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Exact and explicit internal water waves at arbitrary latitude with underlying currents

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Abstract. We present an explicit exact solution to the nonlinear governing equations in the β-plane approximation for internal geophysical water waves propagating at an arbitrary latitude. Such a solution describes oceanic waves in the presence of a constant underlying current, which propagate eastward above the thermocline and beneath the near-surface layer where wind effects are confined.

Contents

1. Introduction

In this paper, we present an explicit exact solution to the geophysical governing equations in the β-plane approximation for internal water waves in a relatively narrow ocean strip at an arbitrary latitude in the presence of an underlying current. Geophysical fluid dynamics is the study of fluid motion where the Earth's rotation plays a significant role, the Coriolis forces are incorporated into the governing Euler equations, and applies to a wide range of oceanic and atmospheric flows [**18, 23, 42**]. Geophysical fluid dynamics contains high complexity, which leads to an inherent mathematical intractability (see the discussions in [**14, 16**]). In order to mitigate the complexity, it is natural and common to derive simpler approximate equations of Euler equations. There are two approximate models which have been

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typically employed in the oceanographic considerations to avoid the intricacies of spherical geometry (see the discussion in [**15**]). One is the β-plane approximation, which introduces a linear variation with latitude of the Coriolis parameter to allow for the variation of the Coriolis force from point to point. It applies in regions within 5° latitude, either side of the Equator [**18, 23**]. For some modifications of the standard geophysical equatorial β-plane model equations, we refer the reader to [**12, 30, 31, 37**]. Another approximate model is the f-plane approximation, which takes a constant Coriolis parameter into account and does not consider the latitudinal variations. This approximation has been applied to equatorial flows within a restricted meridional of approximately 2° latitude, either side of the Equator [**8, 11, 18, 29**]. An f-plane approximation at an arbitrary latitude was taken into account in [**40, 17**]. In this paper, we will use the idea in [**2**] to consider the β-plane approximation at an arbitrary latitude.

Exact solutions play an important role in the study of geophysical flows because many apparently intangible wave motions can be regarded as the perturbations of them. The approach pioneered by Gerstner [**22**] (see [**3, 24**] for a modern exposition) of finding explicit exact solutions for gravity fluid flows within the Lagrangian framework [**1**], was extended to various geophysical flows too. We refer the reader to [**7, 28, 34, 41, 32**] for equatorial waves, [**11, 13, 17, 21, 26**] for waves in the presence of underlying currents, and [**2, 40**] for surface geophysical waves at an arbitrary latitude. Recently, the study of internal water flows which propagate in the center regions is of great interest and has attracted much attention. In [**9, 10, 38**], an explicit, nonlinear solution for equatorial ocean waves propagating eastward, in the layer above the thermocline and beneath the near-surface layer in which wind effects are noticeable, was described. In [**9**] it was assumed that beneath the thermocline lay an abyssal region of still water, whereas in [**10**] a more physically plausible model was achieved whereby the region beneath the thermocline has been divided into three layers, which transition from the internal wave at the thermocline to an abyssal zone of purely motionless fluid. In [**38**], in the presence of a constant underlying current, Kluczek extended the results obtained in [**10**]. See [**20, 27, 33, 35, 36, 39**] and the references therein for more results on internal equatorial ocean waves.

In this paper, we will consider the internal waves propagating in a relatively narrow ocean strip at an arbitrary latitude, as well as admit underlying-current interactions. Physically, the model we deal with is multi-layered, which is complex, but important and interesting. The rest part of this paper is organized as follows. In Section 2, we present the β-plane governing equations for internal geophysical flows in the presence of an underlying current and then discuss our model. In Section 3, we obtain the explicit solution to such problem. Moreover, we make a detailed discussion of the situations encountered in the Northern Hemisphere and the Southern Hemisphere, for the admissible underlying currents c_0. It is worth mentioning that when we consider the arbitrary latitude $\phi = 0$, the model in this paper will become the ones in [**10, 38**], therefore, the regions we considered are more general and the solutions we obtained can recover the ones in [**10, 38**].

