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A Remark on Attractor Bifurcation

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Abstract. In this paper we present some local dynamic bifurcation results in terms of invariant sets of nonlinear evolution equations. We show that if the trivial solution is an isolated invariant set of the system at the critical value $\lambda = \lambda_0$, then either there exists a one-sided neighborhood I^- of λ_0 such that for each $\lambda \in I^-$, the system bifurcates from the trivial solution to an isolated nonempty compact invariant set K_λ with $0 \notin K_\lambda$, or there is a one-sided neighborhood I^+ of λ_0 such that the system undergoes an attractor bifurcation for $\lambda \in I^+$ from $(0, \lambda_0)$. Then we give a modified version of the attractor bifurcation theorem. Finally, we consider the classical Swift-Hohenberg equation and illustrate how to apply our results to a concrete evolution equation.

Contents

1. Introduction

This note is concerned with the dynamic bifurcation of the nonlinear evolution equation

$$u_t + Au = f_\lambda(u) \tag{1.1}$$

on a Banach space X, where $A : X^\alpha \to X$ is a sectorial operator with compact resolvent for some $0 \leqslant \alpha < 1$, $\lambda \in \mathbb{R}$ is the bifurcation parameter, and $f_\lambda(u)$ is

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a locally Lipschitz continuous mapping from $X^\alpha \times \mathbb{R}$ to X. We also assume that $f_\lambda(0) = 0$ for $\lambda \in \mathbb{R}$ and

$$f_\lambda(u) = Df_\lambda(0)u + g_\lambda(u)$$

with $Df_\lambda(u)$ continuous in (u, λ), and that $g_\lambda(u) = o(\|u\|_{X^\alpha})$ as $\|u\|_{X^\alpha} \to 0$. So $u = 0$ is always a trivial solution of (1.1) for each $\lambda \in \mathbb{R}$.

A quite fundamental result in the dynamic bifurcation theory is the well-known Hopf bifurcation theorem [**10, 12, 16, 18**], which concerns the bifurcation of a closed orbit from an equilibrium point. The Hopf bifurcation theorem plays an important role in the study of nonlinear dynamics, and has been fully developed in the last century. However, it only applies to the case where there are exactly a pair of conjugate eigenvalues of the linearized equation crossing the imaginary axis. In practice, the linearized equation near the equilibrium may have more than two eigenvalues crossing the imaginary axis. To deal with this case, a general dynamic bifurcation theory needs to be developed, and this can be performed in the context of invariant sets [**2, 3, 4, 7, 13, 15, 24, 27**], etc.

A particular but important case of the invariant-set bifurcation is the so-called attractor bifurcation, which was systematically studied by Ma and Wang [**11, 19, 20, 21**] and was further developed into a dynamic transition theory [**22**]. Sanjurjo [**25**] also addressed the attractor bifurcation theory from the point of view of topology. Roughly speaking, the attractor bifurcation theory states that if the trivial solution 0 of (1.1) changes from an attractor to a repeller as λ crosses certain critical value λ_0, then the system bifurcates an attractor from the trivial solution. But in [**19**] etc. it was assumed that the trivial solution is an attractor of the system on the local center manifold when $\lambda = \lambda_0$. Because the system is degenerate when it is restricted on the center manifold, the verification of the condition that the trivial solution is an attractor is often not an easy task.

In this paper we give a modified version of the attractor bifurcation theorem in [**19**], which drops the additional condition mentioned above and makes the theorem more efficient in applications. Specifically, let Φ_λ be the local semiflow generated by (1.1), and let $\mathcal{A}_0 = \{0\}$. Suppose $\mathcal{A}_0$ is an attractor of Φ_λ for each $\lambda < \lambda_0$, and that there is at least one eigenvalue of the linearized equation of (1.1) near the trivial solution crossing the imaginary axis at the critical value $\lambda = \lambda_0$. We prove that if $\mathcal{A}_0$ is an isolated invariant set of Φ_{λ_0}, then there exists $\varepsilon_1 > 0$ such that the system Φ_λ bifurcates an isolated invariant set K_λ with $0 \notin K_\lambda$ for each $\lambda \in [\lambda_0 - \varepsilon_1, \lambda_0)$, or it bifurcates an attractor for each $\lambda \in (\lambda_0, \lambda_0 + \varepsilon_1]$. In particular, if $\mathcal{A}_0$ is the global attractor of Φ_λ for each $\lambda < \lambda_0$, then it immediately follows that the system undergoes an attractor bifurcation on $(\lambda_0, \lambda_0 + \varepsilon_1]$. Note that we do not assume the trivial solution to be an attractor of the system at $\lambda = \lambda_0$.

This work is organized as follows. In Section 2 we introduce some basic concepts and results concerning invariant sets. In Section 3 we first show our main results and then give the proofs of the results. Finally we consider the classical Swift-Hohenberg equation to illustrate our results.

2. Preliminaries

In this section we introduce some basic concepts concerning local semiflows. First, let X be a complete metric space with metric $d(\cdot, \cdot)$.

Definition 2.1. A local semiflow Φ defined on X is a continuous map from an open set $\mathcal{D}(\Phi) \subset \mathbb{R}^+ \times X$ to X and satisfies the following properties:

(1) For each $x \in X$, there exists $0 < T_x \leqslant \infty$ such that

$$(t, x) \in \mathcal{D}(\Phi) \Longleftrightarrow t \in [0, T_x).$$

(2) $\Phi(0, \cdot) = \mathrm{id}_X$, and

$$\Phi(t + s, x) = \Phi(t, \Phi(s, x))$$

for every $x \in X$ and $t, s \geqslant 0$ with $t + s \leqslant T_x$.

The number T_x is called the escape time of $\Phi(t, x)$.

For simplicity, we usually rewrite $\Phi(t, x)$ as $\Phi(t)x$.

Let $I \subset \mathbb{R}$ be an interval. A *trajectory* (or *solution*) of Φ on I is a continuous mapping $\gamma : I \to X$ with

$$\gamma(t) = \Phi(t - s)\gamma(s), \quad \forall t, s \in I,\ t \geqslant s.$$

If $I = \mathbb{R}$, the trajectory γ is called a *full trajectory.*

The *ω-limit set* $\omega(\gamma)$ and *α-limit set* $\alpha(\gamma)$ of a full trajectory γ are defined as

$$\omega(\gamma) = \{y \in X : \text{ there exists } t_n \to \infty \text{ such that } \gamma(t_n) \to y\},$$

$$\alpha(\gamma) = \{y \in X : \text{ there exists } t_n \to -\infty \text{ such that } \gamma(t_n) \to y\}.$$

DEFINITION 2.2. Let $N \subset X$. We say that Φ does not explode in N, if we can infer $T_x = \infty$ from $\Phi([0, T_x))x \subset N$.

DEFINITION 2.3. [**24**] $N \subset X$ is said to be admissible, if for every sequences $x_n \in N$ and $t_n \to \infty$ with $\Phi([0, t_n])x_n \subset N$ for all n, the sequence $\Phi(t_n)x_n$ has a convergent subsequence.

N is said to be strongly admissible, if it is admissible and moreover, Φ does not explode in N.

