Dynamics of PDE, Vol.17, No.2, 165-183, 2020

Almost sure existence of global weak solutions to the Boussinesq equations

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Communicated by Jiahong Wu, received September 4, 2019.

Abstract. In this paper, we show that after a suitable randomization of the initial data in the negative order Sobolev spaces $H^{-\alpha}$ with $0 < \alpha < 1/2$, there exist almost sure global weak solutions to the Boussinesq equations in $\mathbb{R}^d$ and $\mathbb{T}^d$, when $d = 2, 3$. Furthermore, we prove that the global weak solutions are unique in dimension two.

Contents

1991 *Mathematics Subject Classification.* 35Q30 , 76D05.

Key words and phrases. Boussinesq equations, almost sure well-posedness, random data, negative order Sobolev spaces.

1. Introduction

In this paper, we address the almost sure existence of global weak solutions to the Boussinesq equations in the whole space $\mathbb{R}^d$ and the tori $\mathbb{T}^d$ for $d = 2, 3$,

$$u_t - \Delta u + u \cdot \nabla u + \nabla \pi = \rho e_3, \tag{1.1}$$

$$\rho_t - \Delta \rho + u \cdot \nabla \rho = 0, \tag{1.2}$$

$$\nabla \cdot u = 0. \tag{1.3}$$

Here, u is the velocity and ρ represents the density or temperature of the fluid which depends on the physical context. π denotes the pressure and $e_3 = (0, 0, 1)^T$. The Boussinesq system is an important physical model arising particularly in two situations. It is a model for the inhomogeneous Navier–Stokes system [3, 4], which is derived from the full compressible Navier—Stokes system under the low Mach assumption. Under this scenario, u represents the velocity while ρ represents the variation of the density. In the second context, the Boussinesq system is also related to the Rayleigh–Bénard problem, in which case ρ represents the temperature. Let us recall the scaling symmetry for the Boussinesq system

$$u(x, t) \to u_\lambda(x, t) := \lambda u(\lambda x, \lambda^2 t),$$
$$\rho(x, t) \to \rho_\lambda(x, t) := \lambda^3 \rho(\lambda x, \lambda^2 t),$$

where $\lambda > 0$. If (u, ρ) satisfies the Boussinesq system (1.1)–(1.3), then $(u_\lambda, \rho_\lambda)$ is also a solution of the Boussinesq system (1.1)–(1.3). Under such scaling for the Boussinesq system, we have $\|u_\lambda(x, 0)\|_{\dot{H}^{s_c}} = \|u(x, 0)\|_{\dot{H}^{s_c}}$ for $s_c = \frac{d}{2} - 1$, so $\dot{H}^{s_c}$ is a scaling critical space for u. We recall that the exponents s are called critical if $s = s_c$, sub-critical if $s > s_c$, and super-critical if $s < s_c$.

Data in $\dot{H}^s$ with $s < s_c$ (super-critical regime) is rougher than the data of critical regularity with respect to the scaling symmetry. Intuitively, scaling is 'against well-posedness' in this case. Ill-posedness in some cases can be circumvented by an appropriate probabilistic method in some probability space of initial data, i.e., one may hope to establish almost sure local well-posedness with respect to certain probability random data space. This random data approach to well-posedness first appeared in the paper [6] of Bourgain when he studied the invariance of Gibbs measures associated to NLS on tori ($\mathbb{T}$ and $\mathbb{T}^2$). Later, Burq and Tzvetkov [7, 8] obtained well-posedness results with random data in the context of the cubic nonlinear wave equation (NLW) on a 3D compact Riemannian manifold. The random data approach to well-posedness has also been pursued by many authors and applied to various dispersive equations and fluid models in different contexts. In the context of the incompressible Navier–Stokes equations, almost sure local well-posedness and in some instances almost sure global existence results in the context of the Navier–Stokes equations include: [9, 17, 22, 21].

Recently, Nahmod, Pavlović, and Staffilani [17] gave the first construction of almost sure global weak solutions for the Navier–Stokes equations with initial data in $H^{-\alpha}(\mathbb{T}^d)$, where $0 < \alpha < 1/2$ for $d = 2$ and $0 < \alpha < 1/4$ for $d = 3$ in the probabilistic point of view. By suitably randomizing the initial data, doing the energy estimates with the random data perturbations, they proved both existence and uniquness for dimension two and the existence results for dimension three on the torus. Their constructions can be understood as a probabilitical analogue of the global existence construction of the classic Leray weak solutions. In the context of the Navier–Stokes equations, the local in time well posedness for randomized initial data in $L^2(\mathbb{T}^3)$ was proven by Zhang and Fang [23] and by Deng and Cui [9] using similar approach under the mild formulations. In [17], by suitably randomizing the initial data u_0, the authors singled out the linear evolution $e^{t\Delta}u_0^\omega$ and the difference equation for w was identified, where they showed that the energy of w is conserved. Later, J. Wang and K. Wang in [21] extended the global existence results from the periodic domain to $\mathbb{R}^d$, for $d = 2, 3$, and improved the range of the parameter of the negative order Sobolev spaces from $0 < \alpha < 1/4$ to $0 < \alpha < 1/2$ for $d = 3$. By using the approach in [17], Du and Zhang in [10] proved the almost sure global existence of weak solutions for the MHD equations in $\mathbb{T}^d$ and $\mathbb{R}^d$, for $d = 2, 3$, where a uniform bound for the energy of the nonlinear part of the solutions was also obtained.

In recent years, there has been extensive research on the Boussinesq equations. People have been studying the persistence of regularity and global existence since the seminal work of Chae [2] and of Hou and Li [12], who proved the global existence of a unique solution. In [15], Larios et al established global existence and uniqueness in the low regularity space $H^1 \times L^2$. Kukavica and the first author of this paper addressed the persistence of regularity in $W^{s,q} \times W^{s,q}$ for the 2D fractional Boussinesq equations in [13] and the long time behavior of solutions in [14]. For other global results of solutions, see [1, 5, 11, 15, 16, 18, 19, 20]; however, the analogous almost sure existence of global weak solutions is less studied. In fact, to the best knowledge of the authors, these are the first results addressing global existence in the random data setting.

In this paper, we first introduce basic notation together with definitions of randomization. Then, we prove the global existence and uniqueness for the almost-sure weak solutions in 2D and the existence for 3D. The paper is organized as follows. In Section 2, we state our key lemmas and the main results. Section 3 contains a lemma on the estimates in terms of random data. Section 4 contains the energy estimates. In section 5, we construct the weak solutions and we prove that the solutions are unique in 2D in section 6. We finish the proof in section 7.

2. Notation and the main results

In this section, we introduce basic notations and state our main results. We first define the Leray projector $\mathbb{P}$

$$\mathbb{P} = I + \nabla(-\Delta)^{-1}\nabla\cdot$$

to be a bounded operator into divergence-free vector fields. The Leray projector $\mathbb{P}$ may also be defined via the Fourier transform

$$\widehat{(\mathbb{P}u)}_j(\xi) = \left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)\hat{u}_k(\xi), \qquad j = 1, 2, 3.$$

We apply the Leray projector to the equation (1.1) and the pressure π vanishes. Then we have

$$u_t - \Delta u + \mathbb{P}\nabla\cdot(u \otimes u) = \mathbb{P}(\rho e_3).$$

We now define

$$H = \text{the closure of } \{u \in C^\infty : \nabla\cdot u = 0\} \text{ in } L^2$$

and

$$V = \text{the closure of } \{u \in C^\infty : \nabla\cdot u = 0\} \text{ in } H^1.$$

We next introduce the construction of random initial data in the whole space $\mathbb{R}^d$ for $d \geq 1$, which was first introduced by Burq and Tzvetkov [7]. In $\mathbb{R}^d$, we divide the frequency space by using the Wiener decomposition. For $n \in \mathbb{Z}^d$, let Q_n be the unit cube $Q_n = n + (-\frac{1}{2}, \frac{1}{2}]^d$. Then we have

$$\mathbb{R}^d = \bigcup_n Q_n.$$

Note that $Q_n \cap Q_m = \emptyset$ if $m \neq n$ and $\sum_n \chi_{Q_n}(\xi) = 1$. Hence, we have the decomposition

$$f(x) = \sum_{n\in\mathbb{Z}^d} \mathcal{F}^{-1}(\chi_{Q_n}\hat{f}).$$

Define a nonnegative and even smooth function ϕ such that $\phi(\xi) = 1$ for $\xi \in (-\frac{1}{2}, \frac{1}{2})^d$ and $\phi(\xi) = 0$ for $\xi \in ([-1, 1]^d)^c$, and let

$$\varphi(\xi) = \frac{\phi(\xi)}{\sum_n \phi(\xi - n)}.$$

Note that $\sum_n \varphi(\xi - n) = 1$. Define

$$\varphi(D - n)f = \int_{\mathbb{R}^d} \hat{f}(\xi)\varphi(\xi - n)e^{i2\pi x\cdot\xi}\, d\xi,$$

where

$$D = \sqrt{-\Delta}.$$

Then f has a smooth version for the Wiener decomposition:

$$f(x) = \sum_{n\in\mathbb{Z}^d} \varphi(D - n)f.$$

For any real-valued function f, we obtain

$$\overline{\varphi(D+n)f} = \varphi(D-n)f$$

and $\sum_n \varphi(D-n)f$ is also a real-valued function. In the $\mathbb{T}^d$ case, the frequencies of functions are in $\mathbb{Z}^d$, so we can divide the frequency space into the integer points. To keep the consistence of the notations, we denote the decomposition operator $\phi(D-n)f = \hat{f}(n)e^{i2\pi x\cdot n}$.