2. The governing equations in the β-plane approximation

It is appropriate to consider the Earth as a sphere of radius $R = 6371$ km, rotating with constant rotational speed of $\Omega = 7.29 \times 10^{-5}$rad/s round the polar axis toward the east. We choose a rotating framework with the origin at a point on the Earth's surface with latitude ϕ, $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$, with the x axis pointing horizontally due east, the y axis pointing horizontally due north (in the tangent plane), and with the z axis corresponding to the local vertical and oriented upward. In this coordinate system, $\mathbf{\Omega} = (0, \Omega\cos\phi, \Omega\sin\phi)$ is the rotation vector of the Earth along the polar axis. The Coriolis parameters are defined by

$$f := 2\Omega\sin\phi, \quad \hat{f} := 2\Omega\cos\phi.$$

These parameters depend on the variable latitude ϕ. At the North Pole we have $f = 2\Omega$, $\hat{f} = 0$ and at the Equator $f = 0$, $\hat{f} = 2\Omega$. The values $f = \hat{f} = 10^{-4}\ s^{-1}$ are appropriate to 45° latitude in the Northern Hemisphere, while $f = -\hat{f} = -10^{-4}\ s^{-1}$ are appropriate to 45° latitude in the Southern Hemisphere (See [**23**]). For internal water waves propagating zonally in a relatively narrow ocean strip less than a few degrees of latitude wide, it is adequate to use the f-plane or β-plane approximation. Within the f-plane approximation the Coriolis parameters are treated as constants, but within the β-plane approximation $\hat{f}$ is constant and for f a linear variation with the latitude is introduced, that is, $f + \beta y$, with

$$\beta = \frac{\hat{f}}{R} = \frac{2\Omega\cos\phi}{R},$$

at the fixed latitude ϕ. In the regions close to the line of the Equator, $\beta = \frac{2\Omega}{R} = 2.28 \cdot 10^{-11}\mathrm{m}^{-1}\mathrm{s}^{-1}$.

In terms of the Cartesian coordinate system (x, y, z), the full governing equations for geophysical ocean waves in the β-plane approximation are the Euler equations

$$\begin{cases} u_t + uu_x + vu_y + wu_z + \hat{f}w - (f + \beta y)v = -\frac{1}{\rho}P_x, \\ v_t + uv_x + vv_y + wv_z + (f + \beta y)u = -\frac{1}{\rho}P_y, \\ w_t + uw_x + vw_y + ww_z - \hat{f}u = -\frac{1}{\rho}P_z - g, \end{cases}$$

together with the equation of mass conservation

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0,$$

and with the condition of incompressibility

$$u_x + v_y + w_z = 0.$$

Here t is the time, (u, v, w) is the fluid velocity, P is the pressure, $g = 9.8\mathrm{ms}^{-2}$ is the standard gravitational acceleration at the Earth's surface and ρ is the water's density, which we take to be constant.

We are interested in the eastward propagation at the constant speed c of geophysical waves with vanishing meridional velocity ($v \equiv 0$), confined to the region beneath the near-surface layer $L(t)$, where wind effects are important. If $z = \eta_+(x - ct, y)$ is the lower boundary of the near-surface layer $L(t)$, beneath it we distinguish four different regions in terms of the flow characteristics (see Figure 1). First, there is the uppermost region $M(t)$ of density ρ_0, with the thermocline $z = \eta_0(x - ct, y)$, which interacts with a uniform underlying current c_0, as its lower boundary, whose oscillations propagate in the longitudinal direction at constant speed c. Beneath the thermocline the water's density is $\rho_+ > \rho_0$, where $(\rho_+ - \rho_0)/\rho_0$ is a real number associated with the latitude. For oceanic equatorial waves, indicative values for the density difference are given by $(\rho_+ - \rho_0)/\rho_0 \approx 4 \times 10^{-3}$ [**10**]. We distinguish three regions beneath the thermocline: in the layer bounded above

