Dynamics of PDE, Vol.18, No.4, 257-278, 2021

An invasive-invaded species dynamics with a high order diffusion operator

José Palencia

Communicated by Yuxi Zheng, received April 14, 2021.

Abstract. The introduction of the Landau-Ginzburg free energy provides a framework to generalize the diffusion beyond the classical fickian approach. The analysis shows the existence and uniqueness of solutions with a priori bounds and making use of the Fixed Point Theorem to a suitable abstract evolution. Asymptotic solutions are provided with the Hamilton-Jacobi operator and a positivity condition is formulated based on an asymptotic positive kernel. Further, the positive region is characterized and a precise assessment is provided. Afterwards, the problem is analyzed in the Travelling Waves domain to show the phenomena of waves synchronization and to provide linear manifolds in the proximity of the critical points. Finally, numerical TW profiles are obtained and the amplitude of a positive region in the TW domain is provided as a function of the TW-speed.

Contents

1. Problem description and objectives

The classical order two diffusion typically comes from a Fick law in which the gradient of a substance provokes a flux or a movement. Other approaches,

1991 *Mathematics Subject Classification.* 35K92, 35K91, 35K55.
Key words and phrases. High order diffusion, Fisher-KPP problem, instabilities, existence, geometric perturbation theory, asymptotic, positivity.

based on random walk, have been followed ([5] together with references there). In addition and to consider a further precise definition of diffusion, it is possible to admit the Landau-Ginzburg (LG) energy concept [3], [6]. In [3], the authors obtained an expression to the LG energy to model the spatial patter formation in non-homogeneous diffusion. In the same study, they proved that the LG energy depends on the gradient of a concentration, i.e. $\frac{1}{2}k(\nabla u)^2$. Considering such LG energy expression together with an appropriate potential, the authors end in a degenerate diffusion involving a order four spatial derivative [3].

Reaction-diffusion models were formally introduced by Fisher [2] and Kolmogorov, Petrovskii and Piskunov [4] to study the interacction of genes and the behaviour of flames in combustion theory respectively. The approach followed by the cited authors was based on a fickian diffusion and a non-linear reaction term $f(u) = u(1-u)$. The problem was tracked with Travelling Waves (TW) solutions to understand the behaviour of diffusion acting along a wave tip and a front. The authors introduced the concept of TW solutions to understand the propagation features of each specie involved. The Fisher - KPP model has been widely applied in different scenarios and with remarkable success, see for example [9], [7] and [11]. The model has been studied with fractional operators [12], high order [24] and with a p-Laplacian Porous Medium Equation [8].

As an alternative to the free energy approach, the fourth order operators may be understood as a perturbation to an order two diffusion. In [13] and [19], the Fisher-Kolmogorov order two equation was extended with a fourth order operator to model the observed instabilities near degenerate points. Furthermore, in [20] and [21], it is shown the existence of oscillatory spatial patterns for the mentioned Extended Fisher-Kolmogorov equation.

The invaded-invasive interaction has been source of research in biomathematics recently. In [10], the system derived was intended to describe the haptotactic cell invasion in a model for melanoma. In addition, [14] examines the spectral stability of TW for the haptotaxis model studied in cancer invasion. The model has been analyzed making use of Evans function to a linearised operator. In these cited cases, the proposed models introduced a general diffusion to predict the invasive-invaded dynamic induced by the haptotactic evolution. Nonetheless, the involved spatial derivatives are of order two and monotony properties in the operator apply for positive solutions.

To derive the reaction-absorption dynamic in an invasive-invaded interaction, consider the definition of biological invasion provided by the Convention of Biological Diversity (p. 1, Ch. 1 [1]) "...those alien species which threaten ecosystems, habitats or species". Hence, let consider u as the invasive specie and v the invaded. If the quantity of invaded is high, the temporal increasing rate in the invaded (u_t) shall be high, otherwise the invasion will not succeed. As the invasive specie proliferates over time, there shall be a limit in its concentration. In addition, the invaded specie evolution shall be decreasing with a rate absolutely higher for increasing values of the invasive u. The proposed invasive-invaded system aims to characterize the behaviour of solutions induced by the non-homogeneous diffusion in the proximity of the equilibrium at $u = 1$ and $v = 0$. The degenerate diffusion is considered to be of fourth order to account for instabilities close the equilibrium points. This permits to model accurately the centre space manifolds (in the sense of oscillatory) arising as perturbations when the species reach the stationary solutions. In the same manner, it is possible to think on a global population energy related with the species random walk in the media that depends on the gradient of each particular species. Indeed, the species will provide further movement in areas of higher spacial gradient as those zones are easier for species to prosper and expand. Following this philosophy the diffusion ends in a four order operator [3]. Consequently, the model reads:

$$\begin{aligned} u_t &= -\Delta^2 u + v(1-u), \\ v_t &= -\Delta^2 v - uv, \\ u_0(x), v_0(x) &\in L^1_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap H^n(\mathbb{R}), \end{aligned} \tag{1.1}$$

typically, $n = 4$. Note that the minus sign in the bi-laplacian term $(-\Delta^2)$ is set to account for an asymptotic stable kernel as it will become apparent during the analysis.

2. A priori assessments on Regularity, Existence and Uniqueness

Consider the following norm:

$$\|u\|_\rho = \int_{\mathbb{R}} \rho(z) \sum_{k=0}^{4} |D^k u(z)| dz, \tag{2.1}$$

where $D = \frac{d}{dz}$, $u \in H^4_\rho(\mathbb{R}) \subset L^1_\rho(\mathbb{R}) \subset L^1(\mathbb{R})$ and the weight ρ is defined as (see [24]):

$$\rho(z) = e^{a_0|z|^{\frac{4}{3}}}, \tag{2.2}$$

$a_0 > 0$ is a small constant.

LEMMA 2.1. *The functional space of* $u \in H^4_\rho(\mathbb{R}) \subset L^1_\rho(\mathbb{R}) \subset L^1(\mathbb{R})$ *with norm* $\|u\|_\rho$ *is a Banach space.*

PROOF. Given two functions $u, v \in H^4_\rho \subset L^1_\rho \subset L^1$:

$$\begin{aligned} \|u+v\|_\rho &= \int_{\mathbb{R}} \rho(z) \sum_{k=0}^{4} |D^k(u+v)(z)| dz \leq \int_{\mathbb{R}} \rho(z) \sum_{k=0}^{4} [|D^k(u)(z)| + |D^k(v)(z)|] dz \\ &= \int_{\mathbb{R}} \rho(z) \sum_{k=0}^{4} |D^k(u)(z)| dz + \int_{\mathbb{R}} \rho(z) \sum_{k=0}^{4} |D^k(v)(z)| dz = \|u\|_\rho + \|v\|_\rho. \end{aligned} \tag{2.3}$$

To show completeness, define a sequence $\{u_n(z) : n \in N\} \in H^4_\rho$. Fix $\epsilon \geq 0$ and consider that the defined sequence is Cauchy under the norm $\|\cdot\|_\rho$. There exists $\nu \in \mathbb{N}$ such that for every $n, m > \nu$, $\|u_n - u_m\|_\rho \leq \epsilon$. The convergence is shown as follows:

$$\begin{aligned} |u_n(z) - u_m(z)| &= |(u_n - u_m)(z)| \leq |u_n - u_m||z| \leq \sum_{k=0}^{4} |D^k(u_n - u_m)(z)||z| \\ &\leq \rho(z) \sum_{k=0}^{4} |D^k(u_n - u_m)(z)||z| \leq \int_R \rho(z) \sum_{k=0}^{4} |D^k(u_n - u_m)(z)| dz |z| \\ &= \|u_n - u_m\|_\rho |z| \leq \epsilon |z|. \end{aligned} \tag{2.4}$$

The function $\rho(z) \geq 1$, then for any arbitrary $\epsilon \to 0$:

$$|u_n(z) - u_m(z)| \to 0. \tag{2.5}$$

□

2.1. A priori bounds. Define $L = -\Delta^2$ the spatial operator and consider the problem $u_t = Lu$ for the invasive specie. Note that similarly occurs of the invaded v. Consider in this case $u_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap H^n(\mathbb{R})$. The following lemma holds.