For simplicity, we will skip the $\mathbb{T}$ or $\mathbb{R}^d$ in the spacial function spaces in the following context (e.g. L^2 means $L^2(\mathbb{T}^d)$ or $L^2(\mathbb{R}^d)$). We will specify them when it is necessary. We now introduce the randomization of elements in negative order Sobolev spaces $\dot{H}^{-\alpha}$.

Definition 2.1. Let $(l_n(\omega))_{n\in\mathbb{Z}^d}$ be a sequence of real, 0-mean, independent random variables on a probability space (Ω, A, p) with associated sequence of distributions $(\mu_n)_{n\in\mathbb{Z}^d}$ so that there exists $c>0$, for all $\gamma\in\mathbb{R}$ and for all $n\in\mathbb{Z}^d$ we have that

$$\left|\int_{-\infty}^{\infty} e^{\gamma x}\, d\mu_n(x)\right| \le e^{c\gamma^2}. \tag{2.1}$$

For $f\in\dot{H}^{-\alpha}(\mathbb{R}^d)$ or $f\in H^{-\alpha}(\mathbb{T}^d)$, we define the map from (Ω, A) to $H^{-\alpha}$ by

$$\omega \to f^\omega$$

where

$$f^\omega = \sum_{n\in\mathbb{Z}^d} l_n(\omega)\phi(D-n)f, \tag{2.2}$$

where $\phi(D-n)f$ is defined as before. We call such a map randomization.

To state the main theorems (Theorem 2.1 and Theorem 2.2), we introduce the following definition:

Definition 2.2. For $T>0$ and $d=2,3$, let $u_0, \rho_0 \in \dot{H}^{-\alpha}(\mathbb{R}^d)$ or $H^{-\alpha}(\mathbb{T}^d)$ and $\nabla\cdot u_0 = 0$. We say (u,ρ) is a *weak solution* of the initial value problem (1.1)-(1.3) with initial datum (u_0,ρ_0), if

- $u,\rho \in L^2_{loc}([0,T];\dot{H}^1)\cap L^\infty_{loc}([0,T];L^2)\cap C_w([0,T];\dot{H}^{-\alpha})$ satisfying $\frac{du}{dt}, \frac{d\rho}{dt} \in L^2([0,T];H^{-1}_{loc})$;
- the map $t\in(0,T)\mapsto u(t,\cdot)$ and $\rho(t,\cdot)$ are continuous from $(0,T)$ to $\dot{H}^{-\alpha}(\mathbb{R}^d)$ or $H^{-\alpha}(\mathbb{T}^d)$ and $\lim_{t\to0^+}(u(t,\cdot),\rho(t,\cdot)) = (u_0,\rho_0)$;
- for all ζ,η in Schwartz space with $\nabla\cdot\zeta = 0$, we have

$$\langle u_t - \Delta u + \mathbb{P}\nabla\cdot(u\otimes u) - \rho e_3, \zeta\rangle = 0,$$

  and

$$\langle \rho_t - \Delta\rho + u\cdot\nabla\rho, \eta\rangle = 0.$$

The following are the main results of this paper.

THEOREM 2.1 (Existence and uniqueness in 2D). Fix $T > 0$, $0 < \alpha \leq 1/2$. Let $u_0, \rho_0 \in \dot{H}^{-\alpha}(\mathbb{R}^2)$ or $H^{-\alpha}(\mathbb{T}^2)$ and $\nabla \cdot u_0 = 0$. We further suppose u_0 and ρ_0 are mean zero in the periodic case. Then there exists a set $\Sigma \subseteq \Omega$ of probability 1 such that for any $\omega \in \Sigma$ the initial value problem (1.1) – (1.3) with datum $(u_0^\omega, \rho_0^\omega)$ has a unique global weak solution in the sense of Definition 2.2 of the form

$$u = g_1^\omega + v,$$

$$\rho = g_2^\omega + \theta,$$

and

$$v, \theta \in L^\infty([0,T]; L^2) \cap L^2([0,T]; \dot{H}^1),$$

where $(u_0^\omega, \rho_0^\omega)$ are random initial datum in the sense of Definition 2.1, $g_1^\omega = e^{t\Delta} u_0^\omega$ and $g_2^\omega = e^{t\Delta} \rho_0^\omega$.

THEOREM 2.2 (Existence in 3D). Fix $T > 0$, $0 < \alpha \leq 1/2$. Let $u_0, \rho_0 \in \dot{H}^{-\alpha}(\mathbb{R}^3)$ or $H^{-\alpha}(\mathbb{T}^3)$ and $\nabla \cdot u_0 = 0$. We further suppose u_0 and ρ_0 are mean zero in the periodic case. Then there exists a set $\Sigma \subseteq \Omega$ of probability 1 such that for any $\omega \in \Sigma$ the initial value problem (1.1) – (1.3) with datum $(u_0^\omega, \rho_0^\omega)$ has a global weak solution in the sense of Definition 2.2 of the form

$$u = g_1^\omega + v,$$

$$\rho = g_2^\omega + \theta,$$

and

$$v, \theta \in L^\infty([0,T]; L^2) \cap L^2([0,T]; \dot{H}^1),$$

where $(u_0^\omega, \rho_0^\omega)$ are random initial datum in the sense of Definition 2.1, $g_1^\omega = e^{t\Delta} u_0^\omega$ and $g_2^\omega = e^{t\Delta} \rho_0^\omega$.

Note that in fact the later construction will show $u, \rho \in L^\infty([\delta_0, T]; L^2) \cap L^2([\delta_0, T]; \dot{H}^1)$ for some small time δ_0.

3. A priori estimates on the random data

We now introduce the deterministic estimates for the random initial data and probabilistic estimates for the heat kernel in terms of random data. The following lemma is a standard large deviation property (see Lemma 3.1 in [7]) and it will be used to analyze the heat flow on the randomized data.

LEMMA 3.1 (Lemma 3.1 in [7]). Let $(l_r(\omega))_{r=1}^\infty$ be a sequence of real, 0-mean, independent random variables on a probability space $(\Omega, A, \mathbf{P})$ with associated sequence of distributions $(\mu_r)_{r=1}^\infty$. Assume that there exists $c > 0$ such that $\forall \gamma \in \mathbb{R}$,

$\forall r \geq 1$ we have

$$\left| \int_{-\infty}^{\infty} e^{\gamma x} \, d\mu_r(x) \right| \leq e^{c\gamma^2}.$$

Then there exists $\alpha > 0$ such that for every $\lambda > 0$, every sequence $(a_r)_{r=1}^{\infty} \in \ell^2$ of real numbers,

$$\mathbf{P}\left(\omega : \left|\sum_{r=1}^{\infty} a_r l_r(\omega)\right| > \lambda\right) \leq C \exp\left(-\frac{\alpha\lambda^2}{\|a_r\|_{\ell_r^2}^2}\right).$$

As a consequence, for every $q \geq 2$ and $(a_r^2)_{r=1}^{\infty} \in \ell^2$,

$$\left\|\sum_{r=1}^{\infty} a_r l_r(\omega)\right\|_{L^q(\Omega)} \leq C\sqrt{q}\,\|a_r\|_{\ell_r^2}.$$

We next recall another classical result for a sequence of real, mean-0, independent random variables.

LEMMA 3.2. Let $\{l_n(\omega)\}_{n\in\mathbb{Z}^d}$ be a sequence of real, mean-0, independent random variables satisfy Definition 2.1 on a probability space $(\Omega, A, \mathbf{P})$. Then given $\epsilon, \delta > 0$, there exists a subset $\Omega_\delta \subset \Omega$ satisfying $\mathbf{P}(\Omega_\delta^c) \lesssim e^{-\frac{1}{\delta^\epsilon}}$, such that for all $\omega \in \Omega_\delta$

$$|l_n(\omega)| \lesssim \frac{1}{\delta^\epsilon} \log(\langle n\rangle + 1)$$

where $\langle n\rangle = \sqrt{|n|^2 + 1}$.

