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A new approach to study constant vorticity water flows in the β-plane approximation with centripetal forces

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Abstract. In this paper, we study three-dimensional equatorial flows with constant vorticity beneath a wave train and above a flat bed in the β-plane approximation with centripetal forces by adopting higher order approximation about the Coriolis terms. We show that the new approximation about the Coriolis forces is also applicable for certain problem, which help us to derive the same result.

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1. Introduction

The modelling of geophysical flows in the equatorial region is a complex subject which has witnessed much attention because of its wide applications in the theory of oceanography and atmospheric science [**1, 15, 25**]. Due to the Earth's rotation, the Coriolis forces are incorporated into the governing Euler equation, leading to the high complexity of the geophysical fluid dynamics.

In order to alleviate the complexity, the β-plane approximation has been typically employed in oceanographic consideration, in this way the Earth's curved surface is approximated by a tangent plane. There has been a large number of recent

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mathematical process in deriving and analysing the β-plane approximation. For three-dimensional equatorial flows, an essential modification of the Gerstner wave solution was constructed in [**3**] in the β-plane approximation. A modified equatorial β-plane approximation modeling nonlinear wave-current interactions was derived in [**19**], they also got an exact solution whose form is explicit in the Lagrangian framework. The readers can refer to the work [**6, 7, 12, 13, 17, 18**] for more details about explicit exact solutions for geophysical flows and refer to the work [**4, 8**] for the results on equatorial flows.

The motion of equatorial flows is captured by the vorticity function, which is defined as the curl of the velocity field. The nonzero vorticity is an essential tool for describing the interactions of waves with non-uniform currents, and the constant nonzero vorticity serves as the simplest rotational setting. The importance about the assumption of the vorticity in modeling the ocean flows is not only because the mathematical analysis, but also the physical perspective, see [**9, 20, 21, 22, 23, 24**].

A realistic modelling of the water flows occurring in the Earth's oceans must contain the vorticity effects and Coriolis effects in the oceanographic context. The change of sign of the Coriolis force across the Equator produces an effective waveguide and forces an azimuthal propagation, which is not the case for other ocean regions like those of the western boundary currents, see [**5, 26**]. Although the centripetal forces are typically neglected for the reasons that they are relatively much smaller than Coriolis forces, they play an important role in certain situations. In particular, the relatively small-scale centripetal force is essential in admitting currents of any physically plausible magnitude, see [**18**]. Three-dimensional equatorial flows with constant vorticity beneath a wave train and above a flat bed in the β-plane approximation with centripetal forces was presented by [**14**], their results were much different from the flows without centripetal forces in [**24**].

In this paper, we follow some idea in [**14, 24**] and develop the techniques to show that the equatorial flow is necessarily irrotational and the free surface is necessarily flat if the flow exhibiting a constant vorticity vector. Although this result reminds of that in [**14**], the main difference is that we adapt a new approach to derive the vorticity of the equatorial flow. We aim to illustrate that the previous approximation $\sin\phi \approx \phi, \cos\phi \approx 1$ used to simplify the counting process is reasonable and the associated approximations given by Taylor expression about the Coriolis forces which contain the high order terms do not affect the final results. It is also applicable for certain problem by using the approximations $\sin\phi \approx \phi - \frac{\phi^3}{6} + \cdots + (-1)^{n-1}\frac{\phi^{2n-1}}{(2n-1)!}$ and $\cos\phi \approx 1$, to study the constant vorticity water flows. Moreover, we want to explain why the whole approach was pursued in the equatorial setting. Firstly, equatorial flows are special in that propagate azimuthally , see [**10, 11**]. Secondly, the β-plane approximation is known to be consistent only in equatorial regions, see [**16**].

