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Asymptotic behavior of solutions for a class of two-coupled nonlinear fractional Schrödinger equations

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ABSTRACT. In the current issue, we consider two coupled weakly dissipative fractional Schrödinger equations with cubic nonlinearities that reads

$$\begin{cases} u_t - i(-\Delta)^{\frac{\alpha}{2}} u + i\left(|u|^2 + |v|^2\right) u + \gamma u = f \\ v_t - i(-\Delta)^{\frac{\alpha}{2}} v + i\left(|u|^2 + |v|^2\right) v + \delta v_x + \gamma v = g \end{cases}$$

We will prove that the asymptotic dynamics of the solutions will be described by the existence of a regular compact global attractor in the phase space with finite fractal dimension.

CONTENTS

1. Introduction

The study of systems of coupled nonlinear Schrödinger type equations in mathematical and physical aspects are of comparative interest and has attracted much attention for many researchers as they appear in different sides of applications to various branches of physics. Lately, interest in dissipative systems increased even more because of intensive elaboration of strange attractors. For a given dissipative dynamical system, the first question that arises is determining the existence of

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a bounded (or even compact) attracting set, the **global attractor** that contains much of the relevant information about the flow, on which one may reduce the qualitative study of the system. We refer the reader to [**34**], [**30**], [**9**] and [**29**] for general frameworks of this theory.

In optics, systems of coupled nonlinear Schrödinger equations are used to describe the propagation of light along birefringent optical fiber [**25**], [**40**] and [**41**]. In hydrodynamics, when studying the modulation instability of gravity waves in fluids in great depths [**8**], a coupling nonlinear Schrödinger system was considered. Appearing also in the study of electromagnetic waves [**28**], the coupling nonlinear Schrödinger systems arise also when studying either Feshbach resonances in atomic Bose-Einstein condensate systems [**35**] or atmospheric gravity waves [**26**]. With the appearance of memory materials, a great attention has been focused on the study of problems involving the fractional space derivative.

In this article we investigate the asymptotic behaviour of solutions for an infinite dimensional dynamical system generated by a generalized class of two coupled nonlinear fractional dissipative Schrödinger equations that reads

$$\begin{cases} u_t - i(-\Delta)^{\frac{\alpha}{2}} u + i\left(|u|^2 + |v|^2\right) u + \gamma u = f \\ v_t - i(-\Delta)^{\frac{\alpha}{2}} v + i\left(|u|^2 + |v|^2\right) v + \delta v_x + \gamma v = g \end{cases} \tag{1.1}$$

supplemented by

$$u(t=0) = u_0\,,\ v(t=0) = v_0 \tag{1.2}$$

where $\alpha \in (1, 2]$, $\delta \in \mathbb{R}$ and $\gamma > 0$ is the damping parameter. The unknown $u = u(t, x)$ and $v = v(t, x)$ map $\mathbb{R}_+ \times \mathbb{R}$ into $\mathbb{C}$. At $t = 0$, the initial conditions u_0 and v_0 belong to the fractional Sobolev space $H^{\frac{\alpha}{2}}(\mathbb{R})$ that will be specified later. The functions f and g, that belong to $L^2(\mathbb{R})$, are given source terms that are independent of time.

The system (1.1) is derived from the governing equation for atmospheric gravity waves [**26**] and it appears also in nonlinear optics [**6**] and in the study of large-scale Rossby waves [**33**]. When $\delta = 0$ and $\alpha = 2$ it was shown that the resultant system, known as Manakov system [**28**], models the evolution of two orthogonal pulse envelopes in birefringent optical fibers and it can be derived from the governing equation of atmospheric gravity waves when studying collision interactions of envelopes Rossby solitons in a barotropic atmosphere [**33**].

From mathematical point of view, the study of fractional nonlinear Schrödinger equations has attracted a lot of attention among researchers. The prototypal example is the Fractional NLS with cubic nonlinearity introduced by N. Laskin [**21, 22**] as follows

$$u_t - i\,(-\Delta)^{\frac{\alpha}{2}}\, u + i|u|^2 u = 0\,. \tag{1.3}$$

This equation was recently studied by several authors among which stand out B. Guo and Z. Huo [**17**] and Y. Hong and Y. Sire [**19**] regarding the question of the initial value problem in Sobolev spaces. Taking into account, in some physical context, an external forcing term and some damping effect, a similar equation was considered in [**15**] in which the issues of existence and regularity of the global attractor were addressed.

Now let us go back to the matter at hand. In the conservative case when $\gamma = f = g = 0$, the study of existence and uniqueness of global smooth solutions for (1.1) was established in [**20**], [**18**] and [**38**] in bounded domains. However, existence and stability of standing waves was considered in [**32**]. While in the dissipative case where, in some physical contexts, external forcing term and some damping effects have been taken into account, it should be cited that G. Li and C. Zhu have considered the system (1.1), with $\alpha = 2$ and in a one dimensional bounded domain. They have established in [**23**] the existence of a compact global attractor with finite fractal dimension and their strategy rely on the work of J. Ghidaglia [**13**]. The asymptotic dynamics for N-coupled Fractional dissipative NLS was studied in [**7**] where M. Cheng has proved the existence of a compact global attractor in the phase space. This result was improved in [**4**] where the author has established the regularity and the finite fractal dimension of the global attractor following [**3**].

2. Mathematical framework and Notations

2.1. Notations and main results. Before giving the main results and the layout of this article, we introduce briefly some definitions and notations.

We initially recall that in what follows we use the notation $(.,.)$ for the usual scalar product in $L^2(\mathbb{R})$ defined by

$$(u,v) = \Re e\left(\int_{\mathbb{R}} u(x)\,\overline{v(x)}\,dx\right).$$

The one dimensional space Fourier transform will be denoted as follows

$$\mathscr{F}(u)(\xi) = \int_{\mathbb{R}} u(x)e^{-ix\xi}dx\,.$$

Now, for a given fractional exponent $s \in (0,1)$, we recall (see [**11**] and [**31**]) that $(-\Delta)^s$ is considered as the homogeneous fractional pseudo-differential operator defined, for $u \in \mathscr{S}(\mathbb{R})$, by

$$(-\Delta)^s u(x) = C_s\ p.v.\int_{\mathbb{R}^2} \frac{u(x)-u(y)}{|x-y|^{1+2s}}\,dy = \mathscr{F}^{-1}\left(|\xi|^{2s}\mathscr{F}(u)\right)(x)\,,$$

where $C_s = \left(\int_{\mathbb{R}} \frac{1-\cos(\zeta)}{|\zeta|^{1+2s}}d\zeta\right)^{-1}$, $\mathscr{S}(\mathbb{R})$ denotes the Schwarz class and "p.v." for principal value.

The fractional Sobolev space $H^s(\mathbb{R})$ defined as follows

$$H^s(\mathbb{R}) = \left\{u \in L^2(\mathbb{R}) \quad \text{such that} \quad \int_{\mathbb{R}} (1+|\xi|^{2s})|\mathscr{F}(u)(\xi)|^2 d\xi < +\infty\right\}$$

as an intermediary Banach space between $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ is a Hilbert space endowed, via the Fourier transform approach, with the norm

$$||u||_{H^s(\mathbb{R})} = \left(||u||^2_{L^2} + \left|\left|(-\Delta)^{\frac{s}{2}}u\right|\right|^2_{L^2}\right)^{\frac{1}{2}}$$

and the associate scalar product denoted by

$$(u,v)_{H^s(\mathbb{R})} = (u,v) + \left((-\Delta)^{\frac{s}{2}}u, (-\Delta)^{\frac{s}{2}}v\right)\,, \quad u,v \in H^s(\mathbb{R})\,.$$

For the sake of simplicity and in order to remove any ambiguity in what follows, we will extensively use the following vectorial notations.

For $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2$ we denotes

$$U.V = u_1 v_1 + u_2 v_2 \quad \text{and} \quad |U|^2 = U.\overline{U}$$

For a given $p \in [2, +\infty)$, the space denoted

$$\mathbb{L}^p = (L^p(\mathbb{R}))^2 \quad \text{endowed with the norm} \quad ||U||_{\mathbb{L}^p} = \left(||u_1||^p_{L^p(\mathbb{R})} + ||u_2||^p_{L^p(\mathbb{R})} \right)^{\frac{1}{p}}$$

is a Banach space. When $p = 2$, $\mathbb{L}^2$ is the Hilbert space endowed with the usual scalar product

$$((U, V)) = (u_1, v_1) + (u_2, v_2)\,, \quad \forall\, U, V \in \mathbb{L}^2\,.$$

Whereas for the limiting case $p = +\infty$, we denote

$$||U||_{\mathbb{L}^\infty} = \sup\left(||u_1||_{L^\infty(\mathbb{R})}, ||u_2||_{L^\infty(\mathbb{R})} \right)\,.$$

In what follows the Hilbert spaces $\mathbb{H}^s = (H^s(\mathbb{R}))^2$, $s \geq 0$, are endowed with the norm

$$||U||^2_{\mathbb{H}^s} = ||U||^2_{\mathbb{L}^2} + \left|\left|(-\Delta)^{\frac{s}{2}} U\right|\right|^2_{\mathbb{L}^2}\,.$$

For convenience use, we denote $\wp$ the canonical projector defined by

$$\wp : U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \wp(U) = \begin{pmatrix} 0 \\ u_2 \end{pmatrix}.$$

Hence, the system $(1.1) - (1.2)$ will be equivalently written as follows

$$U_t - i(-\Delta)^{\frac{\alpha}{2}} U + i|U|^2 U + \gamma U + \delta \wp(U_x) = F \;\in \mathbb{L}^2 \tag{2.1}$$

$$U(t = 0) = U_0 \quad \in \mathbb{H}^{\frac{\alpha}{2}}\,. \tag{2.2}$$

where $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $F = \begin{pmatrix} f \\ g \end{pmatrix}$.

The first result established in this paper states as follows:

THEOREM 2.1. *Let* $\alpha \in (1, 2]$ *and* $F \in \mathbb{L}^2$. *Then the problem* $(2.1) - (2.2)$ *is well posed in* $\mathbb{H}^{\frac{\alpha}{2}}$ *and defines an infinite dimensional dissipative dynamical system that possesses a compact global attractor* $\mathscr{A}_\alpha$ *in* $\mathbb{H}^{\frac{\alpha}{2}}$.

