Dynamics of PDE, Vol.17, No.1, 75-95, 2020

Symmetric and uniform analytic solutions in phase space for Navier-Stokes equations

Qixiang Yang

Communicated by Jiahong Wu, received July 16, 2019.

Abstract. For incompressible Navier-Stokes equations, Leray believed that, for a blow up solution, the initial data and the solution should both have the same special structure for different time and proposed to consider self-similar solutions. Necas-Ruzicka-Sverak proved that self-similar solution has to be zero in 1996. In the study of ill-posedness, Yang-Yang-Wu find symmetry property plays an important role. In this paper, we consider two categories of symmetry properties. On the one hand, we found some kinds of symmetric solution has to be zero as self-similar solution. On the other hand, we prove that three kinds of symmetry initial data can have uniform analytic and symmetric solution in the general Fourier-Herz spaces. We use symmetric and uniform analytic functions to approximate the solution. We take two steps: (i) For these kinds of symmetry of initial data, we prove that the solution has also the same symmetric structure. (ii) We prove that the uniform analyticity is equivalent to the convolution inequality on Herz spaces.

Contents

2010 *Mathematics Subject Classification.* Primary 42B35; 35Q30; Secondary 76D03.

Key words and phrases. Navier-Stokes equations, Symmetry about the component of velocity field, Symmetry and anti-symmetry for the independent variable of velocity field, Uniform analyticity, Fourier-Herz space.

1. Introduction

We consider the incompressible Navier-Stokes equations on the half-space $\mathbb{R}_+ \times \mathbb{R}^3$,

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u - \nabla p = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \nabla \cdot u = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

Since Kato-Fujita [**16**] found mild solution for Navier-Stokes equations (1.1) in 1962, many works have considered the mild solution. Among all the known results on global well-posedness in different initial spaces, BMO^{-1}, studied by Koch-Tataru [**19**] in 2001, is the biggest critical initial spaces in the real version. $P^{-1}_{1,1}$, studied by Giga-Inui-Mahalov-Saal [**12**] in 2008, is the biggest critical initial spaces in the Fourier transform version. The other works consider the wellposedness in solution spaces with particular structure. See [**5, 6, 11, 13, 15, 17, 19, 21, 22, 23, 24, 25, 26, 32, 33, 34, 35, 36, 37, 38, 41, 42**]. Most of these works were finished after [**12**] and [**19**], but they have considered particular structure solutions and are still valued. So we try to consider the above equations in some new points and search special structure solutions.

Leray believed that, for a blow up solution, the initial data and the solution should both have the same special structure for different time and proposed to consider self-similar solutions. But in 1996, Necas-Ruzicka-Sverak [**31**] proved the only possible self-similar solutions are zero. Further, **Yang-Yang-Wu** [**43**] **found that the symmetry property plays an important role in the study of ill-posedness.** The study of special structure both initial data and solution have is relative to the rigidity of the equations and contribute to the study of Navier-Stokes equations (1.1). In this paper, we search **symmetric and uniform analytic solution**. Because of the structure of equations (1.1), we found **many kinds of symmetric initial data can not produce symmetric solution and have rigidity.** But we can prove still, for $m = 1, 2, 3$, if $u_0(x)$ satisfies symmetry property X_m and belongs to some critical Fourier Herz space $P^{2-\frac{3}{p}}_{p,q}$, then there exists global solution $u(t,x)$ such that $u(t,x)$ satisfies still symmetry property X_m and multiply by an exponential growth factor, its uniform norm $U_e(\xi) = \sup\limits_{t>0} e^{t^{\frac{1}{2}}|\xi|}|\hat{u}(t,\xi)|$ belongs to Herz space $H^{2-\frac{3}{p}}_{p,q}$. Such uniform norm for t means uniform analyticity.

Herz spaces $H^{\alpha}_{p,q}$ have been studied heavily in harmonic analysis. See Lu-Yang [**27**] and their references therein. Phase space $P^{\alpha}_{p,q}$ is defined by the functions whose Fourier transform belong to the Herz space $H^{\alpha}_{p,q}$. For $p = q$, Herz space becomes the weighted Lebesgue space $W_{\alpha,p}$.

Definition 1.1. For $\alpha \in \mathbb{R}, 1 \le p,q \le \infty$. We have

(i) $f(\xi) \in H^{\alpha}_{p,q}$, if $\{\sum\limits_{j\in\mathbb{Z}} 2^{qj\alpha}(\int_{2^j \le |\xi| \le 2^{j+1}} |f(\xi)|^p d\xi)^{\frac{q}{p}}\}^{\frac{1}{q}} < \infty$.

(ii) $f(x) \in P^{\alpha}_{p,q}$, if $\hat{f} \in H^{\alpha}_{p,q}$.

If $\alpha = 2 - \frac{3}{p}$, then the initial data space $P^{\alpha}_{p,q}$ becomes critical phase spaces.

There are two categories of symmetry of a vector field u: (i) symmetry about the component of velocity field. (ii) symmetry and anti-symmetry for the independent variable of velocity field. Divergence zero property and non-linearity restrict the possibility of symmetric solutions. We will show that a party of such symmetric initial data has only zero symmetric solution. Up to this paper, we found only the following two kinds of symmetric initial values can produce symmetric solution.

DEFINITION 1.2. We define the symmetry property of u according to the symmetry of its Fourier transform $\hat{u}$:

(i) u satisfies the symmetry property X_1, if

$$\hat{u}_1(\xi_1, \xi_2, \xi_3) = \hat{u}_2(\xi_2, \xi_1, \xi_3). \tag{1.2}$$

(ii) u satisfies the symmetry property X_2, if $\hat{u}$ has the same symmetry as the following polynomial function

$$\begin{pmatrix} \xi_2\xi_3 + & i\xi_1 \\ \xi_1\xi_3 + & i\xi_2 \\ \xi_1\xi_2 + & i\xi_3 \end{pmatrix} \tag{1.3}$$

(iii) We say u satisfies symmetry property X_3, if u satisfies both X_1 and X_2.

For equations (1.1), we have the following symmetric and uniform analytic solution result:

THEOREM 1.3. *Given* $p > \frac{3}{2}, 1 < q < \infty, \alpha = 2 - \frac{3}{p}$. *For each* $m = 1, 2, 3$, *assume that divergence zero* u_0 *satisfies symmetry property* X_m. *There exists constant* C_m *such that, if* $\|\hat{u}_0\|_{(H^{\alpha}_{p,q})^3} \le C_m$, *then equations* (1.1) *have a unique solution* $u(t, x)$ *which satisfies symmetry property* X_m *and*

$$\sup_{t>0} e^{t^{\frac{1}{2}}|\xi|} |\hat{u}(t, \xi)| \in H^{\alpha}_{p,q}. \tag{1.4}$$

We will prove this theorem by constructing symmetric and uniform analytic functions to approach the solution. Three reasons lead us to consider symmetric solutions:

REMARK 1.4. (i) Symmetry property plays an important role in the study of ill-posedness in [**43**].

(ii) Symmetric structure has relation to the rigidity of equations like self-similar solution does. A party of symmetric solution has only zero solution.

(iii) The symmetry property is applied to the dimensionality reduction, see Abidi [**1**] and Abidi-Zhang [**2**]. The study of symmetric solution is helpful to understand how the symmetric solution result in dimensionality reduction, see also Corollary 5.1.

For the two categories of symmetry, we note

REMARK 1.5. (i) As to the symmetry on the component of velocity field, the symmetry property X_1 is a generalization of sub-radial property which has been considered in many papers. See Abidi [**1**] and Abidi-Zhang [**2**]. For the incompressible Navier-Stokes equations (1.1), radial solution has to be zero. But there exists sub-radial solution. Such solution is related to the axisymmetric initial data with zero swirl. See also Remark 5.2. X_1 makes the vorticity of the third axi to be zero.

(ii) The independent variables of vector field u have symmetry property means, the real part and imaginary part of each component of Fourier transform all have axi-symmetry or anti-axi-symmetry. Since a real function has $2^3 = 8$ kinds of aximmetry and anti-axi-symmetry property and a complex function has $8^2 = 64$ kinds of aximmetry and anti-axi-symmetry property, **hence a vector function has $64^3 = 262144$ kinds of possible symmetry property.** Divergence zero and non-linearity restrict the possibility of symmetric solution. **I have taken a long time to check many kinds of symmetry properties and finally, we found symmetry property X_2 can produce symmetric solution.** See both subsection 5.1.1 and remark 5.9. **I did not found any paper which has considered such symmetric solution.** In further study, I have found in another paper [**40**], there is only one kind of symmetry real valued initial data $u_0(x)$ can generate symmetric solution.

As we know, the general symmetric solutions have never been considered before and the analytic solutions for initial data in general Fourier-Herz spaces $P^{\alpha}_{p,q}$ have never been considered before. In fact,

REMARK 1.6. (i) Without considering analyticity, $P^{-1}_{1,1}$ in [**12**] is the biggest initial spaces among all the critical phase spaces $P^{\alpha}_{p,q}$. **We guess $P^{2-\frac{3}{p}}_{p,\infty} (p > \frac{3}{2})$ is the biggest critical initial data spaces which allows uniform analytic solution**, but we did not know how to verify this.

(ii) The following work considered the well-posedness without considering any analyticity: Cannone-Karch [**7**] considered the case $P^{2}_{\infty,\infty}$. Giga-Inui-Mahalov-Saal [**12**] and Lei-Lin [**21**] considered the case $P^{-1}_{1,1}$. Cannone-Wu [**8**] considered the case $P^{-1}_{1,q}$. Further, Xiao-Chen-Fan-Zhou [**39**] considered the general case.

(iii) As to the analyticity, see [**11**] and [**30**]. Biswas considered Fourier version weighted Lebesgue space $(W_{\alpha,p})^3$ which is the particular Fourier-Herz space $(P^{\alpha}_{p,p})^3$. Applying Theorem 3.2 in [**3**] to the equations (1.1), there exists constant C such that $\|u_0\|_{(W_{\alpha,p})^3} \leq C$, then there exists a unique solution $u(t,x) \in C([0,\infty),(W_{\alpha,p})^3)$ satisfying

$$\sup_{0<t<\infty} \|\hat{u}\|_{G_{\theta_0}(t)} \leq 2C, \tag{1.5}$$

where $\theta_0 = \frac{3}{p'} - 1$ and

$$\|v\|_{G_\theta(t)} = \Big\{ \int_{\mathcal{R}^n} e^{t^{\frac{1}{2}} p|\xi|} \|\xi|^{\theta p} |v(t,\xi)|^p d\xi \Big\}^{\frac{1}{p}}.$$

Since (1.4) **is stronger than** (1.5)**, so our result naturally implies the above result. Further, their methods can not be applied to consider the uniform analyticity.**

(iv) As to the uniform analyticity, Le Jan-Sznitman [**20**] consider the case $p = q = \infty$ by applying convolution inequality. For $p = q = 2$, which can be found at Chapter 24 in the book of Lemarié 2002 [**22**], whose proof is made by applying paraproduct and space $\dot{B}^{-1,\infty}_{\infty}$. **In this paper, we can consider uniform analytic solution for general p and q. Our main idea is convolution inequality on Herz spaces.**

The structure of the rest paper is as follows: In section 2, we introduce some history on mild solution and present a result on uniform analytic solution. In section 3, we show how to transform the wellposedness result to the convolution inequality. In section 4, we prove the convolution inequality on Herz spaces. In section 5, we present first how divergence zero exerts influence on symmetry property. Then we show how the iterative algorithm (2.6) inherits the two kinds of symmetry properties. At the end of section 5, we narrate a corollary on how the symmetry property reduce the number of induction function in the iterative algorithm (2.6). In the last section 6, we prove our main theorem on symmetric and uniform analytic solution.

