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Ergodicity effects on transport-diffusion equations with localized damping

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Abstract. The main objective of this paper is to study the time decay of transport-diffusion equation with inhomogeneous localized damping in the multi-dimensional torus. The drift is governed by an autonomous Lipschitz vector field and the diffusion by the standard heat equation with small viscosity parameter ν. In the first part we deal with the inviscid case and show some results on the time decay of the energy using in a crucial way the ergodicity and the unique ergodicity of the flow generated by the drift. In the second part we analyze the same problem with small viscosity and provide quite similar results on the exponential decay uniformly with respect to the viscosity in some logarithmic time scaling of the type $t \in [0, C_0 \ln(1/\nu)]$.

Contents

1. Introduction

In this short note we aim at exploring the time decay of the energy for the transport-diffusion equation with localized damping in the multi-dimensional torus $\mathbb{T}^d$,

(1)
$$\begin{cases} \partial_t\theta(t,x) + v(x)\cdot\nabla\theta(t,x) - \nu\Delta\theta(t,x) = -\phi(x)\,\theta(t,x) \quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^d, \\ \theta(0,x) = \theta_0(x), \end{cases}$$

where $\phi : \mathbb{T}^d \to \mathbb{R}_+$ is a given time-independent positive function and $\nu \geq 0$ is a viscosity parameter. The vector field $v : \mathbb{T}^d \to \mathbb{R}^d$ is assumed to be autonomous,

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solenoidal, that is $\operatorname{div} v = 0$, and belongs to the Lipschitz class. When ϕ is bounded away from zero, meaning that one may find a constant $\mu > 0$ such that $\phi(x) \geq \mu$ almost everywhere, then performing L^2-energy estimate it is quite easy to get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta(t)\|_{L^2}^2 + \nu\|\nabla\theta(t)\|_{L^2}^2 &= -\int_{\mathbb{T}^d}\phi(x)|\theta(t,x)|^2dx \\ &\leq -\mu\|\theta(t)\|_{L^2}^2. \end{aligned} \tag{2}$$

This implies in particular the following time decay

$$\forall\, t \geq 0, \quad \|\theta(t)\|_{L^2} \leq \|\theta_0\|_{L^2}e^{-\mu t}.$$

The main task here is to investigate the time decay when ϕ is not uniformly distributed on the torus and its support may be localized in a small domain. In the absence of the drift $v = 0$ and the viscosity $\nu = 0$ the stabilization can not occur because in this case the solution is explicitly recovered through the formula,

$$\theta(t,x) = \theta_0(x)e^{-t\phi(x)}.$$

Thus the energy is conserved for initial data whose supports do not intersect the support of ϕ.

Now let us discuss and specify the main feature of the full equation (1) which is mainly governed by two mechanisms, the advection represented by the velocity field v and the diffusion given by the viscous part. First, let us neglect the diffusion part and focus on the inviscid equation (1) with the vanishing viscosity $\nu = 0$, then one can guess that the transport structure with special vector fields would allow to advect the mass in a recurrent way to the activity zone of the damping that could guarantee the time decay. Thus we expect the ergodicity to be a crucial phenomenon for flowing and spreading the mass and de facto for stabilizing the inviscid system. As we shall see in Theorem 2, we will make this intuitive idea more pertinent and rigorous in the framework of ergodic flows. We emphasize that in our case the ergodicity allows to get the time decay, however for the exponential decay we need to deal with a special class of ergodic flows, known in the literature by the uniquely ergodic flows. We shall give later some details about this terminology. Notice that in the periodic setting lot of results around the ergodicity of continuous and discrete dynamical system are well-developed in the literature, we can refer for instance to [**5, 6, 8, 10**].

In [**4**] Lebeau characterized the value of the best decay rate for the damped wave equation in terms of two quantities: the spectral abscissa and the mean value of the damping coefficient along the rays of geometrical optics, i.e., the Birkhoff limit of the damping coefficient, which illustrate the ergodicity effect on the dynamics. In other hand in [**9**] Sjöstrand give an asymptotic distribution of the eigenfrequencies associated to the damped wave operator, and show that the spectra is confined to a band determined by the Birkhoff limits of the damping coefficient and that certain averages of the imaginary parts converge to the average of the damping coefficient. Later and based on the work of Sjöstrand [**9**], Asch and Lebeau [**1**] give an asymptotic expression for the distribution of the imaginary parts of the eigenvalues for a radially symmetric geometry and establish the close links between the eigenfunctions and the rays of geometrical optics. For the one dimensional models, we can see [**3**] and [**2**].

