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The Liouville type theorem for the stationary magnetohydrodynamic equations in weighted mixed-norm Lebesgue spaces

Huiying Fan and Meng Wang

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Abstract. In this paper, we are concentrated on demonstrating the Liouville type theorem for the stationary Magnetohydrodynamic equations in mixed-norm Lebesgue spaces and weighted mixed-norm Lebesgue spaces. In particular, we show that, under some sufficient conditions in (weighted) mixed-norm Lebesgue spaces, the solution of stationary MHDs are identically zero. Precisely, we investigate solutions of MHDs that may decay to zero in different rates as $|x| \to \infty$ in different directions. In un-mixed norm case, the result recovers available results. With some additional geometric assumptions on the supports of solutions in weighted mixed-norm Lebesgue spaces, this work also provides several other important Liouville type theorems of solutions in weighted mixed-norm Lebesgue spaces.

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1. Introduction

In this paper, we are concerned with the stationary Magnetohydrodynamic equations(MHDs).

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=(\mathbf{h}\cdot\nabla)\mathbf{h}+\Delta\mathbf{u}, \forall x\in\mathbb{R}^3, \\ div\ \mathbf{u}=0, \\ (\mathbf{u}\cdot\nabla)\mathbf{h}-(\mathbf{h}\cdot\nabla)\mathbf{u}=\Delta\mathbf{h}, \forall x\in\mathbb{R}^3, \\ div\ \mathbf{h}=0, \end{cases} \tag{1.1}$$

equipped with the uniform decay condition at spatial infinity,

$$\mathbf{u}(x)\to 0, \mathbf{h}(x)\to 0 \ \text{ as } \ |x|\to\infty. \tag{1.2}$$

As per usual, $\mathbf{u}=(u_1,u_2,u_3)$ is the velocity field of the fluid flows, $\mathbf{h}=(h_1,h_2,h_3)$ is the magnetic field, and $p=p(x)$ is the pressure of the flows. We refer to the paper [**1**] for the details on the mathematical and physical background of the equations.

Obviously, $(\mathbf{u},\mathbf{h},p)$ with $\mathbf{u}=\mathbf{h}=0$ and $p=constant$ is a trivial solution to (1.1)-(1.2). An important question is if there is other nontrivial solutions. This uniqueness problem, or equivalently Liouville type problem is now hot issue in the community of mathematical fluid mechanics. In general, it is customary to assume the following finiteness of the Dirichlet integral

$$\int_{\mathbb{R}^3}|\nabla\mathbf{u}|^2+|\nabla\mathbf{h}|^2\mathrm{d}x<+\infty. \tag{1.3}$$

Note that if $\mathbf{h}=0$, then the system (MHDs) reduces to the usual stationary Navier-Stokes system. In this paper, we study the Liouville type theorems for the MHD system. The study is motivated by the similar Liouville problem for the stationary Navier-Stokes equations, which is an active research area in the community of mathematical fluid mechanics [**2, 6, 7, 11, 18, 20, 21**]. Very recently, several Liouville type theorems for 3D stationary Navier-Stokes equations are established with the case the solutions are in (weighted) mixed-norm Lebesgue spaces [**18**], which is completely new direction to study the uniqueness problem and use (weighted) mixed-norm Lebesgue spaces to measure those kinds of solutions for Navier-Stokes system.

Admittedly, there are also many developments on the Liouville type theorem for the stationary incompressible MHD systems. In particular, Chae [**3**] generalized Galdi's result for the Navier-Stokes equations to the Hall-MHD and MHD equations under the assumption $(\mathbf{u},\mathbf{h})\in L_{\frac{9}{2}}(\mathbb{R}^3)\cap L_\infty(\mathbb{R}^3)$ with finite Dirichlet integral. Another interesting result [**19**] presented that the condition $(\mathbf{u},\mathbf{h})\in L_p(\mathbb{R}^3)\cap BMO^{-1}(\mathbb{R}^3)$, with $p\in(2,6]$ is sufficient to guarantee the triviality, which is the first challenge without the requirements $(\nabla\mathbf{u},\nabla\mathbf{h})\in L_2(\mathbb{R}^3)$.There are other numerous partial results and references therein proving the triviality of solution to (1.1) under various sufficient conditions [**4, 5, 10, 15, 16, 17, 21, 22**].