LEMMA 2.2. *Given $u_0 \in L^1(\mathbb{R}^N)$, then:*

$$\|u\|_{L^1} \leq \|u_0\|_{L^1}. \tag{2.6}$$

Given $n \in \mathbb{R}^+$ and $u_0 \in H^n(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$:

$$\|u\|_{H^n} \leq \|u_0\|_{H^n}, \tag{2.7}$$

and

$$\|u\|_{H^n} \leq \|u_0\|_{L^1}, \quad for\ \ t \geq \frac{n}{2}. \tag{2.8}$$

In addition,

$$\|u\|_\rho \leq \kappa \|u\|_{H^n} \leq \kappa \|u_0\|_{H^n}, \quad \kappa = 4\ \max\{D^1 u, D^2 u, D^3 u, D^4 u\} \tag{2.9}$$

PROOF. The solution to the problem $u_t = Lu$ reads:

$$u(x,t) = e^{tL} u_0(x), \tag{2.10}$$

and making use of the Fourier transformation:

$$\hat{u}(\omega, t) = e^{t(-\omega^4)}\, \hat{u}_0(\omega). \tag{2.11}$$

Now:

$$\|u\|_{L^1} = \int_{-\infty}^{\infty} |e^{-t\omega^4}|\, |\hat{u}_0(\omega)| d\omega \leq \sup_{\omega \in \mathbb{R}} (e^{-\omega^4 t}) \int_{-\infty}^{\infty} |\hat{u}_0(\omega)| d\omega = \|u_0\|_{L^1}. \tag{2.12}$$

Then, $\|u\|_{L^1} \leq \|u_0\|_{L^1}$, as intended to show. Assume now the following weighted norm in a Sobolev space for $n \in \mathbb{R}^+$ and $0 \leq t < \infty$:

$$\|u\|_{H^n} = \int_{-\infty}^{\infty} e^{n\omega^2}\, |\hat{u}(\omega, t)| d\omega, \tag{2.13}$$

where the weight $e^{n\omega^2}$ satisfies the A_p-condition $(p = 1)$ [18]. Then:

$$\begin{aligned} \|u\|_{H^n} &= \int_{-\infty}^{\infty} e^{n\omega^2} |\hat{u}(\omega, t)| d\omega = \int_{-\infty}^{\infty} e^{n\omega^2} |e^{t(-\omega^4)}|\, |\hat{u}_0(\omega)| d\omega \\ &\leq \sup_{\omega \in \mathbb{R}} (e^{-\omega^4 t}) \int_{-\infty}^{\infty} e^{n\omega^2} |\hat{u}_0(\omega)| d\omega = \|u_0\|_{H^n}. \end{aligned} \tag{2.14}$$

Assuming now $u_0 \in L^1(\mathbb{R}^N)$:

$$\|u\|_{H^n} = \int_{-\infty}^{\infty} e^{n\omega^2} |\hat{u}(\omega, t)| d\omega \leq \sup_{\omega \in \mathbb{R}} (e^{n\omega^2}\, e^{-\omega^4 t}) \int_{-\infty}^{\infty} |\hat{u}_0(\omega)| d\omega. \tag{2.15}$$

An elementary assessment leads to:

$$\|u\|_{H^n} \leq \left(\frac{n}{2t}\right)^{1/2} \|u_0\|_{L^1}, \tag{2.16}$$

so that

$$\|u\|_{H^n} \leq \|u_0\|_{L^1}, \tag{2.17}$$

for $t \geq \frac{n}{2}$, as intended to show. Finally:

$$\begin{aligned} \|u\|_\rho &= \int_{\mathbb{R}} \rho(z) \sum_{k=0}^{4} |D^k u(z)| dz \leq \int_{\mathbb{R}} e^{nz^2} \sum_{k=0}^{4} |D^k u(z)| dz \\ &\leq \kappa \int_{\mathbb{R}} e^{nz^2} |u(z)| dz \leq \kappa \|u\|_{H^n}, \end{aligned} \tag{2.18}$$

being $\kappa = 4 \max\{D^1 u, D^2 u, D^3 u, D^4 u\}$.

To be highlighted that the obtained bounds apply similarly for v.

□

Now. the single parameter (t) representation for the homogeneous equation $u_t = -\Delta^2 u$ is as follows (ídem for v):

$$g(x,t) = e^{-\Delta^2 t}. \tag{2.19}$$

The operator $-\Delta^2$ is an infinitesimal generator of a strongly continuous semi-group for $t > 0$. Hence, the following abstract evolution holds:

$$u(t) = e^{-\Delta^2 t} u_0 + \int_0^t \left[e^{-\Delta^2 (t-s)} v(s)(1 - u(s)) \right] ds. \tag{2.20}$$

$$v(t) = e^{-\Delta^2 t} v_0 - \int_0^t \left[e^{-\Delta^2 (t-s)} v(s) u(s) \right] ds. \tag{2.21}$$

In the transformed domain, the homogeneous $u_t = -\Delta^2 u$ with $u(x,0) = \delta(x)$ solution is:

$$\tilde{u}(t) = e^{-\omega^4 t} \tilde{u}_0. \tag{2.22}$$

Similarly $\tilde{v}(t) = e^{-\omega^4 t} \tilde{v}_0$. Then, the kernel is obtained as:

$$g(x,t) = F^{-1}(e^{-\omega^4 t}) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\omega^4 t - i\omega x} d\omega = \int_{\mathbb{R}} e^{-\omega^4 t} cos(\omega x) d\omega, \tag{2.23}$$

which exists upon integration with ω in $\mathbb{R}$. Then (2.20) and (2.21) can be rewritten in terms of the existing kernel. To this end, consider the following operators in $H^4_\rho(\mathbb{R})$:

$$T_{u_0,t} : H^4_\rho(\mathbb{R}) \to H^4_\rho(\mathbb{R}); \quad G_{v_0,t} : H^4_\rho(\mathbb{R}) \to H^4_\rho(\mathbb{R}) \tag{2.24}$$

defined as:

$$T_{u_0,t}(u) = g(x,t) * u_0(x) + \int_0^t g(x, t-s) * v(x,s)(1 - u(x,s)) ds, \tag{2.25}$$

$$G_{v_0,t}(u) = g(x,t) * v_0(x) - \int_0^t g(x, t-s) * v(x,s) u(x,s) ds, \tag{2.26}$$

so that the following lemma holds.

LEMMA 2.3. *The single parameter (t) operators $T_{u_0,t}$ and $G_{v_0,t}$ are bounded in $H^4_\rho(\mathbb{R})$ with the norm* (2.1).

Proof. Previous to start the formal proof, the following inequality is shown for u (ídem for v):

$$b_0\|u_0\|_\rho \leq \|u\|_\rho. \tag{2.27}$$

To this end:

$$\begin{aligned}\|u\|_\rho &= \int_{\mathbb{R}} \rho(\omega)\sum_{k=0}^{4}|D^k\hat{u}(\omega)|d\omega = \int_{\mathbb{R}} \rho(\omega)\sum_{k=0}^{4}|D^k\left[e^{-\omega^4 t}\,\hat{u}_0\right]|d\omega \\ &\geq \int_{\mathbb{R}} \rho(\omega)\sum_{k=0}^{4}|D^k\left[e^{-\omega^4 t}\right]|\sum_{k=0}^{4}|D^k\hat{u}_0|d\omega \geq b_0\int_{\mathbb{R}} \rho(\omega)\sum_{k=0}^{4}|D^k\hat{u}_0|d\omega = b_0\|u_0\|_\rho\end{aligned} \tag{2.28}$$

where

$$b_0 = \inf_{\omega\in B_r}\{\sum_{k=0}^{4}|D^k\left[e^{-\omega^4 t}\right]|\} > 0 \tag{2.29}$$

and sufficiently small in $B_r = \{\omega,\ |\omega| < r\}$, for $r > 0$.

Considering the operator $T_{u_0,t}$:

$$\begin{aligned}\|T_{u_0,t}(u)\|_\rho = \|T_{u_0,t}\|_\rho\,\|u\|_\rho &\leq \|g\|_\rho\,\|u_0\|_\rho + \int_0^t \|g\|_\rho\,\|v\|_\rho\,\|1-u\|_\rho ds \\ &\leq \left[\|g\|_\rho\,\frac{1}{b_0} + \int_0^t \|g\|_\rho\, b_0\kappa\|u_0\|_{H^n} ds\right]\ \|u\|_\rho,\end{aligned} \tag{2.30}$$

where, it has been considered $\|v\|_\rho \leq \|u\|_\rho$ for $t > \tau > 0$. Then:

$$\|T_{u_0,t}\|_\rho \leq \|g\|_\rho\,\frac{1}{b_0} + \int_0^t \|g\|_\rho\, b_0\kappa\|u_0\|_{H^n} ds \tag{2.31}$$

which is bounded for each value $t > 0$ given the bound properties in Lemma 2.2.