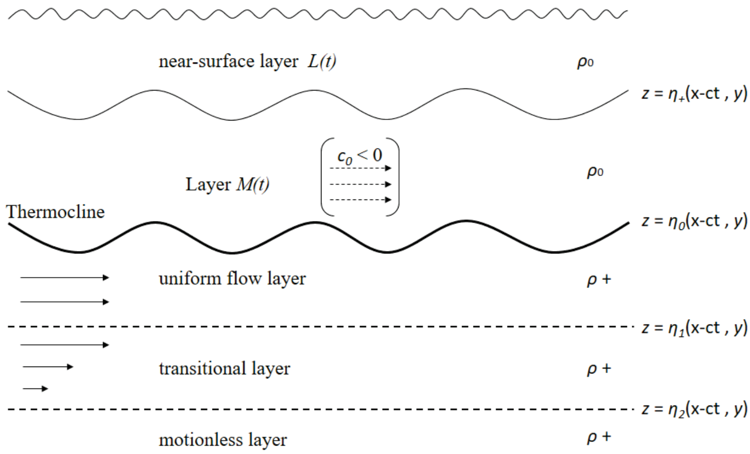

FIGURE 1. Depiction of the main flow regions at each fixed latitude y. Thermocline is described by a trochoid propagating eastward at constant speed. The thermocline separates two layers of different densities $\rho_0 < \rho_+$ in stable stratification (with the denser fluid below) and c_0 represents the constant underlying current term in $M(t)$.

by the thermocline and below by the surface $z = \eta_1(x - ct, y)$ with $\eta_1 < \eta_0$, the flow is uniform with velocity $(c - c_0, 0, 0)$; beneath the surface $z = \eta_2(x - ct, y)$ with $\eta_2 < \eta_1$, the water is completely still; the region between $z = \eta_1(x - ct, y)$ and $z = \eta_2(x - ct, y)$ is a transitional layer where the fluid motion decreases. Consequently, we seek solutions

$$u(x - ct, y, z), \quad w(x - ct, y, z), \quad \eta_+(x - ct, y), \quad \eta_i(x - ct, y) \quad \text{with} \quad i = 0, 1, 2,$$

of the following governing equations: the Euler equations in the form

$$\left\{\begin{array}{l} u_t + uu_x + wu_z + \hat{f}w = -\frac{1}{\rho_0}P_x, \\ (f + \beta y)u = -\frac{1}{\rho_0}P_y, \\ w_t + uw_x + ww_z - \hat{f}u = -\frac{1}{\rho_0}P_z - g, \end{array}\right. \quad \text{in} \quad \eta_0(x - ct, y) < z < \eta_+(x - ct, y), \tag{2.1}$$

and

$$\left\{\begin{array}{l} u_t + uu_x + wu_z + \hat{f}w = -\frac{1}{\rho_+}P_x, \\ (f + \beta y)u = -\frac{1}{\rho_+}P_y, \\ w_t + uw_x + ww_z - \hat{f}u = -\frac{1}{\rho_+}P_z - g, \end{array}\right. \quad \text{in} \quad \eta_i(x - ct, y) < z < \eta_{i-1}(x - ct, y), \tag{2.2}$$

for $i = 1, 2$, respectively, with the incompressibility condition

$$u_x + w_z = 0, \quad \text{in} \quad \eta_2(x - ct, y) < z < \eta_+(x - ct, y), \tag{2.3}$$

and with the kinematic boundary conditions

$$w = (\eta_i)_t + u(\eta_i)_x, \quad \text{on} \quad z = \eta_i(x - ct, y) \quad \text{for} \quad i = 0, 1, 2, \tag{2.4}$$

together with the continuity of the pressure across each interface $z = \eta_i(x - ct, y)$ with $i = 0, 1, 2$. With the velocity field described in Section 3 being discontinuous at the thermocline, in this case the constraint $w = (u - c)(\eta_0)_x$ must hold along $z = \eta_0(x - ct, y)$ for the velocities in both layers that are separated by the thermocline.