Acknowledgement (i) The reviewers have given many good suggestions to me, I would like to express my sincere gratitude to them.

(ii) This paper is financially supported by the China National Science Foundation (No.11571261).

2. Mild solution and uniform analyticity

The notion of mild solution was pioneered by Kato-Fujita [**16**] in 1960s. Denote

$$\mathbb{P}\nabla(u\otimes u)=\sum_{l}\partial_l(u_l u)-(-\Delta)^{-1}\sum_{l,l'}\partial_l\partial_{l'}\nabla(u_l u_{l'}). \tag{2.1}$$

A solution of the above Cauchy problem (1.1) is then obtained via the following integral equation

$$u(t,x)=e^{t\Delta}u_0(x)-B(u,u)(t,x), \tag{2.2}$$

where

$$B(u,v)(t,x)\equiv\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla(u\otimes v)ds. \tag{2.3}$$

The equation (2.2) can be solved by a fixed-point method whenever the convergence is suitably defined in some function space. Denote

$$\begin{array}{lcl} u^0(t,x) & = & e^{t\Delta}u_0\ ; \\ u^{\tau+1}(t,x) & = & u^0(t,x)-B(u^\tau,u^\tau)(t,x),\quad \forall\tau=0,1,2,\cdots. \end{array} \tag{2.4}$$

For $u_0\in X_0^3$, if there exists X^3 such that $u^0(t,x)\in X^3$ and iterative quantity $u^\tau(t,x)$ converge to some function $u(t,x)\in X^3$, then $u(t,x)$ is the solution of (2.2). $u(t,x)$ is the mild solutions of (1.1). The initial data space X_0^3 is called to be critical for (1.1) in the sense that the space is invariant under the scaling

$$u_0^\lambda=\lambda u_0(\lambda x).$$

Note that (1.1) is invariant under the scaling

$$u_\lambda=\lambda u(\lambda^2 t,\lambda x)$$

and

$$p_\lambda(t,x)=\lambda^2 p(\lambda^2 t,\lambda x).$$

Hence critical spaces occupied important position.

During the latest decades, many important results about the mild solutions to (1.1) have been established. See for example, Cannone [**5, 6**], Germain-Pavlovic-Staffilani [**11**], Giga-Miyakawa [**13**], Kato [**15**], Kato-Ponce [**17**], Koch-Tataru [**19**], Taylor [**32**], Wu [**33, 34, 35, 36**] and also the book [**22**]. Extending Koch-Tataru's $\mathrm{BMO}^{-1}(\mathbb{R}^n)$ space in [**19**], Xiao [**37, 38**] introduced the Q-spaces $Q^{-1}_{0<\alpha<1}(\mathbb{R}^n)$ to investigate the global existence and uniqueness of the classical Navier-Stokes system. Further, applying the wavelets in [**29**], we have extended these results to many function spaces: Trieble-Lizorkin spaces, Besov Morrey spaces and Trieble-Lizorkin Morrey spaces (see [**23, 24, 25, 26, 41, 42**]).

Recently, there were also many people considered solution in phase spaces. Taking Fourier transform for Bilinear operator $B(u,v)$ in (2.3), we get

$$\tilde{B}(\hat{u},\hat{v})(t,\xi)=\int_0^t e^{-(t-s)|\xi|^2}\sum_l \xi_l \hat{u}_l*\hat{v}ds+\xi\int_0^t e^{-(t-s)|\xi|^2}\sum_{l,l'}\frac{\xi_l\xi_{l'}}{|\xi|^2}\hat{u}_l*\hat{v}_{l'}ds. \tag{2.5}$$

The above iterative process in (2.4) is equivalent to the following

$$\hat{u}^{\tau+1}(t,\xi)=e^{-t|\xi|^2}\hat{u}_0(\xi)+i\tilde{B}(\hat{u}^\tau,\hat{u}^\tau)(t,\xi),\forall\tau\geq 0. \tag{2.6}$$

In this paper, we consider initial value u_0 belongs to phase space $(P^\alpha_{p,q})^3$. Without considering analyticity, Giga-Inui-Mahalov-Saal [**12**] and Lei-Lin [**21**] obtained a global existence result for small initial data in $P^{-1}_{1,1}$. Xiao-Chen-Fan-Zhou [**39**] considered the general cases. For uniform analytic solution, only two cases of Fourier-weighted Lebesgue spaces have been considered. $P^2_{\infty,\infty}$ has been studied in Le Jan-Sznitman [**20**] (see also [**4**] and [**22**]). $P^{\frac{1}{2}}_{2,2}$ has been studied by Fujita-Kato [**10**] (see also [**22**]). Fourier-weighted Lebesgue spaces are special Fourier-Herz spaces $P^\alpha_{p,p}$.

Among all the known results of global well-posedness, BMO^{-1}, studied by Koch-Tataru in 2001, is the biggest critical initial spaces in the real version. $P^{-1}_{1,1}$, studied by Giga-Inui-Mahalov-Saal in 2008, is the biggest critical initial spaces in the Fourier transform version. Further, A great party of these works were finished after [**12**] and [**19**], but they have considered particular structure solutions and are still valued. So we try to consider the above equations in some new point and search special structure solutions. For $u_0\in X_0^3$ or $\hat{u}_0(\xi)\in X_0^3$,

Definition 2.1. (i) The solution $u(t,x)$ is analytic means there exists $0<\gamma<1$ such that

$$\sup_{t>0}\|e^{-t^\gamma(-\Delta)^\gamma}u(t,x)\|_{X_0^3}<\infty. \tag{2.7}$$

(ii) The solution $u(t,x)$ is uniform analytic means there exists $0<\gamma<1$ such that

$$\|\sup_{t>0}e^{-t^\gamma(-\Delta)^\gamma}u(t,x)\|_{X_0^3}<\infty, \tag{2.8}$$

or

$$\|\sup_{t>0}e^{-t^\gamma|\xi|^{2\gamma}}\hat{u}(t,\xi)\|_{(X_0)^3}<\infty. \tag{2.9}$$

The earliest study of space analyticity can be found in Masuda [**28**]. Foias-Teman [**9**] have shown the space analyticity of solution for period function in Sobolev space. Germain-Pavlović-Staffilani [**11**] constructed the solutions with space analyticity for small initial data in BMO^{-1} (see also [**30**]). Biswas [**3**] consider Gevrey regularity. Uniform analyticity (2.8) or (2.9) mean uniform norm for t, which is stronger than analyticity (2.7). Analyticity has been considered by many people, but there exist only a few works which considered uniform analyticity. See Le Jan-Sznitman [**20**] and Lemarie [**22**]. See also [**11**], [**28**] and [**30**]. Uniform norm for t means that the dilation does not bring influence to the solution.

For a scale function $f(t,x)$, denote its Fourier transform by $\hat{f}(t,\xi)$, its uniform quantity for t denote by $F(\xi)=\sup_{t>0}|\hat{f}(t,\xi)|$ and whose uniform exponential decay quantity denote by $F_e(\xi)=\sup_{t>0}e^{t^{\frac{1}{2}}|\xi|}|\hat{f}(t,\xi)|$. Herz spaces have been studied heavily in harmonic analysis, see [**27**]. We introduce the definition of uniform analyticity in Herz spaces.

DEFINITION 2.2. $\hat{f}(t,\xi) \in S^{\alpha}_{p,q}$, if $F_e(\xi) \in H^{\alpha}_{p,q}$.

In this section, we consider uniform analytic solution in Fourier-Herz spaces. Our main skills are to prove that the existence of uniform analytic solution is equivalent to the boundedness of convolution inequality on Herz spaces.

THEOREM 2.3. *Given $p > \frac{3}{2}, 1 < q < \infty$ and $\alpha = 2 - \frac{3}{p}$. There exists constant C such that for $u_0 \in P^{\alpha}_{p,q}$ and $\|\hat{u}_0\|_{(H^{\alpha}_{p,q})^3} \leq C$, there exists a unique solution $u(t,x)$ such that*

$$\hat{u}(t,\xi) \in (S^{\alpha}_{p,q})^3. \tag{2.10}$$

Such uniform analyticity in phase space will be considered in subsection 3.3 by the estimate of convolution inequality.

3. From well-posedness to convolution inequality

3.1. Bilinear operator $\tilde{\tilde{B}}$ in phase space. Denote $|\hat{u}(t,\xi)| = \sup\limits_{l=1,2,3} |\hat{u}_l(t,\xi)|$ and

$$\tilde{\tilde{B}}(u,v)(t,\xi) = |\xi| \int_0^t e^{-(t-s)|\xi|^2} (|\hat{u}(s,\cdot)| * |\hat{v}(s,\cdot)|)(\xi) ds \tag{3.1}$$

THEOREM 3.1. *Given $1 \leq p, q \leq \infty$ and $\alpha \in \mathbb{R}$.*

$$\tilde{\tilde{B}} \text{ is bounded from } S^{\alpha}_{p,q} \times S^{\alpha}_{p,q} \text{ to } S^{\alpha}_{p,q} \tag{3.2}$$

implies

$$\tilde{B} \text{ is bounded from } (S^{\alpha}_{p,q})^3 \times (S^{\alpha}_{p,q})^3 \text{ to } (S^{\alpha}_{p,q})^3. \tag{3.3}$$

PROOF. For $k, l, l' \in \{1,2,3\}$, denote

$$\tilde{B}_{l,l'}(\hat{u}_l, \hat{v}_{l'})(t,\xi) = \int_0^t e^{-(t-s)|\xi|^2} \sum_l \xi_l \hat{u}_l * \hat{v}_{l'} ds$$

and

$$\tilde{B}_{k,l,l'}(\hat{u}_l, \hat{v}_{l'})(t,\xi) = \int_0^t e^{-(t-s)|\xi|^2} \sum_{l,l'} \frac{\xi_k \xi_l \xi_{l'}}{|\xi|^2} \hat{u}_l * \hat{v}_{l'} ds$$

We note that

$$\begin{aligned} w(t,\xi) &\equiv \sup_{l,l'=1,2,3} |\tilde{B}_{l,l'}(\hat{u}_l, \hat{v}_{l'})(t,\xi)| + \sup_{k,l,l'=1,2,3} |\tilde{B}_{k,l,l'}(\hat{u}_l, \hat{v}_{l'})(t,\xi)| \\ &\leq C\tilde{\tilde{B}}(u,v)(t,\xi). \end{aligned} \tag{3.4}$$

and $\hat{u}(t,\xi) \in (S^{\alpha}_{p,q})^3$ implies $|\hat{u}(t,\xi)| \in S^{\alpha}_{p,q}$. That is to say, if $\tilde{\tilde{B}}$ satisfies (3.2), then

$$w(t,\xi) \in S^{\alpha}_{p,q}. \tag{3.5}$$

The equations (3.4) and (3.5) imply

$$\tilde{B}(\hat{u},\hat{v})(t,\xi) \in (S^{\alpha}_{p,q})^3. \tag{3.6}$$

The equation (3.6) means $\tilde{B}$ satisfied the condition (3.3). $\square$

3.2. From boundedness of $\tilde{\tilde{B}}$ to the convolution inequality. In this subsection, we transfer the boundedness of bilinear operator $\tilde{\tilde{B}}$ to the estimation of the following convolution inequality on Herz spaces $H^{\alpha}_{p,q}$.