2. Preliminary

We assume that the earth is a perfect sphere of radius $R = 6378$ km and $\Omega = 7.29 \times 10^{-5} rad/s$ is the angular of velocity vector of the earth's rotation. We choose the $\{x, y, z\}$-coordinates such that the x-axis is pointing horizontally due east, the y-axis horizontally due north and the z-axis vertically upward. In this coordinate system, the natural choose in the approximation is the β-plane effect:

under the assumption that the meridian distance from the equator is moderate, we can use the approximations $\sin\phi \approx \phi, \cos\phi \approx 1$ to get the equatorial β-plane approximations in which the Coriolis force

$$2\Omega \begin{pmatrix} w\cos\phi - v\sin\phi \\ u\sin\phi \\ -u\cos\phi \end{pmatrix} \tag{2.1}$$

is replaced by

$$\begin{pmatrix} 2\Omega w - \beta y v \\ \beta y u \\ -2\Omega u \end{pmatrix} \tag{2.2}$$

with $\beta = \frac{2\Omega}{R} = 2.28 \times 10^{-11} m^{-1} s^{-1}$ and $y = \phi R$ (see [**3**]). The governing equations about the equatorial flows in the β-plane approximations with Coriolis term and centripetal forces are (see [**14**])

$$\begin{cases} u_t + uu_x + vu_y + wu_z + 2\Omega w - \beta y v = -\frac{1}{\rho}P_x, \\ v_t + uv_x + vv_y + wv_z + \beta y u + \Omega^2 y = -\frac{1}{\rho}P_y, \\ w_t + uw_x + vw_y + ww_z - 2\Omega u - \Omega^2 R = -\frac{1}{\rho}P_z - g, \end{cases} \tag{2.3}$$

together with the equation for incompressibility

$$u_x + v_y + w_z = 0,$$

where $\mathbf{u} = (u, v, w)$ is the fluids velocity, ρ is the constant density of the fluid, the terms of the order Ω denote the Coriolis terms, and the terms of the order Ω^2 represent the centripetal forces, $P(x, y, z, t)$ is the pressure within the fluid and $g = 9.8m/s^2$ is the constant gravitational acceleration.

If we use the approximations

$$\sin\phi \approx \phi - \frac{\phi^3}{6} + \cdots + (-1)^{n-1}\frac{\phi^{2n-1}}{(2n-1)!}, \ \cos\phi \approx 1,$$

then we can get the following new approximation

$$2\Omega \begin{pmatrix} w\cos\phi - v\sin\phi \\ u\sin\phi \\ -u\cos\phi \end{pmatrix}, \tag{2.4}$$

which is replaced by

$$\begin{pmatrix} 2\Omega w - \beta v f(y) \\ \beta u f(y) \\ -2\Omega u \end{pmatrix} \tag{2.5}$$

with $f(y) = \sum\limits_{k=1}^{n} (-1)^{k-1} \frac{y^{2k-1}}{(2k-1)! R^{2k-2}}$ and $y = \phi R$. Then the governing equations about the equatorial flows in our approximations with Coriolis term and centripetal forces become

$$\begin{cases} u_t + uu_x + vu_y + wu_z + 2\Omega w - \beta v f(y) = -\frac{1}{\rho}P_x, \\ v_t + uv_x + vv_y + wv_z + \beta u f(y) + \Omega^2 f(y) = -\frac{1}{\rho}P_y, \\ w_t + uw_x + vw_y + ww_z - 2\Omega u - \Omega^2 R = -\frac{1}{\rho}P_z - g, \end{cases} \tag{2.6}$$

together with the equation of mass conservation

$$u_x + v_y + w_z = 0. \tag{2.7}$$

We will study regular wave trains of water waves propagating steadily in the direction of the horizontal x-axis, L-period in the variable x, and presents no variation in the y-direction. The fluid domain is bounded below by the impermeable flat bed $z = -d$, and above by the free surface $z = \eta(x - ct)$, where $c > 0$ is the wave speed. We also suppose that the wave crest is located at $x = 0$, accordingly $\eta(0) > 0$.

Complementing the equations of motion is the dynamics boundary condition

$$P = P_{atm} \text{ on } z = \eta(x - ct), \tag{2.8}$$

with P_{atm} being the constant atmosphere pressure, decouples the motion of the water flow from the motion of the air above it. In addition, we have the following kinematic boundary conditions

$$u = (u - c)\eta_x \text{ on } z = \eta(x - ct), \tag{2.9}$$

and

$$w = 0 \text{ on } z = -d, \tag{2.10}$$

The vorticity vector $\mathbf{\Upsilon}$ is a good choice to grasp the swirling motion of the water, which is defined as the curl of the velocity field $\mathbf{u}$, that is

$$\mathbf{\Upsilon} = (\Upsilon_1, \Upsilon_2, \Upsilon_3) = (w_y - v_z, u_z - w_x, v_x - u_y). \tag{2.11}$$