Now let us focus on the second mechanism of the equation (1) and analyze the viscosity effects. We point out that without the zero-mean condition there is no hope to get a time decay, to be convinced it suffices to work with θ_0 as a constant function and $\phi = 0$ then the solution is a constant. However, the spatial average $\int_{\mathbb{T}^d} \theta(t, x)dx$ is not conserved in general due to the inhomogeneous damping term. Nevertheless, when ϕ is even $\phi(-x) = \phi(x)$ then odd symmetry is preserved by the dynamics. Consequently if we impose to θ_0 to be odd then the solution keeps this symmetry for any time and therefore the spatial average is always zero. With this special symmetry one may use Poincaré inequality in the torus and deduce from (2) that

$$\frac{1}{2}\frac{d}{dt}\|\theta(t)\|_{L^2}^2 + \nu\|\theta(t)\|_{L^2}^2 \leq 0.$$

From this we infer the exponential decay: $\forall\, t \geq 0, \quad \|\theta(t)\|_{L^2} \leq \|\theta_0\|_{L^2} e^{-\nu t}$, which is of course not uniform for vanishing viscosity $\nu \to 0$. It is worthy to point out that without the zero average, the exponential decay is no longer satisfied. In this case the transfer of mass to low frequencies could happen preventing the energy from any decay rate. Now what happens if we combine both mechanisms and deal with the full equation (1). In this setting, several questions of important interest emerge. For instance, how the ergodicity is affected by the diffusion and in particular what about the recurrence behavior, which is essential to guide the trajectories to the damping zone ? Is it altered or reinforced by the diffusion? Another connected problem is whether or not the time decay occurs uniformly for the vanishing viscosity. It is so hard to give a full and a complete answer to those problems. Here we shall only provide a partial answer by looking to the regime where the ergodicity is not altered by the diffusion. More precisely, we are able to exhibit a logarithmic time scaling $0 \leq t \leq \log \frac{1}{\nu}$ where the exponential time decay continue to survive uniformly with respect to small viscosity. A precise statement will be established in Theorem 3.

In order to formulate carefully our main results, we need to recall some tools and fix some terminology from the theory of dynamical systems. First, let us briefly see how to define the flow on the torus associated to a given autonomous vector field $v : \mathbb{T}^d \to \mathbb{R}^d$. Here the multi-dimensional torus $\mathbb{T}^d$ is identified to the factor space $\mathbb{R}^d/\mathbb{Z}^d$ and any function defined on the torus can be identified with its lift defined on $\mathbb{R}^d$. Now let x be an arbitrary point of the set $\mathbb{T}^d$ and define the orbit $t \mapsto X(t) \in \mathbb{R}^d$ as the unique solution of the ODE

$$\dot{X}(t) = v(X(t)),\ X(0) = x.$$

By projecting down this orbit on the torus using the natural projection $\pi : \mathbb{R}^d \to \mathbb{R}^d/\mathbb{Z}^d$, allows to cover the analogous orbit on the torus. By this way we can generate a flow on the torus that we denote throughout this paper by ψ. To avoid the difficulties related to the blow-up phenomenon, we assume that the vector field is complete meaning that all the orbits are globally defined in time. A sub-class of those vector fields is given by Lipschitz class which will be a canonical assumption in the paper. Therefore the flow ψ can be viewed as a mapping $\psi : \mathbb{R} \times \mathbb{T}^d \to \mathbb{T}^d$ and still satisfies the ODE

$$\left\{ \begin{array}{l} \partial_t \psi(t, x) = v(\psi(t, x)) \quad (t, x) \in \mathbb{R} \times \mathbb{T}^d, \\ \psi(0, x) = x \in \mathbb{T}^d. \end{array} \right. \tag{3}$$