Once the global attractor is obtained, the question arises if it has special regularity properties or if it has finite-dimensional character. This will be the subject of the second main result. We state

THEOREM 2.2. *Let* $\alpha \in (1, 2]$ *and* $F \in \mathbb{L}^2$. *Then the global attractor* $\mathscr{A}_\alpha$ *associated to the dynamical system generated by* (2.1) *is a compact subset of* $\mathbb{H}^\alpha$ *with finite fractal dimension in* $\mathbb{H}^{\frac{\alpha}{2}}$.

These results established in this paper extend those in [**23**] on the real line and present significant improvements of those in [**15**]. This article is organized as follows: in the subsection below (in the current Section 2), we establish some helpful results that play a crucial role in what follows. In section 3 we prove the well-posedness of the problem $(2.1) - (2.2)$ as well as the existence of a compact global attractor in $\mathbb{H}^{\frac{\alpha}{2}}$. The regularity of the global attractor $\mathscr{A}_\alpha$ for $(2.1) - (2.2)$ will be discussed in Section 4. Finally, the section 5 we be dedicated to establish the finite fractal dimension of $\mathscr{A}_\alpha$ using a new idea introduced in [**3**].

In the end of this section it should be noted that throughout this article, the constants Cs are numerical positive constants that vary from one line to another and $A \lesssim B$ means the existence of $C > 0$ such that $A \leq CB$.

2.2. Preliminary results. To begin with, we shall briefly introduce some tools from functional analysis that will be used extensively in the sequel.

We recall a fractional Gagliardo-Nirenberg type inequality (see [**12**] for instance)

LEMMA 2.3. *Let $\alpha > 1$. Then for every $p \in [2, +\infty]$, $\exists\, C = C(\alpha, p) > 0$ such that*

$$||u||_{L^p(\mathbb{R})} \leq C\, ||u||_{L^2}^{1-\frac{2}{\alpha}(\frac{1}{2}-\frac{1}{p})} \left|\left|(-\Delta)^{\frac{\alpha}{4}} u\right|\right|_{L^2}^{\frac{2}{\alpha}(\frac{1}{2}-\frac{1}{p})}, \quad \forall\, u \in H^{\frac{\alpha}{2}}.$$

PROOF OF LEMMA 2.3. For the sake of completeness, we give a simple proof. Using the Hausdorff-Young inequality and the Hölder inequality, it leads that for $p > 2$

$$\begin{aligned} ||u||_{\mathbb{L}^p} &\lesssim ||\mathscr{F}(u)||_{\mathbb{L}^{p'}} \\ &\lesssim \left(\int_{\mathbb{R}^2} (1 + |\xi|^{\alpha})\, |\widehat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{d\xi}{(1 + |\xi|^{\alpha})^{\frac{p'}{2-p'}}}\right)^{\frac{2-p'}{2p'}} \end{aligned}$$

where p' denotes the conjugate exponent of p.

Hence

$$||u||_{\mathbb{L}^p} \leq C(\alpha, p) \left(||u||_{L^2(\mathbb{R})} + \left|\left|(-\Delta)^{\frac{\alpha}{4}} u\right|\right|_{L^2(\mathbb{R})}\right). \tag{2.3}$$

Replacing u by $x \longmapsto u(\lambda x)$, $\lambda \in \mathbb{R}$ in (2.3), it leads that

$$||u||_{\mathbb{L}^p} \leq C(\alpha, p) \left(\frac{1}{\lambda^{\frac{1}{2}-\frac{1}{p}}} ||u||_{L^2(\mathbb{R})} + \lambda^{\frac{\alpha}{2}-(\frac{1}{2}-\frac{1}{p})} \left|\left|(-\Delta)^{\frac{\alpha}{4}} u\right|\right|_{L^2(\mathbb{R})}\right).$$

Minimizing the right hand side of the previous equation with respect to λ achieves the proof of the lemma. □

For later use we recall the following result

LEMMA 2.4. *Let $s \in [1, 2]$. Then there exists $C > 0$ such that for all $p \in [2, +\infty)$,*

$$||u||_{L^p} \leq C\, \sqrt{p}\, ||u||_{H^{\frac{s}{2}}}, \quad \forall\, u \in H^{\frac{s}{2}}.$$

PROOF OF LEMMA 2.4. The proof follows by applying Theorem 8.5 in [**24**] then using an interpolation argument between $H^{\frac{1}{2}}$ and H^1. □

We give now a commutator estimate (see [**15**]) that states as follows

LEMMA 2.5. *Let $u \in \mathscr{S}(\mathbb{R})$ and $v \in H^{\alpha}$. Then $\exists\, C_\alpha > 0$ such that*

$$\begin{aligned} &\left|\left|(-\Delta)^{\frac{\alpha}{2}}(uv) - u(-\Delta)^{\frac{\alpha}{2}} v\right|\right|_{L^2} \\ &\leq C_\alpha \left(||v||_{L^2}\, ||\, |\xi|^{\alpha} \mathscr{F}(u)||_{L^1} + ||v||_{H^{\alpha-1}}\, ||\, |\xi|\, \mathscr{F}(u)||_{L^1}\right). \end{aligned}$$

Proof of Lemma 2.5. Using the Fourier transform, one has

$$\begin{aligned}
&|\mathscr{F}((-\Delta)^{\frac{\alpha}{2}}(uv))(\xi) - \mathscr{F}(u(-\Delta)^{\frac{\alpha}{2}}v)(\xi)| \\
&\le \int_{\mathbb{R}} \big| |\xi-\eta|^{\alpha} - |\xi|^{\alpha}\big| \, |\mathscr{F}(u)(\eta)| \, |\mathscr{F}(v)(\xi-\eta)| \, d\eta \\
&\lesssim \int_{\mathbb{R}} |\eta|^{\alpha} \, |\mathscr{F}(u)(\eta)| \, |\mathscr{F}(v)(\xi-\eta)| \, d\eta \\
&+ \int_{\mathbb{R}} |\eta| \, |\mathscr{F}(u)(\eta)| \, |\xi-\eta|^{\alpha-1} |\mathscr{F}(v)(\xi-\eta)| \, d\eta
\end{aligned}$$

where we have used $\big| |\xi-\eta|^{\alpha} - |\xi|^{\alpha}\big| \lesssim |\eta|^{\alpha} + |\xi-\eta|^{\alpha-1}|\eta|$. Hence, the desired estimate yields thanks to the Minkowski inequality and the Plancherel Theorem. □

3. Well posedness of the problem and existence of the global attractor

3.1. The initial value problem.

Proposition 3.1. *Let $U_0 \in \mathbb{H}^{\frac{\alpha}{2}}$. Then the problem* $(2.1)-(2.2)$ *has a unique solution*

$$U \in \mathscr{C}_b([0,+\infty), \mathbb{H}^{\frac{\alpha}{2}}) \cap \mathscr{C}_b^1([0,+\infty), \mathbb{H}^{-\frac{\alpha}{2}})$$

and the maps $\mathbb{S}_\alpha(t) : U_0 \mapsto U(t)$ are continuous on $\mathbb{H}^{\frac{\alpha}{2}}$ with $\mathscr{C}_b([0,+\infty), \mathbb{H}^{\frac{\alpha}{2}})$ denotes the space of continuous bounded functions which take values in $\mathbb{H}^{\frac{\alpha}{2}}$.

Proof of Proposition 3.1. The proof is very standard and then briefly sketch since similar to that in [**7**] and [**2**]. We proceed into two steps:

First step: A local in time solution.

Let $T > 0$, then applying Duhamel's formula to (2.1) for $0 \le t \le T$, leads to

$$\begin{aligned}
U(t) =& e^{-\gamma t} e^{it[(-\Delta)^{\frac{\alpha}{2}} + i\delta\partial_x]} U_0 \\
&+ \int_0^t e^{-\gamma(t-s)} e^{i(t-s)[(-\Delta)^{\frac{\alpha}{2}} + i\delta\partial_x]} \left[F - i|U(s)|^2 U(s)\right] \, ds \, .
\end{aligned}$$

Since the operator $e^{it(-\Delta)^{\frac{\alpha}{2}} - \delta t \partial_x}$ is unitary on H^s, $s \ge 0$ and the Hilbert space $\mathbb{H}^{\frac{\alpha}{2}}$ is an algebra (due to Lemma 2.3), the mapping $\Psi : U \mapsto |U|^2 U$ is locally lipschitz on bounded subsets of $\mathbb{H}^{\frac{\alpha}{2}}$. This ensures, by a standard fixed point argument, the existence of a local in time solution U in $\mathscr{C}([0,T^*), \mathbb{H}^{\frac{\alpha}{2}})$ for the problem $(2.1)-(2.2)$. Moreover,

$$\text{either} \quad T^* = +\infty \quad \text{or} \quad ||U(t)||_{\mathbb{H}^{\frac{\alpha}{2}}} \underset{t \to T^*, t < T^*}{\longrightarrow} +\infty \, .$$

Second step: The solution is global in time. To begin with, we establish the following a priori estimate that reads

Lemma 3.2. *Let $U_0 \in \mathbb{H}^{\frac{\alpha}{2}}$. Then there exists $C > 0$ depending only on α and $||F||_{\mathbb{L}^2}$ such that*

$$\sup_{t \in [0,T^*)} ||U(t)||^2_{\mathbb{H}^{\frac{\alpha}{2}}} \le \frac{3}{2} e^{-\gamma t} ||U_0||^2_{\mathbb{H}^{\frac{\alpha}{2}}} + C \, .$$

Proof of Lemma 3.2. On the one hand, the scalar product of (2.1) by U, leads to

$$\frac{d}{dt} ||U(t)||^2_{\mathbb{L}^2} + \gamma ||U(t)||^2_{\mathbb{L}^2} = ((F, U)) \, .$$

Hence, applying the Cauchy-Schwarz and Young's inequalities, it can be deduced by the Gronwall's Lemma that

$$||U(t)||_{\mathbb{L}^2}^2 \leq ||U_0||_{\mathbb{L}^2}^2 e^{-\gamma t} + \frac{||F||_{\mathbb{L}^2}^2}{2\gamma^2}(1 - e^{-\gamma t}) \,. \tag{3.1}$$

On the other hand, the scalar product of (2.1) by $-i(U + \gamma U)$ leads to the following energy equation that reads

$$\frac{1}{2}\frac{d}{dt}J(U(t)) + \gamma J(U(t)) = K(U(t)) \tag{3.2}$$

where

$$J(U(t)) = \left|\left|(-\Delta)^{\frac{\alpha}{4}} U(t)\right|\right|_{\mathbb{L}^2}^2 + \delta\left((iU, \wp(U_x))\right) - \frac{1}{2}\left(|U|^2, |U|^2\right) - 2((iF, U(t))) \tag{3.3}$$

$$K(U(t)) = \frac{\gamma}{2}\left(|U|^2, |U|^2\right) - \gamma((iF, U(t))) \tag{3.4}$$

