Dynamics of PDE, Vol.18, No.2, 81-100, 2021

Asymptotic behavior of global solutions to one-dimension quasilinear wave equations

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Communicated by S.-T. Yau, received November 25, 2020.

ABSTRACT. The asymptotic behavior of solutions is a significant subject in the theory of wave equations. In this paper we are concerned with the asymptotic behavior of the unique global solution to the Cauchy problem for one-dimension quasilinear wave equations with null conditions. By applying the small-data-global-existence result and exploiting the strength of weights, we not only provide sharper convergence from the quasilinear case to the linear case but also study the rigidity aspect of the scattering problem for quasilinear waves.

CONTENTS

1. Introduction

It is an important issue and has a long history to study the long time behavior of solutions to nonlinear wave equations [**3**, **23**, **24**], which arise naturally in many physical fields. For quasilinear wave equations in one space dimension, it is known from [**21**, **22**, **28**] that small data lead to global solutions if the null condition is satisfied. This paper is devoted to the further study of the asymptotic behavior of global solutions based on the small-data-global-existence result.

The small-data-global-existence type results of nonlinear wave equations have attracted considerable attention in the past four decades. The approach is to use the

1991 *Mathematics Subject Classification.* Primary 35L05; Secondary 35L72, 35B40.

Key words and phrases. Quasilinear wave equation, null condition, weight function, asymptotic behavior.

decay mechanism of linear waves and treat the nonlinear problem as a perturbation of the linear case. It is well-known that d-dimension linear waves enjoy a uniform decay rate $(1+t)^{-\frac{d-1}{2}}$. In space dimensions at least four, i.e., $d \geqslant 4$, this decay rate is integrable in time and therefore small-data-global-existence type theorems hold for generic quadratic nonlinearities [13]. By contrast, in lower space dimensions, the slower decay rate fails to be integrable in time. In such a case, the nonlinearity controls the behavior of the solution, which may result in the blow up of the solution in finite time with the lifespan explicitly determined by the initial data; see, for example, [9, 16]. A bit later, Klainerman [14] introduced the celebrated null condition on the nonlinearity to compensate for the slower decay rate. Based on some special cancellations provided by this condition, Christodoulou [5] and Klainerman [15] independently proved the small-data-global-existence results for three-dimension quasilinear wave equations with null conditions. More precisely, regarding methods of proof, Christodoulou adopted the conformal method while Klainerman adopted the commuting vector field method. We point out that alternative energy estimates for the three dimensional case are given in [18,19] which use less of the special structure and apply to a more general class of equations. Along this line, Alinhac [2] found the small-data-global-existence result still hold for a certain class of two-dimension quasilinear wave equations with a more restricted type of null conditions. Then some thorough studies are made by Katayama [11] and Zha [27] independently. As we have seen, the small data theory for quasilinear wave equations in space dimensions at least two is rather classical, for which we also refer the readers to [3, 6, 8, 10, 12, 23–25] with references therein for more details.

It is worth mentioning that the proofs above are based on the decay mechanism of linear waves. However, one-dimension linear waves do not decay. Nevertheless, there is a new decay mechanism in one space dimension case that the interaction of waves with different speeds will lead to the decay of nonlinear terms. To display this mechanism, Luli, Yang and Yu [21] developed a kind of weighted energy estimate with positive weights and thus proved the small-data-global-existence result for one-dimension semilinear wave equations with null conditions. Their result indeed improves a former one of Nakamura [22] with limited decay information on the solution, which is based on an integrated local energy estimate, i.e., a spacetime integral with negative weights. Recently, Zha [28] has extended such result to the small-data quasilinear setting and proved that the global solution is asymptotically free in the energy sense. Here we refer the readers to Chapter 6 in [11] for several definitions and examples. Other contributions to the asymptotic behavior of solutions can be found in the works such as [1, 4, 7, 8, 12, 20, 26]. Among these studies, Lindblad and Tao [20] proved that finite energy solutions to one-dimension defocusing nonlinear wave equation enjoy an average L^∞ decay and thus behave differently from linear solutions. We remark here that their result has been improved by Wei and Yang [26]. Furthermore, Abbrescia and Wong [1] showed the global stability of traveling wave solutions to one-kind of one-dimension variational and scalar quasilinear wave equations with cubic nonlinearity satisfying null conditions, based on L^∞ estimates. Later on, Cha and Shao [4] showed the global stability of traveling wave solutions to any systems of quasilinear wave equations satisfying null conditions. We point out that the result in [4] is much sharper then the corresponding one in [1]. It is natural to see from these results that the asymptotic behavior of solutions has not been elucidated very much in the one-dimension case.

The aim of the present paper is to further study the global-in-time behavior of small-data smooth solutions from two standpoints. One is to provide sharper convergence from quasilinear waves to linear waves (free waves), the other is to prove the rigidity from infinity for quasilinear waves.

1.1. First sight of asymptotic behavior: compared with the linear case. For this purpose, we first need to review several concepts. On $\mathbb{R}^{1+1}$, the null coordinates (ξ, η) are defined as

$$\xi = \frac{t+x}{2}, \quad \eta = \frac{t-x}{2}. \tag{1.1}$$

In the null coordinates, we have two null vector fields

$$\partial_\xi = \partial_t + \partial_x, \quad \partial_\eta = \partial_t - \partial_x. \tag{1.2}$$

For brevity, we denote $u_\xi = \partial_\xi u$ and $u_\eta = \partial_\eta u$. In addition, we will use the weight functions in a uniform manner, that is,

$$\langle \xi \rangle = \left(1 + |\xi + a|^2\right)^{\frac{1}{2}}, \quad \langle \eta \rangle = \left(1 + |\eta + a|^2\right)^{\frac{1}{2}}, \tag{1.3}$$

where $a \in \mathbb{R}$ is a constant to be determined in Section 4 and can be taken as $a = a_0 = 0$ for now. We call a the position parameter which indeed tracks the centers of traveling waves and has the similar flavor as the position parameter in [**17**]. We remark here that the choice of weights will be essential to the proof of asymptotic behavior.

Throughout this paper, we consider the Cauchy problem for the following one-dimension system of quasilinear wave equations

$$\begin{aligned} u_{\xi\eta} &= Q(\partial u, \partial^2 u) \\ &:= A_1(u, u_\xi, u_\eta) u_{\xi\eta} + A_2(u, u_\xi, u_\eta) u_{\xi\xi} + A_3(u, u_\xi, u_\eta) u_{\eta\eta} + F(u, u_\xi, u_\eta) \end{aligned} \tag{1.4}$$

with small initial data

$$(u, u_t)|_{t=0} = (\phi, \psi) \in H^3(\mathbb{R}) \times H^2(\mathbb{R}). \tag{1.5}$$

Here, $u = u(t,x) : \mathbb{R}^{1+1} \to \mathbb{R}^n$ is an unknown function, $u_{\xi\eta} = u_{tt} - u_{xx}$, A_i : $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ $(i = 1, 2, 3)$ are given symmetric, smooth and matrix valued functions, and $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a given smooth and vector valued function. We assume that the system (1.4) satisfies the null condition [**4**,**14**,**28**], i.e., it holds near the origin in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ that

$$A_1(u, u_\xi, u_\eta) = \mathcal{O}(|u| + |u_\xi| + |u_\eta|), \tag{1.6}$$

$$A_2(u, u_\xi, u_\eta) = \mathcal{O}(|u_\eta|), \tag{1.7}$$

$$A_3(u, u_\xi, u_\eta) = \mathcal{O}(|u_\xi|), \tag{1.8}$$

$$F(u, u_\xi, u_\eta) = \mathcal{O}(|u_\xi||u_\eta|). \tag{1.9}$$

It implies that any polynomial on u in A_2, A_3, F is allowed. Under these conditions, [**28**] proved that for all $0 < \delta < 1$, there exists a positive constant ε_0 such that for any $0 < \varepsilon \leqslant \varepsilon_0$, if

$$\sum_{l=0}^{2} \left(\left\| \langle x \rangle^{1+\delta} \partial_x^l \partial_x \phi \right\|_{L_x^2(\mathbb{R})} + \left\| \langle x \rangle^{1+\delta} \partial_x^l \psi \right\|_{L_x^2(\mathbb{R})} \right) \leqslant \varepsilon, \tag{1.10}$$

then the system (1.4)-(1.5) admits a unique global solution u. Moreover, if u_{f} is the unique global solution to the Cauchy problem for the linear wave equation

$$u_{\mathrm{f}\,\xi\eta} = 0 \tag{1.11}$$

with initial data

$$(u_{\mathrm{f}}, u_{\mathrm{f}\,t})|_{t=0} = (\phi_{\mathrm{f}}, \psi_{\mathrm{f}}) \in \dot{H}^1(\mathbb{R}) \times L^2(\mathbb{R}),$$

then the global solution u satisfies

$$\lim_{t\to\infty} \Big(\big\| u_\xi - u_{\mathrm{f}\,\xi} \big\|_{L^2_x(\mathbb{R})} + \big\| u_\eta - u_{\mathrm{f}\,\eta} \big\|_{L^2_x(\mathbb{R})} \Big) = 0. \tag{1.12}$$