THEOREM 3.2. *If $p > \frac{3}{2}, \alpha = 2 - \frac{3}{p}$ and $1 < q < \infty$, then*

$$\||\xi|^{-1}(U * V)(\xi)\|_{H^{\alpha}_{p,q}} \leq C\|U\|_{H^{\alpha}_{p,q}}\|V\|_{H^{\alpha}_{p,q}}. \tag{3.7}$$

REMARK 3.3. The weighted Lebesgue cases have been considered in Kerman [**18**]. (3.7) extends Kerman's result to Herz spaces.

In the rest of this subsection, we show how the above convolution inequality implies the boundedness of $\tilde{\tilde{B}}$.

THEOREM 3.4. *Given $p > \frac{3}{2}, \alpha = 2 - \frac{3}{p}$ and $1 < q < \infty$. If the inequality* (3.7) *is true, then $\tilde{\tilde{B}}$ is bounded from $S^{\alpha}_{p,q} \times S^{\alpha}_{p,q}$ to $S^{\alpha}_{p,q}$.*

PROOF. We claim the following inequality

$$e^{-(t-s)|\xi|^2} e^{-s^{\frac{1}{2}}|\xi-\eta|} e^{-s^{\frac{1}{2}}|\eta|} \leq e^2 e^{-t^{\frac{1}{2}}|\xi|} e^{-\frac{1}{2}(t-s)|\xi|^2}. \tag{3.8}$$

In fact, using the triangle inequality gives

$$e^{-(t-s)|\xi|^2} e^{-s^{\frac{1}{2}}|\xi-\eta|} e^{-s^{\frac{1}{2}}|\eta|} \leq e^{-\frac{1}{2}(t-s)|\xi|^2 + (t^{\frac{1}{2}} - s^{\frac{1}{2}})|\xi|} e^{-t^{\frac{1}{2}}|\xi|} e^{-\frac{1}{2}(t-s)|\xi|^2}.$$

Let $I = -\frac{1}{2}(t-s)|\xi|^2 + (t^{\frac{1}{2}} - s^{\frac{1}{2}})|\xi|$, then it suffices to prove

$$I \leq 2.$$

Note that

$$x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1).$$

Hence, we have

$$I = (t^{\frac{1}{2}} - s^{\frac{1}{2}})|\xi|\Big(1 - \frac{1}{2}(t^{\frac{1}{2}} + s^{\frac{1}{2}})|\xi|\Big).$$

On the one hand, if $(t^{\frac{1}{2}} + s^{\frac{1}{2}})|\xi| \geq 2$, then we have $I \leq 0 < 2$. On the other hand, if $(t^{\frac{1}{2}} + s^{\frac{1}{2}})|\xi| \leq 2$, then we have $I \leq t^{\frac{1}{2}}|\xi| \leq 2$.

Denote $w_e(t,\xi) = e^{t^{\frac{1}{2}}|\xi|} w(t,\xi)$, $u_e(t,\xi) = e^{t^{\frac{1}{2}}|\xi|} u(t,\xi)$ and $v_e(t,\xi) = e^{t^{\frac{1}{2}}|\xi|} v(t,\xi)$. Hence (3.1) and (3.8) imply the following inequality:

$$w_e(t,\xi) \leq e^2|\xi| \int_0^t e^{-\frac{1}{2}(t-s)|\xi|^2} ds(|u_e(t,\cdot)| * |v_e(t,\cdot)|)(\xi) ds. \tag{3.9}$$

Let $W_e(\xi) = \sup_{t>0} |w_e(t,\xi)|$, $U_e(\xi) = \sup_{t>0} |u_e(t,\xi)|$ and $V_e(\xi) = \sup_{t>0} |v_e(t,\xi)|$. Taking supremum for s and t in the equation (3.1), we get

$$W_e(\xi) \leq C|\xi| \sup_{t>0} \int_0^t e^{-\frac{1}{2}(t-s)|\xi|^2} ds(U_e * V_e)(\xi) \leq C|\xi|^{-1}(U_e * V_e)(\xi).$$

Hence the convolution inequality (3.7) implies the boundedness of $\tilde{\tilde{B}}$ form $S^{\alpha}_{p,q} \times S^{\alpha}_{p,q}$ to $S^{\alpha}_{p,q}$. $\square$

3.3. Proof of theorem 2.3. In this subsection, we prove theorem 2.3. For a distribution u_0, Picard's iterative process is to find out a mild solution near some function u^0. See Cannone [**6**] and Koch-Tataru [**19**]; see also Li-Yang [**24, 25**] and Lin-Yang [**26**].

PROOF. For small initial value u_0 in some space X_0^3, if there exists certain space X^3 such that

$$\text{the operator } e^{t\Delta} \text{ is continuous from } X_0^3 \text{ to } X^3 \tag{3.10}$$

and

$$B \text{ is bounded operator from } X^3 \times X^3 \text{ to } X^3, \tag{3.11}$$

then $B(u^\tau, u^\tau)(t,x)$ defined in the equation (2.4) is an error term. Picard's contraction principle tells that u^τ converges to the unique solution of the above Navier-Stokes equation (1.1).

Take $X_0^3 = (P^\alpha_{p,q})^3$ and $X^3 = (S^\alpha_{p,q})^3$. It is easy to see

THEOREM 3.5. *Given $1 \leq p, q \leq \infty, \alpha \in \mathbb{R}$. We have*

$$e^{t\Delta} \textit{ is continuous from } (P^\alpha_{p,q})^3 \textit{ to } (S^\alpha_{p,q})^3.$$

By theorems 3.1, 3.2, 3.4 and 3.5, we know Theorem 2.3 is true. □

4. The proof of convolution inequality

For $1 < p, q \leq \infty$, denote $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$. To simplify the proof, we prove only the case $\frac{3}{2} < p < 3$ in this section. For $p \geq 3$, we have to use the duality of Herz spaces. For $1 < p, q < \infty, \alpha \in \mathbb{R}$, we have $(H^\alpha_{p,q})' = H^{-\alpha}_{p',q'}$. If we use auxiliary function $h \in H^{-\alpha}_{p',q'}$ and the similar idea in this section, we can prove that $\int |\xi|^{-1} |U(\xi)| * |V(\xi)||h(\xi)||\xi|^{-\alpha} d\xi$ is bounded, and the above result is true for all $p > \frac{3}{2}$.

Denote $W_{U,V} = \sum\limits_{j\in\mathbb{Z}} 2^{qj(1-\frac{3}{p})} \{\int_{2^j \leq |\xi| \leq 2^{j+1}} (|U(\xi)| * |V(\xi)|)^p d\xi\}^{\frac{q}{p}}$. We decompose the integration of ξ to the dyadic ring and get

$$W_{U,V} \leq C \sum_{j\in\mathbb{Z}} 2^{qj(1-\frac{3}{p})} \{\int_{2^j \leq |\xi| \leq 2^{j+1}} (\int |U(\xi-\eta)||V(\eta)| d\eta)^p d\xi\}^{\frac{q}{p}}.$$

We decompose then the integration of η to three parts and get

$$\begin{aligned} W_{U,V} \leq\ & C \sum_{j\in\mathbb{Z}} 2^{qj(1-\frac{3}{p})} \{\int_{2^j \leq |\xi| \leq 2^{j+1}} (\int_{|\eta| \leq 2^{j-1}} |U(\xi-\eta)||V(\eta)| d\eta)^p d\xi\}^{\frac{q}{p}} \\ & + C \sum_{j\in\mathbb{Z}} 2^{qj(1-\frac{3}{p})} \{\int_{2^j \leq |\xi| \leq 2^{j+1}} (\int_{|\eta| \geq 2^{j+2}} |U(\xi-\eta)||V(\eta)| d\eta)^p d\xi\}^{\frac{q}{p}} \\ & + C \sum_{j\in\mathbb{Z}} 2^{qj(1-\frac{3}{p})} \{\int_{2^j \leq |\xi| \leq 2^{j+1}} (\int_{2^{j-1} \leq |\eta| \leq 2^{j+2}} |U(\xi-\eta)||V(\eta)| d\eta)^p d\xi\}^{\frac{q}{p}} \\ \equiv\ & I_1 + I_2 + I_3. \end{aligned}$$

For I_1, I_2 and I_3, we apply different way to enlarge the convolution part and for $i = 1, 2, 3$, we prove $I_i \leq C \|U\|^q_{H^\alpha_{p,q}} \|V\|^q_{H^\alpha_{p,q}}$.

For I_1, we have

$$\begin{aligned} I_1 &\leq C\sum_{j\in\mathbb{Z}} 2^{qj(1-\frac{3}{p})}\{\int_{2^j\leq|\xi|\leq 2^{j+1}}\int_{|\eta|\leq 2^{j-1}}|U(\xi-\eta)|^p|V(\eta)|d\eta d\xi \\ &\quad (\int_{|\eta|\leq 2^{j-1}}|V(\eta)|d\eta)^{p-1}\}^{\frac{q}{p}} \\ &\leq C\sum_{j\in\mathbb{Z}} 2^{qj(1-\frac{3}{p})}\{\int_{2^{j-1}\leq|\xi|\leq 5\cdot 2^j}|U(\xi)|^p d\xi(\int_{|\eta|\leq 2^{j-1}}|V(\eta)|d\eta)^p\}^{\frac{q}{p}} \\ &= C\sum_{j\in\mathbb{Z}} 2^{qj(1-\frac{3}{p})}\{\int_{2^{j-1}\leq|\xi|\leq 5\cdot 2^j}|U(\xi)|^p d\xi\}^{\frac{q}{p}}(\int_{|\eta|\leq 2^{j-1}}|V(\eta)|d\eta)^q. \end{aligned}$$

Further,

$$\begin{aligned} \int_{|\eta|\leq 2^{j-1}}|V(\eta)|d\eta &= \sum_{j'\leq j-2}\int_{2^{j'}\leq|\eta|\leq 2^{j'+1}}|V(\eta)|d\eta \\ &\leq C\sum_{j'\leq j-2} 2^{\frac{3j'}{p'}}\{\int_{2^{j'}\leq|\eta|\leq 2^{j'+1}}|V(\eta)|^p d\eta\}^{\frac{1}{p}} \\ &= C\sum_{j'\leq j-2} 2^{j'\alpha}\{\int_{2^{j'}\leq|\eta|\leq 2^{j'+1}}|V(\eta)|^p d\eta\}^{\frac{1}{p}} 2^{j'} \\ &\leq C(\sum_{j'\leq j-2} 2^{qj'\alpha}\{\int_{2^{j'}\leq|\eta|\leq 2^{j'+1}}|V(\eta)|^p d\eta\}^{\frac{q}{p}})^{\frac{1}{q}}(\sum_{j'\leq j-2} 2^{q'j'})^{\frac{1}{q'}} \\ &\leq 2^j(\sum_{j'\leq j-2} 2^{qj'\alpha}\{\int_{2^{j'}\leq|\eta|\leq 2^{j'+1}}|V(\eta)|^p d\eta\}^{\frac{q}{p}})^{\frac{1}{q}}. \end{aligned}$$