Inspired by [**18**], we study the uniqueness problem and extend the mentioned known results for MHD systems to (weighted) mixed-norm Lebesgue spaces. In particular, we investigate solutions of the equations (1.1) that may decay to zero in different rates as $|x|\to\infty$ in different directions. We follow the spirit the work [**14**] to use mixed-norm Lebesgue spaces to measure those kinds of such functions, see also [**8, 9**]. For two given numbers $q,r\in(1,\infty)$, the mixed-norm Lebesgue space

$L_{q,r}(\mathbb{R}^3)$ is the space that is equipped with the following norm

$$\|f\|_{L_{q,r}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |f(x_1,x_2,x_3))|^q \mathrm{d}x_1 \mathrm{d}x_2 \right)^{\frac{r}{q}} \mathrm{d}x_3 \right)^{\frac{1}{r}}, \tag{1.4}$$

with $f : \mathbb{R}^3 \to \mathbb{R}$ is a measurable function. When $q = r$, we just simply write $L_q(\mathbb{R}^3) = L_{q,q}(\mathbb{R}^3)$. In this paper, a function f ia said to be in $L_{q,loc}(\mathbb{R}^3)$ if $f \in L_q(U)$ for every compact set $U \subset \mathbb{R}^3$. In a similar way, we also write $f \in H^1_{loc}(\mathbb{R}^3)$ if $f \in H^1(U)$ for every compact set $U \subset \mathbb{R}^3$.

For completeness, in $\mathbb{R}^3$ we define weak (distributional) solution of the Magnetohydrodynamic equations as follows.

Definition 1.1. *A pair* $(\mathbf{u},\mathbf{h}) \in [H^1_\sigma(\mathbb{R}^3)]^3 \times [H^1_\sigma(\mathbb{R}^3)]^3$ ($[H^1_\sigma(\mathbb{R}^3)]^3 = \{v \in [H^1(\mathbb{R}^3)]^3 : \mathrm{div} v = 0\}$) *is called* $H^1_\sigma(\mathbb{R}^3)$*-weak solution to the MHDs (1.1) if* $(\mathbf{u},\mathbf{h})$ *verify (1.1) in the sense of distribution, i.e.*

$$\int_{\mathbb{R}^3} \nabla\varphi : (\mathbf{u}\otimes\mathbf{u})\mathrm{d}x + \int_{\mathbb{R}^3} p\cdot\nabla\varphi \mathrm{d}x = \int_{\mathbb{R}^3} \nabla\varphi : (\mathbf{h}\otimes\mathbf{h})\mathrm{d}x + \int_{\mathbb{R}^3} \nabla\varphi : \nabla\mathbf{u}\mathrm{d}x,$$

and

$$\int_{\mathbb{R}^3} \nabla\varphi : (\mathbf{h}\otimes\mathbf{u})\mathrm{d}x - \int_{\mathbb{R}^3} \nabla\varphi : (\mathbf{u}\otimes\mathbf{h})\mathrm{d}x = \int_{\mathbb{R}^3} \nabla\varphi : \nabla\mathbf{h}\mathrm{d}x,$$

for all $\varphi \in [C_0^\infty(\mathbb{R}^3)]^3$, *where* $H^1(\mathbb{R}^3)$ *denotes the usual Sobolev space and the pressure* p *is defined as*

$$p = \sum_{i,j=1}^{3} \mathcal{R}_i\mathcal{R}_j(\mathbf{u}_i\mathbf{u}_j - \mathbf{h}_i\mathbf{h}_j), \tag{1.5}$$

in which $\mathcal{R}_i$ *denotes the* i*-th Riesz transform.*

The first result of this paper is the following Liouville theorem for solutions of MHDs (1.1) in mixed-norm Lebesgue spaces.

Theorem 1.2. *Let* $q, r \in [3,\infty]$ *be two numbers satisfying*

$$\frac{2}{q} + \frac{1}{r} \geq \frac{2}{3}. \tag{1.6}$$

Also, let $(\mathbf{u},\mathbf{h}) \in [H^1_{loc}(\mathbb{R}^3)]^3$ *be a weak solution of MHDs (1.1), and assume that* $(\mathbf{u},\mathbf{h}) \in [L_{q,r}(\mathbb{R}^3)]^3$, *then* $\mathbf{u} = \mathbf{h} = 0$ in $\mathbb{R}^3$.