Operating analogously for $G_{v_0,t}$:

$$\|G_{v_0,t}(u)\|_\rho = \|G_{v_0,t}\|_\rho\,\|v\|_\rho \leq \|g\|_\rho\,\|v_0\|_\rho - \int_0^t \|g\|_\rho\,\|u\|_\rho\|v\|_\rho ds \leq \|g\|_\rho\,\frac{1}{b_0}\|v\|_\rho, \tag{2.32}$$

so that:

$$\|G_{v_0,t}\|_\rho \leq \|g\|_\rho\,\frac{1}{b_0} \tag{2.33}$$

which is bounded for any value $t > 0$. □

2.2. Uniqueness. Given two vectors $U, V \in \mathbb{R}^2$ with components $u_1, u_2;\ v_1, v_2 \in H^4_\rho(\mathbb{R})$ respectively. The following weighted inner product is defined:

$$< U, V >_\rho = \int_{\mathbb{R}} \rho(z)\sum_{k=0}^{3} D^k(\nabla\cdot U(z))\overline{D^k(\nabla\cdot V(z))}dz + \int_{\mathbb{R}} \rho(z)\sum_{k=1}^{2}\sum_{j=1}^{2} u_k(z)v_j(z)dz, \tag{2.34}$$

where $D = \frac{d}{dz}$, $\rho(z)$ is as per (2.20) and:

$$\begin{aligned}\nabla\cdot U(z) &= \max\{\tfrac{\partial u_1}{\partial z}\,,\ \tfrac{\partial u_2}{\partial z}\}, \\ \nabla\cdot V(z) &= \max\{\tfrac{\partial v_1}{\partial z}\,,\ \tfrac{\partial v_2}{\partial z}\}.\end{aligned} \tag{2.35}$$

The following weighted norm is defined accordingly:

$$\|U\|_\rho = \int_{\mathbb{R}} \rho(z) \sum_{k=0}^{3} |D^k(\nabla \cdot U(z))|^2 dz + \int_{\mathbb{R}} \rho(z) \sum_{k=1}^{2}\sum_{j=1}^{2} u_k(z) u_j(z) dz. \tag{2.36}$$

The uniqueness is shown on the basis that the mapping

$$T_{u_0,v_0,t} : H^4_\rho(\mathbb{R}) \times H^4_\rho(\mathbb{R}) \to H^4_\rho(\mathbb{R}) \times H^4_\rho(\mathbb{R}) \tag{2.37}$$

defined as:

$$T_{u_0,v_0,t}(u,v) = \begin{pmatrix} b(x,t) * u_0(x) \\ b(x,t) * v_0(x) \end{pmatrix} + \begin{pmatrix} \int_0^t b(x,t-s) * v(x,s)(1-u(x,s))ds \\ -\int_0^t b(x,t-s) * u(x,s)v(x,s)ds \end{pmatrix}, \tag{2.38}$$

has a unique fix point $(u,v) = T_{u_0,v_0,t}(u,v)$. For the sake of simplicity, consider:

$$N(U_1) = \begin{pmatrix} v_1(1-u_1) \\ -u_1 v_1 \end{pmatrix}, \quad N(U_2) = \begin{pmatrix} v_2(1-u_2) \\ -u_2 v_2 \end{pmatrix}, \tag{2.39}$$

then

$$\begin{aligned} &\|T_{u_0,v_0,t}(u,v)(U_1) - T_{u_0,v_0,t}(u,v)(U_2)\|_\rho \\ &\le \int_0^t \| \int_t^s b(x,t-s-r)[N(U_1(t-s-r)) - N(U_2(t-s-r))]\|_\rho dr ds \\ &\le \int_0^t \int_t^s \|b(x,t-s-r)[N(U_1(t-s-r)) - N(U_2(t-s-r))]\|_\rho dr ds \\ &= \int_0^t \int_t^s \|b(x,t-s-r)\|_\rho \|N(U_1(t-s-r)) - N(U_2(t-s-r))\|_\rho dr ds \\ &\le M \int_0^t \int_t^s \|N(U_1(t-s-r)) - N(U_2(t-s-r))\|_\rho dr ds, \end{aligned} \tag{2.40}$$

where $M = \sup\{\|b(x,t-s-r)\|_\rho \ , \forall t > 0\}$ and independently of s, r. Now

$$\begin{aligned} \|N(U_1) - N(U_2)\|_\rho = &\int_{\mathbb{R}} \rho(z) \sum_{k=0}^{3} |D^k(\nabla \cdot (N(U_1) - N(U_2))|^2 dz \\ &+ \int_{\mathbb{R}} \rho(z) \sum_{k=1}^{2}\sum_{j=1}^{2} (N(U_1) - N(U_2))_k (N(U_1) - N(U_2))_j dz, \end{aligned} \tag{2.41}$$

so that

$$\begin{aligned} N(U_1) - N(U_2) &= \begin{pmatrix} v_1(1-u_1) - v_2(1-u_2) \\ u_2 v_2 - u_1 v_1 \end{pmatrix} \le \begin{pmatrix} K_u(u_1-u_2) \\ K_v(v_1-v_2) \end{pmatrix} \\ &\le K \begin{pmatrix} u_1 - u_2 \\ v_1 - v_2 \end{pmatrix} = K(U_1 - U_2), \end{aligned} \tag{2.42}$$

where K_u and K_v are the Lipschitz constant with respect to u and v and $K = \max\{K_u, K_v\}$. Now in (2.41)

$$\begin{aligned} \|N(U_1) - N(U_2)\|_\rho = \int_{\mathbb{R}} \rho(z) \sum_{k=0}^{3} |D^k(\nabla \cdot K(U_1 - U_2)|^2 dz \\ + \int_{\mathbb{R}} \rho(z) \sum_{k=1}^{2}\sum_{j=1}^{2} K(U_1 - U_2)_k K(U_1 - U_2)_j dz. \end{aligned} \tag{2.43}$$

Finally,

$$\|T_{u_0,v_0,t}(U_1) - T_{u_0,v_0,t}(U_2)\|_\rho \leq MK^2 \int_0^t \int_t^s \|U_1 - U_2\|_\rho ds dr = MK^2 t(t-s)\|U_1 - U_2\|_\rho. \tag{2.44}$$

For any ball centered in t with radium proportional to $t - s$ a finite $MK^2 t(t - s)$ holds. Hence, the uniqueness is shown in the limit with $\|U_1 - U_2\|_\rho \to 0$ so that there is a local unique fix point in the operator $T_{u_0,v_0,t}$ given by U_1.

2.3. Characterization of a local positive inner region. The objective now is to find an inner spatial region (inner means $|x| \in B_{r(t)}$, where $r(t)$ shall be assessed) where solutions are positive. On the contrary, for $|x| > r(t)$ any solution to (1.1) is oscillatory. The following lemma holds:

LEMMA 2.4. *There exists a spatial inner region characterized by the ball $B_{r(t)}$ such that for $|x| << r(t)$ any solution to (1.1) exhibits a positive behaviour (i.e. non-oscillatory). The value for $r(t)$ is sharply assessed:*

$$r(t) = t^{1/4}|\ln t|, \tag{2.45}$$

for $t \to 0^+$.

PROOF. Let consider the following variable scaling [**24**]:

$$z = \frac{x}{t^{1/4}}; \quad \tau = \ln t \to -\infty \quad if \quad t \to 0^+. \tag{2.46}$$

The equation (1.1) expressed in the new variables with $u(x,t) = w(z,\tau)$ and $v(x,t) = y(z,\tau)$ is:

$$w_\tau = \left(\mathbf{B} - \frac{1}{4}I\right) w + e^\tau y(1 - w), \tag{2.47}$$

where the operator $\mathbf{B} = -D_z^4 + \frac{1}{4} z D_z + \frac{1}{4} I$.

Consider the stationary solutions to:

$$\left(\mathbf{B} - \frac{1}{4}I\right) w_e = 0, \quad w_e(\infty) = 1, \quad w_e(-\infty) = 0. \tag{2.48}$$

Note that the pseudo-boundary conditions at $-\infty$ and ∞ represent a step-like Heaviside function connecting the two stationary solutions for the invaded specie that departs from $w = 0$ and ends in $w = 1$ after the invasion occurs.

Any solution is expressed as:

$$w(z,\tau) = w_e(z) + \alpha(z,\tau). \tag{2.49}$$

In the proximity of the stationary solution $|\alpha| << 1$. Replacing this form of solution into (2.47):

$$\alpha_\tau = \left(\mathbf{B} - \frac{1}{4}I\right)\alpha + e^\tau y_e(1 - w_e). \tag{2.50}$$

Admit the following asymptotic separation of variables:

$$\alpha(z,\tau) = \phi(z)\psi(z), \tag{2.51}$$

and operating in (2.50):

$$\frac{\psi'}{\psi} = \frac{\left(\mathbf{B} - \frac{1}{4}I\right)\phi + e^\tau y_e(1 - w_e)\,/\,\psi}{\phi} = K, \tag{2.52}$$

then:

$$\psi(\tau) = e^\tau, \tag{2.53}$$

with $K = 1$ for the sake of simplicity.