DEFINITION 2.4. Φ is said to be asymptotically compact on X, if each bounded subset B of X is strongly admissible.

From now on, we always assume that

(**AC**) Φ is asymptotically compact on X.

DEFINITION 2.5. A set $\mathcal{A} \subset X$ is said to be positively invariant (resp., invariant) for Φ, if $\Phi(t)\mathcal{A} \subset \mathcal{A}$ (resp., $\Phi(t)\mathcal{A} = \mathcal{A}$) for each $t \geqslant 0$. An invariant set $\mathcal{A}$ is called an attractor of Φ, if it is compact and attracts a neighborhood U of itself, that is,

$$\lim_{t \to \infty} \mathrm{dist}_H(\Phi(t)U, \mathcal{A}) = 0.$$

The attraction basin of $\mathcal{A}$, denoted by $\Omega(\mathcal{A})$, is defined as

$$\Omega(\mathcal{A}) = \{x \in X : \lim_{t \to \infty} \mathrm{dist}_H(\Phi(t)x, \mathcal{A}) = 0\},$$

where dist_H denotes the Hausdorff semi-distance.

Suppose M is a compact invariant set. Then the restriction Φ_M of Φ on M is also a semiflow. A compact set $\mathcal{A} \subset M$ is called an *attractor* of Φ in M, which means that $\mathcal{A}$ is an attractor of Φ_M in M.

Let $\mathcal{A}$ be an attractor of Φ in M. Define

$$\mathcal{R} = \{x \in M : \omega(x) \cap \mathcal{A} = \emptyset\}.$$

Then $\mathcal{R}$ is called the *repeller* of Φ in M dual to $\mathcal{A}$, and $(\mathcal{A}, \mathcal{R})$ is called an *attractor-repeller pair* of Φ in M.

LEMMA 2.6. *Let $\mathcal{R} \subset M$ be a nonempty compact invariant set. Suppose that there exists an open neighborhood W of $\mathcal{R}$ in M such that for each $x \in W \setminus \mathcal{R}$ and each complete trajectory γ in M through x, one has*

$$\alpha(\gamma) \subset \mathcal{R}, \quad \omega(\gamma) \cap W = \emptyset.$$

Denote $\mathcal{A}$ the maximal compact invariant set in $M \backslash W$. Then $(\mathcal{A}, \mathcal{R})$ is an attractor-repeller pair.

PROOF. By the definition of $\mathcal{A}$, it is clear that $\mathcal{A}$ and $\mathcal{R}$ are disjointed. Let $x \in M \setminus (\mathcal{A} \cup \mathcal{R})$, and let γ be a complete trajectory in M through x. We claim that there exists a $t_0 \in \mathbb{R}$ such that $\gamma(t_0) \in W$. Indeed, if this was not the case, then one would have $\gamma(t) \subset M \setminus W$ for all $t \in \mathbb{R}$. Therefore by the definition of $\mathcal{A}$, we find that γ is contained in $\mathcal{A}$. This leads to a contradiction (as $\gamma(0) = x \notin \mathcal{A}$).

Now by the assumption of the lemma, one easily verifies that

$$\alpha(\gamma) \subset \mathcal{R}, \quad \omega(\gamma) \subset \mathcal{A},$$

and the conclusion of the lemma follows immediately from Theorems 1.7 and 1.8 on Morse decompositions of invariant sets in [**24**], Chap. III. □

Next, we recall some basic concepts and results concerning the Conley index theory. One can refer to [**5, 17, 24**], etc. for details.

Let N, E be two closed subsets of X. We say that E is an *exit set* of N, if it satisfies

(1) E is N-positively invariant, that is, if for any $x \in E$ and $t \geqslant 0$,

$$\Phi([0,t])x \subset N \Longrightarrow \Phi([0,t])x \subset E;$$

(2) For any $x \in N$, if $\Phi(t_1)x \notin N$ for some $t_1 > 0$, then there exists a $t_0 \in [0, t_1]$ such that $\Phi(t_0)x \in E$.

A compact invariant set $\mathcal{A}$ of Φ is said to be *isolated*, if there exists a bounded closed neighborhood N of $\mathcal{A}$ such that $\mathcal{A}$ is the maximal invariant set in N. Consequently, N is called an *isolating neighborhood* of $\mathcal{A}$.

Let $\mathcal{A}$ be a compact isolated invariant set. A pair of closed subsets (N, E) is said to be an *index pair* of $\mathcal{A}$, if it satisfies the following conditions:

(1) $N \setminus E$ is an isolating neighborhood of $\mathcal{A}$;

(2) E is an exit set of N.

DEFINITION 2.7. Let (N, E) be an index pair of $\mathcal{A}$. Then the homotopy Conley index of $\mathcal{A}$ is defined to be the homotopy type $[(N/E, [E])]$ of the pointed space $(N/E, [E])$, denoted by $h(\Phi, \mathcal{A})$.

Next we present an important result on the continuation property of Conley index, which plays an important role in the proof of our invariant sets bifurcation results.

Let Φ_λ be a family of semiflows with parameter $\lambda \in \Lambda$, where Λ is a connected compact metric space. Suppose that $\Phi_\lambda(t)x$ is continuous in (t, x, λ). Define the *skew-product flow* $\widetilde{\Phi}$ of the family Φ_λ on $X \times \Lambda$ by

$$\widetilde{\Phi}(t)(x, \lambda) = (\Phi_\lambda(t)x, \lambda), \qquad (x, \lambda) \in X \times \Lambda.$$

LEMMA 2.8. [**15**] *Let $\widetilde{\Phi}$ satisfy the assumption* (**AC**) *on $X \times \Lambda$. Suppose $\mathcal{M}$ is a compact isolated invariant set of $\widetilde{\Phi}$. Then $h(\Phi_\lambda, M_\lambda)$ is constant for $\lambda \in \Lambda$, where $M_\lambda = \{x : (x, \lambda) \in \mathcal{M}\}$ is the λ-section of $\mathcal{M}$.*

Suppose that $B \subset X$ is a bounded closed set. $x \in \partial B$ is said to be a *strict egress* (resp., *strict ingress*, *bounce-off*) *point* of B, if for each trajectory $\gamma : [-\delta, \tau] \to X$ with $\gamma(0) = x$, where $\delta \geqslant 0, \tau > 0$, the following two properties hold.

(1) There exists $0 < s < \tau$ so that

$$\gamma(t) \notin B \quad (\text{resp.,} \quad \gamma(t) \in \text{int}B, \quad \text{resp.,} \quad \gamma(t) \notin B), \qquad \forall t \in (0, s);$$

(2) If $\delta > 0$, then there exists $0 < \beta < \delta$ such that

$$\gamma(t) \in \text{int}B \quad (\text{resp.,} \quad \gamma(t) \notin B, \quad \text{resp.,} \quad \gamma(t) \notin B), \qquad \forall t \in (-\beta, 0).$$

Denote the set of all strict egress (resp. strict ingress, bounce-off) points of the closed set B by B^e(resp. B^i, B^b), and set $B^- = B^e \cup B^b$. For convenience in statement, if B^- is the exit set, we call B^- the *boundary exit set* of B.