Observe that due to the Parseval identity and the Hölder inequality one has

$$\begin{aligned} |((iU, \wp(U_x)))| &= |((\, i\mathscr{F}(U), \wp(\mathscr{F}(U_x))\,))| \\ &\leq \int_{\mathbb{R}} |\xi|\, |\mathscr{F}(U)(\xi)|^{\frac{2}{\alpha}} |\mathscr{F}(U)(\xi)|^{2(1-\frac{1}{\alpha})} d\xi \\ &\leq \left(\int_{\mathbb{R}} |\xi|^{\alpha}\, |\mathscr{F}(U)(\xi)|^2 d\xi\right)^{\frac{1}{\alpha}} \left(\int_{\mathbb{R}} |\mathscr{F}(U)(\xi)|^2 d\xi\right)^{1-\frac{1}{\alpha}} . \end{aligned}$$

Hence,

$$|((iU, \wp(U_x)))| \leq \left|\left|(-\Delta)^{\frac{\alpha}{4}} U\right|\right|_{\mathbb{L}^2}^{\frac{2}{\alpha}} ||U||_{\mathbb{L}^2}^{2(1-\frac{1}{\alpha})} . \tag{3.5}$$

Consequently, thanks to the previous inequality, Lemma 2.3, the Young's inequality and (3.1) one easily obtain the existence of a non negative reel constant C that depends only on $||F||_{\mathbb{L}^2}$ and γ such that

$$\frac{1}{2}\left|\left|(-\Delta)^{\frac{\alpha}{4}} U(t)\right|\right|_{\mathbb{L}^2}^2 - C \leq J(U(t)) \leq \frac{3}{2}\left|\left|(-\Delta)^{\frac{\alpha}{4}} U(t)\right|\right|_{\mathbb{L}^2}^2 + C \tag{3.6}$$

$$K(U(t)) \leq \frac{\gamma}{4}\left|\left|(-\Delta)^{\frac{\alpha}{4}} U(t)\right|\right|_{\mathbb{L}^2}^2 + C \,. \tag{3.7}$$

Gathering (3.6), (3.7) and (3.2) we obtain

$$\frac{d}{dt}J(U(t)) + \gamma J(U(t)) \leq C \,.$$

This concludes the proof thanks to the Gronwall Lemma, (3.6) and (3.1). □

Lemma 3.2 ensure that $T^* = +\infty$ and then a global in time solution is obtained. This step is therefore concluded as well as the proof of the current proposition. □

3.2. Construction of the compact global attractor. For the sake of completeness and clarity we recall the definition of the global attractor (see for instance [**9**] and [**34**]).

Definition 3.3. Let $(\mathscr{H}, (S(t))_{t\geq 0})$ be a continuous dynamical system where $\mathscr{H}$ denotes the phase space (complete metric space). We say that a nonempty subset $\mathscr{A}$ of $\mathscr{H}$ is a global or universal attractor for the semigroup $(S(t))_{t\geq 0}$ if and only if $\mathscr{A}$ enjoys the following properties:

(1) $\mathscr{A}$ is compact.
(2) $\mathscr{A}$ is invariant i.e: $S(t)\mathscr{A} = \mathscr{A}, \forall\, t \geq 0$.
(3) $\mathscr{A}$ attracts the bounded sets of $\mathscr{H}$; i.e: $d_h(S(t)B, \mathscr{A}) \to 0$ as $t \to +\infty$, $\forall\, B$ bounded set in $\mathscr{H}$ with d_h stands for the Hausdorff semi-distance (between two sets) defined by

$$d_h(A, B) = \sup_{x\in A} \inf_{y\in B} ||x - y||_{\mathscr{H}}\,.$$

A mere consequence of Lemma 3.2 states as follows

Proposition 3.4. *The semigroup $(\mathbb{S}_\alpha(t))_{t\in\mathbb{R}_+}$ possesses a bounded absorbing ball $\mathscr{B}_\alpha$ in $\mathbb{H}^{\frac{\alpha}{2}}$. i.e: for any bounded subset $B \subseteq \mathbb{H}^{\frac{\alpha}{2}}$ there exists $t(B) > 0$ such that*

$$\mathbb{S}_\alpha(t)B \subseteq \mathscr{B}_\alpha\,, \quad \forall\, t \geq t(B)\,.$$

We claim now to prove the existence of the global attractor $\mathscr{A}_\alpha$. To do this, one only has, thanks to Theorem 1.1 and Remark 1.4 in [**34**] and Theorem 5.1 in [**9**], to prove the asymptotic compactness of the semi-group $(\mathbb{S}_\alpha(t))_{t\in\mathbb{R}_+}$, that is

Proposition 3.5. *The semi-group $(\mathbb{S}_\alpha(t))_{t\in\mathbb{R}_+}$ is asymptotically compact in $\mathbb{H}^{\frac{\alpha}{2}}$ i.e., for every bounded sequence $(U_k)_k$ in $\mathbb{H}^{\frac{\alpha}{2}}$ and every sequence $t_k \longrightarrow +\infty$, $(\mathbb{S}_\alpha(t_k)U_k)_k$ is relatively compact in $\mathbb{H}^{\frac{\alpha}{2}}$.*

Proof of Proposition 3.5. We start, as a first step, by proving the following statement

Lemma 3.6. *The semi-group $(\mathbb{S}_\alpha(t))_{t\in\mathbb{R}_+}$ is continuous on bounded subsets of $\mathbb{H}^{\frac{\alpha}{2}}$ for the strong topology of $\mathbb{L}^2$.*

Proof of Lemma 3.6. Let $(\Phi_n)_n$ be a bounded sequence in $\mathbb{H}^{\frac{\alpha}{2}}$ that converges towards $U_0 \in \mathbb{H}^{\frac{\alpha}{2}}$ for the strong topology of $\mathbb{L}^2$. We denote

$$Z(t) = \mathbb{S}_\alpha(t)\Phi_n - \mathbb{S}_\alpha(t)U_0 = U_n(t) - U(t)$$

the difference of two solutions of $(2.1) - (2.2)$ issued respectively from Φ_n and U_0. Then $Z(t)$ satisfies

$$\frac{1}{2}\frac{d}{dt}||Z(t)||^2_{\mathbb{L}^2} + \gamma||Z(t)||^2_{\mathbb{L}^2} \leq C_0 \int_{\mathbb{R}} \left(|U_n(x)|^2 + |U(x)|^2\right)|Z(x)|^2 dx\,. \tag{3.8}$$

Now we use an argument due to M. Vladimirov [**36**]. We consider $p \in (2, +\infty)$. Thanks to the Hölder inequality

$$\int_{\mathbb{R}} \left(|U_n(x)|^2 + |U(x)|^2\right)\, |Z(x)|^2 dx \leq \left(||U_n||^2_{\mathbb{L}^{4p}} + ||U||^2_{\mathbb{L}^{4p}}\right) ||Z||^{\frac{2}{p}}_{\mathbb{L}^4} ||Z||^{\frac{2p-2}{p}}_{\mathbb{L}^2}$$

Since U_n and U remain uniformly bounded in $\mathbb{H}^{\frac{\alpha}{2}}$ (Lemma 3.2), it may be deduced, in accordance with (3.8), Lemma 2.3 and Lemma 2.4, that

$$\frac{d}{dt}||Z(t)||^2_{\mathbb{L}^2} \leq C_0 C_1^{\frac{1}{\alpha p}} p\, ||Z||^{2(1-\frac{1}{2p\alpha})}_{\mathbb{L}^2}\,,$$

which is equivalent to

$$\frac{d}{dt}||Z(t)||_{\mathbb{L}^2}^{\frac{1}{\alpha p}} \leq C_\alpha C_1^{\frac{1}{\alpha p}} . \tag{3.9}$$

Integrating (3.9) on $[0,T]$ for a chosen $T>0$ such that $C_\alpha T<1$, leads to

$$\sup_{t\in[0,T]} ||\mathbb{S}_\alpha(t)\Phi_n - \mathbb{S}_\alpha(t)U_0||_{\mathbb{L}^2} \leq \left(C_\alpha C_1^{\frac{1}{\alpha p}} T + ||\Phi_n - U_0||_{\mathbb{L}^2}^{\frac{1}{2p\alpha}}\right)^{p\alpha} . \tag{3.10}$$

Therefore

$$\limsup_{n\to+\infty} \sup_{t\in[0,T]} ||\mathbb{S}_\alpha(t)\Phi_n - \mathbb{S}_\alpha(t)U_0||_{\mathbb{L}^2} \leq C_1 \left(C_\alpha T\right)^{\alpha p} \tag{3.11}$$

and then the desired result follows by letting $p \longrightarrow +\infty$. □

Next, we show that the semigroup $(\mathbb{S}_\alpha(t))_{t\in\mathbb{R}_+}$ is asymptotically compact in $\mathbb{L}^2$ on bounded subsets of $\mathbb{H}^{\frac{\alpha}{2}}$ following the idea in [**27**].