A solution to $\phi(z)$ is obtained under the asymptotic condition $w_e(\infty) = 1$, then:

$$\left(\mathbf{B} - \frac{1}{4}I\right)\phi = \phi. \tag{2.54}$$

The operator $\mathbf{B}$ possesses a discrete set of eigenfunctions in the space $H^4_\rho \subset L^2_\rho$ [**26**], therefore any spanned solution ϕ converges in H^4_ρ. Hence, the searched solutions are of the form:

$$\phi(z) = e^{\gamma z}, \tag{2.55}$$

replacing into (2.54) and making the balancing in each term:

$$\gamma^4 = -1, \tag{2.56}$$

provided that $\frac{1}{4}z << 1$, or equivalently:

$$t^{\frac{1}{4}} \geq \frac{1}{4}|x|. \tag{2.57}$$

This last expression shall be understood as the region of validity for the exponential representation (2.55). Consider now the two main real roots in γ:

$$\phi_+ = e^{\gamma z}, \quad z \to -\infty\,; \quad \phi_- = e^{-\gamma z}, \quad z \to \infty, \tag{2.58}$$

so that:

$$\alpha(z,\tau) = e^\tau(e^{\gamma z} + e^{-\gamma z}). \tag{2.59}$$

The solution (2.49) reads:

$$w(z,\tau) = w_e(z) + e^\tau(e^{\gamma z} + e^{-\gamma z}). \tag{2.60}$$

Returning to the original variables (x,t):

$$u(x,t) = w_e\left(\frac{x}{t^{1/4}}\right) + t(e^{\gamma\frac{x}{t^{1/4}}} + e^{-\gamma\frac{x}{t^{1/4}}}). \tag{2.61}$$

The condition $|\alpha| << 1$ means that whenever $x \to \infty$:

$$|te^{-\gamma\frac{x}{t^{1/4}}}| << 1 \;\Rightarrow\; |x| >> t^{1/4}\ln t. \tag{2.62}$$

Note that $\ln t < 0$, then:

$$|x| << t^{1/4}|\ln t| = r(t), \tag{2.63}$$

which includes the validity region in (2.57), i.e.

$$|x| < 4t^{1/4} << t^{1/4}|\ln t| \tag{2.64}$$

for $t \to 0^+$. □

Finally, the same process can be repeated for any $t = t_0 > 0$ with the simple rescaling $\tau = \ln(t - t_0)$. Hence, given any $t = t_0 > 0$, the inner region, where positivity in the solution holds, is defined as:

$$|x| << (t - t_0)^{1/4}|\ln(t - t_0)|. \tag{2.65}$$

A simple estimation can be obtained assuming that $t \sim 2t_0$ for t_0 sufficiently small:

$$|x| << t_0^{1/4}|\ln t_0|. \tag{2.66}$$

3. Asymptotic solutions and evolution of a maximal profile

In the asymptotic approach, assume $|v| < \epsilon << 1\,, \epsilon > 0$, so that v is a perturbation parameter in (1.1), i. e.

$$u_t = -\Delta^2 u + \epsilon(1 - u), \;\; u_0(x) \in L^1_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap H^n(\mathbb{R}), \tag{3.1}$$

Consider, now, the non-linear scaling given by

$$u = e^w. \tag{3.2}$$

The oscillating character in the solutions induced by the homogeneous operator $\{\frac{\partial}{\partial t} + \Delta^2\}$ leads to define w as a complex map: $w : X \times [0, T] \to \mathbb{C}$.

Note that w satisfies a Hamilton-Jacobi equation [22] of the form:

$$w_t = H_4\left(w, \frac{\partial w}{\partial x}\right) + P_4\left(w, \frac{\partial^i w}{\partial x^i}\right), \quad i = 2, 3, 4, \tag{3.3}$$

where

$$H_4(w) = -w_x^2 w_x^2 + \epsilon(1 - e^w), \tag{3.4}$$

and

$$\begin{aligned} P_4(w) \;\; = \;\; & -\Delta^2 w - \Delta(\nabla w \cdot \nabla w) - 2\nabla w \cdot \nabla \Delta w \\ & -2(\nabla w \cdot \nabla w)\Delta w - 2\nabla w \cdot \nabla(\nabla w \cdot \nabla w) - (\Delta w)^2. \end{aligned}$$

Consider ψ is a sufficiently smooth function so that $P_4(\psi)$ is power order 3 and the Hamilton-Jacobi operator is order 4. For a sufficiently large λ:

$$|P_4(\lambda\psi)| = Order(\lambda^3) << |H_4(\lambda\psi)| = Order(\lambda^4). \tag{3.5}$$

Consequently and keeping the leading terms:

$$w_t = -w_x^2 w_x^2 + \epsilon(1 - e^w). \tag{3.6}$$

Note that (3.6) is a non-linear first order equation that admits a standard separation of variables [22], indeed:

$$w(x, t) = (T + t)^{-\frac{1}{3}}\phi(x), \tag{3.7}$$

for $T < t < \infty$. Upon substitution in (3.6) with t sufficiently large but keeping the perturbation condition $0 < \epsilon(T + t)^{\frac{4}{3}} << 1$, the following leading terms hold:

$$-\frac{1}{3}\phi = \phi_x^4, \tag{3.8}$$

so that, a solution is:

$$\phi(x) = 3\left(\frac{1}{4}C(i)x\right)^{\frac{4}{3}}, \tag{3.9}$$

where $C(i) = (-1)^{\frac{1}{4}}$. Then

$$w(x,t) = 3t^{-\frac{1}{3}}\left(\frac{1}{4}C(i)x\right)^{\frac{4}{3}}. \tag{3.10}$$

Finally and after considering the non-linear scaling (3.2), the solution profiles can be compiled as:

$$u(x,t) = e^{3t^{-\frac{1}{3}}\left(\frac{1}{4}C(i)x\right)^{\frac{4}{3}}}, \tag{3.11}$$

Note the existence of oscillations in the leading front as per the complex number $C(i)$.

Similarly, consider the equation for the invaded v in (1.1). In the asymptotic approach, the invasive $u \sim 1$ while the invaded $|v| < \epsilon << 1, \epsilon > 0$, so that:

$$v_t = -\Delta^2 v - \epsilon \leq -\Delta^2 v. \tag{3.12}$$

Hence, a maximal evolution holds in accordance with the homogeneous problem:

$$v_t = -\Delta^2 v, \quad v_0(x) \in L^1_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap H^n(\mathbb{R}) \tag{3.13}$$

Note that the intention is to assess the invaded behaviour in the proximity of the critical point. Hence a Heaviside initial condition $v_0(x) = H(-x)$ is considered as it satisfies $|v| < \epsilon << 1, \epsilon > 0$ in the null tail.

The high order operator $(-\Delta^2)$ does not admit a comparison principle in general due to the instabilities. Then, the intention is to make an asymptotic approach to find a maximal positive and pure monotone evolution for which a comparison principle holds. For this purpose, consider the self-similar scaling:

$$b(x,t) = t^{-\frac{1}{4}} e^t f(\eta), \quad \eta = \frac{x}{t^{\frac{1}{4}}}. \tag{3.14}$$

Introducing the expression (3.14) into the equation (3.13), the following elliptic ODE holds:

$$- f^4 + \frac{1}{4} f'\eta + \frac{1}{4} f = 0; \quad \int_{\mathbb{R}} f(\eta)d\eta = 1. \tag{3.15}$$

Then, [23] :

$$|f(\eta)| \leq D_0 F(\eta), \quad F(\eta) = \omega_1 e^{-d_0|\eta|^{\frac{4}{3}}} > 0, \quad \omega_1 = \left(\int_R e^{-d_0|\eta|^\alpha} d\eta\right)^{-1}. \tag{3.16}$$

The normalizing constant ω_1 guarantees that the maximal positive kernel F satisfies the normalization condition in (3.15). The parameter $D_0 > 0$ is the order deficiency in the high order operator and shall be selected sufficiently large so that $D_0 F(\eta) > f(\eta)$. Figure 1 provides values for D_0 and d_0 to keep the maximal evolution of F.