A closed set $B \subset X$ is called an *isolated block* [**24**] if B^- is closed and $\partial B = B^i \cup B^-$.

Suppose that $K \subset X$ is a compact isolated invariant set and the isolating block B is the isolating neighborhood of K. If B^- is B-positively invariant, then B is called the *index neighborhood* of K.

By the definition of index neighborhood, we have the following result.

THEOREM 2.9. *Let $K \subset X$ be a compact isolated invariant set and N be the isolating neighborhood of K. Then there exists an isolating block B in N such that B is the index neighborhood of K.*

PROOF. By Chapter 1, Theorem 5.1 in [**24**], we deduce that there exists a bounded closed set $B \subset N$ with $K \subset B$ such that B is an isolating block. Moreover, from [**24**], it holds that if B is a bounded isolating block, then (B, B^-) is an index pair of the maximal compact invariant K in B. Thus the result follows from the definition of index neighborhood. □

3. Invariant-set/attractor bifurcation

In this section, we establish some local bifurcation results in terms of invariant sets.

3.1. Main results. It is well known that (see e.g. [**8, 26**]) that the Cauchy problem of (1.1) is well-posed in X^α under the assumptions in Section 1. Specifically, for any initial value $u_0 \in X^\alpha$, the equation (1.1) has a unique continuous solution $u(t) \in X^\alpha$ with $u(0) = u_0$ on a maximal existence interval $[0, T)$ for some $T > 0$. Let Φ_λ be the local semiflow generated by the equation (1.1) on X^α.

Let $L_\lambda = A - Df_\lambda(0)$. Assume there exist a neighborhood $J_0 = (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ of λ_0 and a positive constant β such that the following hypotheses (**A**1)-(**A**3) are satisfied.

(**A**1) The spectrum $\sigma(L_\lambda)$ has a decomposition $\sigma(L_\lambda) = \bigcup_{1 \leqslant i \leqslant 2} \sigma_\lambda^i$ such that

$$\max_{\mu \in \sigma_\lambda^1} |\mathbf{Re}(\mu)| \leqslant \beta \quad \text{and} \quad \min_{\mu \in \sigma_\lambda^2} \mathbf{Re}(\mu) \geqslant 2\beta, \quad \forall \lambda \in J_0.$$

(**A**2) For every $\lambda \in J_0$, X^α has a decomposition $X^\alpha = X_\lambda^1 \oplus X_\lambda^2$ with respect to the spectral decomposition in (**A**1), where $X_\lambda^i (i = 1, 2)$ are L_λ-invariant subspaces of X^α. Moreover,

$$1 \leqslant \dim(X_\lambda^1) < \infty.$$

(**A**3) The projection operator $P_\lambda^1 : X^\alpha \to X_\lambda^1$ is continuous in λ.

For simplicity, from now on we drop the subscript "λ_0" and rewrite $X^i = X_{\lambda_0}^i$. Let $E = X^\alpha$, and

$$E^i = E \cap X^i, \qquad i = 1, 2.$$

Then $E = E^1 \oplus E^2$. Furthermore, since X^1 is finite dimensional, we have $E^1 = X^1$ and $n := \dim(E^1) \geqslant 1$.

By virtue of Proposition 2 in Appendix A, there is a family of isomorphisms $T = T_\lambda$ ($\lambda \in J_0$) on X depending continuously on λ with $T_{\lambda_0} = I$, such that

$$TX_\lambda^i = X_{\lambda_0}^i := X^i, \qquad i = 1, 2.$$

Next we introduce the definition of invariant sets bifurcation, and then we state and prove the local invariant sets bifurcation result.

Definition 3.1. Let I be a subset of $\mathbb{R}$. We say that the equation (1.1) undergoes an invariant-set bifurcation on I from the trivial solution $(0, \lambda_0)$, if there is a sequence $\lambda_n \in I$, $\lambda_n \to \lambda_0$ as $n \to \infty$, such that Φ_{λ_n} has an invariant set $\mathcal{A}_{\lambda_n}$ with $\mathcal{A}_{\lambda_n} \setminus \{0\} \neq \emptyset$; furthermore,

$$\lim_{n\to\infty} \mathrm{dist}_H(\mathcal{A}_{\lambda_n}, 0) = 0.$$

If each invariant set $\mathcal{A}_{\lambda_n}$ is an attractor of the system with $0 \notin \mathcal{A}_{\lambda_n}$, then we say that (1.1) undergoes an attractor bifurcation on I from $(0, \lambda_0)$.

Theorem 3.2. *Let the assumptions* (**A**1)-(**A**3) *hold true. Suppose*

$$(\mathbf{A}4) \qquad \min_{\mu\in\sigma_\lambda^1} \mathbf{Re}(\mu) > 0 \ (\lambda \in (\lambda_0 - \varepsilon, \lambda_0)), \qquad \max_{\mu\in\sigma_\lambda^1} \mathbf{Re}(\mu) < 0 \ (\lambda \in (\lambda_0, \lambda_0 + \varepsilon)).$$

If $\mathcal{A}_0 := \{0\}$ is an isolated invariant set of Φ_{λ_0}, then there exist $\varepsilon_1 > 0$ and a closed neighborhood W of 0 in E such that one of the alternatives holds.

(1) *The system undergoes an invariant-set bifurcation on $I^- = [\lambda_0 - \varepsilon_1, \lambda_0)$ from $(0, \lambda_0)$. More precisely, for any $\lambda \in I^-$, Φ_λ has a nonempty compact invariant set K_λ with K_λ in $W \setminus \{0\}$ such that*

$$\lim_{\lambda\to\lambda_0} \mathrm{dist}_H(K_\lambda, 0) = 0. \tag{3.1}$$

(2) *The system undergoes an attractor bifurcation on $I^+ = (\lambda_0, \lambda_0 + \varepsilon_1]$ from $(0, \lambda_0)$. Specifically, for any $\lambda \in I^+$, Φ_λ has an attractor K_λ with K_λ in $W \setminus \{0\}$ such that (3.1) holds. Furthermore, K_λ contains an invariant topological sphere $\mathbb{S}^{n-1}$.*

As a simple consequence of the above theorem, we have the following attractor bifurcation result.

Theorem 3.3. *Let the assumptions* (**A**1)-(**A**4) *hold true. Suppose also that $\mathcal{A}_0 := \{0\}$ is a global attractor for each $\lambda < \lambda_0$.*

Then either $\mathcal{A}_0$ is not isolated with respect to Φ_{λ_0}, or the system undergoes an attractor bifurcation on $I^+ = (\lambda_0, \lambda_0 + \varepsilon_1]$ from $(0, \lambda_0)$ for some $\varepsilon_1 > 0$. Specifically, for any $\lambda \in I^+$, Φ_λ has an attractor K_λ with K_λ in $W \setminus \{0\}$ for some closed *neighborhood W of 0 in E such that* (3.1) *holds and K_λ contains an invariant topological sphere $\mathbb{S}^{n-1}$.*

REMARK 3.4. The above result drops an additional assumption that the trivial solution $\mathcal{A}_0$ is an attractor on the local center manifold with respect to the system Φ_{λ_0}, just as what is expressed in Ma and Wang [**19**] and only assumes that the trivial solution $\mathcal{A}_0$ is an isolated invariant set of Φ_{λ_0}. In this sense, our result is more general and extends the attractor bifurcation result obtained in [**19**].