Inspired by the asymptotically free property (1.12), we expect to derive the convergence from quasilinear waves to linear waves given the smoothness of data

$$(u, u_t)|_{t=0} = (\phi, \psi) \in H^2(\mathbb{R}) \times H^1(\mathbb{R}) \tag{1.13}$$

and

$$(u_{\mathrm{f}}, u_{\mathrm{f}\,t})|_{t=0} = (\phi_{\mathrm{f}}, \psi_{\mathrm{f}}) \in H^2(\mathbb{R}) \times H^1(\mathbb{R}) \tag{1.14}$$

respectively, and therefore improve the asymptotic behavior result for the unique global solution u to the Cauchy problem (1.4) and (1.13). We will provide sharper convergence from the quasilinear case to the linear case by making use of the full strength of the weights. More precisely, our first main result is stated as follows:

THEOREM 1.1. *Assume that the system (1.4) satisfies the null condition. Then for all $0 < \delta < 1$, there exists a positive constant ε_0 such that for any $0 < \varepsilon \leqslant \varepsilon_0$, if*

$$\big\| \langle \xi \rangle^{1+\delta} u_\xi(0, 2\xi) \big\|_{H^2_\xi(\mathbb{R})} + \big\| \langle \eta \rangle^{1+\delta} u_\eta(0, -2\eta) \big\|_{H^2_\eta(\mathbb{R})} \leqslant \varepsilon, \tag{1.15}$$

then the Cauchy problem (1.4) and (1.13) admits a unique global solution u. Moreover, the unique global solution u to the Cauchy problem (1.4) and (1.13) satisfies

$$\lim_{t\to\infty} \Big(\big\| \langle \xi \rangle^{1+\delta} \left(u_\xi - u_{\mathrm{f}\,\xi} \right) \big\|_{H^1_\xi(\mathbb{R})} + \big\| \langle \eta \rangle^{1+\delta} \left(u_\eta - u_{\mathrm{f}\,\eta} \right) \big\|_{H^1_\eta(\mathbb{R})} \Big) = 0, \tag{1.16}$$

where u_{f} is the unique global solution to the Cauchy problem (1.11) and (1.14).

REMARK 1.2. Actually, Theorem 1.1 can be regarded as an extension of Theorem 1.1 in [**28**] due to the following two aspects:

(i) It is clear from (1.1), (1.2) and (1.5) that

$$u_\xi(0, 2\xi) = u_\xi(0, x) = \psi(x) + \phi'(x), \quad u_\eta(0, -2\eta) = u_\eta(0, x) = \psi(x) - \phi'(x).$$

Thus the condition (1.15) can also be replaced by the condition (1.10).

(ii) Since $\langle \xi \rangle \geqslant 1$ and $\langle \eta \rangle \geqslant 1$ by (1.3), we have a direct consequence of (1.16) as

$$\lim_{t\to\infty} \Big(\big\| u_\xi - u_{\mathrm{f}\,\xi} \big\|_{H^1_x(\mathbb{R})} + \big\| u_\eta - u_{\mathrm{f}\,\eta} \big\|_{H^1_x(\mathbb{R})} \Big) = 0. \tag{1.17}$$

In particular, for zero order derivative, (1.17) coincides with the asymptotically free property (1.12).

REMARK 1.3. In fact, preliminary lemmas in Section 2 ensure that Theorem 1.1 is independent of the choice of the position parameter a. Without loss of generality, we can assume $a = 0$ in the proof of Theorem 1.1 so that we can construct the global solution by exactly the same argument as in [**28**].

1.2. Another perspective on asymptotic behavior: concerning the rigidity aspect. We turn to define the scattering fields of the global solution $u(t,x)$ on infinities and discuss the energy spaces where they live. In what follows, $\Sigma_{t_0} := \{(t,x)\,|\,t = t_0\}$ for any $t_0 \in \mathbb{R}$. We first note that a given point $(0,x_0) \in \Sigma_0$ determines uniquely a left-traveling line with ξ being constant and a right-traveling line with η being constant:

$$l_\xi = \{(t,\xi)\,|\,\xi = \tfrac{1}{2}x_0\}, \quad l_\eta = \{(t,\eta)\,|\,\eta = -\tfrac{1}{2}x_0\}.$$

Then we denote the sets of left-traveling lines and right-traveling lines towards infinite time as

$$\mathcal{C}_\xi = \{l_\xi\,|\,\xi \in \mathbb{R},\ t \to \infty\}, \quad \mathcal{C}_\eta = \{l_\eta\,|\,\eta \in \mathbb{R},\ t \to \infty\},$$

and call them the left infinity and the right infinity respectively. These infinities can be regarded as Euclidean spaces with the corresponding coordinates as depicted in Figure 1.

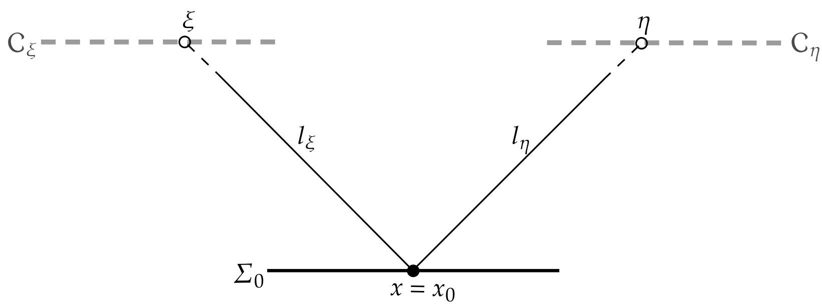

FIGURE 1. Traveling lines and infinities

By virtue of symmetry, we only need consider the left-traveling line l_ξ. It emanates from Σ_0 at the point $(0,x_0)$, travels along the $\partial_\eta = \partial_t - \partial_x$ direction and hits $\mathcal{C}_\xi$ at the point $\xi = \frac{1}{2}x_0$ with $t \to \infty$. Thus, integrating (1.4) over the line segment of l_ξ between $(0,x_0)$ and (t,x_0-t) leads us to

$$u_\xi(t,x_0-t) = u_\xi(0,x_0) + \int_0^t Q(\partial u, \partial^2 u)(s,x_0-s)ds.$$

It is not surprising to expect that the above expression converges as $t \to \infty$ and generates the following explicit formula for the left scattering field on the infinity $\mathcal{C}_\xi$:

$$u_\xi(\infty;\xi) := \lim_{t\to\infty} u_\xi(t,2\xi-t) = u_\xi(0,2\xi) + \int_0^\infty Q(\partial u, \partial^2 u)(s,2\xi-s)ds. \tag{1.18}$$

Analogously, we can define the right scattering field $u_\eta(\infty;\eta)$ on the infinity $\mathcal{C}_\eta$ for $\eta = -\frac{1}{2}x_0$:

$$u_\eta(\infty;\eta) := \lim_{t\to\infty} u_\xi(t,-2\eta+t) = u_\eta(0,-2\eta) + \int_0^\infty Q(\partial u, \partial^2 u)(s,-2\eta+s)ds. \tag{1.19}$$

More precisely, our second main result is stated as follows:

Theorem 1.4. *Under the same assumptions as in Theorem 1.1, the scattering fields given by* (1.18) *and* (1.19) *satisfy*

$$\lim_{t\to\infty}\Big(\big\|\langle\xi\rangle^{1+\delta}\left(u_\xi(\infty;\xi)-u_\xi(t,2\xi-t)\right)\big\|_{H^1_\xi(\mathbb{R})} + \big\|\langle\eta\rangle^{1+\delta}\left(u_\eta(\infty;\eta)-u_\eta(t,-2\eta+t)\right)\big\|_{H^1_\eta(\mathbb{R})}\Big)=0. \tag{1.20}$$

Moreover, if the scattering fields vanish at infinities, i.e., $u_\xi(\infty;\xi)\equiv 0$ *on* $\mathcal{C}_\xi$ *and* $u_\eta(\infty;\eta)\equiv 0$ *on* $\mathcal{C}_\eta$, *then the solution itself vanishes identically, i.e.,* $u(t,x)\equiv 0$ *on* $\mathbb{R}^{1+1}$.

Remark 1.5. First of all, (1.20) tells us that the large-time traveling waves converges to the scattering fields in the corresponding weighted Sobolev spaces, which is equivalent to the statement that the scattering fields are well defined in the weighted Sobolev spaces based on real analysis. Secondly, since the scattering fields are indeed the waves detected by far-away observers, the moreover part of Theorem 1.4 has the following physical intuition: if no waves are detected by far-away observers, then there are no waves at all emanating from the initial time slice. This conclusion reflects the rigidity from infinity for quasilinear waves in the sense of uniqueness, and can also be viewed as a model for the results in Li and Yu [**17**], where the rigidity from infinity has been proved for nonlinear Alfvén waves governed by the MHD system.

Remark 1.6. Since the position parameter a tracks the centers of left-traveling waves and right-traveling waves, the choice of a will play a fundamental role in the proof for the rigidity part.

An outline of this paper is as follows. In Section 2, we present necessary results for the proof of asymptotic behavior. The proof of Theorem 1.1 is presented in Section 3, while the proof of Theorem 1.4 is given in Section 4. The key idea is to describe the pointwise convergence by the weighted L^2 convergence for technical convenience of weighted energies. In particular, we deal with the higher order derivatives based on carefully using the quasilinear null structure and the null coordinates.

