lesssim \int_{B(0,2)} |(\mathrm{S}f)(1,x)|^2\,dx + \int_0^1 \int_{B(0,2)} |(\mathrm{S}f)(t,x)|\,|f(t,x)|\,dxdt \\
&\quad + \int_0^1 \int_{B(0,2)} |(\mathrm{S}f)(t,x)|^2\,dxdt \\
&\lesssim \|\mathrm{S}f\|^2_{L^\infty([0,1]\times B(0,2))} + \|\mathrm{S}f\|_{L^\infty([0,1]\times B(0,2))} \int_0^1 \int_{B(0,2)} |f(t,x)|\,dxdt \\
&\lesssim \|f\|^2_{Y^l_\alpha(\mathbb{R}^{1+n}_+)},
\end{aligned}$$

thereby validating (4.1). □

4.2. Proof of Theorem 1.6. Suppose now that $u : \mathbb{R}^{1+n}_+ \to \mathbb{R}^l$ is a mild solution of (1.2). Then it satisfies

$$u(t,x) = v_0(t,x) + \mathrm{S}(\mathrm{II}(u)(\nabla u,\nabla u))(t,x),$$

where

$$\begin{cases} v_0(t,x) = \int_{\mathbb{R}^n} \mathcal{G}(t,x-y)u_0(y)\,dy; \\ \mathrm{S}(\mathrm{II}(u)(\nabla u,\nabla u))(t,x) = \int_0^t \int_{\mathbb{R}^n} \mathcal{G}(t-s,x-y)\mathrm{II}(u)(\nabla u,\nabla u)(s,y)\,dyds. \end{cases}$$

Define

$$\mathrm{T} : X^l_\alpha(\mathbb{R}^{1+n}_+) \to X^l_\alpha(\mathbb{R}^{1+n}_+)$$

by

$$\mathrm{T}u(t,x) = v_0(t,x) + \mathrm{S}(\mathrm{II}(u)(\nabla u,\nabla u))(t,x) \ \ \forall\ \ u \in X^l_\alpha(\mathbb{R}^{1+n}_+).$$

And let

$$\mathcal{B}_\delta = \left\{u \in X^l_\alpha(\mathbb{R}^{1+n}_+) :\ |||u - v_0|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)} \le \delta\right\}$$

be the ball in $X^l_\alpha(\mathbb{R}^{1+n}_+)$ with center v_0 and radius δ. We are about to show that

$$\exists\ \varepsilon > 0 \ \text{ such that if }\ \|u_0\|_{((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^l} \le \varepsilon$$

then

† $\mathrm{T} : \mathcal{B}_\varepsilon \to \mathcal{B}_\varepsilon$;
‡ $\exists\ \beta \in (0,1)$ such that $|||\mathrm{T}u - \mathrm{T}v|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)} \le \beta\,|||u-v|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)} \quad \forall\ u,v \in \mathcal{B}_\varepsilon$.

For †, if $u \in \mathcal{B}_\varepsilon$, then by the triangle inequality, the essence of $(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)$ mentioned in the proof of Theorem 1.2(i):

$$\|u_0\|_{((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^l} \approx \sup_{(r,x)\in\mathbb{R}^{1+n}_+} \left(\int_0^r \int_{B(x,r)} |\nabla v_0(t,y)|^2 \left(\frac{t}{r}\right)^{n-2\alpha} \frac{dydt}{t^{n-1}}\right)^{\frac{1}{2}}$$

and the following standard gradient estimate

$$\begin{aligned}\sup_{(r,x)\in\mathbb{R}^{1+n}_+}\sqrt{r}|\nabla v_0(r,x)| &\lesssim \left(\int_0^r\int_{B(x,r)}|\nabla v_0(t,y)|^2\left(\frac{t}{r}\right)^{n-2\alpha}\frac{dydt}{t^{n-1}}\right)^{\frac{1}{2}}\\ &\lesssim \|u_0\|_{((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^l},\end{aligned}$$

we use the triangle inequality to obtain

$$\begin{aligned}\|u\|_{X^l_\alpha(\mathbb{R}^{1+n}_+)} &\lesssim \varepsilon + \sup_{(r,x)\in\mathbb{R}^{1+n}_+}\sqrt{r}|\nabla v_0(r,x)| + \|u_0\|_{((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^l}\\ &\lesssim \varepsilon + \|u_0\|_{((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^l}\\ &\lesssim \varepsilon.\end{aligned}$$