Remark 1.3. *It is very interesting to observe from (1.6) that either* q *or* r *can be taken to be sufficiently large. Consequently, the solution* $(\mathbf{u},\mathbf{h})$ *could decay to zero sufficiently slow as* $|x| \to \infty$ *and it may not be in neither* $L_{\frac{9}{2}}(\mathbb{R}^3)$ *nor* $L_6(\mathbb{R}^3)$. *Therefore, Theorem 1.2 covers the cases that are not covered in the known work such as* [**5, 15, 16, 19**]. *It may be also of great interest to find in below Remark 4.1 for a variant of Theorem 1.2 in weighted mixed-norm Lebesgue spaces. Note that Theorem 1.2 holds for* $q = r = \frac{9}{2}$ *and therefore it recovers the result established in* [**3, 10, 22**]. *In the cases that* $q = r$ *and* $q \in [3, \frac{9}{2}]$, *Theorem 1.2 also recovers the recent results obtained in* [**21**].

In some special cases where we have additional geometric assumptions on the supports of solutions, the ranges of the numbers q and r in Theorem 1.2 are improved significantly. Our next two results are Liouville type theorems for (1.1) in this spirit. To introduce the results, we need to introduce the weighted mixed-norm Lebesgue spaces. For two given numbers $q, r \in (1, \infty)$, for two given weight functions

$$w_1 : \mathbb{R}^2 \to \mathbb{R}, \quad w_2 : \mathbb{R}^2 \to \mathbb{R},$$

the measurable function $f : \mathbb{R}^3 \to \mathbb{R}$ is said to be in the weighted mixed-norm Lebesgue space $L_{q,r}(\mathbb{R}^3, w)$ if

$$\|f\|_{L_{q,r}(\mathbb{R}^3,w)} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |f(x_1, x_2, x_3)|^q w_1(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 \right)^{\frac{r}{q}} w_2(x_3) \mathrm{d}x_3 \right)^{\frac{1}{r}} < \infty,$$

where $w(x) = w_1(x_1, x_2) w_2(x_3)$ for a.e. $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Our second result is the following Liouville type theorem in weighted mixed-norm Lebesgue spaces for solutions whose supports are in strips in $\mathbb{R}^3$.

THEOREM 1.4. *Let $q \in [3, 6], r \in [3, \infty), \alpha = \frac{6-q}{3}$ be fixed numbers and let*

$$w_1(x') = (1 + |x'|)^{-\alpha}, x' = (x_1, x_2) \in \mathbb{R}^2$$

and $w_2(x_3) = 1, x_3 \in \mathbb{R}$. Also, let $(\mathbf{u}, \mathbf{h}) \in [H^1_{loc}(\mathbb{R}^3)]^3$ be a weak solution of (1.1), and assume that $(\mathbf{u}, \mathbf{h}) \in [L_{q,r}(\mathbb{R}^3, w)]^3$, and there exists $R_0 > 0$ such that

$$\operatorname{supp}(\mathbf{u}, \mathbf{h}) \subset \mathbb{R}^2 \times [-R_0, R_0],$$

then $\mathbf{u} = \mathbf{h} = 0$.

REMARK 1.5. *Observe that Theorem 1.4 holds with $q = r = 6$. This result demonstrate a possibility that Liouville theorem for the MHDs (1.1) holds for $L_6(\mathbb{R}^3)$-solutions. See also a similar result in* [**19**] *in which the solutions are assumed to be in $(\mathbf{u}, \mathbf{h}) \in L_6(\mathbb{R}^3) \cap BMO^{-1}(\mathbb{R}^3)$.*

The last result in this paper is a Liouville type theorem in weighted mixed-norm Lebesgue spaces for solutions whose supports are in cylinders in $\mathbb{R}^3$. For this purpose, for each $R > 0$, we denote the cylinder along the x_3-axis in $\mathbb{R}^3$ of radius R by

$$C_R = B'_R \times \mathbb{R},$$

where B'_R is the ball in $\mathbb{R}^2$ centered at the origin with radius R. Our result is the following Liouville type theorem for solutions in weighted mixed-norm Lebesgue spaces in $\mathbb{R}^3$.

THEOREM 1.6. *Let $q, r \in [3, \infty], \alpha \in [0, 1)$ be fixed numbers and let $w_1(x') = 1, x' = (x_1, x_2) \in \mathbb{R}^2$ and $w_2(x_3) = (1 + |x_3|)^{-\alpha}, x_3 \in \mathbb{R}$. Also, let $(\mathbf{u}, \mathbf{h}) \in [H^1_{loc}(\mathbb{R}^3)]^3$ be a weak solution of (1.1), and assume that $(\mathbf{u}, \mathbf{h}) \in [L_{q,r}(\mathbb{R}^3, w)]^3$, and there exists $R_0 > 0$ such that*

$$\operatorname{supp}(\mathbf{u}, \mathbf{h}) \subset C_{R_0},$$

then $\mathbf{u} = \mathbf{h} = 0$.