Hence

$$I_1 \leq C\|U\|^q_{H^\alpha_{p,q}}\|V\|^q_{H^\alpha_{p,q}}.$$

Then we consider I_3. Take $\alpha < \lambda < \frac{3}{p'}$. We have

$$\begin{aligned} &I_{U,V} = \int_{2^j\leq|\xi|\leq 2^{j+1}}(\int_{2^{j-1}\leq|\eta|\leq 2^{j+2}}|U(\xi-\eta)||V(\eta)|d\eta)^p d\xi \\ \leq\ &\int_{2^j\leq|\xi|\leq 2^{j+1}}\int_{2^{j-1}\leq|\eta|\leq 2^{j+2}}|\xi-\eta|^{\lambda p}|U(\xi-\eta)|^p|V(\eta)|^p d\eta \\ &(\int_{2^{j-1}\leq|\eta|\leq 2^{j+2}}|\xi-\eta|^{-\lambda p'}d\eta)^{p-1}d\xi \\ \leq\ &\int_{2^j\leq|\xi|\leq 2^{j+1}}\int_{2^{j-1}\leq|\eta|\leq 2^{j+2}}|\xi-\eta|^{\lambda p}|U(\xi-\eta)|^p|V(\eta)|^p d\eta \\ &(\int_{|\xi-\eta|\leq 3\cdot 2^{j+1}}|\xi-\eta|^{-\lambda p'}d\eta)^{p-1}d\xi \\ \leq\ &2^{(3-\lambda p')(p-1)j}\int_{2^j\leq|\xi|\leq 2^{j+1}}\int_{2^{j-1}\leq|\eta|\leq 2^{j+2}}|\xi-\eta|^{\lambda p}|U(\xi-\eta)|^p|V(\eta)|^p d\eta d\xi \\ \leq\ &2^{(3-\lambda p')(p-1)j}\int_{|\xi|\leq 2^{j+3}}|\xi|^{\lambda p}|U(\xi)|^p d\xi\int_{2^{j-1}\leq|\eta|\leq 2^{j+2}}|V(\eta)|^p d\eta. \end{aligned}$$

(i) For $q > p$, we have

$$\begin{aligned} &\int_{|\xi|\leq 2^{j+3}}|\xi|^{\lambda p}|U(\xi)|^p d\xi = \sum_{j'\leq j+2}\int_{2^{j'}\leq|\xi|\leq 2^{j'+1}}|\xi|^{\lambda p}|U(\xi)|^p d\xi \\ \leq\ &C\sum_{j'\leq j+2} 2^{p\lambda j'}\int_{2^{j'}\leq|\xi|\leq 2^{j'+1}}|U(\xi)|^p d\xi \\ =\ &C\sum_{j'\leq j+2} 2^{p(\lambda-\alpha)j'}2^{p\alpha j'}\int_{2^{j'}\leq|\xi|\leq 2^{j'+1}}|U(\xi)|^p d\xi \\ \leq\ &C\{\sum_{j'\leq j+2} 2^{q\alpha j'}(\int_{2^{j'}\leq|\xi|\leq 2^{j'+1}}|U(\xi)|^p d\xi)^{\frac{q}{p}}\}^{\frac{p}{q}}\{\sum_{j'\leq j+2} 2^{\frac{pq}{q-p}(\lambda-\alpha)j'}\}^{\frac{q-p}{q}} \\ \leq\ &2^{pj-(3-\lambda p')(p-1)j}\{\sum_{j'\leq j+2} 2^{qj'\alpha}(\int_{2^{j'}\leq|\xi|\leq 2^{j'+1}}|U(\xi)|^p d\xi)^{\frac{q}{p}}\}^{\frac{p}{q}}. \end{aligned}$$

Hence

$$I_{U,V} \leq 2^{pj}\{\sum_{j'\leq j+2} 2^{qj'\alpha}(\int_{2^{j'}\leq|\xi|\leq 2^{j'+1}}|U(\xi)|^p d\xi)^{\frac{q}{p}}\}^{\frac{p}{q}}\int_{2^{j-1}\leq|\eta|\leq 2^{j+2}}|V(\eta)|^p d\eta.$$

So we get

$$I_3 \leq C \sum_{j\in\mathbb{Z}} 2^{q\alpha j} \sum_{j'\leq j+2} 2^{qj'\alpha} \Big(\int_{2^{j'}\leq|\xi|\leq 2^{j'+1}} |U(\xi)|^p d\xi\Big)^{\frac{q}{p}} \Big\{\int_{2^{j-1}\leq|\eta|\leq 2^{j+2}} |V(\eta)|^p d\eta\Big\}^{\frac{q}{p}}.$$

Hence

$$I_3 \leq C\|U\|^q_{H^\alpha_{p,q}} \|V\|^q_{H^\alpha_{p,q}}.$$

(ii) For $q \leq p$, we have

$$\begin{aligned} & \{\textstyle\int_{|\xi|\leq 2^{j+3}} |\xi|^{\lambda p} |U(\xi)|^p d\xi\}^{\frac{q}{p}} \\ = & \{\textstyle\sum_{j'\leq j+2} \int_{2^{j'}\leq|\xi|\leq 2^{j'+1}} |\xi|^{\lambda p} |U(\xi)|^p d\xi\}^{\frac{q}{p}} \\ \leq & C \textstyle\sum_{j'\leq j+2} 2^{q\lambda j'} \{\int_{2^{j'}\leq|\xi|\leq 2^{j'+1}} |U(\xi)|^p d\xi\}^{\frac{q}{p}}. \end{aligned}$$

Hence

$$\begin{aligned} \{I_{U,V}\}^{\frac{q}{p}} \leq\ & 2^{qj(\frac{3}{p'}-\lambda)} \sum_{j'\leq j+2} 2^{qj'\lambda} \Big(\int_{2^{j'}\leq|\xi|\leq 2^{j'+1}} |U(\xi)|^p d\xi\Big)^{\frac{q}{p}} \\ & \{\textstyle\int_{2^{j-1}\leq|\eta|\leq 2^{j+2}} |V(\eta)|^p d\eta\}^{\frac{q}{p}} \\ \leq\ & 2^{qj(\frac{3}{p'}-\alpha)} \sum_{j'\leq j+2} 2^{qj'\alpha} \Big(\int_{2^{j'}\leq|\xi|\leq 2^{j'+1}} |U(\xi)|^p d\xi\Big)^{\frac{q}{p}} \\ & \{\textstyle\int_{2^{j-1}\leq|\eta|\leq 2^{j+2}} |V(\eta)|^p d\eta\}^{\frac{q}{p}}. \end{aligned}$$

Therefore, we have

$$I_3 \leq C \sum_{j\in\mathbb{Z}} 2^{q\alpha j} \sum_{j'\leq j+2} 2^{qj'\alpha} \Big(\int_{2^{j'}\leq|\xi|\leq 2^{j'+1}} |U(\xi)|^p d\xi\Big)^{\frac{q}{p}} \Big\{\int_{2^{j-1}\leq|\eta|\leq 2^{j+2}} |V(\eta)|^p d\eta\Big\}^{\frac{q}{p}}.$$

Hence

$$I_3 \leq C\|U\|^q_{H^\alpha_{p,q}} \|V\|^q_{H^\alpha_{p,q}}.$$

In the end of this section, we estimate I_2. Take ρ, δ, δ' satisfying that $\frac{p}{3} < \rho \leq \min(1, p-1), 0 < \delta' < \delta < \frac{3\rho}{p} - 1$. We have

$$\begin{aligned} I_2 \leq\ & \sum_{j\in\mathbb{Z}} 2^{qj(1-\frac{3}{p})} \{\textstyle\int_{2^j\leq|\xi|\leq 2^{j+1}} \sum_{\tau\geq 0} 2^{p\tau\delta'} (\int_{2^{j+2+\tau}\leq|\eta|\leq 2^{j+3+\tau}} \\ & |U(\xi-\eta)||V(\eta)|d\eta)^p d\xi\}^{\frac{q}{p}} \\ \leq\ & \sum_{j\in\mathbb{Z},\tau\geq 0} 2^{qj(1-\frac{3}{p})+q\delta\tau} \{\textstyle\int_{2^j\leq|\xi|\leq 2^{j+1}} (\int_{2^{j+2+\tau}\leq|\eta|\leq 2^{j+3+\tau}} \\ & |U(\xi-\eta)||V(\eta)|d\eta)^p d\xi\}^{\frac{q}{p}} \\ =\ & \sum_{j\in\mathbb{Z},\tau\geq 0} 2^{qj(1-\frac{3}{p})+q\delta\tau} \{\textstyle\int_{2^j\leq|\xi|\leq 2^{j+1}} (\int_{2^{j+2+\tau}\leq|\eta|\leq 2^{j+3+\tau}} \\ & |U(\xi-\eta)|^{1-\rho}|V(\eta)||U(\xi-\eta)|^{\rho} d\eta)^p d\xi\}^{\frac{q}{p}} \\ \leq\ & \sum_{j\in\mathbb{Z},\tau\geq 0} 2^{qj(1-\frac{3}{p})+q\tau\delta} \{\textstyle\int_{2^j\leq|\xi|\leq 2^{j+1}, 2^{j+2+\tau}\leq|\eta|\leq 2^{j+3+\tau}} \\ & |U(\xi-\eta)|^{p(1-\rho)}|V(\eta)|^p d\eta d\xi\}^{\frac{q}{p}} \\ \times\ & \sup_{2^j\leq|\xi|\leq 2^{j+1}} \{\textstyle\int_{2^{j+2+\tau}\leq|\eta|\leq 2^{j+3+\tau}} |U(\xi-\eta)|^{p'\rho} d\eta\}^{\frac{q}{p'}}. \end{aligned}$$

Since we have

$$\begin{aligned} & \textstyle\int_{2^j\leq|\xi|\leq 2^{j+1}} |U(\xi-\eta)|^{p(1-\rho)} d\xi \\ \leq\ & \{\textstyle\int_{2^j\leq|\xi|\leq 2^{j+1}} |U(\xi-\eta)|^p d\xi\}^{1-\rho} \{\int_{2^j\leq|\xi|\leq 2^{j+1}} d\xi\}^{\rho} \\ \leq\ & C2^{3j\rho} \{\textstyle\int_{2^j\leq|\xi|\leq 2^{j+1}} |U(\xi-\eta)|^p d\xi\}^{1-\rho}, \end{aligned}$$

hence

$$\begin{aligned} I_2 \quad \leq \quad & \sum_{j\in\mathbb{Z},\tau\geq 0} 2^{qj(1-\frac{3}{p})+\frac{3q\rho j}{p}+q\tau\delta}(\int_{2^{j+\tau+2}-2^{j+1}\leq|\xi|\leq 2^{j+\tau+3}+2^{j+1}}|U(\xi)|^{p'\rho}d\xi)^{\frac{q}{p'}} \\ \times \quad & \{\int_{2^{j+2+\tau}\leq|\eta|\leq 2^{j+3+\tau}}|V(\eta)|^p d\eta(\int_{2^j\leq|\xi|\leq 2^{j+1}}|U(\xi-\eta)|^p d\xi)^{1-\rho}\}^{\frac{q}{p}}. \end{aligned}$$