Then by this way we generate a group of continuous transformation on the torus $\{\psi_t\}_{t\in\mathbb{R}}$, and in particular we have $\psi_t : \mathbb{T}^d \to \mathbb{T}^d$ with $\psi_t \circ \psi_s = \psi_{t+s}$. Moreover, since the vector field v is assumed to be solenoidal or incompressible then the Lebesgue measure denoted by λ or dx is invariant by the flow. This means that for any Borel set $A \subset \mathbb{T}^d$ we have

$$\forall t \in \mathbb{R}, \quad \lambda(\psi_t(A)) = \lambda(A).$$

Remark that the torus $\mathbb{T}^d$ is normalized, that is $\lambda(\mathbb{T}^d) = 1$. A set $A \subset \mathbb{T}^d$ is called *invariant set* if $\psi_t(A) = A, \forall t \in \mathbb{R}$. The one-parameter flow $\{\psi_t\}_{t\in\mathbb{R}}$ is said to be *Ergodic* with respect to Lebesgue measure if any invariant set has either zero measure or full measure. This flow is said *uniquely ergodic* if it admits only one invariant Borel probability measure, which is necessary the Lebesgue measure. A useful tool in our study is the so-called Birkhooff's Ergodic Theorem. Before giving a precise statement we need to introduce for $p \in [1, \infty)$, the standard Lebesgue space $L^p(\mathbb{T}^d)$ which is the set of measurable functions $f : \mathbb{T}^d \to \mathbb{R}$ such that

$$\|f\|_p = \Big(\int_{\mathbb{T}} |f(x)|^p dx\Big)^{\frac{1}{p}} < \infty,$$

with the usual adaptation for the case $p = +\infty$. We also denote by $\langle f \rangle$ the spatial average of f by

$$\langle f \rangle = \int_{\mathbb{T}^d} f(x)dx.$$

The following theorem where we collect several results in classical dynamical systems can be found in standard books dealing with Ergodic Theory. For instance, we refer to [**7, 10**] and the references therein.

THEOREM 1. *Let $\phi \in L^1(\mathbb{T}^d)$ and $\{\psi_t\}_{t\in\mathbb{R}}$ be a one-parameter group of transformations on the torus $\mathbb{T}^d$ preserving Lebesgue measure. Then there exists $\phi_\star \in L^1(\mathbb{T}^d)$ such that*

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \phi(\psi(\tau, x))d\tau = \phi_\star(x), a.e. \tag{4}$$

where $\phi_\star$ satisfies the properties

(i) *For any $t \in \mathbb{R}, \phi_\star(\psi(t, x)) = \phi_\star(x)$, a.e.*
(ii) *If $\phi \in L^p(\mathbb{T}^d)$ then $\phi_\star \in L^p(\mathbb{T}^d)$ with $\|\phi_\star\|_{L^p} \le \|\phi\|_{L^p}$.*
(iii) *If A is invariant by the flow then*

$$\int_A \phi_\star(x)dx = \int_A \phi(x)dx.$$

(iv) *If $\phi \in L^p(\mathbb{T}^d)$ with $p \in [1, \infty)$, then the convergence in* (4) *holds in $L^p(\mathbb{T}^d)$.*
(v) *If the flow $\{\psi_t\}_{t\in\mathbb{R}}$ is ergodic then $\phi_\star(x) = \langle\phi\rangle$ a.e.*
(vi) *If the flow $\{\psi_t\}_{t\in\mathbb{R}}$ is uniquely ergodic and $\phi \in \mathscr{C}(\mathbb{T}^d; \mathbb{R})$, then the convergence in* (1) *to the spatial average $\langle\phi\rangle$ occurs everywhere and is uniform.*

Next we shall give some specific vector fields that generate in the $2d$-torus unique ergodic flows.
Examples of uniquely ergodic flows on the torus. The following result comes from

Saitô [**8**] and Maoan [**5**]. Take a $\mathscr{C}^1$ solenoidal vector field $v=(v^1,v^2)$ without any zero and such that

$$v^1(x,y)=\sum_{n,m\in\mathbb{Z}} a_{mn}e^{2i\pi(mx+ny)},\quad v^2(x,y)=\sum_{n,m\in\mathbb{Z}} b_{mn}e^{2i\pi(mx+ny)},$$

Then the flow associated to v is ergodic if and only if

$$a_{00}b_{00}\neq 0\quad\text{and}\quad \frac{a_{00}}{b_{00}}\notin\mathbb{Q}.$$

Under the same conditions the flow is also uniquely ergodic.