2. Preliminary estimates and global existence

2.1. Weighted energies and basic estimates. To begin with, it is easy to get the following lemma by straightforward calculation.

Lemma 2.1. *Assume* $A_1=A_1(u,u_\xi,u_\eta)$, $A_2=A_2(u,u_\xi,u_\eta)$, $A_3=A_3(u,u_\xi,u_\eta)$ *and* $F=F(u,u_\xi,u_\eta)$ *satisfy* (1.6), (1.7), (1.8), *and* (1.9), *respectively, and*

$$|u|+|u_\xi|+|u_\eta|+|u_{\xi\xi}|+|u_{\xi\eta}|+|u_{\eta\eta}|\leqslant\nu.$$

Then it holds that

$$\begin{aligned}
|\partial_\xi A_1| &\leqslant C(|u_\xi|+|u_{\xi\xi}|+|u_{\xi\eta}|),\\
|\partial_\eta A_1| &\leqslant C(|u_\eta|+|u_{\xi\eta}|+|u_{\eta\eta}|),\\
|\partial_\xi A_2| &\leqslant C(|u_\eta|+|u_{\xi\eta}|),\\
|\partial_\eta A_2| &\leqslant C(|u_\eta|+|u_{\xi\eta}|+|u_{\eta\eta}|),\\
|\partial_\xi A_3| &\leqslant C(|u_\xi|+|u_{\xi\xi}|+|u_{\xi\eta}|),
\end{aligned}$$

$$
\begin{aligned}
&|\partial_\eta A_3| \leqslant C(|u_\xi| + |u_{\xi\eta}|),\\
&|\partial_\xi F| \leqslant C(|u_\xi||u_\eta| + |u_\xi||u_{\xi\eta}| + |u_\eta||u_{\xi\xi}|),\\
&|\partial_\eta F| \leqslant C(|u_\xi||u_\eta| + |u_\xi||u_{\eta\eta}| + |u_\eta||u_{\xi\eta}|),
\end{aligned}
$$

where $C = C(\nu)$ is a constant depending on ν.

We turn to follow [21, 28] to use the following weighted energies

$$
\begin{aligned}
E_1(u(t)) &= \left\|\langle\xi\rangle^{1+\delta} u_\xi\right\|^2_{L^2_x(\mathbb{R})} + \left\|\langle\eta\rangle^{1+\delta} u_\eta\right\|^2_{L^2_x(\mathbb{R})},\\
E_2(u(t)) &= \left\|\langle\xi\rangle^{1+\delta} u_{\xi\xi}\right\|^2_{L^2_x(\mathbb{R})} + \left\|\langle\xi\rangle^{1+\delta} u_{\xi\eta}\right\|^2_{L^2_x(\mathbb{R})}\\
&\quad + \left\|\langle\eta\rangle^{1+\delta} u_{\xi\eta}\right\|^2_{L^2_x(\mathbb{R})} + \left\|\langle\eta\rangle^{1+\delta} u_{\eta\eta}\right\|^2_{L^2_x(\mathbb{R})},\\
E_3(u(t)) &= \left\|\langle\xi\rangle^{1+\delta} u_{\xi\xi\xi}\right\|^2_{L^2_x(\mathbb{R})} + \left\|\langle\xi\rangle^{1+\delta} u_{\xi\xi\eta}\right\|^2_{L^2_x(\mathbb{R})}\\
&\quad + \left\|\langle\eta\rangle^{1+\delta} u_{\xi\eta\eta}\right\|^2_{L^2_x(\mathbb{R})} + \left\|\langle\eta\rangle^{1+\delta} u_{\eta\eta\eta}\right\|^2_{L^2_x(\mathbb{R})},\\
\widetilde{E_3}(u(t)) &= \left\|\langle\xi\rangle^{1+\delta} u_{\xi\eta\eta}\right\|^2_{L^2_x(\mathbb{R})} + \left\|\langle\eta\rangle^{1+\delta} u_{\xi\xi\eta}\right\|^2_{L^2_x(\mathbb{R})},
\end{aligned}
$$

and the following space-time weighted energies

$$
\begin{aligned}
\mathcal{E}_1(u(t)) &= \int_0^t \left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta} u_\xi\right\|^2_{L^2_x(\mathbb{R})} ds + \int_0^t \left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta} u_\eta\right\|^2_{L^2_x(\mathbb{R})} ds,\\
\mathcal{E}_2(u(t)) &= \int_0^t \left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta} u_{\xi\xi}\right\|^2_{L^2_x(\mathbb{R})} ds + \int_0^t \left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta} u_{\xi\eta}\right\|^2_{L^2_x(\mathbb{R})} ds\\
&\quad + \int_0^t \left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta} u_{\xi\eta}\right\|^2_{L^2_x(\mathbb{R})} ds + \int_0^t \left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta} u_{\eta\eta}\right\|^2_{L^2_x(\mathbb{R})} ds,\\
\mathcal{E}_3(u(t)) &= \int_0^t \left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta} u_{\xi\xi\xi}\right\|^2_{L^2_x(\mathbb{R})} ds + \int_0^t \left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta} u_{\xi\xi\eta}\right\|^2_{L^2_x(\mathbb{R})} ds\\
&\quad + \int_0^t \left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta} u_{\xi\eta\eta}\right\|^2_{L^2_x(\mathbb{R})} ds + \int_0^t \left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta} u_{\eta\eta\eta}\right\|^2_{L^2_x(\mathbb{R})} ds,\\
\widetilde{\mathcal{E}_3}(u(t)) &= \int_0^t \left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta} u_{\xi\eta\eta}\right\|^2_{L^2_x(\mathbb{R})} ds + \int_0^t \left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta} u_{\xi\xi\eta}\right\|^2_{L^2_x(\mathbb{R})} ds.
\end{aligned}
$$

We also denote

$$
\begin{aligned}
E(u(t)) &= E_1(u(t)) + E_2(u(t)) + E_3(u(t)),\\
\mathcal{E}(u(t)) &= \mathcal{E}_1(u(t)) + \mathcal{E}_2(u(t)) + \mathcal{E}_3(u(t)).
\end{aligned}
$$

We now present some technical facts concerning the weight functions, the proof of which is based on direct computation.

LEMMA 2.2. *We have the following inequalities:*

$$
\left|\partial_x\langle\xi\rangle^{1+\delta}\right| \leqslant C\langle\xi\rangle^{1+\delta}, \qquad \left|\partial_x\left(\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}\right)\right| \leqslant C\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}, \tag{2.1}
$$

$$
\left|\partial_x\langle\eta\rangle^{1+\delta}\right| \leqslant C\langle\eta\rangle^{1+\delta}, \qquad \left|\partial_x\left(\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}\right)\right| \leqslant C\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}. \tag{2.2}
$$

As an application, we obtain the following pointwise estimates. Though similar estimates have been given in [28], we will present a more detailed proof here since they provide great convenience for the following study of the asymptotic behavior of solutions.

Lemma 2.3. *Let u be a smooth function with sufficient decay at the spatial infinity. Then we have*

$$\begin{aligned}
&\|u\|_{L^\infty_x(\mathbb{R})} \leqslant C\left(E_1(u(t))\right)^{\frac12},\\
&\|\langle\xi\rangle^{1+\delta}u_\xi\|_{L^\infty_x(\mathbb{R})} + \|\langle\eta\rangle^{1+\delta}u_\eta\|_{L^\infty_x(\mathbb{R})} \leqslant C\left(E_1(u(t))\right)^{\frac12} + C\left(E_2(u(t))\right)^{\frac12},\\
&\|\langle\xi\rangle^{1+\delta}u_{\xi\xi}\|_{L^\infty_x(\mathbb{R})} + \|\langle\xi\rangle^{1+\delta}u_{\xi\eta}\|_{L^\infty_x(\mathbb{R})}\\
&+ \|\langle\eta\rangle^{1+\delta}u_{\xi\eta}\|_{L^\infty_x(\mathbb{R})} + \|\langle\eta\rangle^{1+\delta}u_{\eta\eta}\|_{L^\infty_x(\mathbb{R})} \leqslant C\left(E(u(t))\right)^{\frac12} + C\left(\widetilde{E_3}(u(t))\right)^{\frac12},\\
&\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^2_sL^\infty_x} + \left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}u_\xi\right\|_{L^2_sL^\infty_x}\\
&\leqslant C\left(\mathcal{E}_1(u(t))\right)^{\frac12} + C\left(\mathcal{E}_2(u(t))\right)^{\frac12},\\
&\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi}\right\|_{L^2_sL^\infty_x} + \left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_sL^\infty_x}\\
&+ \left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_sL^\infty_x} + \left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}u_{\eta\eta}\right\|_{L^2_sL^\infty_x}\\
&\leqslant C\left(\mathcal{E}(u(t))\right)^{\frac12} + C\left(\widetilde{\mathcal{E}}_3(u(t))\right)^{\frac12}.
\end{aligned}$$