From (2.6) and (2.7), we can get the vorticity equation

$$\mathbf{\Upsilon}_t + (\mathbf{u} \cdot \nabla)\mathbf{\Upsilon} - 2\Omega(u_y, v_y, w_y) - \beta f(y)(u_z, v_z, w_z) + \beta f'(y)(0, 0, v) = (\mathbf{\Upsilon} \cdot \nabla)\mathbf{u}.$$

3. Main results

Suppose that the vorticity vector is constant, then we find

$$(\mathbf{\Upsilon} \cdot \nabla)\mathbf{u} + 2\Omega(u_y, v_y, w_y) + \beta f(y)(u_z, v_z, w_z) - \beta f'(y)(0, 0, v) = 0, \tag{3.1}$$

which equals to the following equalities

$$\Upsilon_1 u_x + (\Upsilon_2 + 2\Omega)u_y + (\Upsilon_3 + \beta f(y))u_z = 0, \tag{3.2}$$

$$\Upsilon_1 v_x + (\Upsilon_2 + 2\Omega)v_y + (\Upsilon_3 + \beta f(y))v_z = 0, \tag{3.3}$$

$$\Upsilon_1 w_x + (\Upsilon_2 + 2\Omega)w_y + (\Upsilon_3 + \beta f(y))w_z - \beta f'(y)v = 0. \tag{3.4}$$

In this paper, we will assume that the components of the vorticity vector are constant and

$$\Upsilon_2 + 2\Omega \neq 0, \tag{3.5}$$

which is a reasonable assumption due to the fact $\Omega \approx 0.73 \times 10^{-4} rad/s$, and the typical value for Υ_2 in the equatorial Pacific is $25 \times 10^{-3} s^{-1}$ (see [**2**]).

LEMMA 3.1. *There is no equatorial flow exhibiting non-zero constant vorticity vector and with a flat surface. Indeed, any equatorial flows with a flat surface and constant vorticity vector must have the vorticity vector vanishing, namely* $\mathbf{\Upsilon} \equiv (0, 0, 0)$.

PROOF. Using the definition of the vorticity vector (2.11), we have

$$(w_y - v_z)_z = 0 \text{ and } (v_x - u_y)_x = 0,$$

that is

$$w_{yz} = v_{zz} \text{ and } v_{xx} = u_{yx},$$

from the equation of mass conservation (2.7), we can infer that

$$\Delta v = v_{xx} + v_{yy} + v_{zz} = u_{yx} + v_{yy} + w_{zz} = (u_x + v_y + w_z)_y = 0.$$

Similarly, we can prove that

$$\Delta u = \Delta w = 0.$$

We claim now that u, v, w are harmonic functions with in the fluid domain and hence the partial derivatives of u, v, w are also harmonic functions.

Noting that v_x, v_y, v_z are harmonic functions, from (3.3) we have

$$\Delta(f(y)v_z) = 0, \tag{3.6}$$

if $k = 1$, the equation (3.6) becomes $v_{zy} = 0$, this case was proved in [**14**] and [**24**]; if $k \geq 2$, (3.6) can be expanded as

$$2v_{zy}f'(y) + v_z f''(y) = 0,$$

then the above equation becomes

$$2(v_z)_y f'(y) + v_z f''(y) = 0, \tag{3.7}$$

If we regard v_z as a function of y, then (3.7) can be written as

$$(v_z)_y + \frac{f''(y)}{2f'(y)} v_z = 0, \tag{3.8}$$

the general solution of the above equation is

$$v_z = C(f'(y))^{-\frac{1}{2}} g(x, z), \tag{3.9}$$

where C is a constant and $g(x, z)$ is a function of variable x and z.