REMARK 1.7. *Note that Theorem 1.6 allow $(\mathbf{u}, \mathbf{h})$ to decay to zero in x_3 at very slow rate. More specially, let us define*

$$\psi(x_3) = \left(\int_{B'_{R_0}} |(\mathbf{u}(x', x_3), \mathbf{h}(x', x_3))|^q \mathrm{d}x' \right)^{\frac{1}{q}}.$$

Assume that there exist $N_0 > 0$ and $\beta > 0$ such that

$$|\psi(x_3)| \leq \frac{N_0}{(1+|x_3|)^{\beta}}, \quad \forall x_3 \in \mathbb{R}.$$

Then, with the choice of $\alpha \in (0,1)$ and sufficiently close to 1 so that $\beta r + \alpha > 1$, we see that $\psi \in L_r(\mathbb{R}, w_2)$ and therefore $(\mathbf{u}, \mathbf{h}) \in L_{q,r}(\mathbb{R}^3, w)$. As β can be sufficiently small, $(\mathbf{u}, \mathbf{h}) \notin L_{\frac{9}{2}}$ and also $(\mathbf{u}, \mathbf{h}) \notin L_6$.

Noting that in Theorem 1.2, we choose the mixed-norm with power in (x_1, x_2) and in x_3 only for convenience as Theorem 1.2 is in some sense consistent with Theorem 1.4 and Theorem 1.6. Of course, Theorem 1.2 can be extended to case of mixed-norm in three different variable directions, that is, Theorem 1.2 still holds with $(\mathbf{u}, \mathbf{h}) \in L_{q_1,q_2,q_3}(\mathbb{R}^3)$ where $q_1, q_2, q_3 \in [3, \infty)$ and

$$\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} \geq \frac{2}{3},$$

where the mixed-norm spaces L_{q_1,q_2,q_3} is defined in the same way as in (1.4). Observing that in the case $q = 6$, Theorem 1.2 only hold with $r = 3$, while Theorem 1.4 and Theorem 1.6 hold for any $r \in [3, \infty)$ with additional conditions on the supports of solutions. When considering MHDs in $\mathbb{R}^2 \times [-R_0, R_0]$ with homogeneous Dirichlet bounded condition on the boundary $\{x_3 = -R_0\} \cup \{x_3 = R_0\}$, Theorem 1.6 is applicable. Similarly, Theorem 1.6 is also applicable for MHDs in C_{R_0}.

The remaining part of the paper is organized as follows. In section 2, we recall the definitions of Muckenhoupt classes of weights, and we introduce a lemma on weighted mixed-norm estimates for the pressure p in (1.1), which is key of our proofs. Then, we will show the complete proofs of Theorems 1.2,1.4,1.6 respectively in Section 3-5. Our approach is based on the combination of the approach used in [**11**, Theorem X.9.5] together with some new results on mixed-norm estimates and weighted mixed-norm estimates.

We employ the letter C to denote by any constant that can be exactly computed in terms of known quantities. The exact value denoted by C may therefore change from line to line in a given computation.

2. Weights and Weighted Mixed-norm Estimates

In this paper we are going to quote some definitions and properties of Muckenhoupt weights we need, which can be found in [**12, 13**]. Let us present some basic facts about them now. For each $q \in [1, \infty)$, a non-negative measurable function $w : \mathbb{R}^n \to \mathbb{R}$ ia said to be in the Muckenhoupt $A_q(\mathbb{R}^n)$-class of weight if $[w]_{A_q} < \infty$ where

$$[w]_{A_q} = \sup_{R>0, x_0 \in \mathbb{R}^n} \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} w(x) \mathrm{d}x \right) \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} w(x)^{-\frac{1}{q-1}} \mathrm{d}x \right)^{q-1}$$

for $q \in (1, \infty)$, and

$$[w]_{A_1} = \sup_{R>0, x_0 \in \mathbb{R}^n} \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} w(x) \mathrm{d}x \right) \|\frac{1}{w}\|_{L_\infty(B_R(x_0))},$$