The next step, in the characterization of the maximal kernel (F), is to obtain a suitable value for d_0. For this purpose, consider an asymptotic approach for the self-similar kernel elliptic ODE (3.15):

$$\eta \to \infty; \; f \to 0 \Rightarrow -f^{(4)} + \frac{1}{4}\eta f' = 0. \tag{3.17}$$

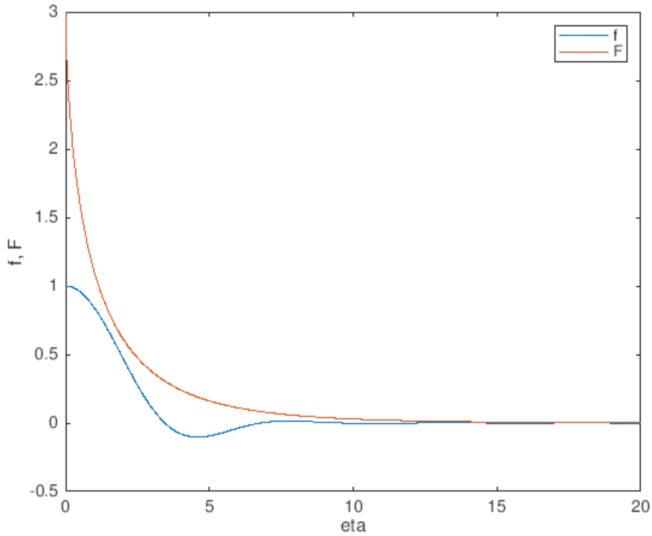

FIGURE 1. The maximal evolution for F is kept for $D_0 = 3$ and $d_0 = 0.5$.

The maximal kernel $F(\eta)$ behaves, asymptotically, as the solution $f(\eta)$ but keeping the global monotone properties. An assessment on d_0 follows a WKB approximation. For this purpose consider the single parameter evolution:

$$e^{-d_0 G(\eta)}, \quad G(\eta) = \eta^{\frac{4}{3}}. \tag{3.18}$$

Now, into (3.17):

$$\begin{aligned} &-\left(d_0 \frac{4}{3}\right)^4 \eta^{4/3} e^{-d_0 \eta^{4/3}} + \left(d_0 \frac{4}{3}\right)^3 e^{-d_0 \eta^{4/3}} + \frac{2}{3}\left(\frac{4}{3}\right)^2 d_0^3 e^{-d_0 \eta^{4/3}} \\ &+ d_0^3 \frac{16}{27} e^{-d_0 \eta^{4/3}} - \frac{1}{4}\eta d_0 \frac{4}{3} \eta^{1/3} e^{-d_0 \eta^{4/3}} = 0. \end{aligned} \tag{3.19}$$

Balancing the leading terms:

$$-\left(d_0 \frac{4}{3}\right)^4 - \frac{1}{4} d_0 = 0, \quad d_0 = Re\left(-\frac{3^3}{2^8}\right)^{\frac{1}{3}}. \tag{3.20}$$

Once a value for d_0 has been obtained, note that two kernels are available at this point given by f and F:

$$b(x,t) = t^{-\frac{1}{4}} e^t f(y), \quad B(x,t) = t^{-\frac{1}{4}} e^t F(y), \quad y = \frac{x}{t^{\frac{1}{4}}}. \tag{3.21}$$

The kernel $B(x,t)$ represents the asymptotic evolution of the kernel $b(x,t)$ and has the positivity property that is lost when operating with the kernel $b(x,t)$ due to the oscillatory behaviour induced by the bi-laplacian term. The following lemma holds:

LEMMA 3.1. *Any solution under the maximal kernel $B(x,t)$ is positive provided the initial condition $v_0(x) \geq 0$. In addition, positivity holds provided $v_0^{med} \geq 0$.*

PROOF. The Duhamel's principle to (3.13) with the maximal kernel $B(x,t)$ reads:

$$\tilde{v}(t) = B(t) * v_0(x) = \int_{\mathbb{R}} B(x-s,t)\, v_0(s)ds, \tag{3.22}$$

where $\tilde{v}(t)$ is the maximal solution to the invaded as a result of the maximal order preserving kernel B.

Making use of the condition $\int_R B = 1$ and the mean value theorem:

$$\tilde{v}(t) = \int_{\mathbb{R}} B(x-s,t)\, v_0(s)ds = \int_{\mathbb{R}} v_0(s)ds \geq 0, \tag{3.23}$$

as intended to show. □

The following is a comparison lemma:

LEMMA 3.2. *Consider $\tilde{v}_0(x) \in L^1_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap H^n(\mathbb{R})$, with $\tilde{v}_0(x) \geq v_0(x)$, then $\tilde{v}(x,t) \geq v(x,t)$.*

PROOF.

$$\begin{aligned}\tilde{v}(t) - v(t) &= B(t) * \tilde{v}_0 - b(t) * v_0 \geq B(t) * \tilde{v}_0 - |b(t)| * |v_0| \\ &\geq B(t) * \tilde{v}_0 - B(t) * |v_0| = B(t)(\tilde{v}_0 - |v_0|).\end{aligned} \tag{3.24}$$

considering that $\tilde{v}_0 \geq |v_0|$:

$$\tilde{v}(t) \geq v(t). \tag{3.25}$$

Consider the spatial variable $x \in \mathbb{R}$ and the boundary value problem in $\mathbb{R}^+ \times \mathbb{R}^+$ with symmetric conditions at the border $x = 0$ towards $x > 0$:

$$v_x(0,t) = v_{xxx}(0,t) = 0, \quad t > 0. \tag{3.26}$$

The same conditions apply for $\tilde{v}$. Then:

$$\begin{aligned}\tilde{v}(x) - v(x) &= B(x) * \tilde{v}_0(t) - b(x) * v_0(t) \geq B(x) * \tilde{v}_0 - |b(x)| * |v_0| \\ &\geq B(x) * \tilde{v}_0 - B(x) * |v_0| = B(x)(\tilde{v}_0 - |v_0|).\end{aligned} \tag{3.27}$$

Then

$$\tilde{v}_0 \geq |v_0| \to \tilde{v}(x) \geq v(x). \tag{3.28}$$

To conclude on $\tilde{v}(x,t) \geq v(x,t)$ as intended to proof.

□

4. Travelling Waves analysis

The problem (1.1) in the TW domain is obtained upon the change $u(x,t) = f(\xi)$, $v(x,t) = g(\xi)$, $\xi = x \cdot n_d - at \in \mathbb{R}$, being n_d the TW-propagation direction, a is the TW-speed and $f : \mathbb{R} \to (0,\infty)$ is the TW profile, $f \in L^\infty(\mathbb{R})$. Note that two TW are equivalent under translation $\xi \to \xi + \xi_0$ and symmetry $\xi \to -\xi$. Without loss of generality, $n_d = (1,0,...,0)$, then $\xi = x - at \in \mathbb{R}$.

The problem (1.1) expressed in the TW variable reads:

$$\begin{gathered} -af' = -f^{(4)} + g(1-f), \\ -ag' = -g^{(4)} - fg, \\ f, g \in L^\infty(\mathbb{R}) \cap H^n(\mathbb{R}), \\ f'(\xi) > 0\,,\ g'(\xi) < 0, \\ f(\infty) = 1\,,\ g(\infty) = 0. \end{gathered} \tag{4.1}$$

Firstly, the intention is to determine the TW behaviour in the proximity of the stationary condition $f = 1,\ g = 0$. For this purpose, consider:

$$
\begin{aligned}
f &= f_1\,; & g &= g_1 \\
f' &= f_1' = f_2\,; & g' &= g_1' = g_2 \\
f'' &= f_1'' = f_2' = f_3\,; & g'' &= g_1'' = g_2' = g_3 \\
f^{(3)} &= f_1^{(3)} = f_2'' = f_3' = f_4\,; & g^{(3)} &= g_1^{(3)} = g_2'' = g_3' = g_4 \\
f^{(4)} &= f_4' = af_2 + f_5(1-f_1)\,; & g^{(4)} &= g_4' = ag_2 - f_5 f_1.
\end{aligned} \tag{4.2}
$$

Then, the set (4.1) is expressed as:

$$
\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix}' = \begin{pmatrix} f_2 \\ f_3 \\ f_4 \\ af_2 + g_1(1-f_1) \\ g_2 \\ g_3 \\ g_4 \\ ag_2 - g_1 f_1 \end{pmatrix} \tag{4.3}
$$

The following lemma holds

Lemma 4.1. *The critical point $f = 1,\ g = 0$ is a degenerate node with:*

- *One null eigenvalue,*
- *Four negative real part eigenvalues,*
- *Three positive real parts eigenvalues,*

For any $a > 0$. Equivalently, in the proximity of the critical point, there exist a 4-D stable family of solutions and a 4-D unstable family of solutions. In addition, the eigenvalues form clusters for increasing values of the TW-speed leading to a synchronization in the oscillating frequencies for the complex branch in the proximity of the critical points.