Proof. For the first inequality, according to the fundamental theorem of calculus and (1.2), we have

$$\|u\|_{L^\infty_x(\mathbb{R})} \leqslant \|u_x\|_{L^1_x(\mathbb{R})} \leqslant \|u_\xi\|_{L^1_x(\mathbb{R})} + \|u_\eta\|_{L^1_x(\mathbb{R})}.$$

Then Hölder inequality gives

$$\begin{aligned}
\|u\|_{L^\infty_x(\mathbb{R})} &\leqslant \left\|\langle\xi\rangle^{-1-\delta}\right\|_{L^2_x(\mathbb{R})}\left\|\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^2_x(\mathbb{R})} + \left\|\langle\eta\rangle^{-1-\delta}\right\|_{L^2_x(\mathbb{R})}\left\|\langle\eta\rangle^{1+\delta}u_\eta\right\|_{L^2_x(\mathbb{R})}\\
&\leqslant C\left(\left\|\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^2_x(\mathbb{R})} + \left\|\langle\eta\rangle^{1+\delta}u_\eta\right\|_{L^2_x(\mathbb{R})}\right)\\
&\leqslant C\left(E_1(u(t))\right)^{\frac12}.
\end{aligned}$$

For the second inequality, by the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, we derive

$$\begin{aligned}
&\left\|\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^\infty_x(\mathbb{R})}\\
&\leqslant C\left\|\langle\xi\rangle^{1+\delta}u_\xi\right\|_{H^1_x(\mathbb{R})} = C\left\|\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^2_x(\mathbb{R})} + C\left\|\partial_x\left(\langle\xi\rangle^{1+\delta}u_\xi\right)\right\|_{L^2_x(\mathbb{R})}\\
&\overset{(2.1)}{\leqslant} C\left\|\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^2_x(\mathbb{R})} + C\left\|\langle\xi\rangle^{1+\delta}\partial_x u_\xi\right\|_{L^2_x(\mathbb{R})}\\
&\overset{(1.2)}{\leqslant} C\left\|\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^2_x(\mathbb{R})} + C\left\|\langle\xi\rangle^{1+\delta}u_{\xi\xi}\right\|_{L^2_x(\mathbb{R})} + C\left\|\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_x(\mathbb{R})}\\
&\leqslant C\left(E_1(u(t))\right)^{\frac12} + C\left(E_2(u(t))\right)^{\frac12}.
\end{aligned}$$

By the symmetry, we also have

$$\left\|\langle\eta\rangle^{1+\delta}u_\eta\right\|_{L^\infty_x(\mathbb{R})} \leqslant C\left(E_1(u(t))\right)^{\frac12} + C\left(E_2(u(t))\right)^{\frac12}.$$

Thus the second inequality is proved.

For the third inequality, using the same manner, we can derive

$$\begin{aligned}
&\left\|\langle\xi\rangle^{1+\delta}u_{\xi\xi}\right\|_{L^\infty_x(\mathbb{R})} \leqslant C\left(E_2(u(t))\right)^{\frac12} + C\left(E_3(u(t))\right)^{\frac12},\\
&\left\|\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^\infty_x(\mathbb{R})} \leqslant C\left(E_2(u(t))\right)^{\frac12} + C\left(E_3(u(t))\right)^{\frac12} + C\left(\widetilde{E_3}(u(t))\right)^{\frac12},
\end{aligned}$$

$$\left\|\langle\eta\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^\infty_x(\mathbb{R})}\leqslant C\left(E_2(u(t))\right)^{\frac12}+C\left(E_3(u(t))\right)^{\frac12}+C\left(\widetilde{E_3}(u(t))\right)^{\frac12},$$
$$\left\|\langle\eta\rangle^{1+\delta}u_{\eta\eta}\right\|_{L^\infty_x(\mathbb{R})}\leqslant C\left(E_2(u(t))\right)^{\frac12}+C\left(E_3(u(t))\right)^{\frac12},$$

which implies the third inequality.

For the last two inequalities, the Sobolev inequality on $\mathbb{R}$ gives

$$\begin{aligned}
&\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^2_sL^\infty_x}\\
\leqslant{}& C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^2_sL^2_x}+C\left\|\partial_x\left(\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\right)\right\|_{L^2_sL^2_x}\\
\overset{(2.1)}{\leqslant}{}& C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^2_sL^2_x}+C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}\partial_xu_\xi\right\|_{L^2_sL^2_x}\\
\overset{(1.2)}{\leqslant}{}& C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^2_sL^2_x}+C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi}\right\|_{L^2_sL^2_x}\\
&+C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_sL^2_x}\\
\leqslant{}& C\left(\mathcal{E}_1(u(t))\right)^{\frac12}+C\left(\mathcal{E}_2(u(t))\right)^{\frac12},
\end{aligned}$$

and similarly,

$$\begin{aligned}
\left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}u_\eta\right\|_{L^2_sL^\infty_x}&\leqslant C\left(\mathcal{E}_1(u(t))\right)^{\frac12}+C\left(\mathcal{E}_2(u(t))\right)^{\frac12},\\
\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi}\right\|_{L^2_sL^\infty_x}&\leqslant C\left(\mathcal{E}_2(u(t))\right)^{\frac12}+C\left(\mathcal{E}_3(u(t))\right)^{\frac12},\\
\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_sL^\infty_x}&\leqslant C\left(\mathcal{E}_2(u(t))\right)^{\frac12}+C\left(\mathcal{E}_3(u(t))\right)^{\frac12}+\left(\widetilde{\mathcal{E}}_3(u(t))\right)^{\frac12},\\
\left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_sL^\infty_x}&\leqslant C\left(\mathcal{E}_2(u(t))\right)^{\frac12}+C\left(\mathcal{E}_3(u(t))\right)^{\frac12}+\left(\widetilde{\mathcal{E}}_3(u(t))\right)^{\frac12},\\
\left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}u_{\eta\eta}\right\|_{L^2_sL^\infty_x}&\leqslant C\left(\mathcal{E}_2(u(t))\right)^{\frac12}+C\left(\mathcal{E}_3(u(t))\right)^{\frac12}.
\end{aligned}$$

Therefore we obtain the last two inequalities. This completes the proof of the lemma. □

We point out that $\widetilde{E_3}(u(t))$ and $\widetilde{\mathcal{E}}_3(u(t))$ are considered separately from the total energies, since they are not compatible with the null structure of the quasilinear part. The following lemma is the key point to treat the quasilinear part, where we refer the reader to Lemma 2.4 in [**28**] for the proof.

LEMMA 2.4. *Let u be a solution to the system (1.4) satisfying null conditions (1.6)-(1.9). Assume that $\sup_{0\leqslant s\leqslant t}E(u(s))$ is sufficiently small. Then we have*

$$\sup_{0\leqslant s\leqslant t}\widetilde{E_3}(u(s))\leqslant C\sup_{0\leqslant s\leqslant t}E(u(s))$$

and

$$\widetilde{\mathcal{E}}_3(u(t))\leqslant C\mathcal{E}(u(t)).$$

Based on the definition of energies and the previous lemmas, we can conclude all the required L^2 estimates and pointwise estimates as follows:

LEMMA 2.5. *Let u be a solution to the system (1.4) satisfying null conditions (1.6)-(1.9). Assume that $\sup_{0\leqslant s\leqslant t} E(u(s))$ is sufficiently small. Then we have*