where $B_R(x_0)$ demotes the ball in $\mathbb{R}^n$ of radius R and centered at $x_0 \in \mathbb{R}^n$. We also recall that two given numbers $q, r \in (1, \infty)$, and for two weight functions

$$w_1 : \mathbb{R}^2 \to \mathbb{R} \quad \text{and} \quad w_2 : \mathbb{R} \to \mathbb{R},$$

the measurable function $f : \mathbb{R}^3 \to \mathbb{R}$ ia said to be in the weighted mixed-norm Lebesgue spaces $L_{q,r}(\mathbb{R}^3, w)$ if

$$\|f\|_{L_{q,r}(\mathbb{R}^3,w)} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |f(x_1,x_2,x_3)|^q w_1(x')\mathrm{d}x_1\mathrm{d}x_2\right)^{\frac{r}{q}} w_2(x_3)\mathrm{d}x_3\right)^{\frac{1}{r}} < \infty,$$

where $x' = (x_1, x_2), w(x) = w_1(x')w_2(x_3)$ for a.e. $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. In this paper, at various contexts, with a given weight function $w : \mathbb{R}^3 \to \mathbb{R}$ we also write $L_q(\mathbb{R}^3, w)$ for the usual weighted Lebasgue space whose norm is defined by

$$\|f\|_{L_q(\mathbb{R}^3,w)} = \left(\int_{\mathbb{R}^3} |f(x)|^q w(x)\mathrm{d}x\right)^{\frac{1}{q}}.$$

We introduce the following lemma on weighted mixed-norm estimates for the pressure p of the system (1.1), which is crucial for the proof of Theorem 1.4 and 1.6.

Lemma 2.1. *Let $q, r \in [2,\infty)$ and $M_0 \geq 1$. Assume that $(\mathbf{u}, \mathbf{h}) \in [L_{q,r}(\mathbb{R}^3, w)]^3$ with $w(x) = w_1(x')w_2(x_3)$ for all a.e. $x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$, and for $w_1 \in A_{\frac{q}{2}}(\mathbb{R}^2)$, $w_2 \in A_{\frac{r}{2}}(\mathbb{R})$ with*

$$[w_1]_{A_{\frac{q}{2}}(\mathbb{R}^2)} \leq M_0, \quad [w_2]_{A_{\frac{r}{2}}(\mathbb{R})} \leq M_0.$$

Then, there exists $N = N(q, r, M_0) > 0$ such that

$$\|p\|_{L_{\frac{q}{2},\frac{r}{2}}(\mathbb{R}^3,w)} \leq N(\|\mathbf{u}\|^2_{L_{q,r}(\mathbb{R}^3,w)} + \|\mathbf{h}\|^2_{L_{q,r}(\mathbb{R}^3,w)}),$$

where p is defined as in (1.5).

Proof. We use the idea developed in [**8**] which makes use of the extrapolation theory due to Rubio de Francia. Recall that

$$p = \sum_{i,j=1}^{3} \mathcal{R}_i\mathcal{R}_j(\mathbf{u}_i\mathbf{u}_j - \mathbf{h}_i\mathbf{h}_j),$$

where $\mathcal{R}_i$ denotes the i-th Riesz transform. Recall also for each $l \in (1,\infty)$, $M_0 \geq 1$ and each $\mu \in A_l(\mathbb{R}^3)$ with $[\mu]_{A_l(\mathbb{R}^3)} \leq M_0$ the map $\mathcal{R}_i\mathcal{R}_j$ is bounded from $L_l(\mathbb{R}^3, \mu) \to L_l(\mathbb{R}^3, \mu)$ and (see [**12**], for example)

$$\|\mathcal{R}_i\mathcal{R}_j(f)\|_{L_l(\mathbb{R}^3,\mu)} \leq N(l, M_0)\|f\|_{L_l(\mathbb{R}^3,\mu)}. \tag{2.1}$$