Proof. Consider $f_1 - 1 = \tilde{f}_1$, then, in the proximity of the critical point, the linearization of (4.1) is as follows:

$$
\begin{pmatrix} \tilde{f}_1 \\ f_2 \\ f_3 \\ f_4 \\ g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\epsilon & a & 0 & 0 & -\epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\epsilon & 0 & 0 & 0 & -\epsilon & a & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{f}_1 \\ f_2 \\ f_3 \\ f_4 \\ g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} \tag{4.4}
$$

where $0 < \epsilon = f_5 << 1$ represents the invaded specie introduced as a perturbation. The computation of a general expression for the eigenvalues and associated eigenvectors is complex in a general case. Then, consider ϵ sufficiently small, i.e. $\epsilon = 0.001$ so that a numerical code is employed for different values of a. Figures 2, 3 and 4 provide evidences on the eigenvalues behaviours. The null eigenvalue is kept for any value of a keeping the $4D$ stable family of solution and the $4D$ unstable. Note that the increasing TW-speed makes the complex eigenvalues to cluster in a single pair of complex conjugate with the same oscillating frequency for both

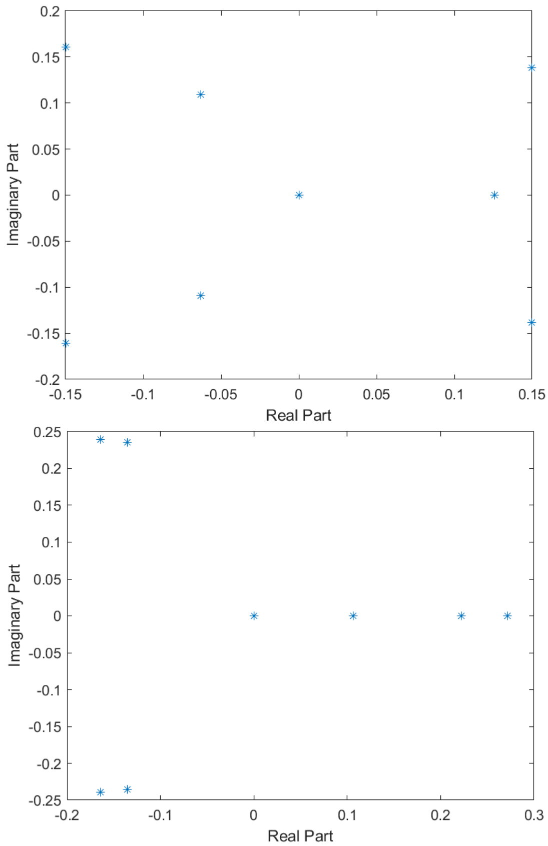

FIGURE 2. Eigenvalues representation for $a = 0.002$ (up figure) and $a = 0.02$ (down). Note that the null eigenvalue is kept, while the eigenvalues with negative real part clusters with the eigenvalues with positive and negative imaginary part. In addition, the eigenvalues with positive real part tends to null imaginary parts.

species. This phenomena is referred as synchronization. The branch of oscillating patterns in the proximity of the critical points are synchronous and depend on the initial conditions and the behaviour of the TW-front.

□

4.1. Geometric Perturbation Theory. Consider as M_0 the 8-dimensional manifold in (4.3) The singular geometric perturbation theory is employed in this section to show the asymptotic behaviour of a perturbed manifold in the proximity of the critical points and defined to make simpler the assessment of a TW analytical profile.

The perturbed manifold M_ϵ close to M_0 is defined as:

(4.5)
$$M_\epsilon = \{f_1, .., f_4, g_1, .., g_4 \ / \ g_1 = \epsilon \, ; \, f_4' = af_2 + \epsilon(1 - f_1) = af_2 - \epsilon\tilde{f}_1 \, ; \, g_4' = ag_2 - g_1 f_1\},$$

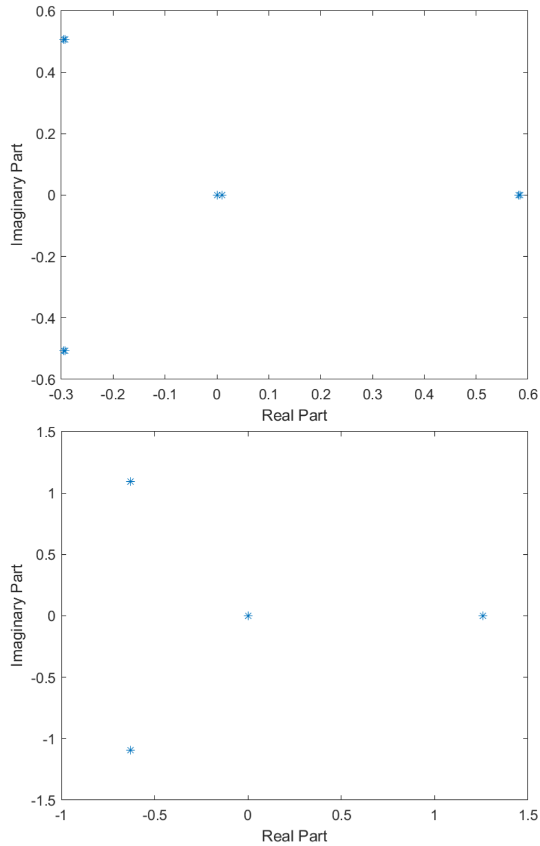

FIGURE 3. Eigenvalues representation for $a = 0.2$ (up figure) and $a = 2$ (down). Note the eigenvalues clustering on the up figure. On the down figure, the eigenvalues are clustered.

being $\tilde{f}_1 = f_1 - 1$. The intention is to use the Fenichel invariant manifold theorem [15], [17] and [16]. For this purpose, M_0 (4.3) shall be a normally hyperbolic manifold, i. e. the eigenvalues of M_0 in the proximity of the critical point, and transversal to the tangent space, have non-zero real part. For this purpose, the eigenvector associated to the zero real part eigenvalue is computed $(1, 0, 0, 0, \epsilon, 0, 0, 0)$ tangent to M_0. Consequently, M_0 is a hyperbolic manifold. The next intention is to show that the manifold M_ϵ is locally invariant under the associated flow. Hence and following [16], for all $R > 0$, for all open interval J with $(a + c) \in J$ and for any value of $i \in \mathbb{N}$, there exists a γ such that for $\epsilon \in (0, \gamma)$ the manifold M_ϵ is invariant. Hence, consider $i \geq 1$ and the functions:

$$\phi_1 = \epsilon, \quad \phi_2 = af_2 + \epsilon(1 - f_1), \quad \phi_3 = ag_2 - g_1 f_1 \tag{4.6}$$

which are $C^i(\overline{B_R(0)} \times \overline{I} \times [0, \gamma])$ in the proximity of the critical point.

A value for $R > 0$ can be chosen considering that $M_0 \cap B_R(0)$ is large enough so as to study the complete TW evolution along the domain. Note that γ is assessed based on computing the distance between flows in M_0 and M_ϵ. For this purpose, assume that the flows are measurable a.e. in $B_R(0)$ with any standard norm including

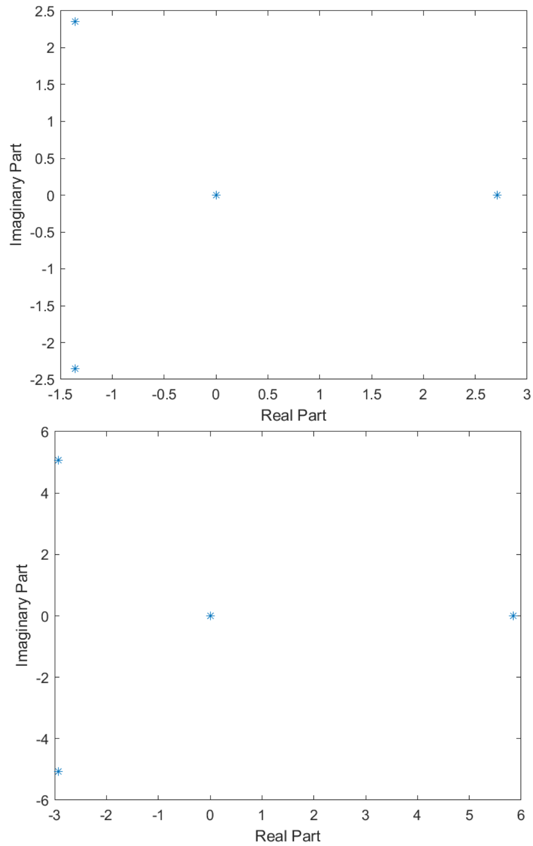

FIGURE 4. Eigenvalues representation for $a = 20$ (up figure) and $a = 200$ (down). For increasing values of the TW speed, the eigenvalues clustering is kept.