3. Exact and explicit solution

Now, we describe the flow in each layer separately.

3.1. The deep motionless water layer. Set $\eta_2(x - ct, y) = -D_\phi + \frac{2fy+\beta y^2}{2\hat{f}}$ for some fixed depth D_ϕ. In the region $z \leq \eta_2(x - ct, y)$, the water is in the hydrostatic state $(u, v, w) = (0, 0, 0)$ with

$$P = P_0 - \rho_+ g z, \quad \text{if} \quad z \leq -D_\phi + \frac{2fy + \beta y^2}{2\hat{f}},$$

for some constant P_0. Here, D_ϕ is a constant that depends on the variable latitude ϕ.

3.2. The transitional layer. Set $\eta_1(x - ct, y) = -d_\phi + \frac{2fy+\beta y^2}{2\hat{f}}$ for some fixed depth $d_\phi < D_\phi$. In the region $\eta_2(x - ct, y) < z < \eta_1(x - ct, y)$, $v = w = 0$, while

$$u(x - ct, y, z) = \frac{c - c_0}{D_\phi - d_\phi}\Big[z - \frac{2fy + \beta y^2}{2\hat{f}} + D_\phi\Big].$$

The second and the third components of (2.2) simplify in this layer to a balance

$$\begin{cases} (f + \beta y)u = -\frac{1}{\rho_+}P_y, \\ -\hat{f}u = -\frac{1}{\rho_+}P_z - g, \end{cases} \tag{3.1}$$

between the Coriolis terms, the pressure gradient terms, and the gravitational term. This yields

$$P(x - ct, y, z) = P_0 - \rho_+ g z + \frac{\hat{f}\rho_+(c - c_0)}{2(D_\phi - d_\phi)}\Big[z - \frac{2fy + \beta y^2}{2\hat{f}} + D_\phi\Big]^2.$$

The wave speed $c > 0$ is to be determined later on (see Section 3.4). Easily, one can check the validity of (2.2) and (2.3) in this transitional layer. Note that P and u are continuous across the interface $z = \eta_2(x - ct, y)$, and (2.4) holds for $i = 2$.

3.3. The layer beneath the thermocline. The determination of $\eta_0(x-ct, y)$ is postponed to Section 3.4, but we anticipate here that at every fixed latitude y the oscillation of the thermocline takes on the form of an eastward-propagating traveling wave with profile $\eta_0(\cdot, y)$. In the region $\eta_1(x - ct, y) < z < \eta_0(x - ct, y)$, the flow is uniform with $(u, v, w) = (c - c_0, 0, 0)$, since the simplified equations (3.1) also hold, then we obtain

$$P(x - ct, y, z) = P_0 - \rho_+ g z + \hat{f}\rho_+(c - c_0)\Big[z - \frac{2fy + \beta y^2}{2\hat{f}}\Big] + \frac{\hat{f}\rho_+(c - c_0)(D_\phi + d_\phi)}{2}. \tag{3.2}$$

It is immediately to validate (2.2) and (2.3) in this layer, P and u are continuous across the interface $z = \eta_1(x - ct, y)$, and (2.4) holds for $i = 1, 0$.

3.4. The layer $M(t)$ above the thermocline. we will present an exact solution in the $M(t)$ layer which represents waves travelling in the longitudinal direction at a constant speed of propagation $c > 0$, in the presence of a constant underlying current c_0. For later considerations, we take, on physical grounds,

$$|c_0| < \frac{g}{\hat{f}}.$$

In contrast to the previously investigated layers, it is convenient to use the Lagrangian framework [**1**] to describe this flow explicitly. In the Lagrangian framework, the Eulerian coordinates of fluid particles $\mathbf{x} = (x, y, z)$ at the time t are expressed as functions of Lagrangian labelling variables (q, s, r) which specify the fluid particle. We will show that an explicit solution to the governing equations (2.1), (2.3), and (2.4) for $i = 0$ is provided by