Since $2^{j+2+\tau}\leq|\eta|\leq 2^{j+3+\tau}$, we enlarge the integration region of $U(\xi-\eta)$ for the variable ξ

$$\int_{2^j\leq|\xi|\leq 2^{j+1}}|U(\xi-\eta)|^p d\xi \leq \int_{2^{j+\tau+2}-2^{j+1}\leq|\xi|\leq 2^{j+\tau+3}+2^{j+1}}|U(\xi)|^p d\xi.$$

Since $\rho\leq p-1$, applying Cauchy-Schwartz inequality, we get

$$\begin{aligned} & \int_{2^{j+\tau+2}-2^{j+1}\leq|\xi|\leq 2^{j+\tau+3}+2^{j+1}}|U(\xi)|^{p'\rho}d\xi \\ \leq \quad & C(\int_{2^{j+\tau+2}-2^{j+1}\leq|\xi|\leq 2^{j+\tau+3}+2^{j+1}}|U(\xi)|^p d\xi)^{\frac{p'\rho}{p}}2^{3(j+\tau)(1-\frac{p'\rho}{p})}. \end{aligned}$$

Hence

$$\begin{aligned} I_2 \quad \leq \quad & \sum_{j\in\mathbb{Z},\tau\geq 0} 2^{qj(1-\frac{3}{p})+q\tau\delta+\frac{3q\rho}{p}j+3q(j+\tau)(\frac{1}{p'}-\frac{\rho}{p})} \\ & \{\int_{2^{j+\tau+2}-2^{j+1}\leq|\xi|\leq 2^{j+\tau+3}+2^{j+1}}|U(\xi)|^p d\xi\}^{\frac{q}{p}} \\ \times \quad & \{\int_{2^{j+2+\tau}\leq|\eta|\leq 2^{j+3+\tau}}|V(\eta)|^p d\eta\}^{\frac{q}{p}} \\ = \quad & \sum_{j\in\mathbb{Z},\tau\geq 0} 2^{qj(2-\frac{3}{p})+q(j+\tau)(2-\frac{3}{p})+q\tau(1+\delta-\frac{3\rho}{p})} \\ & \{\int_{2^{j+\tau+2}-2^{j+1}\leq|\xi|\leq 2^{j+\tau+3}+2^{j+1}}|U(\xi)|^p d\xi\}^{\frac{q}{p}} \\ \times \quad & \{\int_{2^{j+2+\tau}\leq|\eta|\leq 2^{j+3+\tau}}|V(\eta)|^p d\eta\}^{\frac{q}{p}}. \end{aligned}$$

For $\frac{p}{3}<\rho\leq\min(1,p-1)$ and $0<p\delta\leq 3\rho-p$, we have

$$I_2\leq C\|U\|^q_{H^\alpha_{p,q}}\|V\|^q_{H^\alpha_{p,q}}.$$

5. Symmetry property

Leray believed that, for a blow up solution, the initial data and the solution should both have the same special structure for different time. He proposed to consider self-similar solution. Necas-Ruzicka-Sverak [**31**] proved such solution has to be zero. We will show a party of symmetry solution should be zero and a party of symmetry initial data can result in symmetry solution. In this paper, the main important structure of solution is the symmetry property of solution. The divergence zero property and non-linearity property restrict the possibility of symmetry. Symmetry property plays an important role in the study of ill-posedness in [**43**]. If the initial value has certain symmetry, whether the solution has still the same symmetry? The Remark 5.9 tells us this is not always true. **The proofs of the theorems are not complex. But we know there exist 262144 kinds of symmetric property for a vector field function and there exists the differentiability, integrability and product of functions in the non-linear terms, the difficulties exist in how to find non-zero symmetric solution and how to find the relative proof.**

5.1. Divergence zero and iterative algorithm.

5.1.1. *Divergence zero.* We notice that divergence zero implies

LEMMA 5.1. *If* $\nabla(u_1, u_2, u_3) = 0$*, then*

$$\hat{u}_3 = \frac{-(\xi_1 \hat{u}_1 + \xi_2 \hat{u}_2)}{\xi_3}. \tag{5.1}$$

(i) For compressible Navier-Stokes equations, there exists radial solutions. The incompressible Navier-Stokes equations have zero divergence property, so there is not radial solution. But there exists still sub-radial solution. Abidi [**1**] and Abidi-Zhang [**2**] identifies such particular symmetry with a general name axi-symmetry. Sub-radial is just a particular case of the symmetry about the component of velocity field in this paper. See also remark 5.2. The velocity field u satisfies the symmetry property X_1 can produce that the vorticity for the third axi is zero.

(ii) As to the symmetry relative to the independent variable of velocity field, we note, for a scale function $f(\xi) = a(\xi) + ib(\xi)$, $f(\xi)$ has symmetry property, if its real part and imaginary part all have axi-symmetry or anti-axi-symmetry. There are 64 kinds of symmetry property for each scale function. We can label each possibility with polynomial function $\xi^\alpha + i\xi^\beta$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3), \beta = (\beta_1, \beta_2, \beta_3) \in \{0,1\}^3$. We write them in the following way

$$Tf = Ta + iTb = T\xi^\alpha + iT\xi^\beta.$$

For $\sigma = 1, 2, 3$, $\alpha_\sigma = 0$ means function a has symmetry for variable ξ_σ. $\alpha_\sigma = 1$ means function a has antisymmetry for variable ξ_σ. The same thing for function b.

A vector function has $64^3 = 262144$ kinds of such symmetry, but by (5.1), a velocity field u with divergence zero property has 64^2 kinds of symmetry property at most. We found only that the symmetry property X_2 can be preserved by the recursion formula (2.6) up to this paper. In another paper [**40**], I have found there is only one kind of symmetry real valued initial data $u_0(x)$ can generate symmetric solution.

For vector field $u(x)$, if u satisfies the symmetry property X_2, then

$$T(\hat{u}) = \begin{pmatrix} T(\xi_2\xi_3) + & iT\xi_1 \\ T(\xi_1\xi_3) + & iT\xi_2 \\ T(\xi_1\xi_2) + & iT\xi_3 \end{pmatrix} \tag{5.2}$$

5.2. Iterative algorithm. To consider the action of non-linearity to symmetry, we rewrite the quantities in (2.5) and reformulate (2.6). $\forall k = 1, 2, 3$, denote

$$A_k(\hat{u}^\tau, \hat{u}^\tau) = \sum_l \xi_l (\hat{u}_l^\tau * \hat{u}_k^\tau), \tag{5.3}$$

Denote further

$$\begin{array}{rcl} A_0(\hat{u}^\tau, \hat{u}^\tau) & = & \sum\limits_l \sum\limits_{l'} \xi_l \xi_{l'} (\hat{u}_l^\tau * \hat{u}_{l'}^\tau) \\ & = & \xi_1^2 (\hat{u}_1^\tau * \hat{u}_1^\tau) + 2\xi_1\xi_2 (\hat{u}_1^\tau * \hat{u}_2^\tau) + \xi_2^2 (\hat{u}_2^\tau * \hat{u}_2^\tau) \\ & + & 2\xi_1\xi_3 (\hat{u}_1^\tau * \hat{u}_3^\tau) + 2\xi_2\xi_3 (\hat{u}_2 * \hat{u}_3^\tau) + \xi_3^2 (\hat{u}_3^\tau * \hat{u}_3^\tau). \end{array} \tag{5.4}$$

Then equation (2.6) can be written as following:

$$\begin{array}{rcl} \hat{u}_k^{\tau+1}(t,\xi) & = & e^{-t\xi^2} \hat{u}_{0,k}(\xi) + i \int_0^t e^{-(t-s)\xi^2} A_k(\hat{u}^\tau, \hat{u}^\tau) ds \\ & & -\frac{i\xi_k}{\xi^2} \int_0^t e^{-(t-s)\xi^2} A_0(\hat{u}^\tau, \hat{u}^\tau) ds. \end{array} \tag{5.5}$$

In this section, we consider certain symmetry of iterative algorithm (5.5) and reduction of iterative function for solution. The first kind of symmetry is the symmetry of the component of velocity field, which is the generalization of Abidi-Zhang's sub-radial property. The iterative algorithm (2.6) has the heritability for such symmetry. The second symmetry is the symmetry of coordinate variable.

5.3. Symmetry with respect to the component of velocity field. For compressible Navier-Stokes equations, we can consider radial solution. But for incompressible Navier-Stokes equations, because of zero divergence property, there exists not radial solution. But many person have consider sub-radial solution with cylindrical coordinates. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. denote the cylindrical coordinates of x by (r, θ, x_3) where $r(x_1, x_2) \overset{\text{def}}{=} \sqrt{x_1^2 + x_2^2}$ and $\theta(x_1, x_2) \overset{\text{def}}{=} \arctan \frac{x_2}{x_1}$ with $r \in [0, \infty), \theta \in [0, 2\pi]$. Denote $e_r \overset{\text{def}}{=} (\cos\theta, \sin\theta, 0)$, $e_\theta \overset{\text{def}}{=} (-\sin\theta, \cos\theta, 0)$. Abidi and Zhang said a function is axisymmetry, if

$$u(t, x_1, x_2, x_3) = u_r(t, r, x_3)e_r + u_3(t, r, x_3)(0, 0, 1).$$

Such axisymmetry is called to be also sub-radial case. We did not consider the same wellposedness as one did for the subradial initial data in the known papers. The object of this paper is to consider only the rigidity and dimensionality reduction for symmetric structure.

REMARK 5.2. The property (1.2) is more general than subradial case. In fact,

(i) If one takes Fourier transform for $(\cos\theta u_r(t, r, x_3), \sin\theta u_r(t, r, x_3), u_3(t, r, x_3))$ in the ordinary axi system (x_1, x_2, x_3), we can find that subradial function satisfies property (1.2).

(ii) For $t, \tau, \sigma \in \mathbb{R}_+, \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and function $g(t, \sigma, \xi_3)$, we take

$$h(t, \xi_1, \xi_2, \xi_3) = \xi_3^{-1}(\xi_1^2 + \xi_2^2)g(t, |\xi_1|^\tau + |\xi_2|^\tau, \xi_3)$$

and take the Fourier transform of $u(t, x)$ as follows

$$\hat{u}(t, \xi_1, \xi_2, \xi_3) \quad = \quad (\xi_1 g(t, |\xi_1|^\tau + |\xi_2|^\tau, \xi_3), \xi_2 g(t, |\xi_1|^\tau + |\xi_2|^\tau, \xi_3), h(t, \xi_1, \xi_2, \xi_3)),$$

Evidently, $\nabla u(t, x) = 0$ and $u(t, x)$ satisfies property (1.2). But if $\tau \neq 2$, $u(t, x)$ is not a subradial function.

In this paper, we consider the heritability of property (1.2).