$$
\begin{aligned}
&\left\|\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^2_x(\mathbb{R})}+\left\|\langle\xi\rangle^{1+\delta}u_{\xi\xi}\right\|_{L^2_x(\mathbb{R})}+\left\|\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_x(\mathbb{R})}\\
&+\left\|\langle\xi\rangle^{1+\delta}u_{\xi\xi\xi}\right\|_{L^2_x(\mathbb{R})}+\left\|\langle\xi\rangle^{1+\delta}u_{\xi\xi\eta}\right\|_{L^2_x(\mathbb{R})}+\left\|\langle\xi\rangle^{1+\delta}u_{\xi\eta\eta}\right\|_{L^2_x(\mathbb{R})}\\
&+\left\|\langle\eta\rangle^{1+\delta}u_\eta\right\|_{L^2_x(\mathbb{R})}+\left\|\langle\eta\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_x(\mathbb{R})}+\left\|\langle\eta\rangle^{1+\delta}u_{\eta\eta}\right\|_{L^2_x(\mathbb{R})}\\
&+\left\|\langle\eta\rangle^{1+\delta}u_{\xi\xi\eta}\right\|_{L^2_x(\mathbb{R})}+\left\|\langle\eta\rangle^{1+\delta}u_{\xi\eta\eta}\right\|_{L^2_x(\mathbb{R})}+\left\|\langle\eta\rangle^{1+\delta}u_{\eta\eta\eta}\right\|_{L^2_x(\mathbb{R})}\\
&\leqslant C\left(E(u(t))\right)^{\frac12},\\
&\left\|\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^\infty_sL^2_x}+\left\|\langle\xi\rangle^{1+\delta}u_{\xi\xi}\right\|_{L^\infty_sL^2_x}+\left\|\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^\infty_sL^2_x}\\
&+\left\|\langle\xi\rangle^{1+\delta}u_{\xi\xi\xi}\right\|_{L^\infty_sL^2_x}+\left\|\langle\xi\rangle^{1+\delta}u_{\xi\xi\eta}\right\|_{L^\infty_sL^2_x}+\left\|\langle\xi\rangle^{1+\delta}u_{\xi\eta\eta}\right\|_{L^\infty_sL^2_x}\\
&+\left\|\langle\eta\rangle^{1+\delta}u_\eta\right\|_{L^\infty_sL^2_x}+\left\|\langle\eta\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^\infty_sL^2_x}+\left\|\langle\eta\rangle^{1+\delta}u_{\eta\eta}\right\|_{L^\infty_sL^2_x}\\
&+\left\|\langle\eta\rangle^{1+\delta}u_{\xi\xi\eta}\right\|_{L^\infty_sL^2_x}+\left\|\langle\eta\rangle^{1+\delta}u_{\xi\eta\eta}\right\|_{L^\infty_sL^2_x}+\left\|\langle\eta\rangle^{1+\delta}u_{\eta\eta\eta}\right\|_{L^\infty_sL^2_x}\\
&\leqslant C\left(E(u(t))\right)^{\frac12},\\
&\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^2_sL^2_x}+\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi}\right\|_{L^2_sL^2_x}\\
&+\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_sL^2_x}+\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi\xi}\right\|_{L^2_sL^2_x}\\
&+\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi\eta}\right\|_{L^2_sL^2_x}+\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta\eta}\right\|_{L^2_sL^2_x}\\
&+\left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}u_\eta\right\|_{L^2_sL^2_x}+\left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_sL^2_x}\\
&+\left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}u_{\eta\eta}\right\|_{L^2_sL^2_x}+\left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}u_{\xi\xi\eta}\right\|_{L^2_sL^2_x}\\
&+\left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}u_{\xi\eta\eta}\right\|_{L^2_sL^2_x}+\left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}u_{\eta\eta\eta}\right\|_{L^2_sL^2_x}\\
&\leqslant C\left(\mathcal{E}(u(t))\right)^{\frac12},\\
&\|u\|_{L^\infty_x(\mathbb{R})}+\left\|\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^\infty_x(\mathbb{R})}+\left\|\langle\xi\rangle^{1+\delta}u_{\xi\xi}\right\|_{L^\infty_x(\mathbb{R})}+\left\|\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^\infty_x(\mathbb{R})}\\
&+\left\|\langle\eta\rangle^{1+\delta}u_\eta\right\|_{L^\infty_x(\mathbb{R})}+\left\|\langle\eta\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^\infty_x(\mathbb{R})}+\left\|\langle\eta\rangle^{1+\delta}u_{\eta\eta}\right\|_{L^\infty_x(\mathbb{R})}\\
&\leqslant C\left(E(u(t))\right)^{\frac12},\\
&\|u\|_{L^\infty_sL^\infty_x}+\left\|\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^\infty_sL^\infty_x}+\left\|\langle\xi\rangle^{1+\delta}u_{\xi\xi}\right\|_{L^\infty_sL^\infty_x}+\left\|\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^\infty_sL^\infty_x}\\
&+\left\|\langle\eta\rangle^{1+\delta}u_\eta\right\|_{L^\infty_sL^\infty_x}+\left\|\langle\eta\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^\infty_sL^\infty_x}+\left\|\langle\eta\rangle^{1+\delta}u_{\eta\eta}\right\|_{L^\infty_sL^\infty_x}\\
&\leqslant C\left(E(u(t))\right)^{\frac12},\\
&\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^2_sL^\infty_x}+\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi}\right\|_{L^2_sL^\infty_x}+\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_sL^\infty_x}\\
&+\left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}u_\xi\right\|_{L^2_sL^\infty_x}+\left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_sL^\infty_x}+\left\|\langle\xi\rangle^{-\frac{1+\delta}{2}}\langle\eta\rangle^{1+\delta}u_{\eta\eta}\right\|_{L^2_sL^\infty_x}\\
&\leqslant C\left(\mathcal{E}(u(t))\right)^{\frac12}.
\end{aligned}
$$

2.2. Global existence of solution. To end up this section, we restate the small-data-global-existence result for the Cauchy problem (1.4)-(1.5) by the following bootstrap argument, and refer readers to Page 10-17 in [**28**] for a rigorous proof.

Lemma 2.6. *Assume that u is a solution to the Cauchy problem* (1.4)-(1.5)*. Then there exist positive constants ε_0 and A such that*

$$\sup_{0\leqslant s\leqslant t} E(u(s)) + \mathcal{E}(u(t)) \leqslant A^2\varepsilon^2, \tag{2.3}$$

under the assumption

$$\sup_{0\leqslant s\leqslant t} E(u(s)) + \mathcal{E}(u(t)) \leqslant 4A^2\varepsilon^2, \tag{2.4}$$

where $0<\varepsilon\leqslant\varepsilon_0$.

Remark 2.7. The global existence of solution to the Cauchy problem (1.4)-(1.5) is an immediate consequence of Lemma 2.6. In fact, since the constants ε_0 and A are independent of the lifespan $[0,t]$, then the assumption (2.4) will never be saturated so that we can always continue t to ∞, which implies the small-data-global-existence result as the first part of Theorem 1.1.

3. The first perspective on asymptotic behavior

This section is devoted to the proof of Theorem 1.1. It remains to prove (1.16) based on (2.3). Based on the symmetry, it suffices to prove

$$\lim_{t\to\infty} \left\|\langle\xi\rangle^{1+\delta}\left(u_\xi - u_{\mathrm{f}\xi}\right)\right\|_{H^1_\xi(\mathbb{R})} = 0. \tag{3.1}$$

In view of (1.1) and (1.11), we notice that

$$u_\xi(t,x) = u_\xi(t,2\xi-t), \qquad u_{\mathrm{f}\xi}(t,x) = u_{\mathrm{f}\xi}(t,2\xi-t) \equiv u_{\mathrm{f}\xi}(0,2\xi).$$

Therefore, in order to show (3.1), we only need to show

$$\left\|\langle\xi\rangle^{1+\delta}\int_0^\infty Q(\partial u,\partial^2 u)ds\right\|_{H^1_\xi(\mathbb{R})} < \infty.$$

The proof is divided into two steps. The first step deals with the case of zero order derivative. The second step deals with the case of first order derivative. We remark here that owing to (1.1), we will make full use of the following four coordinates on $\mathbb{R}^{1+1}$: (t,x), (t,ξ), (t,η) and (ξ,η).

3.1. The first step of proof. We show that

$$\left\|\langle\xi\rangle^{1+\delta}\int_0^\infty Q(\partial u,\partial^2 u)ds\right\|_{L^2_\xi(\mathbb{R})} < \infty. \tag{3.2}$$

In other words, it suffices to show

$$\mathbf{I} := \int_{\mathbb{R}}\left|\int_0^\infty Q(\partial u,\partial^2 u)ds\right|^2 \langle\xi\rangle^{2+2\delta}d\xi < \infty.$$

By changing coordinates and using Hölder inequality, we can bound $\mathbf{I}$ as follows:

$$\mathbf{I} \leqslant C\int_{\mathbb{R}}\left|\int_{\mathbb{R}} Q(\partial u,\partial^2 u)d\eta\right|^2 \langle\xi\rangle^{2+2\delta}d\xi$$

$$\leqslant C\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\langle\eta\rangle^{-(1+\delta)}d\eta\right)\left(\int_{\mathbb{R}}\langle\eta\rangle^{1+\delta}\left|Q(\partial u,\partial^2 u)\right|^2 d\eta\right)\langle\xi\rangle^{2+2\delta}d\xi$$
$$\leqslant C\int_{\mathbb{R}^{1+1}}\langle\eta\rangle^{1+\delta}\langle\xi\rangle^{2+2\delta}\left|Q(\partial u,\partial^2 u)\right|^2 d\eta d\xi$$
$$\leqslant C\int_{\mathbb{R}^{1+1}}\langle\eta\rangle^{1+\delta}\langle\xi\rangle^{2+2\delta}\left|Q(\partial u,\partial^2 u)\right|^2 dxds. \tag{3.3}$$

LEMMA 3.1. *It holds that*
$$\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}Q(\partial u,\partial^2 u)\right\|_{L_s^2L_x^2}<+\infty.$$

PROOF. According to (1.4), we have
$$\begin{aligned}&\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}Q(\partial u,\partial^2 u)\right\|_{L_s^2L_x^2}\\ \leqslant&\underbrace{\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}A_1u_{\xi\eta}\right\|_{L_s^2L_x^2}}_{\mathbf{I_1}}+\underbrace{\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}A_2u_{\xi\xi}\right\|_{L_s^2L_x^2}}_{\mathbf{I_2}}\\ &+\underbrace{\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}A_3u_{\eta\eta}\right\|_{L_s^2L_x^2}}_{\mathbf{I_3}}+\underbrace{\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}F\right\|_{L_s^2L_x^2}}_{\mathbf{I_4}}.\end{aligned}\tag{3.4}$$

We now estimate these four terms by using null conditions (1.6)-(1.9), Hölder inequality and Lemma 2.5.