Since

$$\mathrm{T}u - v_0 = \int_0^t\int_{\mathbb{R}^n}\mathcal{G}(t-s,x-y)\mathrm{II}(u)(\nabla u,\nabla u)(s,y)\,dyds,$$

we utilize (4.1) to estimate

$$\begin{aligned}|||\mathrm{T}u - v_0|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)} &\lesssim \|\mathrm{II}(u)(\nabla u,\nabla u)\|_{Y^l_\alpha(\mathbb{R}^{1+n}_+)}\\ &\approx \sup_{(r,x)\in\mathbb{R}^{1+n}_+} r|\mathrm{II}(u)(\nabla u,\nabla u)(r,x)|\\ &\quad + \sup_{(r,x)\in\mathbb{R}^{1+n}_+}\int_0^{r^2}\int_{B(x,r)}|\mathrm{II}(u)(\nabla u,\nabla u)(t,y)|\left(\frac{r^2}{t}\right)^\alpha\frac{dydt}{r^n}\\ &\lesssim \left(\sup_{(r,x)\in\mathbb{R}^{1+n}_+}\sqrt{r}|\nabla u(r,x)|\right)^2\\ &\quad + \sup_{(r,x)\in\mathbb{R}^{1+n}_+}\int_0^{r^2}\int_{B(x,r)}|\nabla u(t,y)|^2\left(\frac{r^2}{t}\right)^\alpha\frac{dydt}{r^n}\\ &\lesssim |||u|||^2_{X^l_\alpha(\mathbb{R}^{1+n}_+)}\\ &\lesssim \varepsilon,\end{aligned}$$

by choosing a sufficiently small $\varepsilon > 0$. Thus,

$$\exists\ \varepsilon > 0\ \text{ such that }\ \|u_0\|_{((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^l} \le \varepsilon \Longrightarrow \mathrm{T}:\mathcal{B}_\varepsilon\to\mathcal{B}_\varepsilon.$$

For ‡, if $u, v \in \mathcal{B}_\varepsilon$, then from (4.1) it follows that

$$\begin{aligned}|||\mathrm{T}u - \mathrm{T}v|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)} &= |||\mathrm{S}(\mathrm{II}(u)(\nabla u,\nabla u) - \mathrm{II}(v)(\nabla v,\nabla v))|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)}\\ &\lesssim |||\mathrm{S}\{|\nabla u|^2|u-v| + (|\nabla u|+|\nabla v|)|\nabla(u-v)|\}|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)}\\ &\lesssim |||\nabla u|^2|u-v|||_{Y^l_\alpha(\mathbb{R}^{1+n}_+)} + \|(|\nabla u|+|\nabla v|)|\nabla(u-v)|\,\|_{Y^l_\alpha(\mathbb{R}^{1+n}_+)}\\ &=: \mathsf{D}_1 + \mathsf{D}_2.\end{aligned}$$

Estimate for D_1:

$$\begin{aligned}
\mathsf{D}_1 &= \sup_{(r,x)\in\mathbb{R}^{1+n}_+} r\left|(|\nabla u|^2|u-v|)(r,x)\right| \\
&\quad + \sup_{(r,x)\in\mathbb{R}^{1+n}_+}\int_0^{r^2}\int_{B(x,r)} |(|\nabla u|^2|u-v|)(t,y)|\left(\frac{r^2}{t}\right)^{\alpha}\frac{dydt}{r^n} \\
&\le \left(\sup_{(r,x)\in\mathbb{R}^{1+n}_+}\sqrt{r}\,|\nabla u(r,x)|\right)^2 \sup_{(r,x)\in\mathbb{R}^{1+n}_+}|(u-v)(r,x)| \\
&\quad + \sup_{(r,x)\in\mathbb{R}^{1+n}_+}\left(\int_0^{r^2}\int_{B(x,r)}|\nabla u(t,y)|^2\left(\frac{r^2}{t}\right)^{\alpha}\frac{dydt}{r^n}\right)\sup_{(r,x)\in\mathbb{R}^{1+n}_+}|(u-v)(r,x)| \\
&\lesssim |||u|||^2_{X^l_\alpha(\mathbb{R}^{1+n}_+)}\sup_{(r,x)\in\mathbb{R}^{1+n}_+}|(u-v)(r,x)| \\
&\lesssim \varepsilon^2|||u-v|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)}.
\end{aligned}$$