LEMMA 3.7. *For every bounded sequence $(U_j)_j$ in $\mathbb{H}^{\frac{\alpha}{2}}$ and every nonnegative sequence $t_j \longrightarrow +\infty$, $(\mathbb{S}_\alpha(t_j)U_j)_j$ is relatively compact in $\mathbb{L}^2$.*

PROOF OF LEMMA 3.7. Let $U_0 \in \mathbb{H}^{\frac{\alpha}{2}}$ and $U(t) = \mathbb{S}_\alpha(t)U_0$. Consider a smooth cut-off function θ such that $\theta(x) = 1$ if $|x| \leq 1$ and $\theta(x) = 0$ if $|x| \geq 2$, $x \in \mathbb{R}$. For a given $R > 0$, we set

$$\theta_R(x) = \theta\left(\frac{x}{R}\right) \text{ and } M_R(t) = \int_{\mathbb{R}} |U(t)|^2 \left[1-\theta_R(x)\right]^2 dx .$$

Multiplying (2.1) by $(1-\theta_R)$ then making the scalar product of the resulting equation by $U(1-\theta_R)$ lead to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}M_R(t) + \gamma M_R(t) =& ((i(1-\theta_R)(-\Delta)^{\frac{\alpha}{2}}U, (1-\theta_R)U)) \\ &+(((1-\theta_R)F, (1-\theta_R)U)) \\ &-\delta((\wp[(1-\theta_R)U_x], (1-\theta_R)U)) .\end{aligned}$$

Since

$$((\wp[(1-\theta_R)U_x], (1-\theta_R)U)) = \frac{1}{R}((\wp[\theta'_R U], (1-\theta_R)U)) ,$$

we deduce by the Cauchy-Schwarz inequality that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}M_R(t) + \gamma M_R(t) \\ &\leq \left(||(1-\theta_R)F||_{\mathbb{L}^2} + C\frac{||U||_{\mathbb{L}^2}}{R}\right)\sqrt{M_R(t)} \\ &+ \left|\left|(1-\theta_R)(-\Delta)^{\frac{\alpha}{2}}U - (-\Delta)^{\frac{\alpha}{2}}[(1-\theta_R)U]\right|\right|_{\mathbb{L}^2}\sqrt{M_R(t)} .\end{aligned} \tag{3.12}$$

Thanks to Lemma 2.5,

$$\begin{aligned}&\left|\left|(1-\theta_R)(-\Delta)^{\frac{\alpha}{2}}U - (-\Delta)^{\frac{\alpha}{2}}[(1-\theta_R)U]\right|\right|_{\mathbb{L}^2} \\ &= \left|\left|\theta_R(-\Delta)^{\frac{\alpha}{2}}U - (-\Delta)^{\frac{\alpha}{2}}(\theta_R U)\right|\right|_{\mathbb{L}^2} \\ &\lesssim ||U||_{\mathbb{L}^2}\, ||\,|\xi|^\alpha \mathscr{F}(\theta_R)||_{L^1(\mathbb{R})} + ||U||_{\mathbb{H}^{\alpha-1}}\, ||\,|\xi|\, \mathscr{F}(\theta_R)||_{L^1(\mathbb{R})} .\end{aligned}$$

Remark that $|\xi|^s\mathscr{F}(\theta_R)(\xi) = R|\xi|^s\mathscr{F}(\theta)(R\xi)$, $\forall\, s \in [1,2]$, then it follows that

$$||\,|\xi|^s\mathscr{F}(\theta_R)||_{L^1(\mathbb{R})} = \frac{||\mathscr{F}((-\Delta)^{\frac{s}{2}}\theta)||_{L^1(\mathbb{R})}}{R^s} \leq \frac{C}{R^s} .$$

Hence, since $\alpha \in (1,2]\,(\Rightarrow 0 < \alpha - 1 \le \frac{\alpha}{2})$ and U is uniformly bounded in $\mathbb{H}^{\frac{\alpha}{2}}$, we deduce that

$$(3.13)\qquad \big|\big|(1-\theta_R)(-\Delta)^{\frac{\alpha}{2}}U - (-\Delta)^{\frac{\alpha}{2}}[(1-\theta_R)U]\big|\big|_{\mathbb{L}^2} \lesssim \frac{1}{R^\alpha} + \frac{1}{R}\,.$$

Gathering (3.12) and (3.13) then applying the Young inequality and the Gronwall Lemma lead to

$$(3.14)\quad M_R(t) \le ||U_0||^2_{\mathbb{L}^2(|x|>R)}e^{-\gamma t} + C\left(||F||^2_{\mathbb{L}^2(|x|>R)} + \frac{1}{R^{2\alpha}} + \frac{1}{R^2}\right)(1-e^{-\gamma t})\,.$$

Consider now a bounded sequence $(U_j)_j$ in $\mathbb{H}^{\frac{\alpha}{2}}$ and $t_j \longrightarrow +\infty$.

By splitting $\mathbb{S}_\alpha(t_j)U_j$ as follows

$$\mathbb{S}_\alpha(t_j)U_j = \mathbb{S}_\alpha(t_j)U_j\theta_R + \mathbb{S}_\alpha(t_j)U_j(1-\theta_R) = W_j(t_j) + V_j(t_j)$$

we deduce from (3.14) that for a given $\epsilon > 0$, there exist $R_0 > 0$ such that

$$||V_j(t_j)||_{\mathbb{L}^2} \le \epsilon\,, \quad \forall\, j \ge j_0\,.$$

Moreover, $(W_j(t_j))_j$ remains trapped in a compact set of $\mathbb{L}^2$ which ensure by classical argument that $(\mathbb{S}_\alpha(t_j)U_j)_j$ is relatively compact in $\mathbb{L}^2$ and the proof is achieved. □

Finally, we continue the proof of Proposition 3.5 using the well known John Ball's argument ([**5**] and [**37**]). Let $(U_j)_j$ be a sequence in $\mathbb{H}^{\frac{\alpha}{2}}$ and $t_j \to +\infty$. Since $(\mathbb{S}_\alpha(t_j)U_j)_j$ is bounded in $\mathbb{H}^{\frac{\alpha}{2}}$ and in accordance with Lemma 3.7, we may assume, up to a subsequence extraction, the existence of $U \in \mathbb{H}^{\frac{\alpha}{2}}$ such that

$$(3.15)\qquad \mathbb{S}_\alpha(t_j)U_j \rightharpoonup U \ \text{ weakly in } \mathbb{H}^{\frac{\alpha}{2}} \quad\text{and}\quad \mathbb{S}_\alpha(t_j)U_j \longrightarrow U \ \text{ strongly in } \mathbb{L}^2\,.$$

Thanks to the energy equation (3.2) one has on the one hand

$$(3.16)\quad J(\mathbb{S}_\alpha(t_j)U_j) = J(\mathbb{S}_\alpha(t_j-t)U_j)e^{-2\gamma t} + 2\int_0^t e^{-2\gamma(t-s)}K(\mathbb{S}_\alpha(t_j-t+s)U_j)\,ds$$

and in the other hand,

$$(3.17)\qquad J(U) = J(\mathbb{S}_\alpha(-t)U)e^{-2\gamma t} + \int_0^t e^{-2\gamma(t-s)}K(\mathbb{S}_\alpha(s-t)U)\,ds\,,$$

where J and K are defined by (3.3) and (3.4). Thanks to Lemma 3.6 and Lemma 2.3,

$$(3.18)\qquad \mathbb{S}_\alpha(t_j)U_j \underset{j\to+\infty}{\longrightarrow} U \quad\text{strongly in } \mathbb{L}^p\,,\ p \in [2,+\infty)\,.$$

Moreover, since $\mathbb{S}_\alpha(t_j)U_j$ and U are uniformly bounded in $\mathbb{H}^{\frac{\alpha}{2}}$, we deduce by interpolation that

$$\mathbb{S}_\alpha(t)U_j \underset{j\to+\infty}{\longrightarrow} U \quad\text{strongly in } \mathbb{H}^{\frac{s}{2}}\,,\ s \in [1,\alpha)\,.$$

As a result,

$$(3.19)\qquad ((i\mathbb{S}_\alpha(t_j)U_j, \wp(\mathbb{S}_\alpha(t_j)(U_j)_x))) \underset{j\to+\infty}{\longrightarrow} ((iU, \wp(U_x)))$$

Thanks to the Lebesgue dominated convergence Theorem, Lemma 3.6 and Lemma 3.7 we obtain that $\limsup J(\mathbb{S}_\alpha(t_j)U_j) \le J(U)$ from which we deduce, thanks again to (3.15), (3.18) and (3.19), that

$$||\mathbb{S}_\alpha(t_j)U_j||_{\mathbb{H}^{\frac{\alpha}{2}}} \longrightarrow ||U||_{\mathbb{H}^{\frac{\alpha}{2}}} \ \text{ as } j \longrightarrow +\infty\,.$$

This concludes the proof of the current proposition. □

4. Regularity of the global attractor

Following the strategy in [**15**], let $U(t)$ be the solution of (2.1)-(2.2) which takes values in $\mathbb{H}^{\frac{\alpha}{2}}$. For the sake of simplicity, we may assume that $U(t)$ remains into the $\mathbb{H}^{\frac{\alpha}{2}}-$absorbing set, $\mathscr{B}_\alpha$, for $t \geq 0$.

The first part of our second main result, is stated as follows

THEOREM 4.1. *The global attractor $\mathscr{A}_\alpha$ associated to the semi-group $(\mathbb{S}_\alpha(t))_{t\in\mathbb{R}_+}$ is in fact a compact subset of $\mathbb{H}^\alpha$.*

4.1. The auxiliary problem. We introduce, for a given level $N > 0$, the orthogonal projector P_N acting in $L^2(\mathbb{R})$ by setting

$$P_N(u) = \int_{\mathbb{R}} \chi\left(\frac{\xi}{N}\right) \mathscr{F}(u)(\xi) e^{i\xi x} d\xi = \mathscr{F}^{-1}\left(\xi \mapsto \chi\left(\frac{\xi}{N}\right)\mathscr{F}(u)(\xi)\right). \tag{4.1}$$

where χ is the characteristic function of the interval $[-1, 1]$. Actually, P_N is the projector onto the low-frequencies modes of a given function, at level N. Clearly, $D^\alpha P_N = P_N D^\alpha$. Moreover, P_N and $Q_N = Id - P_N$ are bounded operators from H^s, $s \geq 0$, into itself and satisfy the following statements that read

LEMMA 4.2. *Let $0 \leq s_1 \leq s_2$. Then there exists $C > 0$, that does not depends on N, such that*

$$\left|\left|(-\Delta)^{\frac{s_2}{2}} P_N(u)\right|\right|_{L^2} \leq C\, N^{s_2 - s_1} \left|\left|(-\Delta)^{\frac{s_1}{2}} P_N(u)\right|\right|_{L^2}, \quad \forall\, u \in H^{s_1}.$$

$$\left|\left|(-\Delta)^{\frac{s_1}{2}} Q_N(u)\right|\right|_{L^2} \leq C\, N^{s_1 - s_2} \left|\left|(-\Delta)^{\frac{s_2}{2}} Q_N(u)\right|\right|_{L^2}, \quad \forall\, u \in H^{s_2}.$$

Moreover, for $1 < p < +\infty$, P_N extends to a bounded operator from $L^p(\mathbb{R})$ into itself whose norm does not depends on N.