3.2. Proof of the main results. In this subsection we will give the proof of our main result. For this purpose, we first introduce some lemmas, which are crucial in our proof.

LEMMA 3.5. *Let the assumptions* (**A**1)-(**A**3) *hold true. Then there exist an open convex neighborhood U of 0 in E^1 and a mapping $\xi = \xi_\lambda(v)$ from $U \times J_0$ to E^2, which is continuous in (v, λ) and differentiable in v, such that for any $\lambda \in J_0$,*

$$M_\lambda := T^{-1}\mathcal{M}_\lambda, \qquad \text{where} \quad \mathcal{M}_\lambda := \{v + \xi_\lambda(v), \quad v \in U\}, \tag{3.2}$$

is a local invariant manifold of the system (1.1) *with $\xi_\lambda(0) = D\xi_\lambda(0) = 0$.*

The proof of Lemma 3.1 follows from the standard argument in the geometric theory of PDEs and the uniform contraction principle, one can see Henry [**8**] and Hale [**9**] for details.

LEMMA 3.6. [**15**] *Let the assumptions* (**A**1)-(**A**3) *hold true and M_λ be the local invariant manifold obtained in Lemma 3.1, and Φ^1_λ be the restriction of Φ_λ on M_λ.*

Then there exist a neighborhood U_0 of 0 and $\varepsilon_0 > 0$ such that for each $\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0 + \varepsilon_0]$, $\mathcal{A}_\lambda$ is an isolated invariant set of Φ_λ in U_0 iff it is an isolated invariant set of Φ^1_λ on M_λ. Furthermore,

$$h(\Phi_\lambda, \mathcal{A}_\lambda) = h(\Phi^1_\lambda, \mathcal{A}_\lambda).$$

The proof of Theorem 3.1 is as follows.

PROOF. Because the trivial solution $\mathcal{A}_0$ is isolated with respect to Φ_{λ_0}, two cases my occur.

Case one. If $\mathcal{A}_0$ is not an attractor, next we prove that (1) of Theorem 3.1 holds true.

According to assumption (**A**4), we see that the equilibrium $\mathcal{A}_0 = \{0\}$ is an attractor of Φ_λ for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$.

Set $B_\lambda = TL_\lambda T^{-1}$ and define

$$g_\lambda(v) = T(f_\lambda(T^{-1}v) - Df_\lambda(0)(T^{-1}v)), \qquad v \in E.$$

Then the system (1.1) can be transformed into the following equivalent equation by letting $u = T^{-1}v$,

$$v_t = -B_\lambda v + g_\lambda(v), \tag{3.3}$$

for $\lambda \in J_0$. When system (3.3) is restricted on the local center manifold $\mathcal{M}_\lambda$ defined by (3.2), it reduces to an ODE system on a neighborhood U (independent of λ) of $\mathcal{A}_0$ in E^1:

$$v_t = -B^1_\lambda v + P^1 g_\lambda(v + \xi_\lambda(v)) := G_\lambda(v), \tag{3.4}$$

where $B^1_\lambda = P^1 B_\lambda$, and $P^1 : E = X^\alpha \to E^1$ is the projection operator. Applying Lemma 3.2 to system (3.3) , we conclude that there exists a neighborhood $\mathcal{U}$ of 0 in E and $\varepsilon_0 > 0$ such that for any $\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0 + \varepsilon_0]$, $\mathcal{A}_\lambda$ is an isolated invariant set of system (3.3) in $\mathcal{U}$ iff it is an isolated invariant set of the system restricted on the manifold $\mathcal{M}_\lambda$.

Since $\mathcal{A}_0$ is isolated, we can choose a bounded closed neighborhood W such that W is an isolated neighborhood of $\mathcal{A}_0$ with respect to Φ_{λ_0}. By a simple argument via contradiction, one can easily verify that W is also an isolating neighborhood of the maximal compact invariant set $\mathcal{A}_\lambda$ of Φ_λ in W, provided λ near λ_0. That is to say, there exists $0 < \varepsilon_1 \leqslant \varepsilon_0$ so that W is also an isolated neighborhood of $\mathcal{A}_\lambda$ for every $\lambda \in [\lambda_0 - \varepsilon_1, \lambda_0 + \varepsilon_1]$. It is trivial to check that

$$\lim_{\lambda \to \lambda_0} \mathrm{dist}_H(\mathcal{A}_\lambda, \mathcal{A}_0) = 0. \tag{3.5}$$

Moreover, Lemma 3.2 also shows that $\mathcal{A}_\lambda$ is also an isolated invariant set of the system on the local center manifold $\mathcal{M}_\lambda$ for $\lambda \in [\lambda_0 - \varepsilon_1, \lambda_0 + \varepsilon_1]$. Denote by ϕ_λ the semiflow generated by the system (3.4). Because the topological structure of the solutions of system (3.3) on $\mathcal{M}_\lambda$ is equivalent to that of the system (3.4) on U, $\mathcal{A}_0$ is also an isolated invariant set of ϕ_0 on U. Then one can pick a closed isolated neighborhood U_0 of $\mathcal{A}_0$ satisfying $U_0 \subset U$. By Theorem 2.1, we take a closed neighborhood $N_0 \subset U_0$ of $\mathcal{A}_0$ such that N_0 is an index neighborhood of $\mathcal{A}_0$ with respect to ϕ_0. We may suppose that N_0 is path-connected. By virtue of (3.5), it can be assumed that the N_0 is also an isolating neighborhood of $\mathcal{A}_\lambda$ of ϕ_λ for $\lambda \in [\lambda_0 - \varepsilon_1, \lambda_0 + \varepsilon_1]$. Therefore by the continuation property of Conley index, we have

$$h(\mathcal{A}_\lambda, \phi_\lambda) \equiv const, \qquad \lambda \in [\lambda_0 - \varepsilon_1, \lambda_0 + \varepsilon_1]. \tag{3.6}$$

On the other hand, because $\mathcal{A}_0$ is an attractor of Φ_λ for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$, one deduces that $\mathcal{A}_0$ is also an attractor of ϕ_λ on U for $\lambda \in [\lambda_0 - \varepsilon_1, \lambda_0)$. So there also exists a path-connected closed neighborhood N_1 of $\mathcal{A}_0$ such that N_1 is an isolated neighborhood of $\mathcal{A}_0$ with respect to ϕ_λ for $\lambda \in [\lambda_0 - \varepsilon_1, \lambda_0)$. Note that $\mathcal{A}_0$ is not an attractor for ϕ_0, one concludes that the boundary exit set $N_0^- \neq \emptyset$. By some elementary computations of homology Conley index of $\mathcal{A}_0$, we have

$$\begin{aligned} &H_0(h(\mathcal{A}_0, \phi_0)) = H_0([N_0/N_0^-, [N_0^-]]) = 0, \\ &H_0(h(\mathcal{A}_0, \phi_\lambda)) = H_0([N_1/\emptyset, [\emptyset]]) = H_0([N_1 \cup \{p\}/\{p\}, \{p\}]) = \mathbb{Z}, \end{aligned} \tag{3.7}$$

for $\lambda \in [\lambda_0 - \varepsilon_1, \lambda_0)$, where $p \notin N_1$, which implies $h(\mathcal{A}_0, \phi_0) \neq h(\mathcal{A}_0, \phi_\lambda)$. Therefore, we deduce from (3.6) that for each $\lambda \in [\lambda_0 - \varepsilon_1, \lambda_0)$,

$$h(\mathcal{A}_\lambda, \phi_\lambda) = h(\mathcal{A}_0, \phi_0) \neq h(\mathcal{A}_0, \phi_\lambda).$$