2. Inviscid case

This section is devoted the case $\nu=0$ where the equation (1) reduces to

$$\left\{\begin{array}{l}\partial_t\theta(t,x)+v(x)\cdot\nabla\theta(t,x)=-\,\phi(x)\,\theta(t,x)\quad (t,x)\in\mathbb{R}_+\times\mathbb{T}^d,\\ \theta|_{t=0}=\theta_0,\end{array}\right.\tag{5}$$

Our main result deals with the time decay of the energy.

Theorem 2. *Let $d\geq 1$, $v:\mathbb{T}^d\to\mathbb{R}^d$ be in the Lipschitz class and $\phi:\mathbb{T}^d\to\mathbb{R}$ be a positive and integrable function such that its average $\langle\phi\rangle>0$. Then the following assertions hold true.*

(i) *If the flow $\{\psi_t\}_{t\in\mathbb{R}}$ is ergodic, then for $\theta_0\in L^p(\mathbb{T}^d)$ with $p\in[1,\infty)$, we have*
$$\lim_{t\to+\infty}\|\theta(t)\|_{L^p}=0.$$
(ii) *If the flow $\{\psi_t\}_{t\in\mathbb{R}}$ is uniquely ergodic and $\phi\in\mathscr{C}(\mathbb{T}^d)$. Then for any $\theta_0\in L^p(\mathbb{T}^d)$ with $p\in[1,\infty)$ and for any $\mu\in[0,\langle\phi\rangle)$ there exists $T_0>0$ such that*
$$\forall t\geq T_0,\quad \|\theta(t)\|_{L^p}\leq\|\theta_0\|_{L^p}e^{-\mu t}.$$
(iii) *Assume that $\phi_\star(x)>0, a.e.$ and $\theta_0\in L^p(\mathbb{T}^d)$ with $p\in[1,\infty)$, then*
$$\lim_{t\to+\infty}\|\theta(t)\|_{L^p}=0.$$

Remark 1. *Notice that in the point (iii) we do not assume the ergodicity of the flow, which is substituted by the strict positivity of $\phi_\star$ almost everywhere. From this result, we can deduce that for any non negligible invariant set A, we have necessary* supp $\phi\cap A$ *is a non negligible set. Indeed, using Theorem 1-(iii) we achieve that*

$$\int_A\phi_\star(x)dx=\int_A\phi(x)dx.$$

From the assumption on $\phi_\star$ we easily get $\int_A\phi(x)dx>0$ and this ensures the desired result. This gives an insight where one should localize the control region in order to get the time decay.

Proof. **(i)** Let $\{\psi_t\}_{t\in\mathbb{R}}$ be the flow associated to the velocity field v. Then by setting $\eta(t,x)=\theta(t,\psi(t,x))$ it is quite easy to check that

$$\partial_t\eta(t,x)=-\eta(t,x)\,\phi(\psi(t,x)).$$

Thus we can recover from this ODE the solution as follows,

$$\eta(t,x)=\theta_0(x)e^{-\int_0^t\phi(\psi(\tau,x))d\tau}.\tag{6}$$

Since the flow is assumed to be ergodic then according to Theorem 1-(v) the function $\phi_\star$ given by (4) is constant and coincides with its spatial average, meaning that,

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \phi(\psi(\tau,x))d\tau = \langle\phi\rangle, \quad a.e. \tag{7}$$

Now using the identity (6) combined with the invariance of the Lebesgue measure under the flow we deduce that

$$\begin{aligned}\|\theta(t)\|_{L^p}^p &= \|\eta(t)\|_{L^p}^p \\ &= \int_{\mathbb{T}^d} |\theta_0(x)|^p e^{-p\int_0^t \phi(\psi(\tau,x))d\tau}dx.\end{aligned} \tag{8}$$