Next, we will prove that C is actually equals to 0. Indeed, if $C \neq 0$, then it is easy to obtain that

$$\begin{aligned} v_{zx} &= C(f'(y))^{-\frac{1}{2}} g_x, \\ v_{zy} &= -\frac{1}{2} C(f'(y))^{-\frac{3}{2}} f''(y) g(x, z), \\ v_{zz} &= C(f'(y))^{-\frac{1}{2}} g_z. \end{aligned} \tag{3.10}$$

Differentiating with respect to z in the equation (3.3) and applying the above equalities we get

$$C(f'(y))^{-\frac{1}{2}} \left(\Upsilon_1 g_x - \frac{(\Upsilon_2 + 2\Omega) f''(y)}{2f'(y)} g(x, z) + (\Upsilon_3 + \beta f(y)) g_z \right) = 0. \tag{3.11}$$

Note (3.5) and if $C \neq 0$, then

$$(\Upsilon_2 + 2\Omega) f''(y) g(x, z) + 2\beta f'(y) f(y) g_z + 2f'(y)\Upsilon_1 g_x + 2f'(y)\Upsilon_3 g_z = 0. \tag{3.12}$$

Noting that (3.12) is a second order ordinary differential equation with respect to $f(y)$, then its solution must depend on variables x and z, which is a contradiction, so that $C = 0$. Then, the equation (3.9) and (3.10) become

$$v_z = v_{zy} = v_{zx} = v_{zz} = 0. \tag{3.13}$$

Since $\Upsilon_1 = w_y - v_z$ and the kinematic boundary condition (2.10), we can infer that

$$w_y = 0 \text{ on } z = -d, \tag{3.14}$$

from which we draw a conclusion that

$$w_y \equiv 0 \text{ and } \Upsilon_1 = 0. \tag{3.15}$$

Similarly, from (3.2) we get

$$\Delta(f(y)u_z) = 0.$$

Moreover, we obtain

$$u_z = u_{zy} = u_{zx} = u_{zz} = 0. \tag{3.16}$$

From the definition of Υ_2 and (2.10) we can infer that

$$w_x = 0 \text{ on } z = -d.$$

Therefore, we conclude that

$$w_x \equiv 0 \text{ and } \Upsilon_2 = 0. \tag{3.17}$$

Now, equation (3.2) and (3.3) become

$$2\Omega u_y = 0 \text{ and } 2\Omega v_y = 0, \tag{3.18}$$

which implies that

$$u_y = 0 \text{ and } v_y = 0. \tag{3.19}$$

Recalling that $\Upsilon_1 = \Upsilon_2 = 0$ and $w_x = w_y = 0$, the equation (3.4) becomes

$$(\Upsilon_3 + \beta f(y))w_z - \beta v f'(y) = 0.$$

Differentiating with respect to y in the above equation, due to $v_y = 0$, we get

$$f'(y)w_z - f''(y)v = 0.$$

Differentiating with respect to y in the above equation again, we find

$$f''(y)w_z - f^{(3)}(y)v = 0,$$

from which we infer that for $1 \leq s \leq 2n-2$,

$$f^{(s)}(y)w_z - f^{(s+1)}(y)v = 0.$$

For $s = 2n-2$, we obtain

$$f^{(2n-2)}(y)w_z - f^{(2n-1)}(y)v = \frac{1}{R^{2n-2}}(yw_z - v) = 0.$$

which yields that

$$v = yw_z \text{ and } v_x = yw_{zx} = 0. \tag{3.20}$$

Therefore,

$$\Upsilon_3 = v_x - u_y = 0. \tag{3.21}$$

The proof is complete. □

THEOREM 3.2. *The only bounded solution that satisfies the governing equations (2.6) and their bounded conditions is the one with flat surface, and the velocity and the pressure given by*

$$(u, v, w) = (-\frac{\Omega^2}{\beta}, 0, 0),$$

$$P(x, y, z, t) = \rho(-\frac{2\Omega^3}{\beta} + \Omega^2 R - g)(z - \eta_0) + P_{atm},$$

where η_0 is a constant.

PROOF. From Lemma 3.1 and (3.4) we can infer that $w_z = 0$, which combines (2.10) leading to

$$w = 0 \text{ within the fluid domain.}$$

Using the fact $v_x = v_y = v_z = 0$ and (3.4) we can get $v = 0$. From (2.7) and the fact $v_y = w_z = 0$, we find that $u_x = 0$, from which and the fact $u_y = u_z = 0$ we get

$$u(x, y, z, t) = b(t),$$

for some function b. Now from the Euler equation (2.6) we conclude that

$$\begin{cases} P_x = -\rho b'(t), \\ P_y = -\rho(\beta b(t) f(y) + \Omega^2 f(y)), \\ P_z = \rho(2\Omega b(t) + \Omega^2 R - g). \end{cases}$$