$$\begin{aligned}\mathbf{I_1}\leqslant&\,C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}(|u|+|u_\xi|+|u_\eta|)u_{\xi\eta}\right\|_{L_s^2L_x^2}\\ \leqslant&\,C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}uu_{\xi\eta}\right\|_{L_s^2L_x^2}+C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi u_{\xi\eta}\right\|_{L_s^2L_x^2}\\ &+C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\eta u_{\xi\eta}\right\|_{L_s^2L_x^2}\\ \leqslant&\,C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L_s^2L_x^2}\|u\|_{L_s^\infty L_x^\infty}\\ &+C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L_s^2L_x^2}\left\|\langle\eta\rangle^{1+\delta}u_{\xi\eta}\right\|_{L_s^\infty L_x^\infty}\\ &+C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L_s^2L_x^2}\left\|\langle\eta\rangle^{1+\delta}u_\eta\right\|_{L_s^\infty L_x^\infty}\\ \leqslant&\,C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L_s^2L_x^2}(E(u(t)))^{\frac12}+C\left(\mathcal{E}(u(t))\right)^{\frac12}(E(u(t)))^{\frac12},\\ \mathbf{I_2}\leqslant&\,C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\eta u_{\xi\xi}\right\|_{L_s^2L_x^2}\\ \leqslant&\,C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi}\right\|_{L_s^2L_x^2}\left\|\langle\eta\rangle^{1+\delta}u_\eta\right\|_{L_s^\infty L_x^\infty}\\ \leqslant&\,C\left(\mathcal{E}(u(t))\right)^{\frac12}(E(u(t)))^{\frac12},\\ \mathbf{I_3}\leqslant&\,C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi u_{\eta\eta}\right\|_{L_s^2L_x^2}\\ \leqslant&\,C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L_s^2L_x^2}\left\|\langle\eta\rangle^{1+\delta}u_{\eta\eta}\right\|_{L_s^\infty L_x^\infty}\\ \leqslant&\,C\left(\mathcal{E}(u(t))\right)^{\frac12}(E(u(t)))^{\frac12},\\ \mathbf{I_4}\leqslant&\,C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi u_\eta\right\|_{L_s^2L_x^2}\end{aligned}$$

$$\leqslant C\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\|_{L^2_sL^2_x}\|\langle\eta\rangle^{1+\delta}u_\eta\|_{L^\infty_sL^\infty_x}$$
$$\leqslant C\left(\mathcal{E}(u(t))\right)^{\frac12}\left(E(u(t))\right)^{\frac12}.$$

Combining these estimates with (3.4) and (2.3), we derive

$$\begin{aligned}&\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}Q(\partial u,\partial^2u)\right\|_{L^2_sL^2_x}\\ &\leqslant C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_sL^2_x}\left(E(u(t))\right)^{\frac12}+C\left(\mathcal{E}(u(t))\right)^{\frac12}\left(E(u(t))\right)^{\frac12}\\ &\leqslant CA\varepsilon\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_sL^2_x}+CA^2\varepsilon^2. \qquad (3.5)\end{aligned}$$

It is clear to see from (1.4) and (3.5) that

$$\begin{aligned}\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_sL^2_x}&=\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}Q(\partial u,\partial^2u)\right\|_{L^2_sL^2_x}\\ &\leqslant CA\varepsilon\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_sL^2_x}+CA^2\varepsilon^2.\end{aligned}$$

If ε is sufficiently small, then we have

$$\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_sL^2_x}\leqslant CA^2\varepsilon^2.$$

Together with (3.5), we thus obtain

$$\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}Q(\partial u,\partial^2u)\right\|_{L^2_sL^2_x}\leqslant CA^2\varepsilon^2<\infty.$$

The proof of the lemma is now complete. □

In view of (3.3) and Lemma 3.1, we conclude that

$$\mathbf{I}<\infty.$$

Thus we have proved (3.2).

3.2. The second step of proof. We show that

$$\left\|\partial_\xi\left(\langle\xi\rangle^{1+\delta}\int_0^\infty Q(\partial u,\partial^2u)ds\right)\right\|_{L^2_\xi(\mathbb{R})}<\infty. \qquad (3.6)$$

By virtue of (1.3), it suffices to show that

$$\left\|\langle\xi\rangle^{1+\delta}\partial_\xi\int_0^\infty Q(\partial u,\partial^2u)ds\right\|_{L^2_\xi(\mathbb{R})}<\infty, \qquad (3.7)$$

that is,

$$\mathbf{J}:=\int_{\mathbb{R}}\left|\partial_\xi\int_0^\infty Q(\partial u,\partial^2u)ds\right|^2\langle\xi\rangle^{2+2\delta}d\xi<\infty.$$

According to definition of derivative, Fatou's lemma and the fundamental theorem of calculus, we get

$$\begin{aligned}\mathbf{J}&=\int_{\mathbb{R}}\left|\lim_{h\to0}\int_0^\infty\frac{Q(\partial u,\partial^2u)(s,2(\xi+h)-s)-Q(\partial u,\partial^2u)(s,2\xi-s)}{h}ds\right|^2\langle\xi\rangle^{2+2\delta}d\xi\\ &=\int_{\mathbb{R}}\lim_{h\to0}\left|\int_0^\infty\frac{Q(\partial u,\partial^2u)(s,2(\xi+h)-s)-Q(\partial u,\partial^2u)(s,2\xi-s)}{h}ds\right|^2\langle\xi\rangle^{2+2\delta}d\xi\end{aligned}$$

$$\leqslant \liminf_{h\to 0}\int_{\mathbb{R}}\left|\int_0^\infty \frac{Q(\partial u,\partial^2 u)(s,2(\xi+h)-s)-Q(\partial u,\partial^2 u)(s,2\xi-s)}{h}ds\right|^2\langle\xi\rangle^{2+2\delta}d\xi$$

$$\leqslant C\liminf_{h\to 0}\int_{\mathbb{R}}\left|\int_0^\infty\int_0^1 \partial_\xi Q(\partial u,\partial^2 u)\left(s,2(\xi+\theta h)-s\right)d\theta ds\right|^2\langle\xi\rangle^{2+2\delta}d\xi$$

$$\leqslant C\liminf_{h\to 0}\int_0^1 \underbrace{\int_{\mathbb{R}}\left|\int_0^\infty \left|\partial_\xi Q(\partial u,\partial^2 u)(s,2(\xi+\theta h)-s)\right|ds\right|^2\langle\xi\rangle^{2+2\delta}d\xi}_{\mathbf{J}'}\,d\theta. \tag{3.8}$$

In $\mathbf{J}'$, we note that θ and h do not depend on ξ. By change of variable $\xi\to\widetilde{\xi}=\xi+\theta h$, we obtain

$$\begin{aligned}\mathbf{J}' &\leqslant \int_{\mathbb{R}}\left|\int_0^\infty\left|\partial_\xi Q(\partial u,\partial^2 u)(s,2\widetilde{\xi}-s)\right|ds\right|^2\langle\widetilde{\xi}\rangle^{2+2\delta}d\widetilde{\xi}\\ &=\int_{\mathbb{R}}\left|\int_0^\infty\left|\partial_\xi Q(\partial u,\partial^2 u)(s,2\xi-s)\right|ds\right|^2\langle\xi\rangle^{2+2\delta}d\xi.\end{aligned}$$

By changing coordinates and using Hölder inequality, we can bound $\mathbf{J}'$ as follows:

$$\begin{aligned}\mathbf{J}' &\leqslant C\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\left|\partial_\xi Q(\partial u,\partial^2 u)\right|d\eta\right|^2\langle\xi\rangle^{2+2\delta}d\xi\\ &\leqslant C\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\langle\eta\rangle^{-(1+\delta)}d\eta\right)\left(\int_{\mathbb{R}}\langle\eta\rangle^{1+\delta}\left|\partial_\xi Q(\partial u,\partial^2 u)\right|^2d\eta\right)\langle\xi\rangle^{2+2\delta}d\xi\\ &\leqslant C\int_{\mathbb{R}^{1+1}}\langle\eta\rangle^{1+\delta}\langle\xi\rangle^{2+2\delta}\left|\partial_\xi Q(\partial u,\partial^2 u)\right|^2d\eta d\xi\\ &\leqslant C\int_{\mathbb{R}^{1+1}}\langle\eta\rangle^{1+\delta}\langle\xi\rangle^{2+2\delta}\left|\partial_\xi Q(\partial u,\partial^2 u)\right|^2dxds.\end{aligned}\tag{3.9}$$

LEMMA 3.2. *It holds that*

$$\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}\partial_\xi Q(\partial u,\partial^2 u)\right\|_{L_s^2L_x^2}<\infty.$$

PROOF. According to (1.4), we have

$$\begin{aligned}\partial_\xi u_{\xi\eta} &= \partial_\xi Q(\partial u,\partial^2 u)\\ &=\partial_\xi\left(A_1(u,u_\xi,u_\eta)u_{\xi\eta}+A_2(u,u_\xi,u_\eta)u_{\xi\xi}+A_3(u,u_\xi,u_\eta)u_{\eta\eta}+F(u,u_\xi,u_\eta)\right)\\ &=\partial_\xi A_1u_{\xi\eta}+A_1\partial_\xi u_{\xi\eta}+\partial_\xi A_2u_{\xi\xi}+A_2\partial_\xi u_{\xi\xi}+\partial_\xi A_3u_{\eta\eta}+A_3\partial_\xi u_{\eta\eta}+\partial_\xi F\\ &=A_1u_{\xi\xi\eta}+A_2u_{\xi\xi\xi}+A_3u_{\xi\eta\eta}+\partial_\xi A_1u_{\xi\eta}+\partial_\xi A_2u_{\xi\xi}+\partial_\xi A_3u_{\eta\eta}+\partial_\xi F.\end{aligned}\tag{3.10}$$