THEOREM 5.3. *If u^0 satisfies (1.2), then $\forall \tau \geq 0$, $u^{\tau+1}$ satisfies (1.2).*

Before the proof of the above Theorem 5.3, we introduce some notations and prove two lemmas. By applying zero divergence property, we regroup the term in $A_k(\hat{u}^\tau, \hat{u}^\tau)(k = 0, 1, 2, 3)$. $\forall k = 1, 2$, we divide each $A_k(\hat{u}^\tau, \hat{u}^\tau)$ in (5.3) into two terms:

$$\begin{aligned} A_k^1(\hat{u}^\tau, \hat{u}^\tau) &= \sum_{l=1,2} \xi_l(\hat{u}_l^\tau * \hat{u}_k^\tau), \\ A_k^2(\hat{u}^\tau, \hat{u}^\tau) &= -\xi_3 \sum_{l=1,2} \int \tfrac{\xi_l - \eta_l}{\xi_3 - \eta_3} (\hat{u}_l^\tau(\xi - \eta) * \hat{u}_k^\tau)(\eta) d\eta. \end{aligned}$$

By applying zero divergence property, we regroup $A_0(\hat{u}^\tau, \hat{u}^\tau)$ in (5.4):

$$\begin{aligned} A_0(\hat{u}^\tau, \hat{u}^\tau) = {}& \xi_1^2(\hat{u}_1^\tau * \hat{u}_1^\tau) + \xi_2^2(\hat{u}_2^\tau * \hat{u}_2^\tau) + 2\xi_1\xi_2(\hat{u}_1^\tau * \hat{u}_2^\tau) \\ & -2\xi_1\xi_3 \int \sum_{l=1,2} \tfrac{\xi_l - \eta_l}{\xi_3 - \eta_3} \hat{u}_l^\tau(\xi - \eta)\hat{u}_1(\eta) d\eta \\ & -2\xi_2\xi_3 \int \sum_{l=1,2} \tfrac{\xi_l - \eta_l}{\xi_3 - \eta_3} \hat{u}_l^\tau(\xi - \eta)\hat{u}_2(\eta) d\eta \\ & +\xi_3^2 \int \sum_{l,l'=1,2} \tfrac{\xi_l - \eta_l}{\xi_3 - \eta_3} \hat{\eta_{l'}} \eta_3 \hat{u}_l^\tau(\xi - \eta)\hat{u}_{l'}(\eta) d\eta. \end{aligned}$$

We divide $A_0(\hat u^\tau,\hat u^\tau)$ into the following two terms:

$$\begin{aligned} A^1_{0,0}(\hat u^\tau,\hat u^\tau) &= \xi_1^2(\hat u_1^\tau * \hat u_1^\tau)+\xi_2^2(\hat u_2^\tau * \hat u_2^\tau),\\ A^1_{0,1}(\hat u^\tau,\hat u^\tau) &= 2\xi_1\xi_2(\hat u_1^\tau * \hat u_2^\tau),\end{aligned}$$

and the following four terms:

$$\begin{aligned} A^2_{0,0}(\hat u^\tau,\hat u^\tau) &= -2\xi_3\{\xi_1\int \tfrac{\xi_1-\eta_1}{\xi_3-\eta_3}\hat u_1^\tau(\xi-\eta)\hat u_1(\eta)d\eta+\xi_2\int \tfrac{\xi_2-\eta_2}{\xi_3-\eta_3}\hat u_2^\tau(\xi-\eta)\hat u_2(\eta)d\eta\},\\ A^2_{0,1}(\hat u^\tau,\hat u^\tau) &= -2\xi_3\{\xi_1\int \tfrac{\xi_2-\eta_2}{\xi_3-\eta_3}\hat u_2^\tau(\xi-\eta)\hat u_1(\eta)d\eta+\xi_2\int \tfrac{\xi_1-\eta_1}{\xi_3-\eta_3}\hat u_1^\tau(\xi-\eta)\hat u_2(\eta)d\eta\},\\ A^3_{0,0}(\hat u^\tau,\hat u^\tau) &= \xi_3^2\{\int \tfrac{\xi_1-\eta_1}{\xi_3-\eta_3}\tfrac{\eta_1}{\eta_3}\hat u_1^\tau(\xi-\eta)\hat u_1(\eta)d\eta+\int \tfrac{\xi_2-\eta_2}{\xi_3-\eta_3}\tfrac{\eta_2}{\eta_3}\hat u_2^\tau(\xi-\eta)\hat u_2(\eta)d\eta\},\\ A^3_{0,1}(\hat u^\tau,\hat u^\tau) &= \xi_3^2\{\int \tfrac{\xi_1-\eta_1}{\xi_3-\eta_3}\tfrac{\eta_2}{\eta_3}\hat u_1^\tau(\xi-\eta)\hat u_2(\eta)d\eta+\int \tfrac{\xi_2-\eta_2}{\xi_3-\eta_3}\tfrac{\eta_1}{\eta_3}\hat u_2^\tau(\xi-\eta)\hat u_1(\eta)d\eta\}.\end{aligned}$$

That is to say,

$$\begin{aligned} &A_0(\hat u^\tau,\hat u^\tau)\\ \equiv\ &A^1_{0,0}(\hat u^\tau,\hat u^\tau)+A^1_{0,1}(\hat u^\tau,\hat u^\tau)+A^2_{0,0}(\hat u^\tau,\hat u^\tau)+A^2_{0,1}(\hat u^\tau,\hat u^\tau)\\ &+A^3_{0,0}(\hat u^\tau,\hat u^\tau)+A^3_{0,1}(\hat u^\tau,\hat u^\tau).\end{aligned}$$

First step, for $k=1,2$, we consider the first two terms $A^1_k(\hat u^\tau,\hat u^\tau)(\xi_1,\xi_2,\xi_3)$ and $A^2_k(\hat u^\tau,\hat u^\tau)(\xi_1,\xi_2,\xi_3)$. We have

LEMMA 5.4. *If u^τ satisfies (1.2), then*

$$A^1_2(\hat u^\tau,\hat u^\tau)(\xi_2,\xi_1,\xi_3)=A^1_1(\hat u^\tau,\hat u^\tau)(\xi_1,\xi_2,\xi_3). \tag{5.6}$$

$$A^2_2(\hat u^\tau,\hat u^\tau)(\xi_2,\xi_1,\xi_3)=A^2_1(\hat u^\tau,\hat u^\tau)(\xi_1,\xi_2,\xi_3). \tag{5.7}$$

PROOF. First, we have

$$\begin{aligned} &A^1_2(\hat u^\tau,\hat u^\tau)(\xi_2,\xi_1,\xi_3)\\ =\ &\xi_2\int \hat u_1^\tau(\xi_2-\eta_1,\xi_1-\eta_2,\xi_3-\eta_3)\hat u_1^\tau(\eta_1,\eta_2,\eta_3)d\eta\\ &+\xi_1\int \hat u_2^\tau(\xi_2-\eta_1,\xi_1-\eta_2,\xi_3-\eta_3)\hat u_1^\tau(\eta_1,\eta_2,\eta_3)d\eta\\ =\ &\xi_2\int \hat u_2^\tau(\xi_1-\eta_2,\xi_2-\eta_1,\xi_3-\eta_3)\hat u_1^\tau(\eta_2,\eta_1,\eta_3)d\eta\\ &+\xi_2\int \hat u_1^\tau(\xi_1-\eta_2,\xi_2-\eta_1,\xi_3-\eta_3)\hat u_1^\tau(\eta_2,\eta_1,\eta_3)d\eta\\ =\ &A^1_1(\hat u^\tau,\hat u^\tau)(\xi_1,\xi_2,\xi_3).\end{aligned}$$

Further, we have

$$\begin{aligned} &A^2_2(\hat u^\tau,\hat u^\tau)(\xi_2,\xi_1,\xi_3)\\ =\ &-\xi_3\int \tfrac{\xi_2-\eta_1}{\xi_3-\eta_3}\hat u_1^\tau(\xi_2-\eta_1,\xi_1-\eta_2,\xi_3-\eta_3)\hat u_2^\tau(\eta_1,\eta_2,\eta_3)d\eta\\ &-\xi_3\int \tfrac{\xi_1-\eta_2}{\xi_3-\eta_3}\hat u_2^\tau(\xi_2-\eta_1,\xi_1-\eta_2,\xi_3-\eta_3)\hat u_2^\tau(\eta_1,\eta_2,\eta_3)d\eta\\ =\ &-\xi_3\int \tfrac{\xi_2-\eta_1}{\xi_3-\eta_3}\hat u_2^\tau(\xi_1-\eta_2,\xi_2-\eta_1,\xi_3-\eta_3)\hat u_1^\tau(\eta_2,\eta_1,\eta_3)d\eta\\ &-\xi_3\int \tfrac{\xi_1-\eta_2}{\xi_3-\eta_3}\hat u_1^\tau(\xi_1-\eta_1,\xi_2-\eta_2,\xi_3-\eta_3)\hat u_1^\tau(\eta_2,\eta_1,\eta_3)d\eta\\ =\ &A^2_1(\hat u^\tau,\hat u^\tau)(\xi_1,\xi_2,\xi_3).\end{aligned}$$

□

Second step, we consider the rest six terms. We have

LEMMA 5.5. *If u^τ satisfies (1.2), then for $k=1,2,3$ and $l=0,1$, we have*

$$A^k_{0,l}(\hat u^\tau,\hat u^\tau)(\xi_2,\xi_1,\xi_3)=A^k_{0,l}(\hat u^\tau,\hat u^\tau)(\xi_1,\xi_2,\xi_3). \tag{5.8}$$

PROOF. For the six term of $A_0(\hat u^\tau,\hat u^\tau)(\xi_2,\xi_1,\xi_3)$, we consider only $A^1_{0,0}(\hat u^\tau,\hat u^\tau)$. The other five term

$$A^1_{0,1}(\hat u^\tau,\hat u^\tau),A^2_{0,0}(\hat u^\tau,\hat u^\tau),A^2_{0,1}(\hat u^\tau,\hat u^\tau),A^3_{0,0}(\hat u^\tau,\hat u^\tau)$$

and $A^3_{0,1}(\hat u^\tau,\hat u^\tau)$ can be verified similarly.

For $A^1_{0,0}(\hat{u}^\tau, \hat{u}^\tau)$, we have

$$\begin{array}{ll} & A^1_{0,0}(\hat{u}^\tau, \hat{u}^\tau)(\xi_2, \xi_1, \xi_3) \\ = & \xi_2^2 \int \hat{u}_1^\tau(\xi_2 - \eta_1, \xi_1 - \eta_2, \xi_3 - \eta_3)\hat{u}_1^\tau(\eta_1, \eta_2, \eta_3)d\eta \\ & +\xi_1^2 \int \hat{u}_2^\tau(\xi_2 - \eta_1, \xi_1 - \eta_2, \xi_3 - \eta_3)\hat{u}_2^\tau(\eta_1, \eta_2, \eta_3)d\eta \\ = & \xi_2^2 \int \hat{u}_2^\tau(\xi_1 - \eta_2, \xi_2 - \eta_1, \xi_3 - \eta_3)\hat{u}_2^\tau(\eta_2, \eta_1, \eta_3)d\eta \\ & +\xi_1^2 \int \hat{u}_1^\tau(\xi_1 - \eta_2, \xi_2 - \eta_1, \xi_3 - \eta_3)\hat{u}_1^\tau(\eta_2, \eta_1, \eta_3)d\eta. \end{array}$$

Change η_1 and η_2 in the above last equality, we get that

$$A^1_{0,0}(\hat{u}^\tau, \hat{u}^\tau)(\xi_2, \xi_1, \xi_3) = A^1_{0,0}(\hat{u}^\tau, \hat{u}^\tau)(\xi_1, \xi_2, \xi_3).$$

□

Now we prove theorem 5.3.