Then, the pressure can be given as

$$\begin{aligned} P(x, y, z, t) = &- \rho b'(t)x - \rho(\beta b(t) + \Omega^2) \int_0^y f(s)ds \\ &+ \rho(2\Omega b(t) + \Omega^2 R - g)z + l(t), \end{aligned}$$

and the dynamics boundary condition (2.8) becomes

$$\begin{aligned} &P_{atm} + \rho(\beta b(t) + \Omega^2) \int_0^y f(s)ds \\ &= -\rho b'(t)x + \rho(2\Omega b(t) + \Omega^2 R - g)\eta(x - ct) + l(t), \end{aligned} \tag{3.22}$$

for any x, y, t. We deduce that the coefficient of y has to vanish in the above equation, then we get $b(t) = -\frac{\Omega^2}{\beta}$. Then, (3.22) can be written as

$$P_{atm} = \rho(-\frac{2\Omega^3}{\beta} + \Omega^2 R - g)\eta(x - ct) + l(t) \text{ for any } x, t, \tag{3.23}$$

which can be established only the functions l and η are constants l_0, η_0. Now the pressure function given as

$$P(x, y, z, t) = \rho(-\frac{2\Omega^3}{\beta} + \Omega^2 R - g)(z - \eta_0) + P_{atm}.$$

The proof is complete. □

REMARK 3.3. In general, (2.5) makes the proof of Lemma 3.1 complex, which is not easy to be catched directly. Now we choose $n = 2$, i.e., consider $\sin\phi \approx \phi - \frac{\phi^3}{6}$. Clearly, (2.5) is reduced to

$$\begin{pmatrix} 2\Omega w - \beta y v + \frac{\beta}{6R^2} y^3 v \\ \beta y u - \frac{\beta}{6R^2} y^3 u \\ -2\Omega u \end{pmatrix} \tag{3.24}$$

Then the governing equations about the equatorial flows in this approximations with Coriolis term and centripetal forces become

$$\begin{cases} u_t + uu_x + vu_y + wu_z + 2\Omega w - \beta y v + \frac{\beta}{6R^2} y^3 v = -\frac{1}{\rho} P_x, \\ v_t + uv_x + vv_y + wv_z + \beta y u - \frac{\beta}{6R^2} y^3 u + \Omega^2 y - \frac{\Omega^2}{6R^2} y^3 = -\frac{1}{\rho} P_y, \\ w_t + uw_x + vw_y + ww_z - 2\Omega u - \Omega^2 R = -\frac{1}{\rho} P_z - g, \end{cases} \tag{3.25}$$

together with (2.7), (2.8), (2.9) and (2.10).

From (3.25) and (2.7), we can get the vorticity equation

$$\boldsymbol{\Upsilon}_t + (\mathbf{u}\cdot\nabla)\boldsymbol{\Upsilon} - 2\Omega(u_y, v_y, w_y) - \beta y(u_z, v_z, w_z) + \frac{\beta}{6R^2}y^3(u_z, v_z, w_z) - \frac{\beta}{2R^2}y^2(0,0,v) + \beta(0,0,v) = (\boldsymbol{\Upsilon}\cdot\nabla)\mathbf{u}. \tag{3.26}$$

Next, one can follow the procedure of the proof for Lemma 3.1 to complete the proof. In fact, in order to verity (3.26), we should prove that each component of the above equation is equal, noting that

$$(\mathbf{u}\cdot\nabla)\boldsymbol{\Upsilon} = (u, v, w)\begin{pmatrix}\Upsilon_{1x} & \Upsilon_{2x} & \Upsilon_{3x}\\ \Upsilon_{1y} & \Upsilon_{2y} & \Upsilon_{3y}\\ \Upsilon_{1z} & \Upsilon_{2z} & \Upsilon_{3z}\end{pmatrix}$$

$$= \Big(u(w_y - v_z)_x + v(w_y - v_z)_y + w(w_y - v_z)_z,\; u(u_z - w_x)_x + v(u_z - w_x)_y + w(u_z - w_x)_z,\; u(v_x - u_y)_x + v(v_x - u_y)_y + w(v_x - u_y)_z \Big),$$

and

$$(\boldsymbol{\Upsilon}\cdot\nabla)\mathbf{u} = (\Upsilon_1, \Upsilon_2, \Upsilon_3)\begin{pmatrix}u_x & v_x & w_x\\ u_y & v_y & w_y\\ u_z & v_z & w_z\end{pmatrix}$$

$$= \Big((w_y - v_z)u_x + (u_z - w_x)u_y + (v_x - u_y)u_z,\; (w_y - v_z)v_x + (u_z - w_x)v_y + (v_x - u_y)v_z,\; (w_y - v_z)w_x + (u_z - w_x)w_y + (v_x - u_y)w_z \Big).$$