Then it follows that $\mathcal{A}_\lambda \setminus \mathcal{A}_0 \neq \emptyset$. Since $\mathcal{A}_0$ is an attractor of ϕ_λ, we conclude that the set

$$K_\lambda := \{x \in \mathcal{A}_\lambda : \omega(x) \cap \mathcal{A}_0 = \emptyset\}$$

is a nonempty compact invariant set of ϕ_λ with $(\mathcal{A}_0, K_\lambda)$ being an attractor-repeller pair of $\mathcal{A}_\lambda$; see [**24**], pp.141. Note that $\mathcal{A}_\lambda$ is maximal in N_0. So one can see that K_λ is also the maximal compact invariant set of ϕ_λ in $N_0 \setminus \mathcal{A}_0$ for $\lambda \in [\lambda_0 - \varepsilon_1, \lambda_0)$. Thus, let $I^- = [\lambda_0 - \varepsilon_1, \lambda_0)$ and by (3.5), we see that (1) holds true.

Case two. If $\mathcal{A}_0$ is an attractor, then we prove that (2) of Theorem 3.1 holds true. Indeed, the proof of this case is a slight modification of the one for the corresponding result in Ma and Wang [**19**]. Here we give the details for completeness and the reader' convenience.

Similarly applying Lemma 3.2 to (3.3), we can take a neighborhood $\mathcal{U}$ of 0 in E and $\varepsilon_0 \in (0, \varepsilon)$ such that for any $\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0 + \varepsilon_0]$, $\mathcal{A}_\lambda$ is an isolated invariant set of Φ_λ in $\mathcal{U}$ if and only if it is an isolated invariant set of the system on the manifold $\mathcal{M}_\lambda$. Now we consider the system (3.4) on U, which is restricted on the local center manifold $\mathcal{M}_\lambda$.

Let ϕ_λ be the semiflow generated by system (3.4) on U. Then one deduces that $\mathcal{A}_0$ is also an attractor of ϕ_0. Denote by $\Omega = \Omega(\mathcal{A}_0)$ the attraction basin of $\mathcal{A}_0$ in U with respect to ϕ_0. Owing to the converse Lyapunov theorems on attractors [**14**], we deduce that there is a function $V \in C^\infty(\Omega)$ such that

$$\nabla V(x) \cdot G_{\lambda_0}(x) \leqslant -w(x), \qquad \forall x \in \Omega, \tag{3.8}$$

and satisfies $V(0) = 0$ and $\lim\limits_{x \to \partial\Omega} V(x) = +\infty$, where $w \in C(\Omega)$ and $w(x) > 0$ for $x \neq 0$, G_λ is given by (3.4). Set

$$\mathcal{V} = V_a := \{x \in \Omega : V(x) \leqslant a\}.$$

Then one can easily conclude that $\mathcal{V}$ is a compact positively invariant neighborhood of 0 in E^1 for each $a > 0$. Now we choose two positive numbers a, b sufficiently small such that

$$\tilde{W} := \mathcal{V} \times B_{E^2}(\xi_{\lambda_0}(\mathcal{V}), b) \subset \mathcal{U},$$

where ξ_{λ_0} is the local center manifold mapping obtained in Lemma 3.1, and

$$B_{E^2}(\xi_{\lambda_0}(\mathcal{V}), b)$$

means the b-neighborhood of $\xi_0(\mathcal{V})$ in E^2.

Let $c = \min\limits_{x \in \partial\mathcal{V}} w(x) > 0$. Thanks to (3.8), we have

$$\nabla V(x) \cdot G_{\lambda_0}(x) \leqslant -c, \qquad \forall x \in \partial\mathcal{V}, \tag{3.9}$$

where $\partial\mathcal{V}$ denotes the boundary of $\mathcal{V}$ in E^1. Moreover, by the continuity of G_λ, there exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that for each $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_1]$,

$$\nabla V(x) \cdot G_\lambda(x) \leqslant -\frac{c}{2}, \qquad \forall x \in \partial\mathcal{V}. \tag{3.10}$$

Note that ξ_λ is continuous in λ. Then it can be assumed that the ε_1 is sufficiently small such that

$$\xi_\lambda(\mathcal{V}) \subset B_{E^2}(\xi_{\lambda_0}(\mathcal{V}), b), \qquad \lambda \in [\lambda_0, \lambda_0 + \varepsilon_1].$$

Therefore

$$\mathcal{V} \times \xi_\lambda(\mathcal{V}) \subset \tilde{W} \subset \mathcal{U}, \qquad \lambda \in [\lambda_0, \lambda_0 + \varepsilon_1]. \tag{3.11}$$

From (3.10), one can deduce that $\mathcal{V}$ is an absorbing set of ϕ_λ. So ϕ_λ has an attractor $\mathcal{A}_\lambda$ which is the maximal invariant set of ϕ_λ in $\mathcal{V}$ for $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_1]$. By the upper semicontinuity of attractors [**19**], we have

$$\lim_{\lambda \to \lambda_0} \mathrm{dist}_H(\mathcal{A}_\lambda, 0) = 0.$$

Recalling $\mathbf{Re}\sigma_\lambda^1 < 0$ for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon_1]$, we see that $\mathcal{A}_0 := \{0\}$ is a repeller of ϕ_λ. Thus we conclude from Lemma 2.6 that $\mathcal{A}_\lambda$ has an attractor-repeller pair $(\mathcal{K}_\lambda, \mathcal{A}_0)$ for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon_1]$, where $\mathcal{K}_\lambda$ is the maximal compact invariant set of

ϕ_λ in $\mathcal{A}_\lambda \setminus \{0\}$. By the maximality of $\mathcal{A}_\lambda$, we conclude that $\mathcal{K}_\lambda$ is also the maximal compact invariant set of ϕ_λ in $\mathcal{V} \setminus \{0\}$ for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon_1]$.