It follows that

$$\begin{aligned}&\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}\partial_\xi Q(\partial u,\partial^2 u)\right\|_{L_s^2L_x^2}\\ \leqslant&\underbrace{\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}A_1u_{\xi\xi\eta}\right\|_{L_s^2L_x^2}}_{\mathbf{J_1}}+\underbrace{\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}A_2u_{\xi\xi\xi}\right\|_{L_s^2L_x^2}}_{\mathbf{J_2}}\\ &+\underbrace{\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}A_3u_{\xi\eta\eta}\right\|_{L_s^2L_x^2}}_{\mathbf{J_3}}+\underbrace{\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}\partial_\xi A_1u_{\xi\eta}\right\|_{L_s^2L_x^2}}_{\mathbf{J_4}}\end{aligned}$$

$$+\underbrace{\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}\partial_\xi A_2 u_{\xi\xi}\right\|_{L^2_sL^2_x}}_{\mathbf{J_5}}+\underbrace{\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}\partial_\xi A_3 u_{\eta\eta}\right\|_{L^2_sL^2_x}}_{\mathbf{J_6}}$$

$$+\underbrace{\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}\partial_\xi F\right\|_{L^2_sL^2_x}}_{\mathbf{J_7}}. \tag{3.11}$$

By using null conditions (1.6)-(1.9), Hölder inequality and Lemma 2.5, we can estimate the first three terms as follows:

$$\begin{aligned}
\mathbf{J_1} &\leqslant C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}\left(|u|+|u_\xi|+|u_\eta|\right)u_{\xi\xi\eta}\right\|_{L^2_sL^2_x}\\
&\leqslant C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u u_{\xi\xi\eta}\right\|_{L^2_sL^2_x}+C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi u_{\xi\xi\eta}\right\|_{L^2_sL^2_x}\\
&\quad+C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\eta u_{\xi\xi\eta}\right\|_{L^2_sL^2_x}\\
&\leqslant C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi\eta}\right\|_{L^2_sL^2_x}\|u\|_{L^\infty_sL^\infty_x}\\
&\quad+C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^2_sL^\infty_x}\left\|\langle\eta\rangle^{1+\delta}u_{\xi\xi\eta}\right\|_{L^\infty_sL^2_x}\\
&\quad+C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi\eta}\right\|_{L^2_sL^2_x}\left\|\langle\eta\rangle^{1+\delta}u_\eta\right\|_{L^\infty_sL^\infty_x}\\
&\leqslant C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi\eta}\right\|_{L^2_sL^2_x}(E(u(t)))^{\frac12}+C\left(\mathcal{E}(u(t))\right)^{\frac12}(E(u(t)))^{\frac12},\\
\mathbf{J_2} &\leqslant C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\eta u_{\xi\xi\xi}\right\|_{L^2_sL^2_x}\\
&\leqslant C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi\xi}\right\|_{L^2_sL^2_x}\left\|\langle\eta\rangle^{1+\delta}u_\eta\right\|_{L^\infty_sL^\infty_x}\\
&\leqslant C\left(\mathcal{E}(u(t))\right)^{\frac12}(E(u(t)))^{\frac12},\\
\mathbf{J_3} &\leqslant C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi u_{\xi\eta\eta}\right\|_{L^2_sL^2_x}\\
&\leqslant C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^2_sL^\infty_x}\left\|\langle\eta\rangle^{1+\delta}u_{\xi\eta\eta}\right\|_{L^\infty_sL^2_x}\\
&\leqslant C\left(\mathcal{E}(u(t))\right)^{\frac12}(E(u(t)))^{\frac12}.
\end{aligned}$$

By using Hölder inequality, Lemma 2.1 and Lemma 2.5, we can bound the last four terms as follows:

$$\begin{aligned}
\mathbf{J_4} &\leqslant C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}(|u_\xi|+|u_{\xi\xi}|+|u_{\xi\eta}|)u_{\xi\eta}\right\|_{L^2_sL^2_x}\\
&\leqslant C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi u_{\xi\eta}\right\|_{L^2_sL^2_x}+C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi}u_{\xi\eta}\right\|_{L^2_sL^2_x}\\
&\quad+C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}u_{\xi\eta}\right\|_{L^2_sL^2_x}\\
&\leqslant C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L^2_sL^2_x}\left\|\langle\eta\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^\infty_sL^\infty_x}\\
&\quad+C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi}\right\|_{L^2_sL^2_x}\left\|\langle\eta\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^\infty_sL^\infty_x}\\
&\quad+C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^2_sL^2_x}\left\|\langle\eta\rangle^{1+\delta}u_{\xi\eta}\right\|_{L^\infty_sL^\infty_x}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\left(\mathcal{E}(u(t))\right)^{\frac{1}{2}}\left(E(u(t))\right)^{\frac{1}{2}},\\
\mathbf{J_5} &\leqslant C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}\left(|u_\eta|+|u_{\xi\eta}|\right)u_{\xi\xi}\right\|_{L_s^2L_x^2}\\
&\leqslant C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\eta u_{\xi\xi}\right\|_{L_s^2L_x^2}+C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi}u_{\xi\eta}\right\|_{L_s^2L_x^2}\\
&\leqslant C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi}\right\|_{L_s^2L_x^2}\left\|\langle\eta\rangle^{1+\delta}u_\eta\right\|_{L_s^\infty L_x^\infty}\\
&\quad+C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi}\right\|_{L_s^2L_x^2}\left\|\langle\eta\rangle^{1+\delta}u_{\xi\eta}\right\|_{L_s^\infty L_x^\infty}\\
&\leqslant C\left(\mathcal{E}(u(t))\right)^{\frac{1}{2}}\left(E(u(t))\right)^{\frac{1}{2}},\\
\mathbf{J_6} &\leqslant C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}(|u_\xi|+|u_{\xi\xi}|+|u_{\xi\eta}|)u_{\eta\eta}\right\|_{L_s^2L_x^2}\\
&\leqslant C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi u_{\eta\eta}\right\|_{L_s^2L_x^2}+C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi}u_{\eta\eta}\right\|_{L_s^2L_x^2}\\
&\quad+C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}u_{\eta\eta}\right\|_{L_s^2L_x^2}\\
&\leqslant C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\right\|_{L_s^2L_x^2}\left\|\langle\eta\rangle^{1+\delta}u_{\eta\eta}\right\|_{L_s^\infty L_x^\infty}\\
&\quad+C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi}\right\|_{L_s^2L_x^2}\left\|\langle\eta\rangle^{1+\delta}u_{\eta\eta}\right\|_{L_s^\infty L_x^\infty}\\
&\quad+C\left\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\eta}\right\|_{L_s^2L_x^2}\left\|\langle\eta\rangle^{1+\delta}u_{\eta\eta}\right\|_{L_s^\infty L_x^\infty}\\
&\leqslant C\left(\mathcal{E}(u(t))\right)^{\frac{1}{2}}\left(E(u(t))\right)^{\frac{1}{2}},\\
\mathbf{J_7} &\leqslant C\big\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}(|u_\xi||u_\eta|+|u_\xi||u_{\xi\eta}|+|u_\eta||u_{\xi\xi}|)\big\|_{L_s^2L_x^2}\\
&\leqslant C\big\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi u_\eta\big\|_{L_s^2L_x^2}+C\big\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi u_{\xi\eta}\big\|_{L_s^2L_x^2}\\
&\quad+C\big\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\eta u_{\xi\xi}\big\|_{L_s^2L_x^2}\\
&\leqslant C\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\big\|_{L_s^2L_x^2}\big\|\langle\eta\rangle^{1+\delta}u_\eta\big\|_{L_s^\infty L_x^\infty}\\
&\quad+C\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_\xi\big\|_{L_s^2L_x^2}\big\|\langle\eta\rangle^{1+\delta}u_{\xi\eta}\big\|_{L_s^\infty L_x^\infty}\\
&\quad+C\|\langle\eta\rangle^{-\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi}\big\|_{L_s^2L_x^2}\big\|\langle\eta\rangle^{1+\delta}u_\eta\big\|_{L_s^\infty L_x^\infty}\\
&\leqslant C\left(\mathcal{E}(u(t))\right)^{\frac{1}{2}}\left(E(u(t))\right)^{\frac{1}{2}}.
\end{aligned}
$$

Combining these estimates with (3.11) and (2.3), we derive

$$
\begin{aligned}
&\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}\partial_\xi Q(\partial u,\partial^2u)\right\|_{L_s^2L_x^2}\\
&\leqslant C\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi\eta}\right\|_{L_s^2L_x^2}\left(E(u(t))\right)^{\frac{1}{2}}+C\left(\mathcal{E}(u(t))\right)^{\frac{1}{2}}\left(E(u(t))\right)^{\frac{1}{2}}\\
&\leqslant CA\varepsilon\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi\eta}\right\|_{L_s^2L_x^2}+CA^2\varepsilon^2. \qquad (3.12)
\end{aligned}
$$