From (7) and $\langle\phi\rangle > 0$ we infer that

$$\lim_{t\to+\infty}\int_0^t \phi(\psi(\tau,x))d\tau = +\infty \quad a.e.$$

Hence we get the pointwise convergence

$$\lim_{t\to+\infty} e^{-p\int_0^t \phi(\psi(\tau,x))d\tau} = 0 \quad a.e.$$

Moreover from the positivity of ϕ we get the uniform domination

$$\forall t\geq 0,\ |\theta_0(x)|^p e^{-p\int_0^t \phi(\psi(\tau,x))d\tau} \leq |\theta_0(x)|^p \quad a.e.$$

Thus, by virtue of Lebesgue's dominated convergence theorem we obtain for any $p\in[1,\infty)$,

$$\lim_{t\to+\infty}\|\theta(t)\|_{L^p} = 0.$$

(ii) Since ϕ is continuous and the flow $\{\psi_t\}_{t\in\mathbb{R}}$ is supposed to be uniquely ergodic then using once again Theorem 1-(vi) we get that the convergence in (7) is uniform. Consequently for any $0<\varepsilon<\langle\phi\rangle$ we can find $T_0>0$ such that

$$\forall t\geq T_0,\ \forall x\in\mathbb{T}^d, \quad \langle\phi\rangle-\varepsilon \leq \frac{1}{t}\int_0^t \phi(\psi(\tau,x))d\tau$$

Combined with (8) we find that

$$\forall t\geq T_0, \quad \|\theta(t)\|_{L^p} \leq e^{-(\langle\phi\rangle-\varepsilon)t}\|\theta_0\|_{L^p}.$$

This achieves the proof by taking $\mu = \langle\phi\rangle-\varepsilon$.
(iii) By virtue of Theorem 1 we have

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \phi(\psi(\tau,x))d\tau = \phi_\star(x), a.e.$$

Therefore, we get from the assumption $\phi_\star(x)>0, a.e.$ that

$$\lim_{t\to+\infty}\int_0^t \phi(\psi(\tau,x))d\tau = +\infty, a.e.$$

Then it is enough to apply Lebesgue's dominated convergence theorem with (8) in order to get the desired result. □

3. Viscous case

In this section we shall deal with the viscous case and explore the exponential decay for small viscosity. Our main result reads as follows.

THEOREM 3. *Let $d \geq 1, p \in [2,\infty)$ and $\phi : \mathbb{T}^d \to \mathbb{R}$ be a positive Lipschitz function such that $\langle\phi\rangle > 0$. Assume that the flow $\{\psi_t\}_{t\in\mathbb{R}}$ is uniquely ergodic. Let $\mu \in [0, \langle\phi\rangle)$, then there exists a constant $C_0 > 0$ depending explicitly on the Lipschitz norms of ϕ and v and a constant C_p depending only on p, μ and C_0 such that for any $\nu \in (0,1)$ the solution θ of (1) satisfies*

$$\forall t \in [0, C_0 \ln(1/\nu)], \quad \|\theta(t)\|_{L^p} \leq C_p \|\theta_0\|_{L^p} e^{-\mu t}.$$

We emphasize that neither C_0 nor C_p depends on the small viscosity $\nu \in (0,1)$.

PROOF. Set $\eta(t,x) = \theta(t,\psi(t,x))$ then it satisfies the equation

$$\partial_t \eta(t,x) - \nu(\Delta\theta)(t,\psi(t,x)) = -\eta(t,x)\,\phi(\psi(t,x)).$$

Define $V(t,x) = \int_0^t \phi(\psi(\tau,x))d\tau$ and $h(t,x) = e^{V(t,x)}\eta(t,x)$ then it is easy to check

$$\partial_t h(t,x) - \nu e^{V(t,x)}\,(\Delta\theta)(t,\psi(t,x)) = 0.$$

We multiply this equation by $|h|^{p-2}h$ and we integrate in space variable,

$$\frac{1}{p}\frac{d}{dt}\|h(t)\|_{L^p}^p - \nu\int_{\mathbb{T}^d} e^{V(t,x)}(\Delta\theta)(t,\psi(t,x))|h(t,x)|^{p-2}h(t,x)dx = 0.$$

This energy identity can be written in the form

$$\frac{1}{p}\frac{d}{dt}\|h(t)\|_{L^p}^p - \nu\,\mathcal{I}(t) = 0, \tag{9}$$

with

$$\mathcal{I}(t) := \int_{\mathbb{T}^d} e^{pV(t,x)}(\Delta\theta)(t,\psi(t,x))\big|\theta\big(t,\psi(t,x)\big)\big|^{p-2}\theta\big(t,\psi(t,x)\big)dx.$$