Next we verify that $\mathcal{K}_\lambda$ has an invariant $(n-1)$-dimensional topological sphere. To this end, we consider the inverse flow ϕ_λ^- of ϕ_λ on U generated by the following system

$$w_t = -G_\lambda(w).$$

Thus, we see that $\mathcal{A}_0$ becomes an attractor of ϕ_λ^- for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon_1]$. Let $\Sigma := \Sigma(\mathcal{A}_0)$ denote the attraction basin of $\mathcal{A}_0$ with respect to ϕ_λ^-. It is trivial to check that $\partial\Sigma$ is invariant under ϕ_λ^-. Therefore $\partial\Sigma$ is also an invariant set of ϕ_λ, which implies $\partial\Sigma \subset \mathcal{K}_\lambda$. Now we claim Σ is contractible, so $\partial\Sigma$ is an $(n-1)$-dimensional topological sphere. Indeed, define

$$H(s,x) = \begin{cases} \phi_\lambda^-(\frac{s}{1-s})x, & s \in [0,1), x \in \Sigma; \\ 0, & s = 1, x \in \Sigma. \end{cases}$$

Hence H is a strong deformation retract which shrinks Σ to 0. Finally, we define

$$\tilde{\mathcal{K}}_\lambda = \{v + \xi_\lambda(v) : v \in \mathcal{K}_\lambda\}, \qquad \tilde{S} = \{v + \xi_\lambda(v) : v \in \partial\Sigma\}.$$

Since $\mathcal{K}_\lambda \subset \mathcal{V}$, and from (3.11) we infer that $\tilde{\mathcal{K}}_\lambda \subset \tilde{W} \subset \mathcal{U}$. One can easily see that $\tilde{\mathcal{K}}_\lambda$ is the maximal compact invariant set of (3.3) in $\tilde{W} \setminus \{0\}$. Thus let

$$W = T^{-1}\tilde{W}, \quad K_\lambda = T^{-1}\tilde{\mathcal{K}}_\lambda, \quad \mathbb{S}^{n-1} = T^{-1}\tilde{S}.$$

Then $W, K_\lambda, \mathbb{S}^{n-1}$ satisfy the requirements of (2) in Theorem 3.1. The proof of Theorem 3.1 is complete. □

4. Example

In this section, we give an example to illustrate how to apply our abstract results to a concrete evolution equation.

Consider the initial value problem of the classical Swift-Hohenberg equation as follows:

$$\begin{cases} u_t = -(I+\Delta)^2 u + \lambda u - u^3, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u(0) = u_0, \end{cases} \tag{4.1}$$

where $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is a real-valued function, $\Omega = (0,\pi) \subset \mathbb{R}$, and $\lambda \in \mathbb{R}$ is the bifurcation parameter.

Remark 4.1. Concerning this Swift-Hohenberg equation, in fact, the authors in [**28**] has obtained an attractor bifurcation result of system (4.1) by giving some precise estimates of solutions to prove that the trivial solution $u = 0$ is an attractor of the semiflow Φ_λ at the critical value $\lambda = \lambda_0$. Generally speaking, it is not easy to check that the trivial solution is an attractor of the system Φ_{λ_0}. However in Theorem 3.2, we obtain a more general attractor bifurcation result, which tells us that it suffices to check that the trivial solution is isolated, and then we can obtain the corresponding attractor bifurcation result.

In order to obtain the corresponding attractor bifurcation result of (4.1), we calculate the local center manifold (see e.g. [**6**]) of the trivial solution for the system at $\lambda = \lambda_0$ to check that the trivial solution $u = 0$ is isolated on the local center manifold.

For the mathematical setting, we consider the Hilbert space

$$H = \{\dot{L}^2(\Omega) : u(x,t) = u(x+\pi,t)\}, \quad H_1 = \{\dot{H}^4(\Omega) : u(x,t) = u(x+\pi,t)\},$$

where the dot " $\cdot$ " denotes $\int_0^\pi f\mathrm{d}x = 0$ for $f \in L^2$ or H^4, and equip H with the usual inner product and norm denoted by $(\cdot,\cdot)$, $\|\cdot\|$, respectively.

Let $L_\lambda = A - B_\lambda$, where $A = (I+\Delta)^2$ defined in $H_1 := D(A)$, $B_\lambda = \lambda I$. Then L_λ is a sectorial operator. We see that the eigenvalues and the corresponding eigenvectors of L_λ are as follows:

$$\lambda_k = (1-4k^2)^2 - \lambda, \quad e_{k1}(x) = \sqrt{\frac{2}{\pi}}\sin(2kx), \quad e_{k2}(x) = \sqrt{\frac{2}{\pi}}\cos(2kx) \tag{4.2}$$

for $k \geqslant 1$ associated with the periodic boundary condition:

$$u(x,t) = u(x+\pi,t).$$

Set $H_2 = D(A^{\frac{1}{2}})$ and $g = -u^3, u \in H_2$. Then $g : H_2 \to H$ is a locally Lipschitz continuous mapping, and the system (4.1) can be rewritten as

$$u_t + L_\lambda u = g(u). \tag{4.3}$$

According to [**8, 26**], we deduce that for each $u_0 \in H_2$, the system (4.3) has a unique global strong solution $u(t)$ in H_2 with $u(0) = u_0$.

The system (4.1) is a gradient system and one can check that the system has a natural Lyapunov function $V(u)$,

$$V(u) = \frac{1}{2}\int_\Omega |(I+\Delta)u|^2\mathrm{d}x - \int_\Omega F_\lambda(u)\mathrm{d}x, \qquad \text{where} \quad F_\lambda(s) = \frac{\lambda}{2}s^2 - \frac{1}{4}s^4.$$

Now we consider the case that $\lambda_0 := \lambda = 9$. Then the first eigenvalue of L_λ is $\lambda_1 = 0$, which has the multiplicity 2 and its two eigenvectors are

$$q_1(x) = \sin 2x, \quad q_2(x) = \cos 2x.$$

Let E_1 be the eigenspace spanned by q_1, q_2, that is

$$E_1 = \operatorname{span}\{q_1, q_2\}$$

and $E_2 = E_1^\perp$. Then $H = E_1 \oplus E_2$. The projection $P : H \to E_1$ is defined by

$$P(w_1 + w_2) = (\tilde{w}_1 + \tilde{w}_2)\sin 2x + (\bar{w}_1 + \bar{w}_2)\cos 2x,$$

where

$$\tilde{w}_j = \frac{2}{\pi}\int_0^\pi w_j \sin 2x\mathrm{d}x, \quad \bar{w}_j = \frac{2}{\pi}\int_0^\pi w_j \cos 2x\mathrm{d}x, \qquad j = 1, 2.$$

Let $u = u_1 + u_2, u_1 \in E_1, u_2 \in E_2$ and $u_1 = s_1 \sin 2x + s_2 \cos 2x, s_1, s_2 \in \mathbb{R}$. Then we can rewrite (4.3) in the form

$$\begin{cases} \dot{u}_1 = & Pg(u_1 + u_2), \\ \dot{u}_2 = & -(I-P)L_{\lambda_0}u_2 + (I-P)g(u_1+u_2). \end{cases} \tag{4.4}$$

Denote Φ_λ the semiflow generated by (4.1) and Φ_λ^1 the restriction of Φ_λ on E_1. Then we have the following result.

Lemma 4.2. *There exist positive constants β, ε such that the assumptions* (**A**1)-(**A**4) *hold and when $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$, the trivial solution $u = 0$ of system* (4.1) *is a global attractor of Φ_λ.*

PROOF. It is easy to see that there exist $\beta, \varepsilon > 0$ such that assumptions (**A**1)-(**A**4) hold. The argument of the lemma is standard, we omit the details. One can also see [**23, 28**] for details. □

Now we state and prove our main results on attractor bifurcation of (4.1).