PROOF. For each n and a small $\epsilon > 0$, we have a constant C,

$$\mathbb{E}e^{|l_n(\omega)|} \leq C.$$

Set $M = \frac{1}{\delta^\epsilon}$, and then we have

$$\mathbb{E}\left|\frac{e^{|l_n(\omega)|}}{e^M}\right| \leq Ce^{-\frac{1}{\delta^\epsilon}}$$

Then we obtain,

$$Ce^{-\frac{1}{\delta^\epsilon}} > \mathbb{E}\left|\frac{e^{|l_n(\omega)|}}{e^M}\right| \geq \sum_{j\in\mathbb{Z}^d} \mathbf{P}(e^{|l_j(\omega)|} \geq e^M \langle j\rangle^d) = \sum_{j\in\mathbb{Z}^d} \mathbf{P}(|l_j(\omega)| \geq \frac{1}{\delta^\epsilon} + d\log\langle j\rangle).$$

Excluding $\Omega_\delta^c := \cup_j \{|l_j(\omega)| \geq \frac{1}{\delta^\epsilon} + d\log\langle j\rangle\}$ from Ω, for all $\omega \in \Omega_\delta$, we have

$$|l_n(\omega)| \leq \frac{1}{\delta^\epsilon} + d\log\langle n\rangle \lesssim \frac{1}{\delta^\epsilon} \log(\langle n\rangle + 1), \text{for } n \in \mathbb{Z}^d.$$

with $\mathbf{P}(\Omega_\delta^c) < Ce^{-\frac{1}{\delta^\epsilon}}$. □

REMARK 3.3. For given $\epsilon > 0$ and arbitrary small $\gamma > 0$, it is easy to check the fact that

$$\mathbf{P}(\|f^\omega\|_{H^{-\alpha-\gamma}} > \frac{1}{\delta^\epsilon}\|f\|_{H^{-\alpha}}) \lesssim e^{\frac{1}{\delta^\epsilon}}$$

which implies almost surely $f^\omega \in H^{-\alpha-\gamma}$ for arbitrary small $\gamma > 0$.

LEMMA 3.4. For $0 < \alpha < 1$ and $k \in \mathbb{N}$. Given $\epsilon > 0$ and arbitrary small $\gamma > 0$. If $f \in \dot{H}^{-\alpha}(\mathbb{R}^d)$ or $f \in H^{-\alpha}(\mathbb{T}^d)$ of mean zero. Suppose f^ω is defined as (2.2), then there exists a subset $\Omega_\delta \subset \Omega$ satisfying $\mathbf{P}(\Omega_\delta^c) \lesssim e^{-\frac{1}{\delta^\epsilon}}$, for all $\omega \in \Omega_\delta$ we obtain that

$$\|\nabla^k e^{t\Delta} f^\omega\|_{L^2} \le \frac{1}{\delta^\epsilon}(1 + t^{-\frac{\alpha+\gamma+k}{2}})\|f\|_{\dot{H}^{-\alpha}} \tag{3.1}$$

and

$$\|\nabla^k e^{t\Delta} f^\omega\|_{L^\infty} \le \frac{1}{\delta^\epsilon}\max\{t^{-\frac{1}{2}}, t^{\frac{-(k+\alpha+\gamma+d/2)}{2}}\}\|f\|_{\dot{H}^{-\alpha}}. \tag{3.2}$$

PROOF. Based on the deterministic properties of heat kernel (see Lemma 3.1 in [**17**]), we have

$$\|\nabla^k e^{t\Delta} f^\omega\|_{L^2} \le (1 + t^{-\frac{\alpha+k}{2}})\|f^\omega\|_{\dot{H}^{-\alpha-\gamma}} \tag{3.3}$$

and

$$\|\nabla^k e^{t\Delta} f^\omega\|_{L^\infty} \le \max\{t^{-\frac{1}{2}}, t^{\frac{-(k+\alpha+d/2)}{2}}\}\|f^\omega\|_{\dot{H}^{-\alpha-\gamma}}. \tag{3.4}$$

Applying Remark 3.3 into (3.3) and (3.4), we have (3.1) and (3.2) after excluding a subset of probability $\lesssim e^{-\frac{1}{\delta^\epsilon}}$. □

REMARK 3.5. Given $f \in H^{-\alpha}(\mathbb{T}^d)$ of mean zero, $\|f\|_{H^{-\alpha}(\mathbb{T}^d)}$ is comparable to $(\sum_{k\in\mathbb{Z}^d, k\neq 0}|k|^{-\alpha}|\hat{f}(k)|^2)^{1/2} = \|f\|_{\dot{H}^{-\alpha}(\mathbb{T}^d)}$, since $\hat{f}(0) = 0$. So in the periodic case, it is equivalent to use $H^{-\alpha}$ and $\dot{H}^{-\alpha}$ when the functions are mean zero.

We use the deterministic properties of heat kernel but we still leave the linear evolution $e^{t\Delta} f^\omega(t)$ unbounded when t is near zero. This is also the reason why we can not construct the weak solution of negative regularity deterministically. In the following lemma, we exploit the randomness in the data f^ω and then we bound the linear evolution $e^{t\Delta} f^\omega(t)$ in the small interval around zero in the L^p sense.

LEMMA 3.6. For $p, q \ge 2$, $0 < \alpha < 1$ with $\alpha p \le 2$ and $\delta > 0$. Given some $\epsilon < \frac{1}{p} - \frac{\alpha}{2}$, set

$$E_{\delta,p,q} = \{\omega \in \Omega : \|e^{t\Delta} f^\omega\|_{L^p([0,\delta'],L^q)} > (\delta')^{\frac{1}{p}-\frac{\alpha}{2}-\epsilon}\|f\|_{H^{-\alpha}},\ \forall \delta' \in (0,\delta]\}.$$

Then we have

$$\mathbf{P}(E_{\delta,p,q}) \lesssim e^{-\frac{1}{\delta^\epsilon}}.$$

PROOF. By Minkowski's inequality and large deviation property (Lemma 3.1), for $r \ge p, q$ we can have the following bound (see Lemma 2.4 in [**21**])

$$(\mathbb{E}\|e^{t\Delta} f^\omega\|^r_{L^p([0,\delta'],L^q)})^{1/r} \le C_{p,q}\sqrt{r}(\delta')^{\frac{1}{p}-\frac{\alpha}{2}}\|f\|_{H^{-\alpha}}. \tag{3.5}$$

By Chebyshev's inequality, we have

$$\mathbf{P}(\|e^{t\Delta} f^\omega\|_{L^p([0,\delta'],L^q)} > \lambda) \le C^r_{p,q}\lambda^{-r} r^{\frac{r}{2}}(\delta')^{\frac{r}{p}-\frac{r\alpha}{2}}\|f\|^r_{H^{-\alpha}}$$

for any $r \geq p, q$. When $\lambda / \left((\delta')^{\frac{r}{p} - \frac{r\alpha}{2}} \|f\|_{H^{-\alpha}}^r \right) \geq e^2$, we select

$$r = \frac{\lambda}{(\delta')^{\frac{r}{p} - \frac{r\alpha}{2}} \|f\|_{H^{-\alpha}}^r}.$$

By $r^{-\frac{r}{2}} \leq e^{-r^2}$ when $r \geq e^2$, we have

$$\mathbf{P}(\|e^{t\Delta} f^\omega\|_{L^p([0,\delta'],L^q)} > \lambda) \lesssim_{p,q} \exp\left(-\frac{\lambda^2}{\left((\delta')^{\frac{r}{p} - \frac{r\alpha}{2}} \|f\|_{H^{-\alpha}}^r \right)^2} \right).$$

When $\lambda / \left((\delta')^{\frac{r}{p} - \frac{r\alpha}{2}} \|f\|_{H^{-\alpha}}^r \right) < e^2$, we select $r = \max\{p, q\}$ (WLOG suppose $r = p$). It is easy to check that $C_{p,q}^r \lambda^{-p} p^{\frac{p}{2}} (\delta')^{1 - \frac{p\alpha}{2}} \|f\|_{H^{-\alpha}}^p \lesssim_{p,q} \exp\left(-\frac{\lambda^2}{\left((\delta')^{\frac{r}{p} - \frac{r\alpha}{2}} \|f\|_{H^{-\alpha}}^r \right)^2} \right)$. So we have

$$\mathbf{P}(\|e^{t\Delta} f^\omega\|_{L^p([0,\delta'],L^q)} > \lambda) \lesssim_{p,q} \exp\left(-\frac{\lambda^2}{\left((\delta')^{\frac{r}{p} - \frac{r\alpha}{2}} \|f\|_{H^{-\alpha}}^r \right)^2} \right).$$

By choosing $\lambda = (\delta')^{\frac{1}{p} - \frac{\alpha}{2} - \epsilon} \|f\|_{H^{-\alpha}}$, we prove

$$\mathbf{P}(E_{\delta,\delta',p,q}) \lesssim e^{-\frac{1}{(\delta')^{2\epsilon}}},$$

where

$$E_{\delta,\delta',p,q} = \{\omega \in \Omega : \|e^{t\Delta} f^\omega\|_{L^p([0,\delta'],L^q)} > (\delta'/2)^{\frac{1}{p} - \frac{\alpha}{2} - \epsilon} \|f\|_{H^{-\alpha}}, \text{ where } \delta' \in (0, \delta]\}.$$