Estimate for D_2:

$$\begin{aligned}
\mathsf{D}_2 &= \sup_{(r,x)\in\mathbb{R}^{1+n}_+} r(|\nabla u|+|\nabla v|)|\nabla(u-v)(r,x)| \\
&\quad + \sup_{(r,x)\in\mathbb{R}^{1+n}_+}\int_0^{r^2}\int_{B(x,r)}(|\nabla u|+|\nabla v|)|\nabla(u-v)(t,y)|\left(\frac{r^2}{t}\right)^{\alpha}\frac{dydt}{r^n} \\
&\le \sup_{(r,x)\in\mathbb{R}^{1+n}_+}\sqrt{r}\,(|\nabla u(r,x)|+|\nabla v(r,x)|)\sup_{(r,x)\in\mathbb{R}^{1+n}_+}\sqrt{r}|\nabla(u-v)(r,x)| \\
&\quad + \sup_{(r,x)\in\mathbb{R}^{1+n}_+}\left(\int_0^{r^2}\int_{B(x,r)}\left(|\nabla u(t,y)|^2+|\nabla v(t,y)|^2\right)\left(\frac{r^2}{t}\right)^{\alpha}\frac{dydt}{r^n}\right) \\
&\qquad \times \sup_{(r,x)\in\mathbb{R}^{1+n}_+}\left(\int_0^{r^2}\int_{B(x,r)}|\nabla(u-v)(t,y)|^2\left(\frac{r^2}{t}\right)^{\alpha}\frac{dydt}{r^n}\right) \\
&\lesssim \left(|||u|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)}+|||v|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)}\right)|||u-v|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)} \\
&\lesssim \varepsilon|||u-v|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)}.
\end{aligned}$$

Putting the estimates of D_1 and D_2 together yields that if $\varepsilon > 0$ is sufficiently small then

$$|||\mathrm{T}u-\mathrm{T}v|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)} \lesssim (1+\varepsilon)\varepsilon|||u-v|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)} \le \beta|||u-v|||_{X^l_\alpha(\mathbb{R}^{1+n}_+)}$$

for some $\beta\in(0,1)$. Hence ‡ holds. Now, both † and ‡ are used to produce $\varepsilon > 0$ such that if

$$\|u_0\|_{\left((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\right)^l} \le \varepsilon$$

then there exists a unique solution $u\in\mathcal{B}_\varepsilon$ such that $u = \mathrm{T}u$ (according to the standard fixed point principle), i.e., u solves (1.2).

Next, we will show that there is a constant $\delta_{\mathcal{N}} > 0$ obeying

$$u(\mathbb{R}^{1+n}_+) \subset \mathcal{N}_{\delta_{\mathcal{N}}} = \left\{y\in\mathbb{R}^l:\ \mathrm{dist}(y,\mathcal{N})\le\delta_{\mathcal{N}}\right\}.$$

To do so, we utilize the triangle inequality and

$$\mathrm{c}_{t,x,r} = \fint_{B(0,r)} u_0(x-\sqrt{t}z)\,dz \quad \forall \quad (t,x,r)\in(0,\infty)\times\mathbb{R}^n\times(0,\infty)$$

to derive

$$\mathrm{dist}(u,\mathcal{N}) \le \|u-v_0\|_{(L^\infty(\mathbb{R}^{1+n}_+))^l} + |v_0-\mathrm{c}_{t,x,r}| + \mathrm{dist}(\mathrm{c}_{t,x,r},\mathcal{N}).$$