PROOF OF LEMMA 4.2. The first part follows merely from the very definition of P_N and some well known properties from Fourier analysis (see [**16**]). While for the second part the proof is classical and we refer the reader to [**39**] for more details. □

To highlight the regularity of the global attractor $\mathscr{A}_\alpha$, the strategy consists on splitting the solution as $U(t) = P_N(U) + Q_N(U) = V(t,x) + W(t,x)$ and then, thanks to Lemma 4.2, the regularity of U depends only on the regularity of $W = Q_N(U)$. Thus, we shall focus on the long-time behavior of W, solution for $Q_N(2.1)$ supplemented with initial data $W(0) = Q_N(U(0))$, that will be approximated by a more regular function Z solving the so called **auxiliary problem** that reads

$$Z_t - i(-\Delta)^{\frac{\alpha}{2}} Z + iQ_N\left[|V+Z|^2(V+Z)\right] + \gamma Z + \delta\wp(Z_x) = Q_N(F) \tag{4.2a}$$

$$Z(0) = 0. \tag{4.2b}$$

Now, we state the main result of this subsection.

PROPOSITION 4.3. *There exists $N_0 > 0$ large enough depending only on γ, δ, α and F such that for any $N \geq N_0$, the problem $(4.2a) - (4.2b)$ has a unique local in time solution in $\mathscr{C}(\mathbb{R}_+, \mathbb{H}^\alpha)$ that remains uniformly bounded in $\mathbb{H}^{\frac{\alpha}{2}}$.*

PROOF OF PROPOSITION 4.3. The proof is standard and then sketched for the sake of conciseness. The existence of a local in time solution Z in $\mathscr{C}([0,T^*), \mathbb{H}^\alpha)$ for the problem $(4.2a) - (4.2b)$ is obtained by a fixed point argument on its Duhamel's form thanks to the embedding of $\mathbb{H}^\alpha$ into $\mathbb{L}^\infty$ and the fact that the mapping $\Phi : Z \mapsto Q_M\left(|Z+V|^2(Z+V)\right)$ which is locally lipschitz from $\mathbb{H}^\alpha$ into itself.

Taking the inner product of (4.2a) by $-i(Z_t+\gamma Z)$ we obtain that

$$\frac{1}{2}\frac{d}{dt}\Phi(Z(t)) + \gamma\Phi(Z(t)) = \Psi(Z(t)) \tag{4.3}$$

where we set

$$\begin{aligned}\Phi(Z) = &\left|\left|(-\Delta)^{\frac{\alpha}{4}} Z\right|\right|^2_{\mathbb{L}^2} - \frac{1}{2}\left(|Z+V|^2, |Z+V|^2\right)\\ &+\delta\left((iZ, \wp(Z_x))\right) - 2((iF,Z))\end{aligned} \tag{4.4}$$

$$\begin{aligned}\Psi(Z) =&\frac{\gamma}{2}\left(|Z+V|^2, |Z+V|^2\right) - \gamma((iF,Z))\\ &-\gamma\left(\left(|Z+V|^2(Z+V), V_t+\gamma V\right)\right) \,.\end{aligned} \tag{4.5}$$

For the purpose of establishing upper and lower bounds for $\Phi(Z)$, it should be noted that due to Lemma 2.3, (3.5) and Lemma 4.2 we have

$$||Z||^2_{\mathbb{L}^2} \lesssim \frac{1}{N^\alpha}\left|\left|(-\Delta)^{\frac{\alpha}{4}} Z\right|\right|^2_{\mathbb{L}^2} \tag{4.6}$$

$$||Z||^4_{\mathbb{L}^4} \lesssim \frac{1}{N^{2\alpha-1}}\left|\left|(-\Delta)^{\frac{\alpha}{4}} Z\right|\right|^4_{\mathbb{L}^2} \tag{4.7}$$

$$|((iZ, \wp(Z_x)))| \lesssim \frac{1}{N^{\alpha-1}}\left|\left|(-\Delta)^{\frac{\alpha}{4}} Z\right|\right|^2_{\mathbb{L}^2} \,. \tag{4.8}$$

Now we need the following result

Lemma 4.4. *There exists $C>0$ depending only on γ, δ, α and F such that*

$$||P_N(U)||_{\mathbb{H}^{\frac{\alpha}{2}}} + ||P_N(U_t)||_{\mathbb{H}^{-\frac{\alpha}{2}}} \leq C\,.$$

Proof of Lemma 4.4. Thanks to Lemma 4.2, P_N is uniformly (in N) bounded from $\mathbb{H}^{\frac{\alpha}{2}}$ into itself. Moreover, in accordance with (2.1) one has

$$P_N(U_t) = i(-\Delta)^{\frac{\alpha}{2}}P_N(U) - \delta\wp(P_N(U_x)) - \gamma P_N(U) - iP_N\left[|U|^2U\right] + P_N(F)$$

from which, and the fact that $\mathbb{H}^{\frac{\alpha}{2}}$ is an algebra (Lemma 2.3), we deduce that $P_N(U_t)$ remains uniformly bounded in $\mathbb{H}^{-\frac{\alpha}{2}}$ and the proof is completed. □

Thanks to the Cauchy-Schwarz inequality, (4.6), (4.7), (4.8) and Lemma 4.4 we deduce, for N large enough, the existence of $C_0, C_1>0$ that not depend on N such that

$$\frac{1}{2}\left|\left|(-\Delta)^{\frac{\alpha}{4}} Z\right|\right|^2_{\mathbb{L}^2} - C_0\frac{\left|\left|(-\Delta)^{\frac{\alpha}{4}} Z\right|\right|^4_{\mathbb{L}^2}}{N^{2\alpha-1}} - C_1 \leq \Phi(Z) \leq \frac{3}{2}\left|\left|(-\Delta)^{\frac{\alpha}{4}} Z\right|\right|^2_{\mathbb{L}^2} + C_1 \tag{4.9}$$

Now we derive an upper bound for $\Psi(Z)$. Thanks again to Lemma 2.3 and Lemma 4.2

$$||Z||^2_{\mathbb{L}^\infty} \lesssim \frac{1}{N^{\alpha-1}}\left|\left|(-\Delta)^{\frac{\alpha}{4}} Z\right|\right|^2_{\mathbb{L}^2} \,. \tag{4.10}$$

Independently

$$\begin{aligned}\left|\left(\left(|Z+V|^2(Z+V), V_t\right)\right)\right| &\leq \left|\left|\,|Z+V|^2(Z+V)\right|\right|_{\mathbb{H}^{\frac{\alpha}{2}}} ||V_t||_{\mathbb{H}^{-\frac{\alpha}{2}}}\\ &\lesssim ||Z+V||^2_{\mathbb{L}^\infty}||Z+V||_{\mathbb{H}^{\frac{\alpha}{2}}}||V_t||_{\mathbb{H}^{-\frac{\alpha}{2}}}\end{aligned}$$

which, in accordance with Lemma 4.4 and (4.10), leads to

$$(4.11)\quad |\left(\left(|Z+V|^2(Z+V),V_t\right)\right)| \lesssim \left(1+\frac{\left|\left|(-\Delta)^{\frac{\alpha}{4}}Z\right|\right|_{\mathbb{L}^2}^2}{N^{\alpha-1}}\right)\left(1+\left|\left|(-\Delta)^{\frac{\alpha}{4}}Z\right|\right|_{\mathbb{L}^2}\right).$$

Gathering (4.6), (4.7) and (4.11) it may be deduced, in accordance with the Young inequality, that
(4.12)

$$\Psi(Z) \le C_0 + \frac{\gamma}{4}\left|\left|(-\Delta)^{\frac{\alpha}{4}}Z\right|\right|_{\mathbb{L}^2}^2 + \frac{C_1}{N^{\alpha-1}}\left|\left|(-\Delta)^{\frac{\alpha}{4}}Z\right|\right|_{\mathbb{L}^2}^3 + \frac{C_2}{N^{2\alpha-1}}\left|\left|(-\Delta)^{\frac{\alpha}{4}}Z\right|\right|_{\mathbb{L}^2}^4 .$$

In accordance with (4.3), (4.9) and (4.12), the Gronwall Lemma implies that for N large enough

$$\begin{aligned}\Phi(Z(t)) \le C_0 &+ \frac{C_1}{N^{\alpha-1}}\int_0^t e^{-\gamma(t-s)}\left|\left|(-\Delta)^{\frac{\alpha}{4}}Z(s)\right|\right|_{\mathbb{L}^2}^3 ds\\ &+ \frac{C_2}{N^{2\alpha-1}}\int_0^t e^{-\gamma(t-s)}\left|\left|(-\Delta)^{\frac{\alpha}{4}}Z(s)\right|\right|_{\mathbb{L}^2}^4 ds\end{aligned}$$

As a result, thanks again to (4.9), one easily deduce by classical arguments that

$$\sup_{s\in[0,T^*)}\left|\left|(-\Delta)^{\frac{\alpha}{4}}Z(s)\right|\right|_{\mathbb{L}^2}^2 \le C .$$

This concludes the proof of the current proposition. □

We now prove that Z remains bounded in $\mathbb{H}^\alpha$ with an upper bound that depends on N.

PROPOSITION 4.5. *There exist a reel $K>0$ and $N_0>0$ large enough that depend only on γ, δ, α and F such that for any $N\ge N_0$, the following estimate holds true*

$$\sup_{t\in\mathbb{R}_+}||Z(t)||_{\mathbb{H}^\alpha} \le K\,N^{\frac{\alpha}{2}} .$$

PROOF OF PROPOSITION 4.5. In order to prove that $(-\Delta)^{\frac{\alpha}{2}}Z$ is bounded in $\mathbb{L}^2$ we will prove equivalently that $Y=Z_t$ is bounded in $\mathbb{L}^2$. To do this we will establish some a priori estimates on $||Y||_{\mathbb{H}^{\frac{\alpha}{2}}}$ which can be proved rigorously using some smooth approximation argument. For the sake of simplicity we denote $\Lambda = W+V$.