Since the flow preserves Lebesgue measure and $\psi_t^{-1} = \psi_{-t}$ then change of variables leads to

$$\mathcal{I}(t) = \int_{\mathbb{T}^d} e^{pV(t,\psi(-t,x))}(\Delta\theta)(t,x)\big|\theta\big(t,x\big)\big|^{p-2}\theta\big(t,x\big)dx.$$

Integrating by parts yields

$$\begin{aligned}\mathcal{I}(t) = &-(p-1)\int_{\mathbb{T}^d} e^{pV(t,\psi(-t,x))}|\nabla\theta(t,x)|^2\big|\theta\big(t,x\big)\big|^{p-2}dx\\ &- p\int_{\mathbb{T}^d} e^{pV(t,\psi(-t,x))}\big|\theta\big(t,x\big)\big|^{p-2}\theta\big(t,x\big)\,\nabla\theta(t,x)\cdot\nabla(V(t,\psi(-t,x))dx.\end{aligned}$$

Using the inequality $|ab| \leq \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$, for any $\varepsilon > 0$ with a suitable choice of ε allows to get

$$\begin{aligned}&p\int_{\mathbb{T}^d} e^{pV(t,\psi(-t,x))}\big|\theta\big(t,x\big)\big|^{p-2}\theta\big(t,x\big)\,\nabla\theta(t,x)\cdot\nabla(V(t,\psi(-t,x))dx \leq\\ &\frac{p-1}{2}\int_{\mathbb{T}^d} e^{pV(t,\psi(-t,x))}|\nabla\theta(t,x)|^2\big|\theta\big(t,x\big)\big|^{p-2}dx\\ &+\frac{p^2}{2(p-1)}\int_{\mathbb{T}^d} e^{pV(t,\psi(-t,x))}\big|\theta\big(t,x\big)\big|^{p}|\nabla(V(t,\psi(-t,x))|^2dx.\end{aligned}$$

Consequently

$$\mathcal{I}(t) \le -\frac{p-1}{2}\int_{\mathbb{T}^d} e^{pV(t,\psi(-t,x))}|\nabla\theta(t,x)|^2\big|\theta\big(t,x\big)\big|^{p-2}dx$$
$$+\frac{p^2}{2(p-1)}\|\nabla(V(t,\psi(-t,\cdot))\|_{L^\infty}^2\int_{\mathbb{T}^d} e^{pV(t,\psi(-t,x))}\big|\theta\big(t,x\big)\big|^{p}dx.$$

Notice that from the change of variables $x\mapsto\psi(t,x)$ one gets

$$\int_{\mathbb{T}^d} e^{pV(t,\psi(-t,x))}\big|\theta\big(t,x\big)\big|^{p}dx=\|h(t)\|_{L^p}^p.$$

From the group property of the flow we obtain

$$\begin{aligned} V(t,\psi(-t,x)) &= \int_0^t \phi(\psi(\tau,\psi(-t,x))d\tau\\ &= \int_0^t \phi(\psi(\tau-t,x)d\tau\\ &= \int_{-t}^0 \phi(\psi(\tau,x)d\tau. \end{aligned}$$

Applying the chain rule we infer that for any $t\in\mathbb{R}$

$$|\nabla(V(t,\psi(\tau,\cdot))\|_{L^\infty}\le\|\nabla\phi\|_{L^\infty}\left|\int_{-t}^0\|\nabla\psi(\tau)\|_{L^\infty}d\tau\right|.$$

Coming back to the ODE (3), then it is straightforward to obtain that for any $t\in\mathbb{R}$

$$\|\nabla\psi(t)\|_{L^\infty}\le e^{|t|\|\nabla v\|_{L^\infty}}.$$

It follows that

$$\begin{aligned} |\nabla(V(t,\psi(\tau,\cdot))\|_{L^\infty}^2 &\le\|\nabla\phi\|_{L^\infty}^2\, t^2\, e^{2|t|\|\nabla v\|_{L^\infty}}\\ &\le C_0e^{C_0|t|}. \end{aligned}$$