▷ From $u\in\mathcal{B}_\varepsilon$ it follows that

$$\|u-v_0\|_{(L^\infty(\mathbb{R}^{1+n}_+))^l} \le \varepsilon.$$

▷ For any $\delta>0$ we use

$$u_0:\mathbb{R}^n\to\mathcal{N}\subseteq\mathbb{R}^l$$

to achieve two constants $c_{\mathcal{N}}>0$ and $r=r(\delta,\mathcal{N})$ such that

$$\|u_0\|_{(L^\infty(\mathbb{R}^n))^l}\le c_{\mathcal{N}} \ \ \& \ \ c_{\mathcal{N}}\int_r^\infty \exp\left(-\frac{s^2}{4}\right)s^{n-1}ds\le\delta,$$

as well as

$$\begin{aligned}
|v_0-\mathrm{c}_{t,x,r}| &\le \int_{\mathbb{R}^n}(4\pi)^{-\frac{n}{2}}\exp\left(-\frac{|y|^2}{4}\right)\left|u_0(x-\sqrt{t}y)-\mathrm{c}_{t,x,r}\right|dy\\
&=\left(\int_{B(0,r)}+\int_{\mathbb{R}^n\setminus B(0,r)}\right)(4\pi)^{-\frac{n}{2}}\exp\left(-\frac{|y|^2}{4}\right)\left|u_0(x-\sqrt{t}y)-\mathrm{c}_{t,x,r}\right|dy\\
&\le \int_{B(0,r)}\left|u_0(x-\sqrt{t}y)-\mathrm{c}_{t,x,r}\right|dy+2\|u_0\|_{(L^\infty(\mathbb{R}^n))^l}\int_{\mathbb{R}^n\setminus B(0,r)}\exp\left(-\frac{|y|^2}{4}\right)dy\\
&\lesssim r^n\|u_0\|_{\left((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\right)^l}+c_{\mathcal{N}}\int_r^\infty\exp\left(-\frac{s^2}{4}\right)s^{n-1}ds\\
&\lesssim r^n\|u_0\|_{\left((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\right)^l}+\delta,
\end{aligned}$$

which can be also established by [**26**, (2.10)] and the known inclusion

$$(-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\subseteq BMO(\mathbb{R}^n)\quad\forall\quad\alpha\in[0,1).$$

▷ Using the condition that

$$u_0:\mathbb{R}^n\to\mathcal{N}\ \text{ is in }\ ((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^l,$$

we obtain

$$\begin{aligned}
\mathrm{dist}(\mathrm{c}_{t,x,r},\mathcal{N}) &\le \left|\mathrm{c}_{t,x,r}-u_0(x-\sqrt{t}y)\right|\\
&\le ⨍_{B(0,r)}\left|\mathrm{c}_{t,x,r}-u_0(x-\sqrt{t}y)\right|dy\\
&\lesssim \|u_0\|_{\left((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\right)^l}.
\end{aligned}$$

Accordingly, we have

$$\mathrm{dist}(u,\mathcal{N})\lesssim\varepsilon+\delta+(r^n+1)\|u_0\|_{\left((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\right)^l}$$

thereby finding

$$u(\mathbb{R}^{1+n}_+)\subseteq\mathcal{N}_{\delta_{\mathcal{N}}}$$

by choosing a suitable couple (δ,r).

Finally, upon setting

$$\rho(u)=2^{-1}|u-\Pi(u)|^2,$$

where $\Pi:\mathcal{N}_{\delta_{\mathcal{N}}}\to\mathcal{N}$ is the smooth nearest point projection map, and

$$\mathrm{II}(u)(\nabla u,\nabla u)=-\nabla^2\Pi(u)(\nabla u,\nabla u)\quad\text{on}\quad\mathbb{R}^{1+n}_+.$$

A straightforward calculation derives

$$u \in \mathcal{N}_{\delta_{\mathcal{N}}} \Longrightarrow \begin{cases} \nabla(u - \Pi(u))(v) = (\mathrm{Id} - \nabla\Pi(u))(v) \quad \forall\ v \in \mathbb{R}^l; \\ \nabla^2(u - \Pi(u))(v, w) = -\nabla^2\Pi(u)(v, w) \quad \forall\ v, w \in \mathbb{R}^l. \end{cases}$$