By choosing $\delta' = \delta, \frac{\delta}{2}, \frac{\delta}{4}, \frac{\delta}{8}, \ldots$, we have $\mathbf{P}(\cup_{j=1}^\infty E_{\delta,2^{-j}\delta,p,q}) \leq \sum_{j=1}^\infty \mathbf{P}(E_{\delta,2^{-j}\delta,p,q}) \lesssim e^{-\frac{1}{\delta^\epsilon}}$. It is easy to check $E_{\delta,p,q} \subset \cup_{j=1}^\infty E_{\delta,2^{-j}\delta,p,q}$ yielding (3.5). □

4. Energy estimates for the Boussinesq system

In this section, we give energy estimates for the difference equations. We will use these *a priori* estimates to construct global weak solutions. First, we set

$$u = g_1 + v,$$

$$\rho = g_2 + \theta,$$

where g_1 and g_2 are two functions satisfying some specific conditions (in the following Theorem 4.1). It is equivalent to consider the new perturbed Boussinesq system

$$v_t - \Delta v + \mathbb{P}\nabla \cdot ((g_1 + v) \otimes (g_1 + v)) = \mathbb{P}((g_2 + \theta)e_3), \tag{4.1}$$

$$\theta_t - \Delta\theta + \nabla \cdot ((g_1 + v)(g_2 + \theta)) = 0, \tag{4.2}$$

$$\nabla \cdot v = 0. \tag{4.3}$$

Now we define the energy for v and θ, respectively.

$$E_1(v,t) = \|v\|_{L^2}^2 + \int_0^t \|\nabla v\|_{L^2}^2 \, ds$$

and

$$E_2(\theta,t) = \|\theta\|_{L^2}^2 + \int_0^t \|\nabla \theta\|_{L^2}^2 \, ds.$$

The following theorem establishes energy bounds which will be used in constructing global weak solutions. The proofs of the whole space and the periodic space are similar, so we only present the proof of the whole space. Denote that $f = (u_0, \rho_0)$ and $\|f\|_{\dot{H}^{-\alpha}} = \|u_0\|_{\dot{H}^{-\alpha}} + \|\rho_0\|_{\dot{H}^{-\alpha}}$.

THEOREM 4.1. For any fixed number $T > 0$ and $\alpha \in (0, \frac{1}{2})$ and for any $\delta > 0$ which is small enough. Given $0 < \epsilon < \frac{1}{4} - \frac{\alpha}{2}$ and $\gamma > 0$ which is an arbitrarily small positive number. Consider functions g_1 and g_2 satisfying the following properties, for $i = 1, 2$ and $k = 0, 1$

$$\|\nabla^k g_i\|_{L^2} \le \frac{1}{\delta^\epsilon}(1 + t^{-\frac{\alpha+\gamma+k}{2}})\|f\|_{\dot{H}^{-\alpha}}, \tag{4.4}$$

and

$$\|g_i\|_{L^4([0,\delta],L^4)} + \|g_i\|_{L^4([0,\delta],L^{4^+})} \le \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon}\|f\|_{\dot{H}^{-\alpha}}, \qquad \text{when } d = 2 \tag{4.5}$$

where $\frac{1}{4^+} = \frac{1}{4} - \gamma$ and

$$\|g_i\|_{L^3([0,\delta],L^9)} + \|g_i\|_{L^4([0,\delta],L^4)} \le \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon}\|f\|_{\dot{H}^{-\alpha}}, \qquad \text{when } d = 3. \tag{4.6}$$

Suppose $v, \theta \in L^\infty([0,T]; L^2) \cap L^2([0,T]; \dot{H}^1)$ is a solution of (4.1)-(4.3), then for all $t \in [0,T]$, we have

$$E_1(v,t) + E_2(\theta,t) \le C(T, \alpha, \delta, \|f\|_{\dot{H}^{-\alpha}}). \tag{4.7}$$

Furthermore for any bounded domain B with smooth boundary in $\mathbb{R}^d$ (d=2, 3), we have

$$\|\partial_t v\|_{L^2([0,T],H^{-1}(B))} + \|\partial_t \theta\|_{L^2([0,T],H^{-1}(B))} \lesssim_B C(T, \alpha, \delta, \|f\|_{\dot{H}^{-\alpha}}). \tag{4.8}$$

PROOF OF THEOREM 4.1. We will prove this theorem by separating into two cases $t \in [0,\delta]$ and $t \in [\delta, T]$.

Case 1: $t \in [0,\delta]$. First, by multiplying by v and integrating the resulting equation for $t \in [0,\delta]$ we obtain the following equation:

$$\begin{aligned} E_1(v,t) = &- \int_0^t \int v \cdot \mathbb{P}\nabla \cdot (g_1 \otimes g_1) \, dxds - \int_0^t \int v \cdot \mathbb{P}\nabla \cdot (g_1 \otimes v) \, dxds \\ &- \int_0^t \int v \cdot \mathbb{P}\nabla \cdot (v \otimes g_1) \, dxds - \int_0^t \int v \cdot \mathbb{P}\nabla \cdot (v \otimes v) \, dxds \\ &- \int_0^t \int v \cdot \mathbb{P}(g_2 + \theta) e_3 \, dxds = \sum_{i=1}^5 I_i. \end{aligned}$$

By the divergence-free condition on v and g_1, we have

$$I_3 = I_4 = 0.$$

Indeed, since v is divergence-free, we have

$$\int v \cdot \mathbb{P}\nabla \cdot (v \otimes v)\, dx = \frac{1}{2} \int v \cdot \mathbb{P}\nabla |v|^2\, dx = 0.$$

Therefore, it remains to estimate I_1, I_2 and I_5. For I_1, by Hölder inequality, the definition of E_1 and (4.5) (4.6), we have

$$I_1 \lesssim \|\nabla v\|_{L^2([0,t],L^2)} \|g_1\|^2_{L^4([0,t],L^4)} \lesssim \delta^{\frac{1}{2}-\alpha-2\epsilon} \|f\|^2_{\dot{H}^{-\alpha}} E_1(t)^{\frac{1}{2}}.$$

For I_2, when $d = 2$, by Hölder inequality, (4.5) and the definition of E_1, we have that

$$\begin{aligned} I_2 \lesssim& \|\nabla v\|_{L^2([0,t],L^2)} \|g_1\|_{L^4([0,t],L^{4+})} \|v\|_{L^4([0,t],L^{4-})} \\ \lesssim& \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon} \|f\|_{\dot{H}^{-\alpha}} E_1(t)^{\frac{1}{2}} \|v\|_{L^4([0,t],L^{4-})}, \end{aligned} \tag{4.9}$$

where $\frac{1}{4^-} = \frac{1}{4} + \gamma$. For $\|v\|_{L^4([0,t],L^{4-})}$, by L^p interpolation theory and Sobolev inequality, we have

$$\begin{aligned} \|v\|_{L^4([0,t],L^{4-})} &\lesssim \left(\|v\|_{L^\infty([0,t],L^2)}\right)^{\frac{1}{2}} \left(\|v\|_{L^2([0,t],L^{\infty-})}\right)^{\frac{1}{2}} \\ &\lesssim \left(\sup_{0\le s\le t} E_1(s)\right)^{\frac{1}{4}} \left(\|v\|_{L^2([0,t],H^1)}\right)^{\frac{1}{2}} \\ &\lesssim \left(\sup_{0\le s\le t} E_1(s)\right)^{\frac{1}{4}} \left(\|v\|_{L^2([0,t],L^2)} + \|\nabla v\|_{L^2([0,t],L^2)}\right)^{\frac{1}{2}} \\ &\lesssim \left(\sup_{0\le s\le t} E_1(s)\right)^{\frac{1}{4}} \left(t^{1/2} \sup_{0\le s\le t} E_1(s)^{\frac{1}{2}} + E_1(t)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \end{aligned} \tag{4.10}$$

where $\frac{1}{\infty^-} = \gamma/2$. Combining (4.9) and (4.10), and taking $t = \delta$, we have that for $\delta < 1$

$$I_2 \lesssim \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon} \|f\|_{\dot{H}^{-\alpha}} \sup_{0\le s\le \delta} E_1(s). \tag{4.11}$$

For I_2, when $d = 3$, by Hölder inequality, (4.6) and the definition of E_1, we have that

$$\begin{aligned} I_2 \lesssim& \|\nabla v\|_{L^2([0,t],L^2)} \|g_1\|_{L^3([0,t],L^9)} \|v\|_{L^6([0,t],L^{\frac{18}{7}})} \\ \lesssim& \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon} \|f\|_{\dot{H}^{\alpha}} E_1(t)^{\frac{1}{2}} \|v\|_{L^6([0,t],L^{\frac{18}{7}})}. \end{aligned} \tag{4.12}$$