For the first component in the equation (3.26), we find that

$$\begin{aligned}&(w_y - v_z)_t + u(w_y - v_z)_x + v(w_y - v_z)_y + w(w_y - v_z)_z - 2\Omega u_y\\ &- \beta y u_z + \frac{\beta}{6R^2}y^3 u_z\\ &= (w_y - v_z)u_x + (u_z - w_x)u_y + (v_x - u_y)u_z.\end{aligned} \tag{3.27}$$

From (2.6) we get

$$\begin{aligned}(w_y - v_z)_t &= w_{ty} - v_{tz}\\ &= -uw_{xy} - vw_{yy} - ww_{zy} + 2\Omega u_y - \frac{1}{\rho}P_{zy}\\ &\quad + uv_{xz} + vv_{yz} + wv_{zz} + \beta y u_z - \frac{\beta}{6R^2}y^3 u_z + \frac{1}{\rho}P_{zy}\\ &\quad - u_y w_x - v_y w_y - w_y w_z + u_z v_x + v_z v_y + w_z v_z.\end{aligned}$$

Thus, (3.27) can be simplified to

$$-v_y w_y - w_y w_z + v_z v_y + w_z v_z - w_y u_x + u_x v_z = 0,$$

that is

$$-w_y(u_x + v_y + w_z) + v_z(u_x + v_y + w_z) = 0,$$

from which and (2.7) we can infer that the first component in the equation (3.26) is reasonable. Analogously, for the second and third component, we obtain

$$-u_z(u_x+v_y+w_z)+w_x(u_x+v_y+w_z)=0,$$
$$-v_x(u_x+v_y+w_z)+u_y(u_x+v_y+w_z)=0.$$

Therefore, the vorticity equation (3.26) can be established strictly due to (3.25) and (2.7).

Suppose that the vortivity vector is constant, then we find

$$(\boldsymbol{\Upsilon}\cdot\nabla)\mathbf{u}+2\Omega(u_y,v_y,w_y)+\beta y(u_z,v_z,w_z)-\frac{\beta}{6R^2}y^3(u_z,v_z,w_z)$$
$$+\frac{\beta}{2R^2}y^2(0,0,v)-\beta(0,0,v)=0, \tag{3.28}$$

which equals to the following equalities

$$\Upsilon_1u_x+(\Upsilon_2+2\Omega)u_y+(\Upsilon_3+\beta y-\frac{\beta}{6R^2}y^3)u_z=0, \tag{3.29}$$

$$\Upsilon_1v_x+(\Upsilon_2+2\Omega)v_y+(\Upsilon_3+\beta y-\frac{\beta}{6R^2}y^3)v_z=0, \tag{3.30}$$

$$\Upsilon_1w_x+(\Upsilon_2+2\Omega)w_y+(\Upsilon_3+\beta y-\frac{\beta}{6R^2}y^3)w_z-\beta v+\frac{\beta}{2R^2}y^2v=0. \tag{3.31}$$

For the convenience of the reader, we also give a detail proof for the case of $n=2$. Similarly, using the definition of the vorticity vector (2.11), we can that u,v,w are harmonic functions with in the fluid domain and hence the partial derivatives of u,v,w are also harmonic functions.

Noting that v_x,v_y,v_z are harmonic functions, from (3.30) we have

$$\Delta(yv_z)-\frac{1}{6R^2}\Delta(y^3v_z)=0,$$

which can be expanded as

$$y\Delta(v_z)+2v_{zy}-\frac{1}{6R^2}(\Delta(v_z)+6y^2v_{zy}+6yv_z)=0,$$

from which we obtain that

$$(2R^2-y^2)v_{zy}-yv_z=0, \tag{3.32}$$

If we regard v_z as a function of y, then (3.32) can be written as

$$(v_z)_y-\frac{y}{2R^2-y^2}v_z=0, \tag{3.33}$$

the general solution of the above equation is

$$v_z=C(2R^2-y^2)^{-\frac{1}{2}}f(x,z), \tag{3.34}$$

where C is a constant and $f(x,z)$ is a function of variable x and z.