$$(3.3) \qquad \begin{cases} x = q - c_0 t - \frac{1}{k} e^{-k[r+m(s)]} \sin[k(q - ct)], \\ y = s, \\ z = -d_0 + r - \frac{1}{k} e^{-k[r+m(s)]} \cos[k(q - ct)], \end{cases}$$

here, k is the wave number, c_0 is the constant underlying current term, the travelling speed c and the function m depending on s are determined below. In (3.3), the parameter q covers the real line, while $s \in [-s_0, s_0]$. For every fixed value of $s \in [-s_0, s_0]$, we require $r \in [r_0(s), r_+(s)]$, such that

$$r + m(s) \geq r^* = \min_{s \in [-s_0, s_0]} r_0(s) > 0,$$

where the choice $r = r_0(s) > 0$ defines the thermocline $z = \eta_0(x - ct, y)$ at the latitude $y = s$, while $r = r_+(s) > r_0(s)$ marks at this latitude the lower boundary $z = \eta_+(x - ct, y)$ of the near-surface layer $L(t)$. At the Equator, the indicative value for $(r_+ - r_0)$ is 60 m (see [**19**]). We will prove the existence of such functions $r_0(s)$ and $r_+(s)$ below. As for $d_0 > 0$, it is determined by specifying that $[d_0 - r_0(s)]$ is the mean depth of the thermocline at the latitude of s. The wave speed c is obtained from solving relation (3.11) below, giving us the dispersion relation

$$(3.4) \qquad c = \frac{\rho_+ - \rho_0}{\rho_0} \frac{\sqrt{\hat{f}^2 + \frac{4k\rho_0(\hat{f}c_0 + g)}{\rho_+ - \rho_0}} - \hat{f}}{2k} > 0.$$

We note that the underlying current term c_0 plays a role in the dispersion relation which is highly dependent on the presence of Coriolis terms.

Let us now verify that (3.3) is an exact solution of (2.1) and (2.3) representing internal water waves travelling in the presence of a constant underlying current. Set

$$\xi = k[r + m(s)], \quad \theta = k(q - ct),$$

then we have $e^{-\xi} < 1$ throughout the layer $M(t)$, since $\xi \geq kr^* > 0$. The Jacobian matrix of the map relating the particle positions to the Lagrangian labelling variables is given by

$$J = \begin{pmatrix} \frac{\partial x}{\partial q} & \frac{\partial y}{\partial q} & \frac{\partial z}{\partial q} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 - e^{-\xi}\cos\theta & 0 & e^{-\xi}\sin\theta \\ m_s e^{-\xi}\sin\theta & 1 & m_s e^{-\xi}\cos\theta \\ e^{-\xi}\sin\theta & 0 & 1 + e^{-\xi}\cos\theta \end{pmatrix},$$

therefore, its determinant is time independent, equaling $1 - e^{-2\xi}$, thus the flow defined by (3.3) is volume preserving and (2.3) holds in this layer (see the discussion in [**6**]). Now, let us write (2.1) in the form

$$\begin{cases} \frac{\mathrm{D}u}{\mathrm{D}t} + \hat{f}w - (f+\beta y)v = -\frac{1}{\rho_0}P_x, \\ \frac{\mathrm{D}v}{\mathrm{D}t} + (f+\beta y)u = -\frac{1}{\rho_0}P_y, \\ \frac{\mathrm{D}w}{\mathrm{D}t} - \hat{f}u = -\frac{1}{\rho_0}P_z - g, \end{cases} \tag{3.5}$$

where $\frac{\mathrm{D}}{\mathrm{D}t}$ stands for the material derivative. From (3.3) we can compute the velocity and acceleration of a particle as