For some constant $C_0>0$ depending only on the Lipschitz norms of ϕ and v. Putting together the preceding estimates allows to get

$$\mathcal{I}(t) \le -\frac{p-1}{2}\int_{\mathbb{T}^d} e^{pV(t,\psi(-t,x))}|\nabla\theta(t,x)|^2\big|\theta\big(t,x\big)\big|^{p-2}dx+C_0\frac{p^2}{p-1}e^{C_0|t|}\|h(t)\|_{L^p}^p.$$

Inserting this into (9) implies that for any $t\ge0$,

$$\frac1p\frac{d}{dt}\|h(t)\|_{L^p}^p+\nu\frac{p-1}{2}\int_{\mathbb{T}^d} e^{pV(t,\psi(-t,x))}|\nabla\theta(t,x)|^2\big|\theta\big(t,x\big)\big|^{p-2}dx$$
$$\le C_0\frac{p^2}{p-1}\nu e^{C_0t}\|h(t)\|_{L^p}^p.$$

From Gronwall lemma we deduce that for any $t\ge0$,

$$\|h(t)\|_{L^p}^p+\nu\frac{p(p-1)}{2}\int_0^t\int_{\mathbb{T}^d} e^{pV(\tau,\psi(-\tau,x))}|\nabla\theta(\tau,x)|^2\big|\theta\big(\tau,x\big)\big|^{p-2}dxd\tau\le$$
$$e^{C_0\frac{p^3}{p-1}\nu te^{C_0t}}\|h(0)\|_{L^p}^p$$
$$\le e^{C_0\frac{p^3}{p-1}\nu e^{C_0t}}\|\theta_0\|_{L^p}^p.$$

By choosing $t \geq 0$ such that $\nu e^{C_0 t} \leq 1$, that is, $0 \leq t \leq C_0^{-1} \ln(1/\nu)$ we obtain

$$\|h(t)\|_{L^p}^p + \nu \frac{p(p-1)}{2} \int_0^t \int_{\mathbb{T}^d} e^{pV(\tau,\psi(-\tau,x))} |\nabla\theta(\tau,x)|^2 \big|\theta(\tau,x)\big|^{p-2} dx d\tau \leq$$
$$e^{C\frac{p^3}{p-1}} \|\theta(0)\|_{L^p}^p,$$

In particular, we get that for any $t \in [0, C_0^{-1} \ln(1/\nu)]$ and for any $p \in [2,\infty)$,

$$\|h(t)\|_{L^p} \leq e^{C_0 p} \|\theta(0)\|_{L^p}. \tag{10}$$

Using the unique ergodicity of the flows we obtain by virtue of Theorem 1-(v) that

$$\lim_{t\to+\infty} \left\| \frac{1}{t} V(t,\cdot) - \langle\phi\rangle \right\|_{L^\infty} = 0.$$

Consequently, for any $0 < \mu < \langle\phi\rangle$ there exists $T_0 > 0$, independent of ν and p such that

$$\forall t \geq T_0, \forall x \in \mathbb{T}^d, \quad \mu \leq \frac{1}{t} V(t,x).$$

This allows to get the lower bound: for any $t \geq T_0$

$$\begin{aligned}\|h(t)\|_{L^p}^p =& \int_{\mathbb{T}^d} e^{pV(t,x)} |\theta(t,\psi(t,x))|^p dx \\ \geq& e^{p\mu t} \|\theta(t)\|_{L^p}^p.\end{aligned}$$

Hence

$$\|h(t)\|_{L^p} \geq e^{\mu t} \|\theta(t)\|_{L^p}$$

Combing this estimate with (10) yields for any $t \in [T_0, C_0^{-1} \ln(1/\nu)]$

$$\|\theta(t)\|_{L^p} \leq e^{C_0 p - \mu t} \|\theta_0\|_{L^p}$$

Thus we can find a constant C_p depending only on p and the Lipschitz norms of ϕ and v such that

$$\forall t \in [0, C_0^{-1} \ln(1/\nu)]|, \quad \|\theta(t)\|_{L^p} \leq C_p e^{-\mu t} \|\theta_0\|_{L^p}.$$

This achieves the desired result. □

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