(2.1):

$$\|\phi_2^{M_\epsilon} - \phi_2^{M_0}\| \leq \epsilon + \|g_1\|\|1 - f_1\| \leq \gamma\|1 - f_1\|, \tag{4.7}$$

The distance between the manifolds keeps the normal hyperbolic condition in the proximity of $f_1 \nearrow\searrow 1$ for $\gamma \in [1, \infty)$. In addition,

$$\|\phi_3^{M_\epsilon} - \phi_3^{M_0}\| \leq \epsilon\|f_1\| + \|f_1\| + \|g_1\|\|1 - f_1\| \leq \gamma\|1 - f_1\|, \tag{4.8}$$

$f_1 \nearrow\searrow 1$ for $\gamma \in [2, \infty)$. For simplicity assume $\gamma = 2$ in (4.7) and (4.8).

Analogously, the following manifold M_δ close to M_0 is defined:

$$M_\delta = \{f_1, .., f_4, g_1, .., g_4 \;/\; 1 - f_1 = \delta\,;\; f_4' = af_2 + \delta g_1\,;\; g_4' = ag_2 - g_1\}. \tag{4.9}$$

The Fenichel invariant manifold is applied similarly as for M_ϵ to show the invariant properties under the flow in M_δ. To this end, consider the functions:

$$\phi_1' = \delta, \;\; \phi_2' = af_2 + \delta g_1, \;\; \phi_3' = ag_2 - g_1 \tag{4.10}$$

which are $C^i(\overline{B_R(0)} \times \overline{I} \times [0, \delta])$ in the proximity of the critical point.

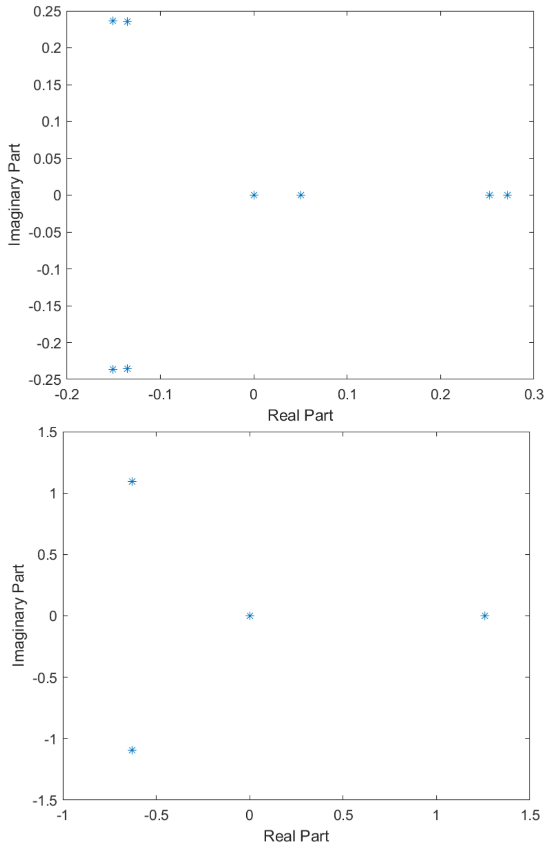

FIGURE 5. Eigenvalues representation for M_ϵ with $a = 0.02$ (up figure) and $a = 2$ (down). Note that the eigenvalues behaviour under M_ϵ is similar to that obtained for M_0 in Figure 3 which permits to validate the analysis done under the geometric perturbation theory.

Note that γ is assessed based on the following flows measurable a.e. in $B_R(0)$:

$$\|{\phi_2'}^{M_\delta} - {\phi_2'}^{M_0}\| \leq 2\kappa \|g_1\|, \quad \|{\phi_3'}^{M_\delta} - {\phi_3'}^{M_0}\| \leq \kappa \|g_1\|, \tag{4.11}$$

M_0 and M_δ are sufficiently close so as to keep the normal hyperbolity in the proximity of the critical points for $\kappa \in (0, \infty)$.

To validate the assessment done, the set of eigenvalues for each of the associated manifolds are provided in Figures 5 and 6 for different values of a.

5. Travelling Waves Profiles and Positivity

Analytical TW profiles can be obtained by operating under the flows in M_ϵ and M_δ with standard means and in the proximity of the equilibrium, i.e. $\epsilon \sim \delta \sim 0$. Nonetheless, the methodology to determine TW profiles and a positivity region is based on numerical exercises for which the Matlab software has been used, in particular the function bvp4c. The numerical method used is based on an implicit Runge-Kutta with an interpolant extension [25]. The collocation method requires

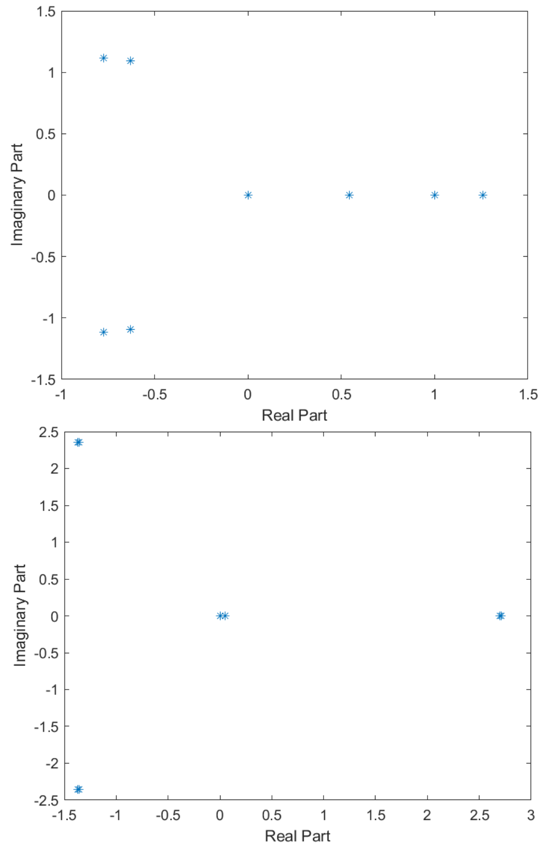

FIGURE 6. Eigenvalues representation for M_δ with $a = 2$ (up figure) and $a = 20$ (down). Note that the eigenvalues behaviour under M_δ is similar to that obtained for M_0 in Figures 3 and 4 for increasing values of a. Again, this behaviour validates the analysis done under the geometric perturbation theory.

to specify the boundary conditions, in this case given by the stationary solutions $f = 1$, $g = 0$. The number of nodes in the integration domain is 10000 and the absolute error fixed at 10^{-6}. The analysis has been done over a sufficiently large ξ interval $[-1000, 1000]$ to avoid the influence of the boundary conditions.

In the classical Fisher-KPP order two problem [4], one of the most remarkable analysis consists on finding a minimal TW profile for which the TW follows a monotone behaviour. This principle does not hold in the problem (1.1). The set of oscillations induced by the high order operator requires a reformulation to find a positive region of monotone behaviour. The objective now is to find an inner region for each TW speed so that outside of this inner region, the oscillatory behavior is given and positivity does not hold.

The TW profiles are provided for different TW-speed values (Figures 7 and 8). An increase in the TW-speed leads to a decreasing positive amplitude. Consider that the positivity region is given by the interval $[0, \xi_+]$. Following an interpolation exercise over the TW-speed interval $[0,0001; 100000]$, a power law has been obtained

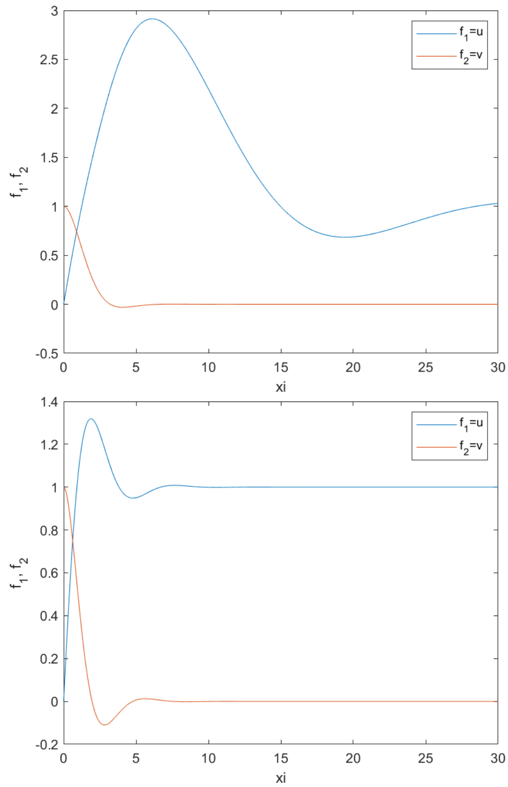

FIGURE 7. TW profiles for $a = 0.002$ (up figure) and $a = 2$ (down). Note the oscillatory behaviour of both interacting species when approaching the equilibrium state. The positivity of solutions is given up to $\xi_+ = 4.451$ (up figure) and $\xi_+ = 1.9502$ (down).