Since $u \in X^l_\alpha(\mathbb{R}^n)$, one has

$$|\nabla u| \in L^\infty([\varepsilon^2, \infty) \times \mathbb{R}^n) \quad \forall \quad \varepsilon > 0,$$

and consequently, the higher order regularity of (1.2) implies that u is of $C^2([\varepsilon^2, \infty) \times \mathbb{R}^n)$. Note that

$$(u - \Pi(u)) \perp T_{\Pi(u)}\mathcal{N} \quad \& \quad \nabla\Pi(u)(\nabla^2\Pi(u)(\nabla u, \nabla u)) \in T_{\Pi(u)}\mathcal{N}.$$

Accordingly,

$$\begin{aligned} &(\partial_t - \Delta)\rho(u) \\ &= \Big\langle (u - \Pi(u)), \nabla(u - \Pi(u))(\partial_t u - \Delta u) - \nabla^2(u - \Pi(u))(\nabla u, \nabla u)\Big\rangle - |\nabla(u - \Pi(u))|^2 \\ &= \Big\langle (u - \Pi(u)), \nabla\Pi(u)(\nabla^2\Pi(u)(\nabla u, \nabla u))\Big\rangle - |\nabla(u - \Pi(u))|^2 \\ &= -|\nabla(u - \Pi(u))|^2 \\ &\le 0. \end{aligned}$$

This, along with $\rho(u)(0, \cdot) = 0$ and the maximal principle, derives $\rho(u) \equiv 0$.

5. Application to liquid crystal flow

5.1. A bilinear estimation. Recall that any mild solution of the system

$$\begin{cases} u_t + u \cdot \nabla u - \Delta u + \nabla \mathrm{p} = -\nabla \cdot (\nabla \mathrm{d} \otimes \nabla \mathrm{d}) & \text{in } \mathbb{R}^{1+n}_+; \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^{1+n}_+; \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}^n, \end{cases} \tag{5.1}$$

can be represented as

$$u(t, x) = \mathrm{T}_1[u, \mathrm{d}](t, x) = v_0(t, x) - \mathrm{V}[u \otimes u + \nabla \mathrm{d} \otimes \nabla \mathrm{d}](t, x),$$

where

$$\begin{cases} v_0(t, x) = \int_{\mathbb{R}^n} \mathcal{G}(t, x - y)u_0(y)\, dy; \\ \mathrm{V}[u \otimes u + \nabla \mathrm{d} \otimes \nabla \mathrm{d}](t, x) = \int_0^t e^{-(t-s)\Delta}\mathrm{P}\nabla(u \otimes u + \nabla \mathrm{d} \otimes \nabla \mathrm{d})(s, x)\, ds; \\ \mathrm{P} = \{\mathrm{P}_{jk}\}_{j,k=1,\cdots,n} = \{\delta_{jk} + \mathrm{R}_j\mathrm{R}_k\}_{j,k=1,\cdots,n}; \\ \delta_{jk} = \text{the Kronecker symbol}; \\ \mathrm{R}_j = \partial_{x_j}(\sqrt{-\Delta})^{-1}. \end{cases}$$

Thus, we have the following bilinear estimation.

LEMMA 5.1. *Let* $(\alpha, n - 1) \in (-\infty, 1) \times \mathbb{N}$. *Then*

$$\begin{aligned} &\|V[u \otimes u + \nabla \mathrm{d} \otimes \nabla \mathrm{d}]\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} \\ &\qquad \lesssim \|u\|^2_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} + \|\mathrm{d}\|^2_{\mathrm{X}^3_\alpha(\mathbb{R}^{1+n}_+)} \ \forall\ (\mathrm{u}, \mathrm{d}) \in \mathrm{Z}^{\mathrm{n}}_\alpha(\mathbb{R}^{1+\mathrm{n}}_+) \times \mathrm{X}^3_\alpha(\mathbb{R}^{1+\mathrm{n}}_+). \end{aligned}$$

PROOF. It follows from an appropriate modification of the proofs of the well-posedness for the incompressible Navier-Stokes system in [**11**, Theorem 1] and [**30**, Theorem 1.4]. □