Proof. By lemmas 5.4 and 5.5, $\forall \tau \geq 0$, if u^τ satisfies (1.2), then the eight terms satisfies equations (5.6), (5.7) and (5.8). Hence $u^{\tau+1}$ satisfies (1.2). □

5.4. Symmetry of independent variable. For a complex function f, if one of its real part or its imaginary part has no axi-symmetry property, then we say that f has no symmetry property. For two function f_1 and f_2, if and only if f_1 and f_2 have the same symmetry, we can consider the symmetry of the sum of f_1 and f_2. Further, $T(f_1 + f_2) = Tf_1 = Tf_2$.

For the product of two real function f and g, it is easy to see

Lemma 5.6. *Given $\alpha, \beta \in \{0, 1\}^3$. If $Tf = T\xi^\alpha$ and $Tg = T\xi^\beta$, then*

$$T(fg) = T\xi^{\alpha+\beta},$$

where $\alpha_k + \beta_k\, (k = 1, 2, 3)$, modulate 2, belongs to $\{0, 1\}$.

For the convolution of two real functions f and g,

Lemma 5.7. *Given $\alpha, \beta \in \{0, 1\}^3$. If $Tf = T\xi^\alpha$ and $Tg = T\xi^\beta$, then $T(f * g) = T\xi^{\alpha+\beta}$.*

Proof. We consider only the axi-symmetry for the first coordinate axis, the other case can be proved in a similar way. That is to say, if $f(\xi_1, \xi_2, \xi_3) = f(-\xi_1, \xi_2, \xi_3)$ and $g(\xi_1, \xi_2, \xi_3) = g(-\xi_1, \xi_2, \xi_3)$, then $f{*}g(\xi_1, \xi_2, \xi_3) = f{*}g(-\xi_1, \xi_2, \xi_3)$.

$$\begin{array}{rcl} f * g(-\xi_1, \xi_2, \xi_3) & = & \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} f(-\xi_1 - \eta_1, \xi_2 - \eta_2, \xi_3 - \eta_3)g(\eta_1, \eta_2, \eta_3)d\eta_1 d\eta_2 d\eta_3 \\ \text{symmetry of } g & = & \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} f(-\xi_1 - \eta_1, \xi_2 - \eta_2, \xi_3 - \eta_3)g(-\eta_1, \eta_2, \eta_3)d\eta_1 d\eta_2 d\eta_3 \\ -\eta_1 \to \eta_1 & = & \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} f(-\xi_1 + \eta_1, \xi_2 - \eta_2, \xi_3 - \eta_3)g(\eta_1, \eta_2, \eta_3)d\eta_1 d\eta_2 d\eta_3 \\ \text{symmetry of } f & = & \int_{-\infty}^{\infty} \int_{\mathbb{R}^2} f(\xi_1 - \eta_1, \xi_2 - \eta_2, \xi_3 - \eta_3)g(\eta_1, \eta_2, \eta_3)d\eta_1 d\eta_2 d\eta_3 \\ & = & f * g(\xi_1, \xi_2, \xi_3). \end{array}$$

□

We get down to the second main result on symmetry.

Theorem 5.8. *If u^0 satisfies (5.2), then $\forall \tau \geq 0$, $u^{\tau+1}$ satisfies (5.2).*

Remark 5.9. Non-linearity may change the symmetry property. Many kinds of symmetry of complex vector field can not be preserved in the iterative algorithm

(2.6). For example, a component has three anti-symmetry. Denote

$$T\hat{u}_0 = \begin{pmatrix} T\xi_2 \\ T\xi_1 \\ T(\xi_1\xi_2\xi_3) \end{pmatrix}$$

If $B(u^0, u^0) \neq 0$, then we can prove that u^0 and $B(u^0, u^0)$ have different symmetry property; further if $u^0 \in \dot{H}^{\frac{1}{2}}$, then we can verify such symmetric solution has only zero solution. That is to say, such kind of symmetric property has rigidity.

Before the proof of Theorem 5.8, we introduce some notations and a lemma. $\forall k = 1, 2, 3$, denote $u_k^\tau = a_k^\tau + i b_k^\tau$, then the real part of $A_k(\hat{u}^\tau, \hat{u}^\tau)(\xi)$ can be written as:

$$A_{k,1}(\hat{u}^\tau, \hat{u}^\tau)(\xi) = \sum_l \xi_l \{a_l^\tau * b_k^\tau + b_l^\tau * a_k^\tau\}.$$

The imaginary part of $A_k(\hat{u}^\tau, \hat{u}^\tau)(\xi)$ can be written as:

$$A_{k,2}(\hat{u}^\tau, \hat{u}^\tau)(\xi) = \sum_l \xi_l \{a_l^\tau * a_k^\tau - b_l^\tau * b_k^\tau\}.$$

The real part of $A_0(\hat{u}^\tau, \hat{u}^\tau)(\xi)$ can be written as the sum of the following two terms:

$$A_{0,1}(\hat{u}^\tau, \hat{u}^\tau)(\xi) = \sum_l \sum_{l'} \xi_l \xi_{l'} a_l^\tau * b_{l'}^\tau$$

and

$$A_{0,2}(\hat{u}^\tau, \hat{u}^\tau)(\xi) = \sum_l \sum_{l'} \xi_l \xi_{l'} b_l^\tau * a_{l'}^\tau.$$

The imaginary part of $A_k(\hat{u}^\tau, \hat{u}^\tau)(\xi)$ can be written as the sum of the following two terms:

$$A_{0,3}(\hat{u}^\tau, \hat{u}^\tau)(\xi) = \sum_l \sum_{l'} \xi_l \xi_{l'} a_l^\tau * a_{l'}^\tau$$

and

$$A_{0,4}(\hat{u}^\tau, \hat{u}^\tau)(\xi) = -\sum_l \sum_{l'} \xi_l \xi_{l'} b_l^\tau * b_{l'}^\tau.$$

Hence we can write the equation (2.6) as follows:

$$\begin{array}{rl} \hat{u}_k^{\tau+1}(t,\xi) = & e^{-t\xi^2}\hat{u}_{0,k}(\xi) - \int_0^t e^{-(t-s)\xi^2} A_{k,1}(\hat{u}^\tau, \hat{u}^\tau)(\xi) ds \\ & + i\int_0^t e^{-(t-s)\xi^2} A_{k,2}(\hat{u}^\tau, \hat{u}^\tau)(\xi) ds \\ & -\frac{\xi_k}{\xi^2}\int_0^t e^{-(t-s)\xi^2} \{A_{0,1}(\hat{u}^\tau, \hat{u}^\tau)(\xi) + A_{0,2}(\hat{u}^\tau, \hat{u}^\tau)(\xi)\} ds \\ & +\frac{i\xi_k}{\xi^2}\int_0^t e^{-(t-s)\xi^2} \{A_{0,3}(\hat{u}^\tau, \hat{u}^\tau)(\xi) + A_{0,4}(\hat{u}^\tau, \hat{u}^\tau)(\xi)\} ds. \end{array} \tag{5.9}$$

We prove first

LEMMA 5.10.

$$TA_0(\hat{u}^\tau, \hat{u}^\tau) = T(\xi_1\xi_2\xi_3) + iT1.$$

PROOF. There exists four term in $A_0(\hat{u}^\tau, \hat{u}^\tau)$. We consider first two real part term:

$$\begin{array}{rl} & T(A_{0,1}(\xi)) \\ = & T\xi_1^2 T(\xi_2\xi_3)T\xi_1 + T\xi_2^2 T(\xi_1\xi_3)T\xi_2 + T\xi_3^2 T(\xi_1\xi_2)T\xi_3 + T(\xi_1\xi_2)T(\xi_2\xi_3)T\xi_2 \\ + & T(\xi_1\xi_3)T(\xi_2\xi_3)T\xi_3 + T(\xi_2\xi_3)T(\xi_1\xi_3)T\xi_3 \\ = & T(\xi_1\xi_2\xi_3). \end{array}$$

And,

$$\begin{array}{rl} & T(A_{0,2}(\xi)) \\ = & T\xi_1^2T(\xi_1)T(\xi_2\xi_3)+T\xi_2^2T\xi_2T(\xi_1\xi_3)+T\xi_3^2T\xi_3T(\xi_1\xi_2)+T(\xi_1\xi_2)T\xi_1T(\xi_1\xi_3) \\ + & T(\xi_1\xi_3)T\xi_1T(\xi_1\xi_2)+T(\xi_2\xi_3)T\xi_3T(\xi_1\xi_2) \\ = & T(\xi_1\xi_2\xi_3). \end{array}$$

Further, we consider the two imaginary part term:

$$\begin{array}{rl} & T(A_{0,3}(\xi)) \\ = & T\xi_1^2T(\xi_2\xi_3)T(\xi_2\xi_3)+T\xi_2^2T(\xi_1\xi_3)T(\xi_1\xi_3)+T\xi_3^2T(\xi_1\xi_2)T(\xi_1\xi_2) \\ + & T(\xi_1\xi_2)T(\xi_2\xi_3)T(\xi_2\xi_3)+T(\xi_1\xi_3)T(\xi_2\xi_3)T(\xi_1\xi_2)+T(\xi_2\xi_3)T(\xi_1\xi_3)T(\xi_1\xi_2) \\ = & T1. \end{array}$$

and

$$\begin{array}{rl} & T(A_{0,4}(\xi)) \\ = & T\xi_1^2T(\xi_1)T\xi_1+T\xi_2^2T\xi_2T\xi_2+T\xi_3^2T\xi_3T\xi_3 \\ + & T(\xi_1\xi_2)T\xi_1T\xi_2+T(\xi_1\xi_3)T\xi_1T\xi_3+T(\xi_2\xi_3)T\xi_2T\xi_3 \\ = & T1. \end{array}$$

□

Now we come to prove the main theorem of this subsection.