According to (3.10) and (3.12), we get

$$
\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}u_{\xi\xi\eta}\right\|_{L_s^2L_x^2}=\left\|\langle\eta\rangle^{\frac{1+\delta}{2}}\langle\xi\rangle^{1+\delta}\partial_\xi u_{\xi\eta}\right\|_{L_s^2L_x^2}
$$

$$= \left\| \langle \eta \rangle^{\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} \partial_\xi Q(\partial u, \partial^2 u) \right\|_{L^2_s L^2_x}$$
$$\leqslant CA\varepsilon \left\| \langle \eta \rangle^{\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} u_{\xi\xi\eta} \right\|_{L^2_s L^2_x} + CA^2\varepsilon^2.$$

Provided ε is sufficiently small, then we derive

$$\left\| \langle \eta \rangle^{\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} u_{\xi\xi\eta} \right\|_{L^2_s L^2_x} \leqslant CA^2\varepsilon^2.$$

Together with (3.12), we thus obtain

$$\left\| \langle \eta \rangle^{\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} \partial_\xi Q(\partial u, \partial^2 u) \right\|_{L^2_s L^2_x} \leqslant CA^2\varepsilon^2 < \infty.$$

It is now obvious that the lemma holds. □

In view of (3.9) and Lemma 3.2, we conclude that

$$\mathbf{J}' < \infty.$$

As an immediate consequence of (3.8), we have

$$\mathbf{J} < \infty.$$

Thus we have proved (3.7) and hence (3.6). To sum up, we obtain (3.1).

The proof of Theorem 1.1 is now complete.

4. The second perspective on asymptotic behavior

Our task reduces to prove Theorem 1.4 based on (2.4).

4.1. The proof of (1.20)**.** We shall adopt the same procedure as in the proof of Theorem 1.1. By the symmetry consideration, it suffices to prove that

$$\lim_{t\to\infty} \left\| \langle \xi \rangle^{1+\delta} \left(u_\xi(\infty;\xi) - u_\xi(t, 2\xi - t) \right) \right\|_{H^1_\xi(\mathbb{R})} = 0.$$

According to (1.18), it suffices to prove that

$$\lim_{t\to\infty} \left\| \langle \xi \rangle^{1+\delta} \int_t^\infty Q(\partial u, \partial^2 u) ds \right\|_{H^1_\xi(\mathbb{R})} = 0.$$

Using a similar argument as in the last section, we only need to prove the following (4.1) and (4.2).

Firstly, we show that

$$\lim_{t\to\infty} \left\| \langle \xi \rangle^{1+\delta} \int_t^\infty Q(\partial u, \partial^2 u) ds \right\|_{L^2_\xi(\mathbb{R})} = 0. \tag{4.1}$$

In fact, consider

$$\mathbf{K} := \int_{\mathbb{R}} \left| \int_t^\infty Q(\partial u, \partial^2 u) ds \right|^2 \langle \xi \rangle^{2+2\delta} d\xi.$$

In the sequel, $\chi_{s\geqslant t}$ represents the characteristic function defined on $\mathbb{R}^{1+1}$ with $s \geqslant t$. By changing coordinates and using Hölder inequality, we can bound $\mathbf{K}$ as follows:

$$\mathbf{K} \leqslant C \int_{\mathbb{R}} \left| \int_{\mathbb{R}} Q(\partial u, \partial^2 u) \cdot \chi_{s\geqslant t}(\xi,\eta) d\eta \right|^2 \langle \xi \rangle^{2+2\delta} d\xi$$
$$\leqslant C \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \langle \eta \rangle^{-(1+\delta)} d\eta \right) \left(\int_{\mathbb{R}} \langle \eta \rangle^{1+\delta} \left| Q(\partial u, \partial^2 u) \right|^2 \cdot \chi_{s\geqslant t}(\xi,\eta) d\eta \right) \langle \xi \rangle^{2+2\delta} d\xi$$

$$\leqslant C\int_{\mathbb{R}^{1+1}}\langle\eta\rangle^{1+\delta}\langle\xi\rangle^{2+2\delta}\left|Q(\partial u,\partial^2 u)\right|^2\cdot\chi_{s\geqslant t}(\xi,\eta)d\eta d\xi$$
$$=C\int_t^\infty\int_{\mathbb{R}}\langle\eta\rangle^{1+\delta}\langle\xi\rangle^{2+2\delta}\left|Q(\partial u,\partial^2 u)\right|^2dxds.$$

which is finite by Lemma 3.3. Thus, **K** decays to zero as $t\to\infty$, which implies (4.1).

Secondly, we turn to prove

$$\lim_{t\to\infty}\left\|\langle\xi\rangle^{1+\delta}\partial_\xi\int_t^\infty Q(\partial u,\partial^2 u)ds\right\|_{L^2_\xi(\mathbb{R})}=0. \tag{4.2}$$

In the same manner, we only need show that the following integral

$$\mathbf{L}:=\int_{\mathbb{R}}\left|\partial_\xi\int_t^\infty Q(\partial u,\partial^2 u)ds\right|^2\langle\xi\rangle^{2+2\delta}d\xi$$

is finite. Analogous to the simplification from **J** to **J**′ in the previous section, it suffices to consider

$$\mathbf{L}':=\int_{\mathbb{R}}\left|\int_t^\infty\partial_\xi Q(\partial u,\partial^2 u)ds\right|^2\langle\xi\rangle^{2+2\delta}d\xi.$$

We then proceed by the same method as employed in **K** above:

$$\mathbf{L}'\leqslant C\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\langle\eta\rangle^{-(1+\delta)}d\eta\right)\left(\int_{\mathbb{R}}\langle\eta\rangle^{1+\delta}\left|\partial_\xi Q(\partial u,\partial^2 u)\right|^2\cdot\chi_{s\geqslant t}(\xi,\eta)d\eta\right)\langle\xi\rangle^{2+2\delta}d\xi$$
$$\leqslant C\int_t^\infty\int_{\mathbb{R}}\langle\eta\rangle^{1+\delta}\langle\xi\rangle^{2+2\delta}\left|\partial_\xi Q(\partial u,\partial^2 u)\right|^2dxds.$$

We have already seen in the proof of Lemma 3.9 that this integral is finite. As a consequence, **L** decays to zero as $t\to\infty$, which yields (4.2).

Up to now, (1.20) is evident from what we have proved.

4.2. The proof of the rigidity part. We now assume that the scattering fields vanish at infinities, i.e.,

$$u_\xi(\infty)=0\ \text{ on }\mathcal{C}_\xi,\quad u_\eta(\infty)=0\ \text{ on }\mathcal{C}_\eta.$$

Let $0<\varepsilon\leqslant\varepsilon_0$ be an arbitrarily given small positive constant. By (1.20), there exists a $t_\varepsilon>0$ such that

$$\left\|\langle\xi\rangle^{1+\delta}u_\xi(t_\varepsilon,2\xi-t_\varepsilon)\right\|_{H^1_\xi(\mathbb{R})}+\left\|\langle\eta\rangle^{1+\delta}u_\eta(t_\varepsilon,-2\eta+t_\varepsilon)\right\|_{H^1_\eta(\mathbb{R})}\leqslant\varepsilon.$$

We are now in a position to study the position parameter a. At initial slice Σ_0, the position parameter $a_0=0$ is given so that we have constructed the solution $u(t,x)$. At time slice Σ_{t_ε}, if we take $a=\frac{t_\varepsilon}{2}$ as the new position parameter and regard $u(t_\varepsilon,x)$ as the initial data for the Cauchy problem (1.4)-(1.5), then we can solve the equations backwards in time as follows on account that Theorem 1.1 is independent of the position parameter a.

According to the proof of Theorem 1.1, we conclude that at time slice Σ_0, there holds

$$\left\|\langle\xi\rangle^{1+\delta}u_\xi(0,2\xi)\right\|_{H^1_\xi(\mathbb{R})}+\left\|\langle\eta\rangle^{1+\delta}u_\eta(0,-2\eta)\right\|_{H^1_\eta(\mathbb{R})}\leqslant C^{\frac12}A\varepsilon.$$

Since ε is arbitrary, we arrive at the conclusion that

$$\left\|\langle\xi\rangle^{1+\delta}u_\xi(0,2\xi)\right\|_{H^1_\xi(\mathbb{R})}+\left\|\langle\eta\rangle^{1+\delta}u_\eta(0,-2\eta)\right\|_{H^1_\eta(\mathbb{R})}=0.$$

This implies that the left-traveling waves and the right-traveling waves vanish identically. Therefore, the solution itself vanishes identically, i.e., $u(t,x) \equiv 0$ on $\mathbb{R}^{1+1}$.

The proof of Theorem 1.4 is now complete.

Acknowledgement

The author is greatly indebted to Prof. Pin Yu for the guidance over the past years. Sincere thanks also go to Dr. Ziyao Wang and Dr. Bei Xu for helpful comments. The author would also like to express heartfelt gratitude to the anonymous referees for useful suggestions. Finally, the support of Yau Mathematical Sciences Center at Tsinghua University is gratefully acknowledged.

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