5.2. Proof of Theorem 1.8. Also, note that any mild solution to the system

$$\begin{cases} d_t + u\cdot\nabla d - \Delta d = |\nabla d|^2 d & \text{in } \mathbb{R}^{1+n}_+; \\ d|_{t=0} = d_0 & \text{in } \mathbb{R}^n, \end{cases} \tag{5.2}$$

can be written as

$$d(t,x) = T_2[u,d](t,x) = w_0 + S[-\nabla^2\Pi_{\mathbb{S}^2}(d)(\nabla d,\nabla d) - u\cdot\nabla d](t,x),$$

where

$$\begin{cases} w_0(t,x) = \int_{\mathbb{R}^n} \mathcal{G}(t,x-y)d_0(y)\,dy; \\ \Pi_{\mathbb{S}^2} \in (C^\infty(\mathbb{R}^3))^3; \\ \Pi_{\mathbb{S}^2}(d) = \frac{d}{|d|}:\ \ \mathbb{S}^2_{2^{-1}} \equiv \left\{y\in\mathbb{R}^3:\ \ 2^{-1}\le |y| \le \frac{3}{2}\right\} \to \mathbb{S}^2. \end{cases}$$

So, any mild solution of (1.3) (combining (5.1) & (5.2)) can be written as

$$(u,d) = (T_1[u,d], T_2[u,d]).$$

This suggests us to consider whether the following operator

$$T[u,d] = (T_1[u,d], T_2[u,d])$$

satisfies

$$T:\ \ Z^n_\alpha(\mathbb{R}^{1+n}_+)\times X^3_\alpha(\mathbb{R}^{1+n}_+) \to Z^n_\alpha(\mathbb{R}^{1+n}_+).$$

To deal with this issue, for

$$(v_0,w_0)\in Z^n_\alpha(\mathbb{R}^{1+n}_+)\times X^3_\alpha(\mathbb{R}^{1+n}_+) \quad \& \quad \delta>0$$

let

$$\breve{\mathscr{B}}_\delta = \left\{(u,d)\in Z^n_\alpha(\mathbb{R}^{1+n}_+)\times X^3_\alpha(\mathbb{R}^{1+n}_+) : \|u-v_0\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} + |||d-w_0|||_{X^3_\alpha(\mathbb{R}^{1+n}_+)} \le \delta\right\}.$$

Then, in accordance with the standard fixed point principle, our aim is to demonstrate that we can find a sufficiently small $\varepsilon>0$ to obey the implication that if

$$\|u_0\|_{\left(\left((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\right)^{-1}\right)^n} + \|d_0\|_{((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n))^3} \le \varepsilon$$

then

♮ $T:\breve{\mathscr{B}}_\varepsilon\to\breve{\mathscr{B}}_\varepsilon$;

♯ $\exists\,\beta\in(0,1)$ such that

$$\begin{aligned} &\|T_1[u,d]-T_1[v,e]\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} + |||T_2[u,d]-T_2[v,e]|||_{X^3_\alpha(\mathbb{R}^{1+n}_+)} \\ &\le \beta\left(\|u-v\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} + |||d-e|||_{X^3_\alpha(\mathbb{R}^{1+n}_+)}\right)\ \ \forall\ \ (u,d),\ (v,e)\in\breve{\mathscr{B}}_\varepsilon. \end{aligned}$$

For ♮, assume $(u,d)\in\breve{\mathscr{B}}_\varepsilon$. Then, by the triangle inequality and the boundary condition

$$\begin{cases} u_0(y) = v_0(0,y); \\ d_0(y) = w_0(0,y), \end{cases}$$

with

$$\begin{cases} \|u_0\|_{\left(\left((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\right)^{-1}\right)^n} \approx \sup_{(r,x)\in\mathbb{R}^{1+n}_+}\left(\int_0^{r^2}\int_{B(x,r)} |v_0(t,y)|^2\left(\frac{r^2}{t}\right)^\alpha \frac{dydt}{r^n}\right)^{\frac12}; \\ \|d_0\|_{\left((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\right)^3} \approx \sup_{(r,x)\in\mathbb{R}^{1+n}_+}\left(\int_0^{r^2}\int_{B(x,r)} |\nabla w_0(t,y)|^2\left(\frac{r^2}{t}\right)^\alpha \frac{dydt}{r^n}\right)^{\frac12}, \end{cases}$$