PROOF. First, we have

$$\begin{array}{rl} & T(A_{1,1}(\xi)) \\ = & T\xi_1\{T(\xi_2\xi_3)T\xi_1+T\xi_1T(\xi_2\xi_3)\}+T\xi_2\{T(\xi_1\xi_3)T\xi_1+T\xi_2T(\xi_2\xi_3)\} \\ & +T\xi_3\{T(\xi_1\xi_2)T\xi_1+T\xi_3T(\xi_2\xi_3)\} \\ = & T(\xi_2\xi_3). \end{array} \tag{5.10}$$

and

$$\begin{array}{rl} & T(A_{2,1}(\xi)) \\ = & T\xi_1\{T(\xi_2\xi_3)T\xi_2+T\xi_1T(\xi_1\xi_3)\}+T\xi_2\{T(\xi_1\xi_3)T\xi_2+T\xi_2T(\xi_1\xi_3)\} \\ & +T\xi_3\{T(\xi_1\xi_2)T\xi_2+T\xi_3T(\xi_1\xi_3)\} \\ = & T(\xi_1\xi_3). \end{array} \tag{5.11}$$

Further, by divergence zero property, we have

$$T(A_{3,1}(\xi))=T(\xi_1\xi_2). \tag{5.12}$$

Secondly, we have

$$\begin{array}{rl} & T(A_{1,2}(\xi)) \\ = & T\xi_1\{T(\xi_2\xi_3)T(\xi_2\xi_3)+T\xi_1T\xi_1)\}+T\xi_2\{T(\xi_1\xi_3)T(\xi_2\xi_3)+T\xi_2T\xi_1)\} \\ & +T\xi_3\{T(\xi_1\xi_2)T(\xi_2\xi_3)+T\xi_3T\xi_1\} \\ = & T\xi_1 \end{array} \tag{5.13}$$

and

$$\begin{array}{rl} & T(A_{2,2}(\xi)) \\ = & T\xi_1\{T(\xi_2\xi_3)T(\xi_1\xi_3)+T\xi_1T\xi_3)\}+T\xi_2\{T(\xi_1\xi_3)T(\xi_1\xi_3)+T\xi_2T\xi_2)\} \\ & +T\xi_3\{T(\xi_1\xi_2)T(\xi_1\xi_3)+T\xi_3T\xi_2\} \\ = & T\xi_2. \end{array} \tag{5.14}$$

Further, by divergence zero property, we have

$$T(A_{3,2}(\xi))=T\xi_3. \tag{5.15}$$

Combine the above symmetry (5.10, 5.11, 5.12, 5.13, 5.14, 5.15), with the symmetry in the lemma 5.10, we get the symmetry property of Theorem 5.8.

□

5.5. Reduction of iteration function. For a complex initial data, the iterative algorithm (2.6) is an iteration of four real functions. By Theorems 5.3 and 5.8, if we consider the symmetry property, the number of iterative function will be reduced.

COROLLARY 5.1. (i) If the initial value u_0 satisfies symmetry property (1.2), then the iterative algorithm (2.6) can be reduce to the iteration of a complex function.

(ii) If the initial value u_0 satisfies symmetry property (5.2) and the real part of $\hat{u}_0$ is zero, then the iterative algorithm (2.6) can be reduce to the iteration of two real functions.

(iii) If the initial value u_0 satisfies symmetry properties (1.2) and (5.2) and the real part of $\hat{u}_0$ is zero, then the iterative algorithm (2.6) can be reduce to the iteration of one real functions.

6. Proof of the main theorem

The proof of the main theorem is the corollary of the five Theorems 3.1, 3.2, 3.4, 5.3 and 5.8. In fact, we have

PROOF. We prove the main theorem 1.3 by considering the iterative process (2.6). We search functions with symmetry property X_m and uniform analyticity property to approach the solution. First, if u_0 is a function with symmetry property X_m, then $e^{t\Delta}u_0$ is a function with symmetry property X_m and with uniform analyticity property. According to theorems 5.3 and 5.8, $u^\tau(\tau \geq 1)$ are functions with symmetry property X_m. According to theorems 3.1, 3.2 and 3.4, $u^\tau(\tau \geq 1)$ are functions with uniform analyticity property.

Since u_0 is a small initial value in $(P^{\alpha}_{p,q})^3$, by theorems 3.1, 3.2 and 3.4, u^τ convergence to the solution $u(t,x)$ and $u(t,x)$ is a function with symmetry property X_m and with uniform analyticity property.

□

References

[1] H. Abidi, Résultats de régularité de solutions axisymtriques pour le système de Navier-Stokes, *Bull. Sci. Math.* **132**(2008), 592-624.
[2] H. Abidi, P. Zhang Global smooth axisymmetric solutions of 3-D inhomogeneous incompressible Navier-Stokes system, *Calc. Var. DOI 10.1007/s00526-015-0902-6* **54** (2015), 3251-3276
[3] A. Biswas, Gevrey regularity for a clsss of dissipative equations with applications to decay, *J. Differential Equaitons*, **253**(2012), 2739-2764.
[4] A. Biswas and D. Swanson, Navier-Stokes equaitons and weighted convolution inequalities in groups, *Comm. Partial Differential Equaitons*, **35**(2010), 559-589.
[5] M. Cannone, A generalization of a theorem by Kato on Navier-Stokes equation, *Rev. Mat. Iberoamericana*, **13**(1997), 673-697.
[6] M. Cannone, Harmonic analysis tools for solving the incompressible Navier-Stokes equaitons, *Handbook of Mathematical Fluid Dynamics*, **3**(2005), 161-244.
[7] M. Cannone and G. Karch, Smooth or singular solutions to the Navier-Stokes system, *Journal of Differential Equations*, **vol. 197, 2** (2004), 247-274.
[8] M. Cannone and G. Wu, Global well-posedness for Navier-Stokes equations in critical Fourier-Herz spaces, *Nonlinear Analysis. Theory, Methods & Applications*, **vol. 75, 9** (2012), 3754-3760.
[9] C. Foias and R. Teman, Gevrey class regularity for the solutions of the Navier-Stokes equaitons, *J. Funct. Anal.*, **87**(1989), 359-369.

[10] Y. Fujita and T. Kato, On the Navier-Stokes initial value problem, I, *Arch. Rat. Mech. Anal.*, **16**(1964), 269-315.

[11] P. Germain, N. Pavlovié and G. Staffilani, Regularity of solutions to the Navier-Stokes equaitons evolving from small data in BMO^{-1}, *Int. Math. Res. Not.*, **21**(2007), 227-239.

[12] Y. Giga, K. Inui, A. Mahalov and J. Saal, Uniform global solvability of the rotating Navier-Stokes equations for nondecaying initial data, *Indiana University Mathematical Journal*, **57** (2008), 2775-2791.

[13] Y. Giga and T. Miyakawa, Navier-Stokes flow in $\mathcal{R}^3$ with measures as initial vorticity and Morrey spaces, *Comm. Partial Differential Equations*, **14**(1989), 577-618.

[14] W. Guo, H. Wu, Q. Yang and G. Zhao, Characterization of inclusion relations between wiener amalgam and some classical spaces, *J. Funct. Anal.*, **273**(2017), 404-443.

[15] T. Kato, Strong L^p solutions of the Navier-Stokes equaiton in $\mathcal{R}^n$ with applications to weak solutions, *Math. Z.*, **187**(1984), 471-480.

[16] T. Kato and H. Fujita, On the non-stationary Navier-Stokes system, *Rend. Semin. Mat. Univ. Padova*, **30**(1962), 243-260.

[17] T. Kato and G. Ponce, Commutator estiamtes and the Euler and Navier-Stokes equations, *Comm. Pure Appl. Math.* **XLI**(1988),891-907.

[18] R. A. Kerman, Convolution theorems with weights, *Trans. Amer. Math. Soc.*, **280**(1983), 207-219.

[19] H. Koch and D. Tataru, Well-posedness for the Navier-Stokes equations, *Adv. Math.*, **157**(2001), 22-35.

[20] Y. Le Jan and A. S. Sznitman, Cascades aléatoires et équations de Navier-Stokes, *C. R. Acad. Sci. Paris*, **324**(1997), 823-826.

[21] Z. Lei and Fanghua Lin, Global mild solutions of Navier-Stokes equaitons, *Comm. Pure Appl. Math.* , **64**(2011), 1297-1304.

[22] P. G. Lemarié-Rieusset, *Recent development in the Navier-Stokes problems*, Boca Raton, London, New York, Washington, D. C., CRC Press, 2002.

[23] P. Li, J. Xiao and Q. Yang, Global mild solutions of fractional Navier-Stokes equations with small initial data in critical Besov-Q spaces, *Electron. J. Diff. Equ.*, **185**(2014), 1-37.

[24] P. Li and Q. Yang, Well-posedness of Quasi-Geostrophic Equations with data in Besov-Q spaces, *Nonlinear Analysis*, **94**(2014), 243-258.

[25] P. Li and Q. Yang, Wavelets and the well-posedness of incompressible magneto-hydrodynamic equations in Besov type Q-space, *Journal of Mathematical Analysis and Applications*, **405**(2013), 661-686.

[26] C. Lin and Q. Yang, Semigroup characterization of Besov type Morrey spaces and well-posedness of generalized Navier-Stokes equations, *J. Differential Equations*, **254**(2013), 804-846

[27] S. Lu and D. Yang, The decomposition of weighted Herz space of $\mathbb{R}^n$ and its applications, *Science in China, Series A*, **38 (2)**(1995), 147-158.

[28] K. Masuda, On the analyticity and the unique continuation theorem for solutions of the Navier-Stokes equation, *Proc. Japan Acad.*, **43**(1967), 827-832.

[29] Y. Meyer, *Wavelets and operators*, Cambridge Studies in Advanced Mathematics, **37**, Cambridge: Cambridge University Press, 1922.

[30] H. Miura and O. Sawada, On the regularizing rate estimates of Koch-Tataru's solution to the Navier-Stokes equations, *Asymptot. Anal.*, **49**(2006), 1-15.

[31] J. Necas, M. Ruzicka and V. Sverak, On Leray's self similar solutions of the Navier-Stokes equations, *Acta. Math.* **176** (1996), 283-294.

[32] M. E. Taylor, Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations, *Comm. Partial Differential Equations*, **17** (1992), pp. 1407-1456.

[33] J. Wu, Generalized MHD equations, *J. Differential Equations*, **195**(2003), 284-312.

[34] J. Wu, The generalized imcompressible Navier-Stokes equaitons in Besov spaces, *Dyn. Partial Differ. Equ.*, **1**(2004), 381-400.

[35] J. Wu, Lower bounds for an integral involving fractional Laplacians and the generalzied Navier-Stokes equations in Besov spaces, *Comm. Math. Phys.*, **263**(2005), 803-831.

[36] J. Wu, Regularity criteria for the generalized MHD equations, *Comm. Partial Differential Equations*, **33**(2008), 285-306.

[37] J. Xiao, Homothetic variant of fractional Sobolev space with application to Navier-Stokes system, *Dyn. Partial Differ. Equ.*, **4**(2007), 227-245.

[38] J. Xiao, Homothetic variant of fractional Sobolve space with application to Navier-Stokes system revisited, *Dyn. Partial Differ. Equ.*, **11**(2014), 167-181.

[39] W. Xiao, J. Chen, D. Fan and X. Zhou, Global Well-Posedness and Long Time Decay of Fractional Navier-Stokes Equations in Fourier-Besov Spaces, *Abstract and Applied Analysis* **Volume 2014**, *Article ID 463639, 11 pages, http://dx.doi.org/10.1155/2014/463639*

[40] Q. Yang, Only one kind of symmetry can generate symmetric solution, preprint

[41] Q. Yang and P. Li, Regular wavelets, heat semigroup and application to the magneto-hydrodynamic equaitons with data in critical Triebel-Lizorkin type oscillation spaces, Taiwanese J. Math., **20***(2016), 1335-1376.*

[42] Q. Yang and H. Yang, Y spaces and global smooth solution of fractional Navier-Stokes equaitons with initial value in the critical oscillation spaces, J. Differential Equations, **Vol 264, 7** *(2018), 4402-4424.*

[43] Q. Yang, H. Yang and H. Wu, Ill-posedness of Naiver-Stokes equations and critical Besov-Morrey spaces, arXiv 1706.08120.

Qixiang Yang, School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, P. R. China

Email address: qxyang@whu.edu.cn