Based on the interpolation theory and Sobolev inequality, we have

$$\|v\|_{L^6([0,t],L^{\frac{18}{7}})} \lesssim \|v\|^{\frac{2}{3}}_{L^\infty([0,t],L^2)} \|v\|^{\frac{1}{3}}_{L^2([0,t],\dot{H}^1)} \lesssim \sup_{0\le s\le \delta} E_1(s)^{\frac{1}{2}}. \tag{4.13}$$

Hence when $d = 3$, we have the same bound of I_2 as (4.11). For I_5, by Cauchy inequality and (4.4) we have that

$$
\begin{aligned}
I_5 &\lesssim \|v\|_{L^\infty([0,t],L^2)} \left(t\|\theta\|_{L^\infty([0,t],L^2)} + \|g_2\|_{L^1([0,t],L^2)} \right) \\
&\lesssim \delta \sup_{0\le s\le\delta} E_1(s)^{1/2} \sup_{0\le s\le\delta} E_2(s)^{1/2} + \delta^{1-\frac{\alpha+\gamma}{2}-\epsilon} \sup_{0\le s\le\delta} E_1(s)^{\frac{1}{2}}.
\end{aligned} \tag{4.14}
$$

For the estimate of E_2, first, we multiply by θ and integrate the resulting equation

$$
\begin{aligned}
E_2(\theta,t) &= -\int_0^t \int (\nabla\theta)\cdot(g_1g_2 + g_1\theta + vg_2 + v\theta)\,dxds \\
&= -\int_0^t \int (\nabla\theta)\cdot(g_1g_2) - \int_0^t \int (\nabla\theta)\cdot(vg_2)\,dxds = J_1 + J_2,
\end{aligned}
$$

where we use the fact that both v and g_1 are divergence-free. For J_1, we apply Hölder's inequality obtaining

$$
J_1 \le \|\nabla\theta\|_{L^2([0,t],L^2)} \|g_1\|_{L^4([0,t],L^4)} \|g_2\|_{L^4([0,t],L^4)} \lesssim \delta^{\frac{1}{2}-\alpha-2\epsilon} \|f\|^2_{\dot H^{-\alpha}} E_2(t)^{\frac{1}{2}}.
$$

For J_2, when $d = 2$, applying Hölder's inequality, (4.5) and (4.10) we have

$$
\begin{aligned}
J_2 &\lesssim \|\nabla\theta\|_{L^2([0,t],L^2)} \|g_2\|_{L^4([0,t],L^{4+})} \|v\|_{L^4([0,t],L^{4-})} \\
&\lesssim \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon} \|f\|_{\dot H^{-\alpha}} \sup_{0\le s\le\delta} E_1(s)^{\frac{1}{2}} \sup_{0\le s\le\delta} E_2(s)^{\frac{1}{2}}.
\end{aligned}
$$

For J_2, when $d = 3$, applying Hölder's inequality, (4.6) and (4.14) we have

$$
\begin{aligned}
J_2 &\lesssim \|\nabla\theta\|_{L^2([0,t],L^2)} \|g_2\|_{L^3([0,t],L^9)} \|v\|_{L^6([0,t],L^{\frac{18}{7}})} \\
&\lesssim \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon} \|f\|_{\dot H^{-\alpha}} \sup_{0\le s\le\delta} E_1(s)^{\frac{1}{2}} \sup_{0\le s\le\delta} E_2(s)^{\frac{1}{2}}.
\end{aligned}
$$

Summarizing $\sum_{i=1}^5 I_i$ and $\sum_{i=1}^2 J_i$, when $t \in [0,\delta]$ we have the following bound

$$
\begin{aligned}
\sup_{0\le s\le\delta} (E_1(s) + E_2(s)) \lesssim\, & \delta^{\frac{1}{2}-\alpha-2\epsilon} \|f\|^2_{\dot H^{\alpha}} \left(\sup_{0\le s\le\delta} E_1(s)^{\frac{1}{2}} + E_2(s)^{\frac{1}{2}} \right) \\
& + \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon} \|f\|_{\dot H^{-\alpha}} \sup_{0\le s\le\delta} E_1(s) + \delta^{\frac{1}{2}-\alpha-2\epsilon} \|f\|^2_{\dot H^{-\alpha}} E_2(t)^{\frac{1}{2}} \\
& + \delta \sup_{0\le s\le\delta} E_1(s)^{1/2} \sup_{0\le s\le\delta} E_2(s)^{1/2} + \delta^{1-\frac{\alpha+\gamma}{2}-\epsilon} \sup_{0\le s\le\delta} E_1(s)^{\frac{1}{2}} \\
& + \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon} \|f\|_{\dot H^{-\alpha}} \sup_{0\le s\le\delta} E_1(s)^{\frac{1}{2}} \sup_{0\le s\le\delta} E_2(s)^{\frac{1}{2}}.
\end{aligned} \tag{4.15}
$$

Since $\alpha < \frac{1}{2} - 2\epsilon$, we could choose δ is small enough such that $\delta^{1-\frac{\alpha+\gamma}{2}-\epsilon} \ll 1$ and $\delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon} \|f\|_{\dot H^{-\alpha}} \ll 1$. Then the continuity argument with (4.15) helps us obtain that

$$
\sup_{0\le s\le\delta} (E_1(s) + E_2(s)) \le C(\alpha, \|f\|_{\dot H^{-\alpha}}).
$$

Next we will consider the estimate of $\|\partial_t v\|_{L^2([0,\delta],H^{-1}(B))} + \|\partial_t \theta\|_{L^2([0,\delta],H^{-1}(B))}$. By the equation (4.1) we have

$$
\begin{aligned}
\|\partial_t v\|_{L^2([0,\delta],H^{-1}(B))} \leq & \|v\|_{L^2([0,\delta],H^1(B))} + \|\mathbb{P}\nabla\cdot(g_1\otimes g_1)\|_{L^2([0,\delta],H^{-1}(B))} \\
& + \|\mathbb{P}\nabla\cdot(g_1\otimes v)\|_{L^2([0,\delta],H^{-1}(B))} + \|\mathbb{P}\nabla\cdot(v\otimes g_1)\|_{L^2([0,\delta],H^{-1}(B))} \\
& + \|\mathbb{P}\nabla\cdot(v\otimes v)\|_{L^2([0,\delta],H^{-1}(B))} + \|\mathbb{P}(g_2+\theta)e_3\|_{L^2([0,\delta],H^{-1}(B))}.
\end{aligned} \tag{4.16}
$$

By the definition of the energy, we have $\|v\|_{L^2([0,\delta],H^1(B))} \leq E_1^{\frac{1}{2}}(\delta)$. For the remaining terms in (4.16), using the similar estimates for terms from I_1 to I_5, and hence for $d=2$ we have

$$
\begin{aligned}
\|\partial_t v\|_{L^2([0,\delta],H^{-1}(B))} \leq\ & E_1^{\frac{1}{2}}(\delta) + \|g_1\|^2_{L^4([0,\delta],L^4)} + 2\|g_1\|_{L^4([0,\delta],L^{4+})}\|v\|_{L^4([0,\delta],L^{4-})} \\
& + \|v\|^2_{L^4([0,\delta],L^4)} + \|\theta+g_2\|_{L^2([0,\delta],H^{-1}(B))} \\
\lesssim_B\ & E_1^{\frac{1}{2}}(\delta) + \delta^{\frac{1}{2}-\alpha-2\epsilon}\|f\|^2_{\dot{H}^{-\alpha}} + \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon}\|f\|_{\dot{H}^{-\alpha}} \sup_{0\leq s\leq\delta} E_1^{\frac{1}{2}}(s) \\
& + \sup_{0\leq s\leq\delta} E_1(s) + \sup_{0\leq s\leq\delta} E_2^{\frac{1}{2}}(s) + \delta^{\frac{1-\alpha-\gamma-\epsilon}{2}} \\
\lesssim\ & C(\alpha, \|f\|_{\dot{H}^{-\alpha}}),
\end{aligned} \tag{4.17}
$$

where $L_t^4 L_x^4$ norm of v is bounded by a interpolation of $L_t^\infty L_x^2$ and $L_t^2 \dot{H}^1$ norms of v which are in E_1. Similar for d=3 we have

$$
\begin{aligned}
\|\partial_t v\|_{L^2([0,\delta],H^{-1}(B))} \leq\ & E_1^{\frac{1}{2}}(\delta) + \|g_1\|^2_{L^4([0,\delta],L^4)} + 2\|g_1\|_{L^3([0,\delta],L^9)}\|v\|_{L^6([0,\delta],L^{\frac{18}{7}})} \\
& + \|v\|^2_{L^4([0,\delta],L^4)} + \|\theta+g_2\|_{L^2([0,\delta],H^{-1}(B))} \\
\lesssim_B\ & E_1^{\frac{1}{2}}(\delta) + \delta^{\frac{1}{2}-\alpha-2\epsilon}\|f\|^2_{\dot{H}^{-\alpha}} + \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon}\|f\|_{\dot{H}^{-\alpha}} \sup_{0\leq s\leq\delta} E_1^{\frac{1}{2}}(s) \\
& + \sup_{0\leq s\leq\delta} E_1(s) + \sup_{0\leq s\leq\delta} E_2^{\frac{1}{2}}(s) + \delta^{\frac{1-\alpha-\gamma-\epsilon}{2}} \\
\lesssim\ & C(\alpha, \|f\|_{\dot{H}^{-\alpha}}).
\end{aligned} \tag{4.18}
$$