as well as the following standard gradient estimates

$$\begin{cases}\sup_{(r,x)\in\mathbb{R}^{1+n}_+} \sqrt{r}|v_0(r,x)| \lesssim \|u_0\|_{\left(\left((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\right)^{-1}\right)^n};\\ \sup_{(r,x)\in\mathbb{R}^{1+n}_+} \sqrt{r}|\nabla w_0(r,x)| \lesssim \|\mathrm{d}_0\|_{\left((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\right)^3},\end{cases}$$

we have

$$\begin{aligned}&\|u\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} + \|\mathrm{d}\|_{X^3_\alpha(\mathbb{R}^{1+n}_+)}\\ &\quad\le \|u-v_0\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} + \|\mathrm{d}-w_0\|_{X^3_\alpha(\mathbb{R}^{1+n}_+)} + \|v_0\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} + \|w_0\|_{X^3_\alpha(\mathbb{R}^{1+n}_+)}\\ &\quad\lesssim \varepsilon.\end{aligned}$$

Owing to

$$\mathrm{T}[u,\mathrm{d}] - (v_0,w_0) = \Big(-\mathrm{V}[u\otimes u + \nabla\mathrm{d}\otimes\nabla\mathrm{d}],\ \mathrm{S}[-\nabla^2\Pi_{\mathbb{S}^2}(\mathrm{d})(\nabla\mathrm{d},\nabla\mathrm{d}) - \mathrm{u}\cdot\nabla\mathrm{d}]\Big),$$

we apply Lemma 5.1 and (4.1) to calculate

$$\begin{aligned}&\|\mathrm{T}_1[u,\mathrm{d}] - v_0\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} + |||\mathrm{T}_2[u,\mathrm{d}] - w_0|||_{X^3_\alpha(\mathbb{R}^{1+n}_+)}\\ &\quad\lesssim \|u\|^2_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} + \|\mathrm{d}\|^2_{X^3_\alpha(\mathbb{R}^{1+n}_+)} + \|\nabla^2\Pi_{\mathbb{S}^2}(\mathrm{d})(\nabla\mathrm{d},\nabla\mathrm{d}) - \mathrm{u}\cdot\nabla\mathrm{d}\|_{\mathrm{Y}^3_\alpha(\mathbb{R}^{1+\mathrm{n}}_+)}\\ &\quad\lesssim \Big(\|u\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} + \|\mathrm{d}\|_{\mathrm{X}^3_\alpha(\mathbb{R}^{1+\mathrm{n}}_+)}\Big)^2\\ &\quad\lesssim \Big(\|u-v_0\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} + \|\mathrm{d}-w_0\|_{X^3_\alpha(\mathbb{R}^{1+n}_+)} + \|v_0\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} + \|w_0\|_{X^3_\alpha(\mathbb{R}^{1+n}_+)}\Big)^2\\ &\quad\lesssim \varepsilon,\end{aligned}$$

thereby achieving a sufficiently small $\varepsilon > 0$ such that

$$\|u_0\|_{\left(\left((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\right)^{-1}\right)^n} + \|\mathrm{d}_0\|_{((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^\mathrm{n}))^3} \le \varepsilon \Longrightarrow \mathrm{T}: \breve{\mathscr{B}}_\varepsilon \to \breve{\mathscr{B}}_\varepsilon.$$