Next, we will prove that C is actually equals to 0. Indeed, if $C\neq 0$, then it is easy to obtain that

$$v_{zx}=C(2R^2-y^2)^{-\frac{1}{2}}f_x,$$
$$v_{zy}=-\frac{1}{2}Cy(2R^2-y^2)^{-\frac{3}{2}}f(x,z),$$
$$v_{zz}=C(2R^2-y^2)^{-\frac{1}{2}}f_z. \tag{3.35}$$

Differentiating with respect to z in the equation (3.30) and applying the above equalities we get

$$
\begin{aligned}
&C(2R^2-y^2)^{-\frac{1}{2}}\Big(\Upsilon_1 f_x+(\Upsilon_2+2\Omega)\frac{y}{2(2R^2-y^2)}f(x,z)\\
&\qquad\qquad +(\Upsilon_3+\beta y-\frac{\beta y^3}{6R^2})f_z\Big)=0. \qquad (3.36)
\end{aligned}
$$

Since $C\neq 0$, then

$$\beta y\Big(\frac{(\Upsilon_2+2\Omega)}{2\beta(2R^2-y^2)}f(x,z)+(1-\frac{y^2}{6R^2})f_z\Big)=-\Upsilon_1 f_x-\Upsilon_3 f_z, \qquad (3.37)$$

noting that the terms on the right hand side of this equality don't contain variable y, thus the coefficient of y has to vanish, which means

$$\frac{f(x,z)}{f_z}=\frac{2\beta(y^2-6R^2)(2R^2-y^2)}{6R^2(\Upsilon_2+2\Omega)}.$$

Obviously, the above equation is contradictory because the term $\frac{f(x,z)}{f_z}$ is a function of variable of x and z, which implies that $C=0$. Then, (3.34) and (3.35) become (3.13). Since $\Upsilon_1=w_y-v_z$ and (3.14). Similarly, from (3.29) we get

$$\Delta(yu_z)-\frac{1}{6R^2}\Delta(y^3u_z)=0.$$

Moreover, we obtain (3.16). Next, (3.17) holds clearly. Now, equation (3.29) and (3.30) become (3.18) which implies that (3.19). Recalling that $\Upsilon_1=\Upsilon_2=0$ and $w_x=w_y=0$, the equation (3.31) becomes

$$(\Upsilon_3+\beta y-\frac{\beta}{6R^2}y^3)w_z-\beta v+\frac{\beta}{2R^2}y^2v=0.$$

Differentiating with respect to y in the above equation twice, due to $v_y=0$, we get

$$\frac{\beta}{R^2}(v-yw_z)=0,$$

which yields that (3.20). Thus, (3.21) holds.

Remark 3.4. Although our result about Lemma 3.1 reminds of that in [**14**], the approach we used to derive our result is different from their work. In our paper, using the new approximations $\sin\phi\approx\phi-\frac{\phi^3}{6}+\cdots+(-1)^{n-1}\frac{\Phi^{2n-1}}{(2n-1)!}$, $\cos\phi\approx 1$ the representation of the Coriolis forces and the centripetal forces become more precise than previous work. This is coincided with the possible geometrical approximations proposed by Gallagher and Saint-Raymond [**27**] and presents the feasibility. Although the equations seem more complicated, the explicit expression about v_z, u_z is convenient to compute the component of the vorticity vector. Moreover, the proof and the results in our paper are different from the result without centripetal force in [**24**], Martin has shown that the only flow exhibiting a constant vorticity vector is the stationary flow $(u,v,w)=(0,0,0)$.

References

[1] A. Constantin, Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis, CBMS-NSF Conference Series in Applied Mathematics, vol. **81**, SIAM, Philadelphia, 2011.
[2] A. Constantin, On the modelling of equatorial waves, *Geophys. Res. Lett.*, **39** (2012) L05602.