to determine a relation between the TW-speed and the upper positive limit:

$$\xi_+ = 1.3003\, a^{-0.198}. \tag{5.1}$$

6. Conclusions

The existence of solutions to (1.1) has been shown based on solution bounds together with the bounds for abstract evolution operators. Afterward, uniqueness has been shown based on a suitable norm defined for a vector of solutions and application of the Fixed Point Theorem. The main questions related with positivity and a comparison principle have been answered with the assessment of a positive inner region and an asymptotic positive kernel. Solutions for the invasive and invaded species have been obtained in the proximity of the stationary based on the Hamilton-Jacobi equation to show the existence of oscillations induced by the fourth order diffusion. Finally, the problem (1.1) has been analyzed in the TW domain. The eigenvalues evolution with the TW-speed reflects the phenomena named as synchronization of species to account for the clustering of complex eigenvalues. Two perturbed manifolds have been obtained to determine analytical profiles with standard means. In addition, a numerical assessment has been provided to validate

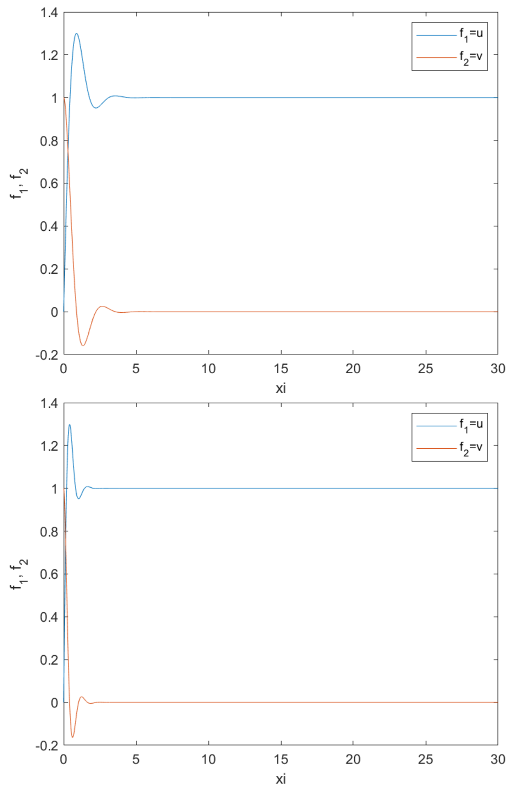

FIGURE 8. TW profiles for $a = 20$ (up figure) and $a = 200$ (down). Note the oscillatory behaviour of both interacting species when approaching the equilibrium state. The positivity of solutions is given up to $\xi_+ = 0.894$ (up figure) and $\xi_+ = 0.455$ (down).

the Geometric Perturbation Theory applied and to determine sharp TW-profiles together with a region of positivity.

7. Conflict of Interest

The author states that there is no conflict of interest.

References

[1] Mooney, H.and Williamson, M. (2010). *The problem of Biological Invasions*. Oxford Scholarship.

[2] Fisher, R.A. (1937). The advance of advantageous genes. *Ann. Eugenics,* 7. pp 355–369.

[3] Cohen, D.S. and Murray, J.D. (1981). A generalized diffusion model for growth and dispersal in a population. *J. Math. Biology* 12, 237–249. https://doi.org/10.1007/BF00276132

[4] Kolmogorov, A.N., Petrovskii I.G. and Piskunov N.S. (1937). Study of the diffusion equation with growth of the quantity of matter and its application to a biological problem. *Byull. Moskov. Gos. Univ.*, Sect. A, 1.

[5] Okubo, A. and Levin, S.A. (2001). *The Basics of Diffusion. In: Diffusion and Ecological Problems: Modern Perspectives.* Interdisciplinary Applied Mathematics, vol 14. Springer, New York, NY. https://doi.org/10.1007/978-1-4757-4978-6.

[6] Coutsias, E.A. (1980). *Some effects of spatial nonuniformities in chemically reacting systems.* California Institute of Technology.

[7] Aronson, D. and Weinberger, H. (1975). Nonlinear diffusion in population genetics, combustion and nerve propagation. *Partial Differential Equations and Related Topics*. Pub., New York, 5–49.

[8] Audrito, A. and Vázquez, J.L. (2017) The Fisher–KPP problem with doubly nonlinear "fast" diffusion *Nonlinear Anal.*, 157, pp 212-248.

[9] Aronson, D. (1980). Density-dependent interaction-diffusion systems. *Proc. Adv. Seminar on Dynamics and Modeling of Reactive System*, Academic Press, New York.

[10] Perumpanani, J. A., Sherratt, J., Norbury and Byrne, H. (1999). A two parameter family of travelling waves with a singular barrier arising from the modelling of extracellular matrix mediated cellular invasion. *Physica D.* 126, 145–159.

[11] Aronson, D. and Weinberger, H. (1978). Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.* 30, pp 33–76.

[12] Cabre, X. and Roquejoffre, J. (2013). The influence of fractional diffusion in Fisher-KPP equations. *Comm. Math. Phys.* 320, pp 679–722.

[13] Rottschäfer, V. and Doelman, A. (1998). On the transition from the Ginzburg-Landau equation to the extended Fisher-Kolmogorov equation. *Physica D.* 118. 261 - 292.

[14] Harley, K., Van Heijster, R., Marangell, R., Pettet, G., Roberts, T. and Wechselberger M. Instability of Travelling Waves in a Model of Haptotaxis. (2019) Preprint.

[15] Fenichel N. (1971). Persistence and smoothness of invariant manifolds for flows, *Indiana Univ. Math. J.* 21, pp 193-226.

[16] Jones C.K. (1995). *Geometric singular perturbation theory in dynamical systems.* Springer-Verlag, Berlín.

[17] Akveld, M.E. and Hulshof, J. (1998). Travelling Wave Solutions of a Fourth-Order Semilinear Diffusion Equation. *Appl. Math. Lett.* Vol. 11, No. 3, pp. 115-120.

[18] Goldshtein, V. and Ukhlov, A. (2009). Weighted Sobolev Spaces and embeddings Theorems. *Trans. Am. Math. Soc.* 361, 3829-3850.

[19] Dee, G.T. and Van Sarloos, W. (1998). Bistable systems with propagating fronts leading to pattern formation. *Phys. Rev. Lett.* Volume 60.

[20] Peletier, L.A. and Troy, W.C.(2001). Spatial Patterns. Higher order models in Physics and Mechanics. *Prog. Nonlinear Differ. Equ. Their Appl.* 45. Université Pierre et Marie Curie.

[21] Bonheure, D. and Sánchez, L. (2006). *Heteroclinics Orbits for some classes of second and fourth order differential equations. Handbook of differential equations.* 3(06), 103-202.

[22] Chavez, M. and Galaktionov, V. (2001). Regional blow up for a higher-order semilinear parabolic equation. *Eur. J. Appl. Math.* 12. 601-623.

[23] Galaktionov, V.A. (2001). On a spectrum of blow-up patterns for a higher-order semilinear parabolic equation. *Proc. R. Soc. Lond.* 457. 1623–1643. http://doi.org/10.1098/rspa.2000.0733

[24] Galaktionov, V. (2012). Towards the KPP–Problem and Log-Front Shift for Higher-Order Nonlinear PDEs I. Bi-Harmonic and Other Parabolic Equations. *Cornwell University arXiv:1210.3513.*

[25] Enright, H. and Muir, P.H. (1993). *A Runge-Kutta type boundary value ODE solver with defect control.* Teh.Rep. 267/93, University of Toronto, Dept. of Computer Sciences. Toronto. Canada.

[26] Egorov, Y., Galaktionov, V., Kondratiev V., Pohozaev, S. (2004). Global solutions of higher-order semilinear parabolic equations in the supercritical range. *Adv. Differ. Equat.* 9. 1009-1038.

Escuela Politécnica Superior, Universidad Francisco de Vitoria, Ctra. Pozuelo-Majadahonda Km 1,800, 28223, Pozuelo de Alarcón, Madrid

Email address: joseluis.diaz@ufv.es