$$\begin{cases} u = \frac{\mathrm{D}x}{\mathrm{D}t} = -c_0 + ce^{-\xi}\cos\theta, \\ v = \frac{\mathrm{D}y}{\mathrm{D}t} = 0, \\ w = \frac{\mathrm{D}z}{\mathrm{D}t} = -ce^{-\xi}\sin\theta, \end{cases}$$

and

$$\begin{cases} \frac{\mathrm{D}u}{\mathrm{D}t} = kc^2e^{-\xi}\sin\theta, \\ \frac{\mathrm{D}v}{\mathrm{D}t} = 0, \\ \frac{\mathrm{D}w}{\mathrm{D}t} = kc^2e^{-\xi}\cos\theta, \end{cases}$$

respectively. We can therefore write (3.5) as

$$\begin{cases} P_x = -\rho_0(kc^2e^{-\xi}\sin\theta - \hat{f}ce^{-\xi}\sin\theta), \\ P_y = -\rho_0(f+\beta s)(-c_0 + ce^{-\xi}\cos\theta), \\ P_z = -\rho_0(kc^2e^{-\xi}\cos\theta + \hat{f}c_0 - \hat{f}ce^{-\xi}\cos\theta + g). \end{cases} \tag{3.6}$$

The change of variables

$$\begin{pmatrix} P_q \\ P_s \\ P_r \end{pmatrix} = J \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}$$

transforms (3.6) into

$$\begin{cases} P_q = -\rho_0(kc^2 - \hat{f}c + \hat{f}c_0 + g)e^{-\xi}\sin\theta, \\ P_s = -\rho_0[m_s(kc^2 - \hat{f}c)e^{-2\xi} + M(s)e^{-\xi}\cos\theta - (f+\beta s)c_0], \\ P_r = -\rho_0[(kc^2 - \hat{f}c)e^{-2\xi} + (kc^2 - \hat{f}c + \hat{f}c_0 + g)e^{-\xi}\cos\theta + \hat{f}c_0 + g], \end{cases} \tag{3.7}$$

where

$$M(s) = fc + c\beta s + \hat{f}c_0 m_s + gm_s.$$

Note that making a natural assumption that the pressure in $M(t)$ has continuous second partial derivatives, and thus $P_{rs} = P_{sr}$, implies that

$$m_s = \frac{fc + c\beta s}{kc^2 - \hat{f}c},$$

which leads to a natural choice

$$m(s) = \frac{\beta}{2(kc - \hat{f})}s^2 + \frac{f}{kc - \hat{f}}s. \tag{3.8}$$

Now for every constant $\widetilde{P}_0$ the gradient of the expression

$$\begin{aligned} P(q,r,s,t) \quad = \quad & \rho_0\frac{kc^2 - \hat{f}c + \hat{f}c_0 + g}{k}e^{-\xi}\cos\theta + \rho_0(kc^2 - \hat{f}c)\frac{e^{-2\xi}}{2k} \\ & -\rho_0(\hat{f}c_0 + g)r + \rho_0(kc - \hat{f})c_0 m(s) + \widetilde{P}_0. \end{aligned} \tag{3.9}$$

with respect to the labeling variables is given by the right-hand side of (3.7). We have showed that the pressure is continuous between each layer in the region under the thermocline. Now we want to have the continuity of the pressure across the thermocline. Evaluating the pressure (3.2), at the thermocline we obtain

$$\begin{aligned} P &= P_0 + \rho_+\Big[g - \hat{f}(c-c_0)\Big]\frac{1}{k}e^{-\xi}\cos\theta - \rho_+\Big[g - \hat{f}(c-c_0)\Big](-d_0+r) \\ &\quad -\hat{f}\rho_+(c-c_0)\frac{2fs+\beta s^2}{2\hat{f}} + \frac{\rho_+\hat{f}(c-c_0)(D_\phi+d_\phi)}{2}. \end{aligned} \tag{3.10}$$