By the equation (4.2) we have

$$
\begin{aligned}
\|\partial_t \theta\|_{L^2([0,\delta],H^{-1}(B))} \leq\ & \|\theta\|_{L^2([0,\delta],H^1(B))} + \|g_1 g_2\|_{L^2([0,\delta],H^{-1}(B))} \\
& + \|g_1\theta\|_{L^2([0,\delta],H^{-1}(B))} + \|v g_2\|_{L^2([0,\delta],H^{-1}(B))} \\
& + \|v\theta\|_{L^2([0,\delta],H^{-1}(B))}.
\end{aligned} \tag{4.19}
$$

By the definition of the energy, we have $\|\theta\|_{L^2([0,\delta],H^1(B))} \leq E_2^{\frac{1}{2}}(\delta)$. For the remaining terms in (4.16), using the similar estimates for terms J_1 and J_2, and hence we have

(4.20)
$$\begin{aligned}\|\partial_t\theta\|_{L^2([0,\delta],H^{-1}(B))} \le & E_2^{\frac12}(\delta) + \delta^{\frac12-\alpha-2\epsilon}\|f\|_{\dot H^{-\alpha}}^2 + \|v\|_{L^4([0,\delta],L^4)}\|\theta\|_{L^4([0,\delta],L^4)}\\ &+\delta^{\frac14-\frac{\alpha}{2}-2\epsilon}\|f\|_{\dot H^{-\alpha}}(\sup_{0\le s\le\delta}E_1^{\frac12}(s)+\sup_{0\le s\le\delta}E_2^{\frac12}(s))\\ &\lesssim C(\alpha,\|f\|_{\dot H^{-\alpha}})\end{aligned}$$

where $L_t^4L_x^4$ norm of v and θ are bounded by a interpolation of $L_t^\infty L_x^2$ and $L_t^2\dot H^1$ norms of v and θ which are in E_1 and E_2.
Case 2: $t\in[\delta,T]$. The previous energy of (v,θ) is bounded at $t=\delta$ which gives $\|v(\delta)\|_{L^2}$ and $\|\theta(\delta)\|_{L^2}$ are bounded by $C(\alpha,\|f\|_{\dot H^{-\alpha}})$. Back to (u,ρ) which is the solution of (1.1)-(1.3), we know that

$$\|u(\delta)\|_{L^2}\le\|v(\delta)\|_{L^2}+\|g_1(\delta)\|_{L^2}\lesssim E_1(\delta)^{\frac12}+\delta^{-\frac{\alpha+\gamma-\epsilon}{2}}\|f\|_{\dot H^{-\alpha}}\le C(\alpha,\delta,\|f\|_{\dot H^{-\alpha}})$$

and

$$\|\rho(\delta)\|_{L^2}\le\|\theta(\delta)\|_{L^2}+\|g_2(\delta)\|_{L^2}\lesssim E_2(\delta)^{\frac12}+\delta^{-\frac{\alpha+\gamma-\epsilon}{2}}\|f\|_{\dot H^{-\alpha}}\le C(\alpha,\delta,\|f\|_{\dot H^{-\alpha}}).$$

By the property of classical L^2 weak solution of (u,ρ), we have that for $t\in[\delta,T]$

$$E_1(u,t)+E_2(\rho,t)\le C(T,\alpha,\delta,\|f\|_{\dot H^{-\alpha}}), \tag{4.21}$$

and

$$\|\partial_t v\|_{L^2([\delta,T],H^{-1}(B))}+\|\partial_t\theta\|_{L^2([\delta,T],H^{-1}(B))}\lesssim_B C(T,\alpha,\delta,\|f\|_{\dot H^{-\alpha}}). \tag{4.22}$$

Hence using (4.21), for the energy of (v,θ) we have that for $t\in[\delta,T]$

$$\begin{aligned}E_1(v,t)+E_2(\theta,t)&\le E_1(g_1,t)+E_1(u,t)+E_2(g_2,t)+E_2(\rho,t)\\ &\lesssim E_1(u,t)+E_2(\rho,t)+\delta^{-\frac{\alpha+\gamma+2\epsilon}{2}}\|f\|_{\dot H^{-\alpha}}+t\delta^{-\frac{\alpha+\gamma+1+2\epsilon}{2}}\|f\|_{\dot H^{-\alpha}}\\ &\lesssim C(T,\alpha,\delta,\|f\|_{\dot H^{-\alpha}}).\end{aligned}$$

Using (4.22) and (4.4), we have

$$\begin{aligned}&\|\partial_t u\|_{L^2([\delta,T],H^{-1}(B))}+\|\partial_t\rho\|_{L^2([\delta,T],H^{-1}(B))}\\ \le\ &\|\partial_t v\|_{L^2([\delta,T],H^{-1}(B))}+\|\partial_t\theta\|_{L^2([\delta,T],H^{-1}(B))}+\|g_1\|_{L^2([\delta,T],H^1)}+\|g_2\|_{L^2([\delta,T],H^1)}\\ \lesssim_B&C(T,\alpha,\|f\|_{\dot H^{-\alpha}})+(T-\delta)^{\frac12}\delta^{-\frac{\alpha+\gamma+1+2\epsilon}{2}}\|f\|_{\dot H^{-\alpha}}\\ \le\ &C(T,\alpha,\delta,\|f\|_{\dot H^{-\alpha}}).\end{aligned}$$

In the end, combining the two cases $t\in[0,\delta]$ and $t\in[\delta,T]$, hence for all $t\in[0,T]$ we prove (4.7) and (4.8). □

5. Construction of the weak solutions to the difference equation of the Boussinesq system

In this section, we shall construct weak solutions to the initial value problem (1.1)–(1.3).

$$\begin{cases} v_t - \Delta v + \mathbb{P}\nabla \cdot ((g_1+v)\otimes(g_1+v)) - \mathbb{P}((g_2+\theta)e_3) = 0, \\ \theta_t - \Delta\theta + \nabla\cdot((g_1+v)(g_2+\theta)) = 0, \\ \nabla\cdot v = 0, \quad v(x,0)=0, \quad \theta(x,0)=0. \end{cases} \tag{5.1}$$

THEOREM 5.1. Fix $T>0$ and $\alpha \in (0,\frac{1}{2})$. Given $0<\epsilon<\frac{1}{4}-\frac{\alpha}{2}$ and $\gamma>0$ can be arbitrarily small. Consider functions g_1 and g_2 satisfying the following properties, for $i=1,2$ and $k=0,1$

$$\|\nabla^k g_i\|_{L^2} \le \frac{1}{\delta^\epsilon}(1+t^{-\frac{\alpha+k+\gamma}{2}})\|f\|_{\dot H^{-\alpha}}$$

and

$$\|g_i\|_{L^4([0,\delta],L^4)} + \|g_i\|_{L^4([0,\delta],L^{4^+})} \le \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon}\|f\|_{\dot H^{-\alpha}}, \qquad \text{when } d=2$$

where $\frac{1}{4^+}=\frac{1}{4}-\gamma$ and

$$\|g_i\|_{L^3([0,\delta],L^9)} + \|g_i\|_{L^4([0,\delta],L^4)} \le \delta^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon}\|f\|_{\dot H^{-\alpha}}, \qquad \text{when } d=3$$

when δ is small enough. Then there exists a weak solution (v,θ) in $[0,T]$ for the initial value problem (5.1).