[3] A. Constantin, An exact solution for equatorial trapped waves, *J. Geophys. Res.,* **117** (2012) C05029.

[4] A. Constantin, Some three-dimensional nonlinear equatorial flows, *J. Phys. Oceanogr.,* **43** (2013) 165-175.

[5] A. Constantin, R. S. Johnson, The dynamics of waves interacting with the Equatorial Undercurrent, *Geophys. Astrophys. Fluid Dyn.,* **109** (2015), 311-358

[6] A. Constantin, R. S. Johnson, An exact, steady, purely azimuthal equatorial flow with a free surface, *J. Phys. Oceanogr.,* **46** (2016) 1417-1447.

[7] A. Constantin, R. S. Johnson, An exact, steady, purely azimuthal flow as a model for the Antarctic Circumpolar Current, *J. Phys. Oceanogr.,* **46** (2016) 3585-3594.

[8] A. Constantin, R.S. Johnson, A nonlinear, three-dimensional model for ocean flows, motived by some observations of the Pacific Equatorial Undercurrent and thermocline, *Phys. Fluids,* **29** (2017) 056604.

[9] A. Constantin, R.S. Johnson, Steady large-scale ocean flows in spherial coordinates, *Oceanography,* **31** (2018) 42-50.

[10] A. Constantin, R. I. Ivanov, Equatorial wave-current interactions, *Comm. Math. Phys.,* **370** (2019), 1-48.

[11] A. Constantin, R. S. Johnson, On the nonlinear, three-dimensional structure of equatorial oceanic flows, *J. Phys. Oceanography,* **49**(2019), 2029-2042.

[12] J. Chu, D. Ionescu-Kruse, Y. Yang, Exact solution and instability for geophysical trapped waves at arbitrary latitude, *Discrete Contin. Dyn. Syst.,* **39** (2019) 4399-4414.

[13] J. Chu, D. Ionescu-Kruse, Y. Yang, Exact solution and instability for geophysical waves with centripetal forces and at arbitrary latitude, *J. Math. Fluid Mech.,* **21** (2019) 19, 16pp.

[14] J. Chu, Y. Yang, Constant vorticity water flows in the β-plane approximation with centripetal forces, *J. Differential Equations,* **269** (2020) 9336-9347.

[15] B. Cushman-Robisin, J.M. Beckers, Introduction to Geophysical Fluid Dynamics: Physical and Numerical Aspects, Academic, Waltham, Mass, 2011.

[16] P. G. Dellar, Variations on a β-plane: derivation of non-traditional β-plane equations from Hamilton's principle on a sphere, *J. Fluid Mech.,* **674** (2011), 174-195.

[17] D. Henry, An exact solution for equatorial geophysical water waves with an underlying current, *Eur. J. Mech. B, Fluids,* **38** (2013) 18-21.

[18] D. Henry, Equatorially trapped nonlinear water waves in a β-plane approximation with centripetal forces, *J. Fluid Mech.,* **804** (2016), 11pp.

[19] D. Henry, A modified equatorial β-plane approxination modelling nonlinear wave-current interactions, *J. Differential Equations,* **263** (2017) 2554-2566.

[20] C.I. Martin, Resonant interactions of capillary-gravity water wave, *J. Math. Fluid Mech.,* **19** (2017) 807-817.

[21] C.I. Martin, Two dimensionality of gravity water flows governed by the equatorial f-plane approximation. *Ann, Math. Pures Appl.,* **196** (2017) 2253-2260.

[22] C.I. Martin, on the vorticity of mesoscale ocean currents, *Oceanography,* **31** (2018) 28-35.

[23] C.I. Martin, Constant vorticity water flows with full Coriolis term, *Nonlinearity,* **32** (2019) 2327-2336.

[24] C.I. Martin, On constant vorticity water flows in the β-plane approximation, *J. Fluid Mech.,* **865** (2019) 762-774.

[25] J. Pedlosky, Geophysical Fluid Dynamics, Springer, New York, 1979.

[26] L. D. Talley, G. L. Pickard, W. J. Emery, J. H. Swift, Descriptive Physical Oceanography: An Introduction, Elsevier, 2011.

[27] I. Gallagher, L. Saint-Raymond, On the influence of the Earth's Rotation on Geophysical Flows, Handbook of Mathematical Fluid Dynamics, 2007.

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