We differentiate the system $(4.2a)$ with respect to t then we take the scalar product of the resulting equation by $-i(Y_t+\gamma Y)$ lead, by mere computations, to

$$(4.13)\qquad \frac{1}{2}\frac{d}{dt}\Upsilon(Y(t)) + \gamma\Upsilon(Y(t)) = \Theta(Y(t))$$

where

$$(4.14)\quad \begin{aligned}\Upsilon(Y) = &\left|\left|(-\Delta)^{\frac{\alpha}{4}}Y\right|\right|_{\mathbb{L}^2}^2 - \left(|\Lambda|^2,|Y|^2\right) - 2\left|\left|\Re e(\overline{\Lambda}.Y)\right|\right|_{\mathbb{L}^2}^2 + \delta\left((iY,\wp(Y_x))\right)\\ &-2\left(|\Lambda|^2,\Re e(\overline{Y}.V_t)\right) - 4\left(\Re e(\overline{\Lambda}.V_t),\Re e(\overline{\Lambda}.Y)\right)\end{aligned}$$

and

$$\begin{aligned}\Theta(Y) =& -2\gamma\left(\Re e(\overline{\Lambda}.V_t), \Re e(\overline{\Lambda}.Y)\right) - (1+\gamma)\left(|\Lambda|^2, \Re e(\overline{Y}.V_t)\right)\\ &- 2\left(\Re e(\overline{\Lambda}.Y), \Re e(\overline{Y}.\Lambda_t)\right) - \left(|Y|^2, \Re e(\overline{\Lambda}.\Lambda_t)\right)\\ &- 2\left(\Re e(\overline{\Lambda}.V_t), \Re e(\overline{Y}.\Lambda_t)\right) - 2\left(\Re e(\overline{\Lambda}.Y), \Re e(\overline{V_t}.\Lambda_t)\right)\\ &- 2\left(\Re e(\overline{\Lambda}.\Lambda_t), \Re e(\overline{Y}.V_t)\right) - 2\left(\Re e(\overline{\Lambda}.V_{tt}), \Re e(\overline{\Lambda}.Y)\right).\end{aligned} \tag{4.15}$$

First of all, thanks to Lemma 2.3, the estimate (3.5), Lemma 4.2 and since Λ is uniformly bounded in $\mathbb{H}^{\frac{\alpha}{2}}$, the following estimates hold

$$\left(|\Lambda|^2, |Y|^2\right) + \left|\left|\Re e(\overline{\Lambda}.Y)\right|\right|_{\mathbb{L}^2}^2 \lesssim ||\Lambda||_{\mathbb{L}^4}^2 ||Y||_{\mathbb{L}^4}^2 \lesssim \frac{1}{N^{\alpha-\frac{1}{2}}}\left|\left|(-\Delta)^{\frac{\alpha}{4}}Y\right|\right|_{\mathbb{L}^2}^2. \tag{4.16}$$

$$|((iY, \wp(Y_x)))| \lesssim \frac{1}{N^{\alpha-1}}\left|\left|(-\Delta)^{\frac{\alpha}{4}}Y\right|\right|_{\mathbb{L}^2}^2. \tag{4.17}$$

$$\begin{aligned}&\left|\left(|\Lambda|^2, \Re e(\overline{Y}.V_t)\right)\right| + \left|\left(\Re e(\overline{\Lambda}.V_t), \Re e(\overline{\Lambda}.Y)\right)\right|\\ &\lesssim ||\Lambda||_{\mathbb{L}^\infty}^2 ||Y||_{\mathbb{L}^2} ||V_t||_{\mathbb{L}^2}\\ &\lesssim \left|\left|(-\Delta)^{\frac{\alpha}{4}}Y\right|\right|_{\mathbb{L}^2}.\end{aligned} \tag{4.18}$$

As a result of (4.16), (4.17), (4.18) and (4.14) it follows, for N large enough, that

$$\frac{1}{2}\left|\left|(-\Delta)^{\frac{\alpha}{4}}Y\right|\right|_{\mathbb{L}^2}^2 - C \leq \Upsilon(Y) \leq \frac{3}{2}\left|\left|(-\Delta)^{\frac{\alpha}{4}}Y\right|\right|_{\mathbb{L}^2}^2 + C \tag{4.19}$$

where $C > 0$ does not depends on N.

We will now proceed to determinate an upper bound for $\Theta(Y)$.

Firstly, thanks to Lemma 4.4 and Proposition 4.3 one easily obtain the existence of $K_0 > 0$ depending only on F, γ, δ and α such that

$$\sup_{t\in\mathbb{R}_+} ||\Lambda(t)||_{\mathbb{H}^{-\frac{\alpha}{2}}} \leq K_0. \tag{4.20}$$

Now by the use of (4.20), Lemma 2.3, Lemma 4.2 and in accordance with Proposition 4.3, we have

$$\left|\left(\Re e(\overline{\Lambda}.Y), \Re e(\overline{Y}.\Lambda_t)\right)\right| + \left|\left(|Y|^2, \Re e(\overline{\Lambda}.\Lambda_t)\right)\right|$$

$$\lesssim \left|\left|\,|Y|^2\Lambda\,\right|\right|_{\mathbb{H}^{\frac{\alpha}{2}}} ||\Lambda_t||_{\mathbb{H}^{-\frac{\alpha}{2}}} \tag{4.21}$$

$$\lesssim ||Y||_{\mathbb{L}^\infty}\left(||Y||_{\mathbb{L}^\infty} + \left|\left|(-\Delta)^{\frac{\alpha}{4}}Y\right|\right|_{\mathbb{L}^2}\right) \tag{4.22}$$

$$\lesssim \frac{1}{N^{\frac{\alpha-1}{2}}}\left|\left|(-\Delta)^{\frac{\alpha}{4}}Y\right|\right|_{\mathbb{L}^2}^2. \tag{4.23}$$

Independently, recalling that $\Lambda_t = Y + V_t$ we easily check that

$$\begin{aligned}&\left(\Re e(\overline{\Lambda}.V_t), \Re e(\overline{Y}.\Lambda_t)\right) + \left(\Re e(\overline{\Lambda}.Y), \Re e(\overline{V_t}.\Lambda_t)\right) + \left(\Re e(\overline{\Lambda}.\Lambda_t), \Re e(\overline{Y}.V_t)\right)\\ &= \left(|Y|^2, \Re e(\overline{\Lambda}.V_t)\right) + 2\left(\Re e(\overline{Y}.V_t), \Re e(\overline{\Lambda}.Y)\right)\\ &+ \left(|V_t|^2, \Re e(\overline{\Lambda}.Y)\right) + 2\left(\Re e(\overline{\Lambda}.V_t), \Re e(\overline{Y}.V_t)\right).\end{aligned} \tag{4.24}$$

Due to Lemma 2.3 and Lemma 4.2 one has

$$\begin{aligned}\left|\left(|V_t|^2, \Re e(\overline{\Lambda}.Y)\right)\right| + \left|\left(\Re e(\overline{\Lambda}.V_t), \Re e(\overline{Y}.V_t)\right)\right| &\lesssim ||Y||_{\mathbb{L}^\infty}||\Lambda||_{\mathbb{L}^\infty}||V_t||_{\mathbb{L}^2}^2\\ &\lesssim N^{\frac{\alpha+1}{2}}\left|\left|(-\Delta)^{\frac{\alpha}{4}}Y\right|\right|_{\mathbb{L}^2}\end{aligned} \tag{4.25}$$

Hence, in the light of (4.23) and (4.24) and (4.25), we deduce that

$$\begin{aligned}&\left|\left(\Re e(\overline{\Lambda}.V_t),\Re e(\overline{Y}.\Lambda_t)\right)+\left(\Re e(\overline{\Lambda}.Y),\Re e(\overline{V_t}.\Lambda_t)\right)\right|\\ &+\left|\left(\Re e(\overline{\Lambda}.\Lambda_t),\Re e(\overline{Y}.V_t)\right)\right|\lesssim N^{\frac{\alpha+1}{2}}\left|\left|(-\Delta)^{\frac{\alpha}{4}}Y\right|\right|_{\mathbb{L}^2}+\frac{\left|\left|(-\Delta)^{\frac{\alpha}{4}}Y\right|\right|_{\mathbb{L}^2}^2}{N^{\frac{\alpha-1}{2}}}\end{aligned} \tag{4.26}$$

Now it only remains to bound $\left(\Re e(\overline{\Lambda}.V_{tt}),\Re e(\overline{\Lambda}.Y)\right)$. Firstly, observe that V_t satisfies $V_{tt}=iD^\alpha V_t-\gamma V_t-\delta\wp(V_{tx})-iP_M\left[|U|^2U_t+2\Re e(\overline{U}.U_t)U\right]$. Consequently, due to the fact that $\mathbb{H}^{\frac{\alpha}{2}}$ is an algebra,

$$\text{either}\quad P_M\left[|U|^2U_t\right]\quad\text{or}\quad P_M\left[\Re e(\overline{U}.U_t)U\right]$$

are uniformly bounded in $\mathbb{H}^{-\frac{\alpha}{2}}$. Hence, thanks to Lemma 4.2 and Lemma 4.4

$$||V_{tt}||_{\mathbb{H}^{-\frac{\alpha}{2}}}\lesssim\left|\left|(-\Delta)^{\frac{\alpha}{2}}V_t\right|\right|_{\mathbb{H}^{-\frac{\alpha}{2}}}+||V_t||_{\mathbb{H}^{-\frac{\alpha}{2}}}+||\partial_xV_t||_{\mathbb{H}^{-\frac{\alpha}{2}}}\lesssim N^\alpha\,. \tag{4.27}$$

This enables us to have

$$\left|\left(\Re e(\overline{\Lambda}.V_{tt}),\Re e(\overline{\Lambda}.Y)\right)\right|\lesssim\left|\left|\,|\Lambda|^2Y\,\right|\right|_{\mathbb{H}^{\frac{\alpha}{2}}}||V_{tt}||_{\mathbb{H}^{-\frac{\alpha}{2}}}\lesssim N^\alpha\left|\left|(-\Delta)^{\frac{\alpha}{4}}Y\right|\right|_{\mathbb{L}^2}\,. \tag{4.28}$$

Gathering (4.16), (4.18), (4.23), (4.26) and (4.28), it may deduced in accordance with the Young inequality that for N large enough

$$\Theta(Y)\leq\frac{\gamma}{4}\left|\left|(-\Delta)^{\frac{\alpha}{4}}Y\right|\right|_{\mathbb{L}^2}^2+C\,N^{2\alpha}$$

which, with (4.19), leads to

$$\frac{d}{dt}\Upsilon(Y(t))+\gamma\Upsilon(Y(t))\leq C\,N^{2\alpha}\,.$$

This completes the proof thanks to Gronwall's Lemma, (4.19) and Lemma 4.2. ☐

4.2. Proof of Theorem 4.1. To begin with, we proceed to the large time comparison between $U=W+V$ and $\Lambda=Z+V$.

PROPOSITION 4.6. *There exist $N_0,C>0$ depending only on γ, δ, α and F such that for any $N\geq N_0$ and for all $t\geq0$,*

$$||U(t)-\Lambda(t)||_{\mathbb{H}^{\frac{\alpha}{2}}}=||W(t)-Z(t)||_{\mathbb{H}^{\frac{\alpha}{2}}}\leq C\,e^{-\gamma t}\,.$$