For $\sharp$, suppose that (u,d) and (v,e) are in $\breve{\mathscr{B}}_\varepsilon$. Using the proof idea of ‡ in §4.2, (4.1) and Lemma 5.1, we produce some $\beta\in(0,1)$ to obey

$$\begin{aligned}&\|\mathrm{T}_1[u,\mathrm{d}] - \mathrm{T}_1[v,\mathrm{e}]\,\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} + |||\mathrm{T}_2[u,\mathrm{d}] - \mathrm{T}_2[v,\mathrm{e}]\,|||_{X^3_\alpha(\mathbb{R}^{1+n}_+)}\\ &\quad= \|\mathrm{V}[u\otimes u + \nabla\mathrm{d}\otimes\nabla\mathrm{d} - v\otimes v - \nabla\mathrm{e}\otimes\nabla\mathrm{e}]\,\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)}\\ &\quad\ + |||\mathrm{S}[-\nabla^2\Pi_{\mathbb{S}^2}(\mathrm{d})(\nabla\mathrm{d},\nabla\mathrm{d}) - u\cdot\nabla\mathrm{d} + \nabla^2\Pi_{\mathbb{S}^2}(\mathrm{e})(\nabla\mathrm{e},\nabla\mathrm{e}) + v\cdot\nabla\mathrm{e}]\,|||_{X^3_\alpha(\mathbb{R}^{1+n}_+)}\\ &\quad\lesssim \|\mathrm{V}\left((|u|+|v|)|u-v| + (|\nabla\mathrm{d}|+|\nabla\mathrm{e}|)|\nabla(\mathrm{d}-\mathrm{e})|\right)\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)}\\ &\quad\ + |||\mathrm{S}((|\nabla\mathrm{d}|+|\nabla\mathrm{e}|+|u|)|\nabla(\mathrm{d}-\mathrm{e})| + |\nabla\mathrm{e}|^2|\mathrm{d}-\mathrm{e}| + |u-v|\,|\nabla\mathrm{e}|)|||_{X^3_\alpha(\mathbb{R}^{1+n}_+)}\\ &\quad\lesssim \varepsilon\Big(\|u-v\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} + |||\mathrm{d}-\mathrm{e}|||_{X^3_\alpha(\mathbb{R}^{1+n}_+)}\Big)\\ &\quad\le \beta\Big(\|u-v\|_{Z^n_\alpha(\mathbb{R}^{1+n}_+)} + |||\mathrm{d}-\mathrm{e}|||_{X^3_\alpha(\mathbb{R}^{1+n}_+)}\Big)\end{aligned}$$

whenever $\varepsilon > 0$ is sufficiently small.

Now, the above-verified $\natural$ and $\sharp$ yield a small number $\varepsilon > 0$ such that if

$$\|u_0\|_{\left(\left((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^n)\right)^{-1}\right)^n} + \|\mathrm{d}_0\|_{((-\Delta)^{-\frac{\alpha}{2}}\mathscr{L}^{2,2\alpha}(\mathbb{R}^\mathrm{n}))^3} \le \varepsilon$$

then there exists a unique $(u,\mathrm{d})\in\breve{\mathscr{B}}_\varepsilon$ solving $(u,\mathrm{d}) = \mathrm{T}[u,\mathrm{d}]$ - namely - (u,d) solves (1.3) with

$$\mathrm{d}_t + u\cdot\nabla\mathrm{d} - \Delta\mathrm{d} = |\nabla\mathrm{d}|^2\mathrm{d}$$

being replaced by

$$\mathrm{d}_t + u\cdot\nabla\mathrm{d} - \Delta\mathrm{d} = -\nabla^2\Pi_{\mathbb{S}^2}(\mathrm{d})(\nabla\mathrm{d},\nabla\mathrm{d}).$$

Last of all, we are required to show $\mathrm{d}(\mathbb{R}^{1+n}_+) \subseteq \mathbb{S}^2$. Nevertheless, this is similar to the argument for Theorem 1.6:

▷ Firstly, we verify

$$u(\mathbb{R}^{1+n}_+) \subseteq \mathbb{S}^2_{2^{-1}}.$$

▷ Secondly, we consider the function

$$\rho(\mathrm{d}) = 2^{-1}|\mathrm{d} - \Pi_{\mathbb{S}^2}(\mathrm{d})|^2,$$

thereby getting

$$(\partial_t + u \cdot \nabla - \Delta)\rho(u) = -|\nabla(\mathrm{d} - \Pi_{\mathbb{S}^2}(\mathrm{d}))|^2 \leq 0.$$

▷ Thirdly, upon taking into account of both $\rho(d)(0, \cdot) = 0$ and the maximal principle, we conclude $\rho(d) \equiv 0$.

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