PROOF. In the construction of weak solutions, we follow Galerkin approximations approach. We first construct the solutions (v^M,θ^M) (where $M>1$) of finite dimensional approximation equations as follows

$$\begin{cases} v^M_t - \Delta v^M + P_M\mathbb{P}\nabla\cdot((P_Mg_1+v^M)\otimes(P_Mg_1+v^M)) - P_M\mathbb{P}((P_Mg_2+\theta^M)e_3) = 0, \\ \theta^M_t - \Delta\theta^M + P_M\nabla\cdot((P_Mg_1+v^M)(P_Mg_2+\theta^M)) = 0, \\ \nabla\cdot v^M = 0, \quad v^M(x,0)=0, \quad \theta^M(x,0)=0. \end{cases} \tag{5.2}$$

Our plan is to obtain the local-in-time well-posedness of the finite approximation equations via the fixed point argument in the space

$$X_\tau = C([0,\tau],L^2_x)\cap L^2([0,\tau],\dot H^1_x).$$

Define

$$\Phi(v^M,\theta^M) = \int_0^t \Delta v^M\,dt - \int_0^t P_M\mathbb{P}\nabla\cdot((P_Mg_1+v^M)\otimes(P_Mg_1+v^M)) \\ -P_M\mathbb{P}((P_Mg_2+\theta^M)e_3)\,dt$$

and

$$\Psi(v^M,\theta^M) = \int_0^t \Delta\theta^M\,dt - P_M\nabla\cdot((P_Mg_1+v^M)(P_Mg_2+\theta^M))\,dt.$$

It is easy to obtain the following estimates

$$\begin{aligned}\|\Phi(v^M,\theta^M)\|_{L_t^\infty L_x^2([0,\tau])} \lesssim\ & M^2\tau\|v^M\|_{L_t^\infty L_x^2} + M^{1+\frac{d}{2}}\tau\|v^M\|_{L_t^\infty L_x^2}^2 \\ &+ M^{1+\frac{d}{2}+}\tau^{1-\frac{\alpha}{2}}\|v^M\|_{L_t^\infty L_x^2} + M\tau^{h(d)-2\gamma}\lambda^2 + \tau^{1-\frac{\alpha}{2}}\lambda \\ &+ \tau\|\theta^M\|_{L_t^\infty L_x^2}.\end{aligned}$$

$$\begin{aligned}\|\Psi(v^M,\theta^M)\|_{L_t^\infty L_x^2([0,\tau])} \lesssim\ & M^2\tau\|\theta^M\|_{L_t^\infty L_x^2} + M^{1+\frac{d}{2}}\tau\|v^M\|_{L_t^\infty L_x^2}\|\theta^M\|_{L_t^\infty L_x^2} \\ &+ M^{1+\frac{d}{2}+}\tau^{1-\frac{\alpha}{2}}(\|\theta^M\|_{L_t^\infty L_x^2} + \|v^M\|_{L_t^\infty L_x^2}) \\ &+ M\tau^{h(d)-2\gamma}\lambda^2 + \tau^{1-\frac{\alpha}{2}}\lambda.\end{aligned}$$

And

$$\begin{aligned}\|\Phi(v^M,\theta^M)\|_{L_t^2\dot H_x^1([0,\tau])} \lesssim\ & M^3\tau^{\frac{3}{2}}\|v^M\|_{L_t^\infty L_x^2} + M^{2+\frac{d}{2}}\tau^{\frac{3}{2}}\|v^M\|_{L_t^\infty L_x^2}^2 \\ &+ M^{2+\frac{d}{2}+}\tau^{-\gamma+\theta(d)}\|v^M\|_{L_t^\infty L_x^2} + M^2\tau^{\rho(d)-2\gamma}\lambda^2 \\ &+ \tau^{1-\frac{\alpha}{2}}\lambda + \tau\|\theta^M\|_{L_t^\infty L_x^2}.\end{aligned}$$

$$\begin{aligned}\|\Psi(v^M,\theta^M)\|_{L_t^2\dot H_x^1([0,\tau])} \lesssim\ & M^3\tau^{\frac{3}{2}}\|\theta^M\|_{L_t^\infty L_x^2} + M^{2+\frac{d}{2}}\tau^{\frac{3}{2}}\|v^M\|_{L_t^\infty L_x^2}\|\theta^M\|_{L_t^\infty L_x^2} \\ &+ M^{2+\frac{d}{2}+}\tau^{-\gamma+\theta(d)}(\|\theta^M\|_{L_t^\infty L_x^2} + \|v^M\|_{L_t^\infty L_x^2}) \\ &+ M^2\tau^{\rho(d)-2\gamma}\lambda^2 + \tau^{1-\frac{\alpha}{2}}\lambda,\end{aligned}$$

where $\lambda = \|f\|_{\dot H^{-\alpha}}$. By taking $\tau = \tau(M, \|f\|_{\dot H^{-\alpha}}, \alpha)$ small enough, we can make sure that $\Psi(v^M,\theta^M)$ and $\Psi(v^M,\theta^M)$ are contraction mappings. Hence a fixed point argument helps us hold the local well-posedness of (v^M,θ^M) in the $[0,\tau]$.

Since $P_M g_i$ satisfies the same assumptions as g_i in Section 4, we can repeat the proof of Theorem 4.1 and obtain the same energy bounds (4.7) and the bound (4.8) for $\partial_t v$ and $\partial_t\theta$ given in Theorem 4.1 for finite dimensional approximation solutions (v^M,θ^M) uniformly in M. As a consequence we can use an iteration argument to advance this solution of (v^M,θ^M) up to time T. By applying a standard compactness argument, together with the fact that $P_M g_i$ strongly converges to g_i for $i = 1,2$, we obtains a weak solution (v,θ) to (5.1) on $[0,T]$. Since T is arbitrary large, we obtain a global weak solution. □

6. Uniqueness in 2D

In this section, we give the proof of uniqueness of 2D global weak solutions.

Theorem 6.1. Suppose g_1 and g_2 satisfy the decay properties in Theorem 4.1. Then, the weak solutions in $L^2([0,T];V)\cap L^\infty([0,T];H)$ are unique when $d = 2$.

Proof of Theorem 6.1. Suppose (v_1,θ_1) and (v_2,θ_2) are two solutions. Then, set

$$\begin{aligned} w &= v_1 - v_2 \\ z &= \theta_1 - \theta_2.\end{aligned}$$

Thus, we obtain the equation in terms of w and z

$$w_t - \Delta w + \mathbb{P}\nabla \cdot (g_1 \otimes w) + \mathbb{P}\nabla \cdot (w \otimes g_1) + \mathbb{P}\nabla \cdot (v_1 \otimes w) + \mathbb{P}\nabla \cdot (w \otimes v_2) = \mathbb{P}(ze_3)$$

and

$$z_t - \Delta z + w \cdot \nabla g_2 + w \cdot \nabla \theta_1 + g_1 \cdot \nabla z + v_1 \cdot \nabla z = 0. \tag{6.1}$$

Now we do the L^2 energy estimates given $w(0) = z(0) = 0$. Take the L^2 inner product on (6.1) with w and we get that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 &\le \|w\|_{L^4}^2\|\nabla v_2\|_{L^2} + \|g_1\|_{L^4}\|w\|_{L^4}\|\nabla w\|_{L^2} + \|z\|_{L^2}\|w\|_{L^2} \\ &\le \|w\|_{L^4}^2\|\nabla v_2\|_{L^2} + \frac{1}{2}\|g_1\|_{L^4}^2\|w\|_{L^4}^2 + \frac{1}{2}\|\nabla w\|_{L^2}^2 \\ &+ \frac{1}{2}\|z\|_{L^2}^2 + \frac{1}{2}\|w\|_{L^2}^2.\end{aligned}$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 \le C(\|\nabla v_2\|_{L^2}^4 + \|g_1\|_{L^4}^2)\|w\|_{L^4}^2 + \frac{1}{2}\|z\|_{L^2}^2.$$

Next, we consider the energy estimates for z by using Holder's inequality and Ladyzhenskaya inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|z\|_{L^2}^2 + \|\nabla z\|_{L^2}^2 &= \int w \cdot \nabla z g_2 \, dx - \int w \cdot \nabla \theta_1 z \, dx \\ &- \int g_1 \cdot \nabla z z \, dx - \int v_2 \cdot \nabla z z \, dx \\ &= \int w \cdot \nabla z g_2 \, dx - \int w \cdot \nabla \theta_1 z \, dx \\ &\le \|w\|_{L^4}\|\nabla z\|_{L^2}\|g_2\|_{L^4} + \|w\|_{L^4}\|\nabla \theta_1\|_{L^2}\|z\|_{L^4} \\ &\le (\frac{1}{2}\|\nabla z\|_{L^2}^2 + \frac{1}{2}\|w\|_{L^4}^2\|g_2\|_{L^4}^2) \\ &+ (\|w\|_{L^2}^{1/2}\|\nabla w\|_{L^2}^{1/2}\|\nabla \theta_1\|_{L^2}\|z\|_{L^2}^{1/2}\|\nabla z\|_{L^2}^{1/2}) \\ &= M_1 + M_2.\end{aligned}$$

For M_1,

$$M_1 \le \frac{1}{2}\|\nabla z\|_{L^2}^2 + C\|w\|_{L^2}^2\|g_2\|_{L^4}^4 + \frac{1}{3}\|\nabla w\|_{L^2}^2.$$

For M_2,

$$\begin{aligned}M_2 &\le C\|\nabla \theta_1\|_{L^2}\|w\|_{L^2}\|z\|_{L^2} + C\|\nabla w\|_{L^2}\|\nabla z\|_{L^2} \\ &\le C\|\nabla \theta_1\|_{L^2}^2\|w\|_{L^2}^2 + C\|z\|_{L^2}^2 + C\|\nabla w\|_{L^2}^2 + \frac{1}{3}\|\nabla z\|_{L^2}^2.\end{aligned}$$

Combining the estimates above yields the uniqueness of the solutions by Gronwall inequality. □

7. Proof of main theorems

We find the solution (u, ρ) by

$$u = g_1^\omega + v,$$

$$\rho = g_2^\omega + \theta,$$

where $g_1^\omega = e^{t\Delta} u_0^\omega$ and $g_2^\omega = e^{t\Delta}\rho_0^\omega$. Then we consider the corresponding system of (v, θ): (4.1)-(4.3).