PROOF OF PROPOSITION 4.6. The proof is classical and then details are omitted. For the sake of simplicity we shall denote $\Theta(t)=U(t)-\Lambda(t)=W(t)-Z(t)$. Then Θ satisfies the following identity that reads

$$\frac{1}{2}\frac{d}{dt}\Xi(\Theta(t))+\gamma\Xi(\Theta(t))=\Gamma(\Theta(t)) \tag{4.29}$$

where we denote

$$\Xi(\Theta)=\left|\left|(-\Delta)^{\frac{\alpha}{4}}\Theta\right|\right|_{\mathbb{L}^2}^2+\delta\left((i\Theta,\wp(\Theta_x))\right)-\left(|U|^2,|\Theta|^2\right)-2\left|\left|\Re e(\overline{\Lambda}.\Theta)\right|\right|_{\mathbb{L}^2}^2 \tag{4.30}$$

and

$$\begin{aligned}\Gamma(\Theta(t))=&\gamma\left(|\Theta|^2,\Re e(\overline{\Lambda}.\Theta)\right)+\left(|\Theta|^2,\Re e(\overline{\Lambda}.\Theta_t)\right)-\left(|\Theta|^2,\Re e(\overline{U}.U_t)\right)\\ &-2\left(\Re e(\overline{\Lambda}.\Theta),\Re e(\overline{\Theta}.\Lambda_t)\right)\,.\end{aligned} \tag{4.31}$$

Since U_t and Λ_t remain uniformly bounded in $\mathbb{H}^{-\frac{\alpha}{2}}$, then applying Lemma 2.3 and Lemma 4.2 by means of which, using identical computations as (4.16) and (4.23), we conclude on the one hand that for N large enough

$$\frac{1}{2}\left|\left|(-\Delta)^{\frac{\alpha}{4}}\Theta\right|\right|_{\mathbb{L}^2}^2\leq\Xi(\Theta)\leq\frac{3}{2}\left|\left|(-\Delta)^{\frac{\alpha}{4}}\Theta\right|\right|_{\mathbb{L}^2}^2 \tag{4.32}$$

and on the other hand

$$\Gamma(\Theta) \lesssim \frac{\left|\left|(-\Delta)^{\frac{\alpha}{4}}\Theta\right|\right|_{\mathbb{L}^2}^2}{N^{\frac{\alpha-1}{2}}} \leq \frac{\gamma}{4}\left|\left|(-\Delta)^{\frac{\alpha}{4}}\Theta\right|\right|_{\mathbb{L}^2}^2 . \tag{4.33}$$

Gathering (4.32), (4.33) and (4.29) completes the proof thanks to the Gronwall Lemma and once again to (4.32). □

Propositions 4.6, 4.3 and 4.5 enable us to prove, in identical manner as in [**1**] or [**14**], that $\mathscr{A}_\alpha$ is a bounded subset of $\mathbb{H}^\alpha$. The proof of the compactness of the global attractor $\mathscr{A}_\alpha$ in $\mathbb{H}^\alpha$, based on the famous J. Ball's argument, is standard and similar to that in [**14**] to which we refer the reader. We omit it for the sake of conciseness and the proof of Theorem 4.1 is therefore completed.

5. Fractal dimension of the global attractor

Later on, this article will examine in some details the specific concept of the fractal dimension of the global attractor. For that purpose we use a new idea recently introduced in [**3**]. Firstly, for the sake of completeness we start by recalling the definition of the fractal dimension.

Definition 5.1. The fractal dimension of a compact subset $\mathscr{M}$ of a metric space $\mathscr{H}$ is defined by $d_f(\mathscr{M}) = \lim\limits_{\epsilon\to 0} \dfrac{\ln N(\mathscr{M},\epsilon)}{\ln\left(\frac{1}{\epsilon}\right)}$, where $N(\mathscr{M},\epsilon)$ denotes the minimal number of closed balls of the radius ϵ which cover the set $\mathscr{M}$.

We now state

Theorem 5.2. *Let $F \in \mathbb{L}^2$. Then the compact global attractor $\mathscr{A}_\alpha$ possesses a finite fractal dimension in $\mathbb{H}^{\frac{\alpha}{2}}$.*

5.1. Some helpful tools. To begin with, we recall a merely deduced result from that given in [**10**].

Theorem 5.3. *Let X be a Banach space and M be a bounded closed set in X. Assume that there exists a mapping $V : M \longrightarrow X$ such that*

(1) *$M \subseteq V(M)$. Moreover, V is Lipschitz on M, i.e, there exists $L > 0$ such that for all $u_1, u_2 \in M$, $||V(u_1) - V(u_2)||_X \leq L\,||u_1 - u_2||_X$.*
(2) *There exist compact semi-norm $||\,.\,||_Y$ on X (i.e: $X \hookrightarrow Y$ is compact) such that $\forall\, u_1, u_2 \in M$,*

$$||V(u_1) - V(u_2)||_X \leq \delta\,||u_1 - u_2||_X + K[||u_1 - u_2||_Y + ||V(u_1) - V(u_2)||_Y] .$$

where $0 < \delta < 1$ and $K > 0$ are constants.

Then M is a compact subset of X and has a finite fractal dimension.

In what follows, we state an important result (see [**3**]) that will enables us to apply Theorem 5.3.

Lemma 5.4. *Let $\mathscr{C}$ be a compact subset of $\mathbb{L}^2(\mathbb{R})$. Then for every $\epsilon > 0$ there exists $R = R(\epsilon) > 0$ such that for all $\vartheta \in \mathscr{C}$ and for all $u \in H^{\frac{\alpha}{2}}$ the following estimate holds*

$$\int_{\mathbb{R}} |\vartheta|\,|u|^2 dx \leq \epsilon ||u||_{H^{\frac{\alpha}{2}}}^2 + R\,||u||_{L^2([-R,R])}^2 .$$

PROOF OF LEMMA 5.4. To begin with, for arbitrary $R > 0$ we write

$$\int_{\mathbb{R}} |\vartheta|\,|u|^2 dx = \int_{[-R,R]\cap\{|\vartheta|\le R\}} |\vartheta|\,|u|^2 dx + \int_{[-R,R]\cap\{|\vartheta|>R\}} |\vartheta|\,|u|^2 dx + \int_{\{|x|>R\}} |\vartheta|\,|u|^2 dx$$

we obtain by applying the Hölder inequality that

$$\int_{\mathbb{R}} |\vartheta|\,|u|^2 dx \le R\,||u||^2_{L^2([-R,R])} + \left(||\vartheta||_{L^2(\{|\vartheta|>R\})} + ||\vartheta||_{L^2(\{|x|>R\})}\right)\,||u||^2_{L^4(\mathbb{R})}$$

by means of which and Lemma 2.3, we deduce that

$$\int_{\mathbb{R}} |\vartheta|\,|u|^2 dx \le R\,||u||^2_{L^2([-R,R])} + C\,\left(||\vartheta||_{\mathbb{L}^2(\{|\vartheta|>R\})} + ||\vartheta||_{\mathbb{L}^2(\{|x|>R\})}\right)\,||u||^2_{H^{\frac{\alpha}{2}}}\,.$$

Since $\mathscr{C}$ is precompact in $L^2(\mathbb{R})$ then there exist finitely many points $\vartheta_1, ..., \vartheta_n \in \mathscr{C}$ such that $\mathscr{C} \subseteq \bigcup_{k=1}^{n} B_k(\vartheta_k, \epsilon)$ where

$$B_k(\vartheta_k, \epsilon) = \{\vartheta \in \mathbb{L}^2(\mathbb{R}) \text{ such that } ||\vartheta - \vartheta_k||_{L^2(\mathbb{R})} \le \epsilon\}\,.$$

Moreover, for a fixed $1 \le k \le n$ and for every $\epsilon > 0$, we deduce that there exists $R_k = R_k(\vartheta_k, \epsilon) > 0$ such that for all $\vartheta \in B_k(\vartheta_k, \epsilon)$ we have

$$\begin{aligned}\int_{\mathbb{R}} |\vartheta|\,|u|^2 dx &\le \int_{\mathbb{R}} |\vartheta_k|\,|u|^2 dx + \int_{\mathbb{R}} |\vartheta - \vartheta_k|\,|u|^2 dx \\ &\lesssim R_k ||u||^2_{\mathbb{L}^2([-R_k,R_k]} + \epsilon ||u||^2_{H^{\frac{\alpha}{2}}}\,.\end{aligned}$$

Choosing $R = \max_{1\le k\le n} R_k$ achieves the proof of the lemma. ☐

5.2. Proof of Theorem 5.2. In order to check the assumptions in Theorem 5.3, one only has to prove

PROPOSITION 5.5. *There exist $t^* > 0$ and $R^* > 0$ that depend on γ, δ, α and F such that for all U_0, $V_0 \in \mathscr{A}_\alpha$,*

$$(5.1)\quad ||\mathbb{S}_\alpha(t^*)U_0 - \mathbb{S}_\alpha(t^*)V_0||^2_{\mathbb{H}^{\frac{\alpha}{2}}} \le \frac{1}{2}\,||U_0 - V_0||^2_{\mathbb{H}^{\frac{\alpha}{2}}} + L\,||U_0 - V_0||^2_{\mathbb{L}^2([-2R^*,2R^*])}\,,$$

where $L > 0$ is a non negative reel constant depending only on t^* and R^*.