Comparing (3.9) and (3.10), the continuity of the pressure across the thermocline holds if and only if

$$\rho_0\Big[kc^2 - \hat{f}c + \hat{f}c_0 + g\Big] = \rho_+\Big[g - \hat{f}(c-c_0)\Big] \tag{3.11}$$

and

$$\begin{aligned} &P_0 + \rho_+ d_0 g + \frac{\hat{f}\rho_+(c-c_0)}{2}\Big[D_\phi + d_\phi - 2d_0\Big] - \rho_+(c-c_0)\frac{2fs+\beta s^2}{2} \\ &= \rho_0(kc^2 - \hat{f}c)\Big[\frac{e^{-2\xi}}{2k} + r\Big] + \frac{\rho_0 c_0(\beta s^2 + 2fs)}{2} + \widetilde{P}_0 \end{aligned} \tag{3.12}$$

holds for $r = r_0(s)$. Solving the quadratic for c in (3.11) leads us directly to the dispersion relation (3.4). In fact, we also have another relation but it is merely mathematical solution so that must be disregarded physically. The dispersion relation (3.4) ensures that $kc - \hat{f} > 0$ holds if

$$k > \frac{\hat{f}^2}{\hat{f}c_0 + g}\Big(1 + \frac{\rho_0}{\rho_+ - \rho_0}\Big).$$

For oceanic equatorial waves with $c_0 = 0$ (see [**10**] for details), it becomes

$$k > \frac{4\Omega^2}{g}\Big(1 + \frac{\rho_0}{\rho_+ - \rho_0}\Big) \approx 5 \times 10^{-7} m^{-1}.$$

Therefore, the choices of (3.4) and (3.8) yield a solution to (2.1), (2.3) and (2.4) with $i = 0$. The oscillations of the thermocline described in the absence of an underlying current $c_0 = 0$ propagate eastwards in the shape of trochoid, and each particle in the layer $M(t)$ describes a counter-clockwise circular vertical closed path of decreasing diameter as we ascend above the thermocline. This situation contrasts with that occurring in irrotational two-dimensional surface waves (Stokes waves), where the particle paths are open loops [**4, 5, 25**].

Writing (3.12) in the form

$$\begin{aligned} P_0 - \widetilde{P}_0 &= \rho_0(kc^2 - \hat{f}c)\Big[\frac{e^{-2\xi}}{2k} + r\Big] + \frac{\rho_0 c_0(\beta s^2 + 2fs)}{2} - \rho_+ d_0 g \\ &\quad - \frac{\rho_+\hat{f}(c-c_0)}{2}\Big[D_\phi + d_\phi - 2d_0\Big] + \rho_+(c-c_0)\frac{2fs+\beta s^2}{2}. \end{aligned} \tag{3.13}$$

For each fixed $s \in [-s_0, s_0]$, the mapping

$$r \to r + \frac{e^{-2k[r + \frac{\beta}{2(kc-\hat{f})}s^2 + \frac{f}{kc-\hat{f}}s]}}{2k}$$

is a strictly increasing diffeomorphism from $(0,\infty)$ onto $(\frac{1}{2k}e^{-2k[\frac{\beta}{2(kc-\hat{f})}s^2 + \frac{f}{kc-\hat{f}}s]}, \infty)$, so for fixed $|s| > 0$, the existence of a unique solution $r_0(s)$ of the equation (3.12) is equivalent to setting

$$\begin{aligned} P_0 - \widetilde{P}_0 \quad > \quad & \frac{1}{2k}\rho_0(kc^2 - \hat{f}c) + \frac{\rho_0 c_0(\beta s^2 + 2fs)}{2} - \rho_+ d_0 g \\ & - \frac{\rho_+ \hat{f}(c - c_0)}{2}[D_\phi + d_\phi - 2d_0] + \rho_+(c - c_0)\frac{2fs + \beta s^2}{2}. \end{aligned}$$