Proof of Theorem 2.1 and 2.2. In Theorem 4.1, Theorem 5.1 and Theorem 6.1, we show that main theorems (Theorem 2.1) and (Theorem 2.2) are true when g_1^ω and g_2^ω satisfy the following conditions for $i = 1, 2$:

$$\|\nabla^k g_i^\omega\|_{L^2} \le \frac{1}{\delta^\epsilon}(1 + t^{-\frac{\alpha+\gamma+k}{2}})\|f\|_{\dot{H}^{-\alpha}}, \tag{7.1}$$

and for all $\delta' \le \delta$

$$\|g_i^\omega\|_{L^4([0,\delta'],L^4)} + \|g_i^\omega\|_{L^4([0,\delta'],L^{4+})} \le (\delta')^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon}\|f\|_{\dot{H}^{-\alpha}}, \qquad \text{when } d = 2, \tag{7.2}$$

where $\frac{1}{4^+} = \frac{1}{4} - \gamma$ and

$$\|g_i^\omega\|_{L^3([0,\delta'],L^9)} + \|g_i^\omega\|_{L^4([0,\delta'],L^4)} \le (\delta')^{\frac{1}{4}-\frac{\alpha}{2}-\epsilon}\|f\|_{\dot{H}^{-\alpha}}, \qquad \text{when } d = 3. \tag{7.3}$$

Define

$$\Omega_\delta^{(1)} = \{\omega \in \Omega : g_1^\omega, g_2^\omega \text{ satifies } (7.1)\}$$

and

$$\Omega_\delta^{(2)} = \{\omega \in \Omega : g_1^\omega, g_2^\omega \text{ satifies } (7.2)\&(7.3)\}.$$

It is easy to see that for any $0 < \delta_1 < \delta_2$, $\Omega_{\delta_2}^{(1)} \subset \Omega_{\delta_1}^{(1)}$ and $\Omega_{\delta_2}^{(2)} \subset \Omega_{\delta_1}^{(2)}$. Suppose $\omega \in \Omega_{good} = (\cup_{\delta>0}\Omega_\delta^{(1)}) \cap (\cup_{\delta>0}\Omega_\delta^{(2)})$, for the initial data u_0^ω and ρ_0^ω we can solve the system (1.1)-(1.3) on $[0, T]$. It remains to show $\mathbf{P}(\Omega_{good}) = 1$. First we have

$$\mathbf{P}(\Omega_{good}) = 1 - \mathbf{P}\left(\cap_{\delta>0}(\Omega_\delta^{(1)})^c \cup \cap_{\delta>0}(\Omega_\delta^{(2)})^c\right).$$

By Lemma 3.4 and Lemma 3.6, we know that $\mathbf{P}((\Omega_\delta^{(1)})^c) \lesssim e^{-\frac{1}{\delta^\epsilon}}$ and $\mathbf{P}((\Omega_\delta^{(2)})^c) \lesssim e^{-\frac{1}{\delta^\epsilon}}$ so we have

$$\mathbf{P}\left(\cap_{\delta>0}(\Omega_\delta^{(1)})^c \cup \cap_{\delta>0}\Omega_\delta^{(2)})^c\right) \le \lim_{\delta\to 0}\mathbf{P}((\Omega_\delta^{(1)})^c) + \mathbf{P}((\Omega_\delta^{(2)})^c) \lesssim \lim_{\delta\to 0} e^{-\frac{1}{\delta^\epsilon}} = 0$$

which shows $\mathbf{P}(\Omega_{good}) = 1$. □

Acknowledgments

The authors would like to thank Professor Nahmod for useful discussions. The authors also express deep gratitude to Professors Li and Wu for handling this manuscript and the anonymous referees for their valuable suggestions that greatly improved this paper. W. Wang was supported in part by the NSF grant DMS-1907992.

References

[1] D. Adhikari, C. Cao, and J. Wu, *Global regularity results for the 2D Boussinesq equations with vertical dissipation*, J. Differential Equations **251** (2011), no. 6, 1637–1655.

[2] D. Chae, *Global regularity for the 2D Boussinesq equations with partial viscosity terms*, Adv. Math. **203** (2006), no. 2, 497—513.

[3] P. Constantin and C. Foias, *Navier-Stokes equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988.

[4] M. Chen and O. Goubet, *Long-time asymptotic behavior of two-dimensional dissipative Boussinesq systems*, Discrete Contin. Dyn. Syst. Ser. S **2** (2009), no. 1, 37—53.

[5] C. Cao and J. Wu, *Global regularity for the two-dimensional anisotropic Boussinesq equations with vertical dissipation*, Arch. Ration. Mech. Anal. **208** (2013), no. 3, 985—1004.

[6] J. Bourgain, *Invariant measures for the 2D defocusing nonlinear Schrödinger equation*, Comm. Math. Phys. **176** (1996), 421–445.

[7] N. Burq and N. Tzvetkov, *Random data Cauchy theory for super-critical wave equation I: Local theory*, Invent. Math. **173** (2008), no. 3, 449—475.

[8] N. Burq and N. Tzvetkov, *Random data Cauchy theory for super-critical wave equation II: A global existence result*, Invent. Math. **173** (2008), no. 3, 477—496.

[9] C. Deng and S. Cui, *Random-data Cauchy problem for the Navier-Stokes equations on* $\mathbb{T}^3$, J. Differential Equations **251** (2011), no. 4-5, 902–917.

[10] L. Du and T. Zhang, *Almost sure existence of global weak solutions for incompressible MHD equations in negative-order Sobolev space*, J. Differential Equations. **263** (2017), no. 2, 1611–1642.

[11] T. Hmidi and S. Keraani, *On the global well-posedness of the two-dimensional Boussinesq system with a zero diffusivity*, Adv. Differential Equations **12** (2007), no. 4, 461-–480.

[12] T.Y. Hou and C. Li, *Global well-posedness of the viscous Boussinesq equations*, Discrete Contin. Dyn. Syst. **12** (2005), no. 1, 1–12.

[13] I. Kukavica and W. Wang, *Global Sobolev persistence for the fractional Boussinesq equations with zero diffusivity*, Pure and Applied Functional Analysis, **5** (2020), no. 1, 27-45.

[14] I. Kukavica and W. Wang, *Long time behavior of solutions to the 2D Boussinesq equations*, J. Dyn. Diff. Equat. (2019). https://doi.org/10.1007/s10884-019-09802-w.

[15] A. Larios, E. Lunasin, and E.S. Titi, *Global well-posedness for the 2D Boussinesq system with anisotropic viscosity and without heat diffusion*, J. Differential Equations **255** (2013), no. 9, 2636–2654.

[16] Z. Li and W. Wang, *Norm inflation for the Boussinesq system*, arXiv 1912.06114.

[17] A. Nahmod, N. Pavlović and G. Staffilani, *Almost Sure Existence of Global Weak Solutions For Supercritical Navier-Stokes Equation*, SIAM J.Math. Anal. **45** (2013), no. 6, 3431–3452.

[18] A. Stefanov and J. Wu, *A global regularity result for the 2D Boussinesq equation with critical dissipation*, to appear, J. Analy. Math. **137** (2019), no. 1, 269—290.

[19] W. Wang, *On the global regularity for a 3D Boussinesq model without thermal diffusion*, Z. Angew. Math. Phys. (2019) 70: 174. https://doi.org/10.1007/s00033-019-1221-0.

[20] W. Wang, *On the global stability of large solutions for the Boussinesq equations with Navier boundary conditions*, submitted.

[21] J. Wang and K. Wang, *Almost sure existence of global weak solutions to the 3D incompressible Navier-Stokes equation*, Journal of Math. and Fluid Mech. **37** (2017), no. 9, 5003–5019.

[22] W. Wang and H. Yue, *Time decay of almost sure weak solutions for the Navier-Stokes and the MHD equations with initial data of low regularity*, submitted.

[23] T. Zhang and D. Fang, *Random data Cauchy theory for the generalized incompressible Navier-Stokes equations*, J. Math. Fluid Mech. **14** (2012), no. 2, 311–324.

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