PROOF OF PROPOSITION 5.5. Firstly, let $U_0, V_0 \in \mathscr{A}_\alpha$ and we denote for the sake of simplicity $Z(t) = \mathbb{S}(t)U_0 - \mathbb{S}(t)V_0 = U(t) - V(t)$ that satisfies

$$(5.2)\qquad Z_t - i(-\Delta)^{\frac{\alpha}{2}} Z + i(|U|^2U - |V|^2V) + \gamma Z + \delta\wp(Z_x) = 0$$

$$(5.3)\qquad Z(t=0) = Z_0 = U_0 - V_0 \quad \in \mathscr{A}_\alpha\,.$$

LEMMA 5.6. *There exist $R_0 > 0$ and $C_1, C_2 > 0$ depending only on γ, δ, α and $\mathscr{A}_\alpha$ such that*

$$(5.4)\quad ||Z(t)||^2_{\mathbb{H}^{\frac{\alpha}{2}}} \le C_1\,||Z(0)||^2_{\mathbb{H}^{\frac{\alpha}{2}}}\,e^{-\gamma t} + C_2 R_0 \int_0^t e^{-\gamma(t-s)}\,||Z(s)||^2_{\mathbb{L}^2([-R_0,R_0])}\,ds\,.$$

PROOF OF LEMMA 5.6. On the one hand, by the scalar product of (5.2) by Z we obtain

$$\frac{1}{2}\frac{d}{dt}||Z||^2_{\mathbb{L}^2}+\gamma||Z||^2_{\mathbb{L}^2}=\left(\Re e\left[(\overline{U+V}).Z\right],\Im m(V.\overline{Z})\right)\,. \tag{5.5}$$

On the other hand, the scalar product of (5.2) by $-i(Z_t+\gamma Z)$ leads to

$$\frac{1}{2}\frac{d}{dt}J(Z)+\gamma J(Z)=K(Z) \tag{5.6}$$

where

$$J(Z)=\left|\left|(-\Delta)^{\frac{\alpha}{4}}Z\right|\right|^2_{\mathbb{L}^2}+\delta\left((iZ,\wp(Z_x))\right)-\left(|U|^2,|Z|^2\right)-2\left|\left|\Re e(\overline{V}.Z)\right|\right|^2_{\mathbb{L}^2} \tag{5.7}$$

and

$$\begin{aligned}K(Z)=&\gamma\left(|Z|^2,\Re e(\overline{V}.Z)\right)+\left(|Z|^2,\Re e(\overline{V}.Z_t)\right)-\left(|Z|^2,\Re e(\overline{U}.U_t)\right)\\&-2\left(\Re e(\overline{V}.Z),\Re e(\overline{Z}.Z_t)\right)\,.\end{aligned} \tag{5.8}$$

since U an V are uniformly bounded in $\mathbb{H}^{\frac{\alpha}{2}}$, we deduce from Lemma 2.3 and the estimate (3.5) the existence of $C_0>1$ depending only on γ, δ, α and $\mathscr{A}_\alpha$ such that

$$\left(|U|^2,|Z|^2\right)+2\left|\left|\Re e(\overline{V}.Z)\right|\right|^2_{\mathbb{L}^2}+\delta\left|((iZ,\wp(Z_x)))\right|\leq C_0||Z||^2_{\mathbb{L}^2}\,. \tag{5.9}$$

Introducing

$$\Psi(Z)=J(Z)+(1+C_0)||Z||^2_{\mathbb{L}^2} \tag{5.10}$$

$$\Phi(Z)=K(Z)+(1+C_0)\left(\Re e\left[(\overline{U+V}).Z\right],\Im m(V.\overline{Z})\right) \tag{5.11}$$

we obtain, in accordance with (5.6), that

$$\frac{1}{2}\frac{d}{dt}\Psi(Z)+\gamma\Psi(Z)=\Phi(Z) \tag{5.12}$$

Hence, it may be easily deduced from (5.9) that on the one hand

$$||Z||^2_{\mathbb{H}^{\frac{\alpha}{2}}}\leq\Psi(Z)\leq(1+C_0)||Z||^2_{\mathbb{H}^{\frac{\alpha}{2}}} \tag{5.13}$$

and on the other hand

$$\Phi(Z)\leq C\int_{\mathbb{R}}\left(|U|^2+|V|^2\right)|Z|^2dx+\int_{\mathbb{R}}\left(|U_t|+|V_t|\right)|Z|^2dx\,. \tag{5.14}$$

On the one hand, thanks to Theorem 4.1 and Lemma 2.3, $\{|\vartheta|^2\,,\ \vartheta\in\mathscr{A}_\alpha\}$ is precompact in $\mathbb{L}^2$. On the other hand, thanks again to Lemma 2.3, the map $U\mapsto F-i(-\Delta)^{\frac{\alpha}{2}}U-i|U|^2U-\delta\wp(U_x)$ is continuous from $\mathbb{H}^\alpha$ into $\mathbb{L}^2$. Thus, we deduce from the equation (2.1) and Theorem 4.1 that $\{\vartheta_t\,,\ \vartheta\in\mathscr{A}_\alpha\}$, is a precompact subset of $\mathbb{L}^2$ where ϑ denotes either U or V.

Hence, in accordance with (5.14) and by the use of Lemma 5.4 we obtain the existence of $R_0>0$ such that

$$|\Phi(z)|\leq\frac{\gamma}{2}||Z||^2_{\mathbb{H}^{\frac{\alpha}{2}}}+C\,R_0||Z||^2_{\mathbb{L}^2([-R_0,R_0])}\,.$$

This implies, thanks to (5.13), that

$$\Phi(z)\leq\frac{\gamma}{2}\Psi(z)+C\,R_0||Z||^2_{\mathbb{L}^2([-R_0,R_0])}\,. \tag{5.15}$$

Gathering (5.12) and (5.15) achieves the proof of the lemma thanks again to (5.13) and the Gronwall Lemma. □

In the next, we shall focus on the right hand side of (5.4).

LEMMA 5.7. *Let $R>0$. Then there exist C, c_1 and $c_2>0$ depending only on γ, α, δ and $\mathscr{A}_\alpha$ such that*

$$\begin{aligned}||Z(t)||^2_{\mathbb{L}^2([-R,R])} \leq & ||Z(0)||^2_{\mathbb{L}^2([-2R,2R])}e^{c_1 t} \\ & + C\, R_0 e^{c_2 t}\left(\frac{1}{R^{2\alpha}}+\frac{1}{R^2}\right)||Z(0)||^2_{\mathbb{H}^{\frac{\alpha}{2}}}\end{aligned} \tag{5.16}$$

where R_0 still denotes the fixed nonnegative reel given by Lemma 5.6.

PROOF OF LEMMA 5.7. Let θ be a cut-off smooth function such that $\theta(x)=1$ if $|x|\leq 1$ and $\theta(x)=0$ if $|x|\geq 2$, $x\in\mathbb{R}$. Now consider for a given $R>0$, $\theta_R : x\mapsto \theta\left(\frac{x}{R}\right)$, $x\in\mathbb{R}$.

We multiply (5.2) by θ_R then we make the scalar product of the resultant equation by $\theta_R Z$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\theta_R Z||^2_{\mathbb{L}^2} + \gamma||\theta_R Z||^2_{\mathbb{L}^2} = & \left(\Re e[(\overline{U+V})\theta_R Z], \Im m[V.(\overline{\theta_R Z})]\right) \\ & + \left(\left(i\theta_R(-\Delta)^{\frac{\alpha}{2}}Z, \theta_R Z)\right]\right) - \delta\left((\wp(\theta_R Z_x), \theta_R Z)\right).\end{aligned} \tag{5.17}$$

Observe that

$$((\wp(\theta_R Z_x), \theta_R Z)) = -\frac{1}{R}((\wp(\theta_R Z), \theta'_R Z)).$$

Hence, since U and V remain uniformly bounded in $\mathbb{H}^{\frac{\alpha}{2}}$ which is continuously embedded in $\mathbb{L}^\infty$, we infer from (5.17) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\theta_R Z||^2_{\mathbb{L}^2} + \gamma||\theta_R Z||^2_{\mathbb{L}^2} \lesssim & ||\theta_R Z||^2_{\mathbb{L}^2} + \frac{1}{R}||Z||_{\mathbb{L}^2}||\theta_R Z||_{\mathbb{L}^2} \\ & + \left|\left|\theta_R(-\Delta)^{\frac{\alpha}{2}}Z - (-\Delta)^{\frac{\alpha}{2}}(\theta_R Z)\right|\right|_{\mathbb{L}^2}||\theta_R Z||_{\mathbb{L}^2}\end{aligned} \tag{5.18}$$

Thanks to Lemma 2.5, the estimate (5.18) becomes

$$\begin{aligned}& \frac{1}{2}\frac{d}{dt}||\theta_R Z||^2_{\mathbb{L}^2} + \gamma||\theta_R Z||^2_{\mathbb{L}^2} \\ & \leq C_1||\theta_R Z||^2_{\mathbb{L}^2} + C_2\left(\frac{1}{R^\alpha}+\frac{1}{R}\right)||Z||_{\mathbb{L}^2}||\theta_R Z||_{\mathbb{L}^2}.\end{aligned} \tag{5.19}$$

Hence, thanks to the Young inequality, we obtain from (5.19) that

$$\frac{d}{dt}||\theta_R Z||^2_{\mathbb{L}^2} \leq C_1||\theta_R Z||^2_{\mathbb{L}^2} + C_2\left(\frac{1}{R^{2\alpha}}+\frac{1}{R^2}\right)||Z||^2_{\mathbb{H}^{\frac{\alpha}{2}}}. \tag{5.20}$$

Independently, we infer from the scalar product of (5.2) by Z we obtain

$$\frac{1}{2}\frac{d}{dt}||Z||^2_{\mathbb{L}^2} + \gamma||Z||^2_{\mathbb{L}^2} = \left(\Re e([\overline{U+V}]Z), \Im m[V.(\overline{Z})]\right) \lesssim ||Z||^2_{\mathbb{L}^2}.$$

This leads, thanks to Gronwall's Lemma and in accordance with Lemma 5.6, to

$$||Z(t)||^2_{\mathbb{H}^{\frac{\alpha}{2}}} \leq C_1||Z(0)||^2_{\mathbb{H}^{\frac{\alpha}{2}}}e^{-\gamma t} + C_2 R_0 e^{Ct}||Z(0)||^2_{\mathbb{L}^2}. \tag{5.21}$$

Gathering (5.20) and (5.21) achieves the proof thanks to the Gronwall Lemma. □

By means of Lemma 5.6 and Lemma 5.7, we can deduce that

$$\begin{aligned}||Z(t)||^2_{\mathbb{H}^{\frac{\alpha}{2}}} \leq & C_1 e^{-\gamma t} + C_2\, R_0\left(\frac{1}{R^{2\alpha}}+\frac{1}{R^2}\right)e^{Ct}||Z(0)||^2_{\mathbb{H}^{\frac{\alpha}{2}}} \\ & + C_3 R_0 e^{C_0 t}||Z(0)||^2_{\mathbb{L}^2([-2R,2R])}.\end{aligned} \tag{5.22}$$

Recalling that R_0 is given by Lemma 5.6, we choose

$$t^* > 0 \text{ such that } C_1 e^{-\gamma t^*} \leq \frac{1}{4}$$

then $R^* > R_0$ large enough such that

$$C_2\, R_0 \left(\frac{1}{(R^*)^{2\alpha}} + \frac{1}{(R^*)^2} \right) e^{Ct^*} \leq \frac{1}{4}$$

achieves the proof of the current proposition as well as the proof of Theorem 5.2 by applying Theorem 5.3 with $V = \mathbb{S}_\alpha(t^*)$, $M = \mathscr{A}_\alpha$ and $Y = \mathbb{L}^2([-2R^*, 2R^*])$. $\square$

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