Since the constant $\widetilde{P}_0$ is arbitrary, we can find a unique solution $r_0(s) > 0$ of (3.13). By the implicit function theorem, $r_0(s)$ is a smooth function of s. Evaluating (3.13) at $r = r_0(s)$ and differentiating the outcome with respect to s yields

$$r_0'(s) = \frac{-\rho_0 c_0 - \rho_+(c - c_0) + \rho_0 c e^{-2\xi}}{\rho_0(kc^2 - \hat{f}c)[1 - e^{-2\xi}]}(f + \beta s).$$

We enforce the restriction that the function $r_0(s)$ is strictly decreasing with $|s| > 0$. It means that

$$\begin{cases} r_0'(s) < 0, & s > 0, \\ r_0'(s) > 0, & s < 0. \end{cases} \tag{3.14}$$

Thus we will look for the geophysical waves satisfying

$$\begin{cases} \Big[- \rho_0 c_0 - \rho_+(c - c_0) + \rho_0 c e^{-2k[r_0(s)+m(s)]}\Big](f + \beta s) < 0, & s > 0, \\ \\ \Big[- \rho_0 c_0 - \rho_+(c - c_0) + \rho_0 c e^{-2k[r_0(s)+m(s)]}\Big](f + \beta s) > 0, & s < 0. \end{cases}$$

We will make a separate case discussion in the Northern Hemisphere and in the Southern Hemisphere.

For the Northern Hemisphere, we have that $\hat{f} > 0$, $f > 0$ and $\beta > 0$. Thus

$$-\frac{g}{\hat{f}} < c_0 < \frac{\rho_+ - \rho_0 e^{-2k[r_0(s)+m(s)]}}{\rho_+ - \rho_0} c = (1 + \varepsilon)c$$

for $s > 0$ or $s < -\frac{f}{\beta}$; and

$$\frac{\rho_+ - \rho_0 e^{-2k[r_0(s)+m(s)]}}{\rho_+ - \rho_0} c = (1 + \varepsilon)c < c_0 < \frac{g}{\hat{f}}$$

for $-\frac{f}{\beta} < s < 0$, where $\varepsilon = \frac{\rho_0(1 - e^{-2k[r_0(s)+m(s)]})}{\rho_+ - \rho_0}$.

For the Southern Hemisphere, we have that $\hat{f} > 0$, $f < 0$ and $\beta > 0$. Thus

$$-\frac{g}{\hat{f}} < c_0 < \frac{\rho_+ - \rho_0 e^{-2k[r_0(s)+m(s)]}}{\rho_+ - \rho_0} c = (1 + \varepsilon)c$$

for $s < 0$ or $s > -\frac{f}{\beta}$; and

$$\frac{\rho_+ - \rho_0 e^{-2k[r_0(s)+m(s)]}}{\rho_+ - \rho_0} c = (1 + \varepsilon)c < c_0 < \frac{g}{\hat{f}}$$

for $0 < s < -\frac{f}{\beta}$. Therefore, for different cases, the above equations provide the bound for the values of the constant underlying current c_0. Note that for equatorial waves, $\hat{f} = 2\Omega$, we recover the results obtained in [**38**], and the results in [**10**] for setting $c_0 = 0$. We remark that (3.14) shows that the mean depth of the thermocline $d_0 - r_0(s)$ at every s increases slightly with the distance from the line of the fixed latitude ϕ, while the boundaries (the upper and lower) of transitional layer ascend.

This is indicative of the fact that at some latitude the two regions will intersect, thus producing a more complex flow pattern.

The upper boundary of the layer $M(t)$ is specified by setting $r = r_+(s)$ at a fixed value of $s \in [-s_0, s_0]$, with $r_+(s) > r_0(s)$, The function $s \mapsto r_+(s)$ has the same features as the function $s \mapsto r_0(s)$. Now, we complete the proof that (3.3) is the exact solution of the governing equation (2.1) of internal water waves propagating in presence of the underlying current c_0.

Remark: In this paper, we assumed that the model describes the water is still in the deep-water layer. However, it is possible to apply our method to deal with the other case that in the deep-water expanse presenting a constant underlying current.

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