THEOREM 4.3. *The trivial solution $u = 0$ is isolated for the system Φ_{λ_0} generated by (4.1) at $\lambda = \lambda_0$. Then there exist a closed neighborhood W of 0 and a one-sided neighborhood $I^+ = (\lambda_0, \lambda_0 + \varepsilon_1]$ such that for each $\lambda \in I^+$, Φ_λ has an attractor K_λ with K_λ in $W \setminus \{0\}$ and*

$$\lim_{\lambda \to \lambda_0} \mathrm{dist}_H(K_\lambda, 0) = 0.$$

Furthermore, K_λ contains an invariant topological sphere $\mathbb{S}^{n-1}$.

PROOF. According to Lemma 3.1, we deduce that there exists a neighborhood $U_1 \subset E_1$ of 0 such that the system (4.4) has a center manifold mapping $u_2 = h(u_1) : U_1 \to E_2$ with $h(0) = h'(0) = 0$. Thus the equation which determines the asymptotic behavior of solutions of (4.4) is the following two-dimensional equation

$$\dot{u}_1 = Pg(u_1 + h(u_1)), \tag{4.5}$$

where $u_1 = s_1 \sin 2x + s_2 \cos 2x, s_1, s_2 \in \mathbb{R}$. By Theorem 3.2, one concludes that it suffices to check that the trivial solution $u_1 = 0$ of system (4.5) is isolated for the semiflow ϕ_{λ_0} generated by (4.5).

In what follows we calculate the local center manifold (see e.g. [**6**]) of $u_1 = 0$ for ϕ_{λ_0} in order to show that the trivial solution $u = 0$ is isolated. Note that

$$u_1^3 = s_1^3 \sin^3 2x + s_2^3 \cos^3 2x + 3s_1^2 s_2 \sin^2 2x \cos 2x + 3s_1 s_2^2 \sin 2x \cos^2 2x. \tag{4.6}$$

So by the definition of P, we obtain

$$Pg(u_1) = -Pu_1^3 = -\frac{3}{4}s_1^3 \sin 2x - \frac{3}{4}s_2^3 \cos 2x - \frac{3}{4}s_1^2 s_2 \cos 2x - \frac{3}{4}s_1 s_2^2 \sin 2x. \tag{4.7}$$

In order to calculate an approximation to $h(u_1)$, we set

$$(\mathcal{M}_1(\psi))(u_1) = \psi'(u_1)Pg(u_1 + \psi(u_1)) + (I-P)L_{\lambda_0}\psi - (I-P)g(u_1 + \psi(u_1)), \tag{4.8}$$

where $\psi : E_1 \to E_2$. To apply Theorem 10 in [**6**], we choose ψ so that $(\mathcal{M}_1(\psi))(u_1) = \mathrm{O}(u_1^5)$. If $\psi(u_1) = \mathrm{O}(u_1^3)$, then

$$Pg(u_1 + \psi(u_1)) = Pg(u_1) + \mathrm{O}(u_1^5)$$

and

$$(\mathcal{M}_1(\psi))(u_1) = (I-P)L_{\lambda_0}\psi - (I-P)g(u_1 + \psi(u_1)) + \mathrm{O}(u_1^5). \tag{4.9}$$

It follows from (4.6), (4.7) that

$$\begin{aligned} &-(I-P)g(u_1 + \psi(u_1)) = (I-P)u_1^3 + \mathrm{O}(u_1^5) \\ &= (-\frac{1}{2}\sin 2x \cos 4x - \frac{1}{4}\sin 2x)s_1^3 + (\frac{1}{2}\cos 2x \cos 4x - \frac{1}{4}\cos 2x)s_2^3 \\ &+ (-\frac{3}{2}\cos 2x \cos 4x + \frac{3}{4}\cos 2x)s_1^2 s_2 \\ &+ (\frac{3}{2}\sin 2x \cos 4x + \frac{3}{4}\sin 2x)s_1 s_2^2 + \mathrm{O}(u_1^5). \end{aligned} \tag{4.10}$$

Let

$$\psi = (\alpha_1 \sin 2x \cos 4x)s_1^3 + (\alpha_2 \cos 2x \cos 4x)s_2^3 + (\alpha_3 \cos 2x \cos 4x)s_1^2 s_2 + (\alpha_4 \sin 2x \cos 4x)s_1 s_2^2.$$

By some elementary computations, one can check $PL_{\lambda_0}\psi = 0$, and so

$$\begin{aligned}(I-P)L_{\lambda_0}\psi &= L_{\lambda_0}\psi \\ &= (608\alpha_1 \sin 2x \cos 4x + 608\alpha_1 \cos 2x \sin 4x)s_1^3 + (608\alpha_2 \cos 2x \cos 4x)s_2^3 \\ &+ (-608\alpha_2 \sin 2x \sin 4x)s_2^3 + (608\alpha_3 \cos 2x \cos 4x - 608\alpha_3 \sin 2x \sin 4x)s_1^2 s_2 \\ &+ (608\alpha_4 \sin 2x \cos 4x + 608\alpha_4 \cos 2x \sin 4x)s_1 s_2^2. \end{aligned} \tag{4.11}$$

Then we conclude from (4.9)-(4.11) that

$$\begin{aligned}(\mathcal{M}_1(\psi))(u_1) &= L_{\lambda_0}\psi - (I-P)g(u_1 + \psi(u_1)) + \mathrm{O}(u_1^5) \\ &= (608\alpha_1 \sin 2x \cos 4x + 608\alpha_1 \cos 2x \sin 4x)s_1^3 + (608\alpha_2 \cos 2x \cos 4x)s_2^3 \\ &+ (-608\alpha_2 \sin 2x \sin 4x)s_2^3 + (608\alpha_3 \cos 2x \cos 4x - 608\alpha_3 \sin 2x \sin 4x)s_1^2 s_2 \\ &+ (608\alpha_4 \sin 2x \cos 4x + 608\alpha_4 \cos 2x \sin 4x)s_1 s_2^2 \\ &+ (-\frac{1}{2}\sin 2x \cos 4x - \frac{1}{4}\sin 2x)s_1^3 + (\frac{1}{2}\cos 2x \cos 4x - \frac{1}{4}\cos 2x)s_2^3 \\ &+ (-\frac{3}{2}\cos 2x \cos 4x + \frac{3}{4}\cos 2x)s_1^2 s_2 \\ &+ (\frac{3}{2}\sin 2x \cos 4x + \frac{3}{4}\sin 2x)s_1 s_2^2 + \mathrm{O}(u_1^5). \end{aligned} \tag{4.12}$$

Therefore, if

$$\alpha_1 = \frac{1}{2432}, \quad \alpha_2 = -\frac{1}{2432}, \quad \alpha_3 = \frac{3}{2432}, \quad \alpha_4 = -\frac{3}{2432}.$$

Then

$$\psi = (\frac{1}{2432}\sin 2x \cos 4x)s_1^3 - (\frac{1}{2432}\cos 2x \cos 4x)s_2^3 + (\frac{3}{2432}\cos 2x \cos 4x)s_1^2 s_2 - (\frac{3}{2432}\sin 2x \cos 4x)s_1 s_2^2,$$

and

$$(\mathcal{M}_1(\psi))(u_1) = \mathrm{O}(u_1^5).$$

By Theorem 10 in [**6**], we have

$$\begin{aligned} h(u_1) &= \psi(u_1) + \mathrm{O}(u_1^5) \\ &= (\frac{1}{2432}\sin 2x \cos 4x)s_1^3 - (\frac{1}{2432}\cos 2x \cos 4x)s_2^3 \\ &+ (\frac{3}{2432}\cos 2x \cos 4x)s_1^2 s_2 - (\frac{3}{2432}\sin 2x \cos 4x)s_1 s_2^2 + \mathrm{O}(u_1^5). \end{aligned} \tag{4.13}$$

Substituting (4.13) into (4.5), we obtain

$$\begin{aligned}\dot{s}_1 &= -\frac{3}{4}s_1^3 - \frac{3}{4}s_1 s_2^2 + \frac{3}{4864}s_1^5 - \frac{9}{4864}s_1^3 s_2^2 + \mathrm{O}(u_1^7), \\ \dot{s}_2 &= -\frac{3}{4}s_2^3 - \frac{3}{4}s_1^2 s_2 + \frac{3}{4864}s_2^5 - \frac{9}{4864}s_1^2 s_2^3 + \mathrm{O}(u_1^7),\end{aligned}$$

from which one can conclude that the trivial solution $u_1 = 0$ is an isolated equilibrium. Hence the trivial solution $u = 0$ of system (4.1) is isolated; see e.g.[**1**].

According to Theorem 3.2, there exist a closed neighborhood W and $\varepsilon_1 > 0$ such that the system (4.1) bifurcates from $(0, \lambda_0)$ an attractor K_λ for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon_1]$, where K_λ is the maximal compact invariant set of Φ_λ in $W \setminus \{0\}$ with

$$\lim_{\lambda \to \lambda_0} \mathrm{dist}_H(K_\lambda, 0) = 0.$$

Furthermore, K_λ contains an invariant topological sphere $\mathbb{S}^{n-1}$. The proof of Theorem 4.1 is complete. $\square$

5. Appendix A: Isomorphisms Induced by Projections

Let X^i_λ, P^i_λ be the same as in Section 3.1. Since $P^2_\lambda = I - P^1_\lambda$, the continuity of P^1_λ implies that P^2_λ is continuous in λ as well.

By (**A**3) we can assume J_0 is chosen sufficiently small so that

$$||P^i_\lambda - P^i_{\lambda_0}|| \leqslant c < 1, \qquad \forall\, \lambda \in J_0,\ i = 1, 2. \tag{A.1}$$

As before, we drop the subscript "λ_0" and rewrite

$$X^i = X^i_{\lambda_0}, \quad P^i = P^i_{\lambda_0}.$$

Proposition A1. *For each $i = 1, 2$, the restriction $P^i|_{X^i_\lambda}$ of P^i on X^i_λ is an isomorphism between X^i_λ and X^i.*

Proof. To prove Proposition A1, let us first verify that $P^i|_{X^i_\lambda}$ are one-to-one mappings.

As $P^2_\lambda = I - P^1_\lambda$, we deduce that

$$||P^2_\lambda - P^2|| = ||P^1_\lambda - P^1|| \leqslant c < 1. \tag{A.2}$$

In what follows we argue by contradiction and suppose $P^i|_{X^i_\lambda}$ fails to be a one-to-one mapping for some i. Then there would exist $x_i \in X^i_\lambda$ with $x_i \neq 0$ such that $P^i x_i = 0$. Further by (A.1) and (A.2) we see that

$$||x_i|| = ||P^i_\lambda x_i|| = ||P^i_\lambda x_i - P^i x_i|| \leqslant c||x_i|| < ||x_i||,$$

a contradiction!

Now we show that $P^i|_{X^i_\lambda}$ are isomorphisms. Since $P^i|_{X^i_\lambda}$ are one-to-one mappings, one immediately concludes that $P^1|_{X^1_\lambda}$ is an isomorphism. So we only need to consider the case $i = 2$.

Let $Q = P^2 + P^1_\lambda$. Then

$$Q|_{X^2_\lambda} = P^2|_{X^2_\lambda} + P^1_\lambda|_{X^2_\lambda} = P^2|_{X^2_\lambda}.$$

Because

$$Q = (I - P^1) + P^1_\lambda = I - (P^1 - P^1_\lambda),$$

and $||P^1 - P^1_\lambda|| < 1$, by the basic knowledge in linear functional analysis, we know that $Q : X \to X$ is an isomorphism. To show that $P^2|_{X^2_\lambda}$ is an isomorphism, there remains to check that $QX^2_\lambda = X^2$. For this purpose, it suffices to show that $Q^{-1}X^2 \subset X^2_\lambda$.

We argue by contradiction and suppose the contrary. There would exist $u \notin X^2_\lambda$ such that $Qu \in X^2$. Let $u = x_\lambda + x^2_\lambda$, where $x_\lambda \in X^1_\lambda$, and $x^2_\lambda \in X^2_\lambda$. Then $x_\lambda \neq 0$. We observe that

$$Qu = (P^2 + P^1_\lambda)u = P^2 u + P^1_\lambda(x_\lambda + x^2_\lambda) = x_\lambda + P^2 u \in X^2.$$

Hence $x_\lambda \in X^2$. Thereby we have $x_\lambda \in X^1_\lambda \cap X^2$. It follows that

$$P^1_\lambda x_\lambda = x_\lambda, \quad P^1 x_\lambda = 0.$$

Thus

$$||x_\lambda|| = ||P^1_\lambda x_\lambda - P^1 x_\lambda|| \leqslant c||x_\lambda|| < ||x_\lambda||.$$

This leads to a contradiction and completes the proof of the proposition. □

Now we define for each $\lambda \in J_0$ a linear operators T_λ on X as follows:

$$T_\lambda u = \Sigma_{1\leqslant j\leqslant 2}(P^j|_{X^j_\lambda} P^j_\lambda)\, u, \qquad u \in X.$$

It is trivial to check that T_λ is an isomorphism with $T_{\lambda_0} = I$. Clearly T_λ is continuous in λ, and

$$T_\lambda X^i_\lambda = \Sigma_{1\leqslant j\leqslant 2}(P^j|_{X^j_\lambda} P^j_\lambda)\, X^i_\lambda = P^i|_{X^i_\lambda} X^i_\lambda = X^i, \qquad i = 1, 2.$$

Thus we have the following conclusion.

Proposition A2. *Under the assumptions* (**A**1)-(**A**3)*, there exists a family of isomorphisms T_λ ($\lambda \in J_0$) on X depending continuously on λ with $T_{\lambda_0} = I$, such that*

$$T_\lambda X^i_\lambda = X^i_{\lambda_0} := X^i, \qquad i = 1, 2. \tag{A3}$$

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