It is sufficient to show that the estimate (2.1) can be extended to the weighted mixed-norm. In particular, we claim that for every $q_1, q_2 \in (1,\infty)$

$$\|\mathcal{R}_i\mathcal{R}_j(f)\|_{L_{q_1,q_2}(\mathbb{R}^3,\mu)} \leq N(q_1, q_2, M_0)\|f\|_{L_{q_1,q_2}(\mathbb{R}^3,\mu)}. \tag{2.2}$$

for $\mu(x) = \mu_1(x')\mu_2(x_3)$ with $x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$ and

$$[\mu_1]_{A_{q_1}} \leq M_0, \quad [\mu_2]_{A_{q_2}} \leq M_0. \tag{2.3}$$

For this purpose, for a fixed $\mu_1 \in A_{q_1}(\mathbb{R}^2)$ as in (2.3), let us define

$$\begin{aligned} \phi(x_3) &= \left(\int_{\mathbb{R}^2} |\mathcal{R}_i\mathcal{R}_j(f)(x', x_3)|^{q_1} \mu_1(x')\mathrm{d}x'\right)^{\frac{1}{q_1}} \\ \psi(x_3) &= \left(\int_{\mathbb{R}^2} |f(x', x_3)|^{q_1} \mu_1(x')\mathrm{d}x'\right)^{\frac{1}{q_1}}, \text{for } a.e.\ x_3 \in \mathbb{R}. \end{aligned}$$

Let $\tilde{\mu}_2 \in A_{q_1}(\mathbb{R})$ be any weight and we denote

$$\tilde{\mu}(x) = \mu_1(x')\tilde{\mu}_2(x_3), \quad x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}.$$

By inspection of $\tilde{\mu} \in A_{q_1}(\mathbb{R}^3)$. Therefore, it follows from (2.1) that

$$\|\phi\|_{L_{q_1}(\mathbb{R},\tilde{\mu}_2)} = \|\mathcal{R}_i\mathcal{R}_j(f)\|_{L_{q_1}(\mathbb{R}^3,\tilde{\mu})} \leq N\|f\|_{L_{q_1}(\mathbb{R}^3,\tilde{\mu})} = N\|\psi\|_{L_{q_1}(\mathbb{R},\tilde{\mu}_2)}.$$

Then, we could use the extrapolation theorem [**8**, Theorem 2.5] to infer that

$$\|\phi\|_{L_{q_2}(\mathbb{R},\mu_2)} \leq N\|\psi\|_{L_{q_2}(\mathbb{R},\mu_2)}$$

for every $q_2 \in (1,\infty)$ and for $\mu_2 \in A_{q_2}(\mathbb{R})$, which implies (2.2) and completes the proof of Lemma 2.1. □

To apply Lemma 2.1 in our proof of Theorem 1.4 and 1.6, we introduce the following simple but important proposition. For a rigorous proof of this proposition the reader is referred to [**18**].

Proposition 2.2. *Let $q \in (1,\infty)$, $\alpha \in [0,n)$, and $w(x) = (1+|x|)^{-\alpha}$ for all $x \in \mathbb{R}^n$. Then, $w \in A_q(\mathbb{R}^n)$.*

3. PROOF OF THEOREM 1.2

Adding the equation $(1.1)_1$ and $(1.1)_3$, $(1.1)_2$ and $(1.1)_4$ respectively, we obtain

$$\begin{cases} \mathbf{T}\cdot\nabla\mathbf{v} - \Delta\mathbf{v} = -\nabla p, \\ div\ \mathbf{v} = 0. \end{cases} \tag{3.1}$$

Notice that it follows from Lemma 2.1 that $p \in L_{\frac{q}{2},\frac{r}{2}}(\mathbb{R}^3, w)$. Observe also that as $(\mathbf{u},\mathbf{h}) \in [H^1_{loc}(\mathbb{R}^3)]^3$, we see that $(\mathbf{u},\mathbf{h}) \in [L_{6,loc}(\mathbb{R}^3)]^3 \subset [L_{3,loc}(\mathbb{R}^3)]^3$ by the Sobolev imbedding, $(\mathbf{u},\mathbf{h}) \in [C^\infty(\mathbb{R}^3)]^3$, namely, $(\mathbf{v},\mathbf{T}) \in [C^\infty(\mathbb{R}^3)]^3$. Therefore, for $\xi \in C_0^\infty(\mathbb{R}^3)$, we could use $\mathbf{v}\xi$ as a test function in (3.1) for the system (1.1) and obtain

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla\mathbf{v}|^2\xi \mathrm{d}x &= \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\cdot\Delta\xi\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}|\mathbf{v}|^2(\mathbf{T}\cdot\nabla\xi)\mathrm{d}x \\ &+ \int_{\mathbb{R}^3} p(\mathbf{v}\cdot\nabla\xi)\mathrm{d}x. \end{aligned} \tag{3.2}$$

Proof of Theorem 1.2. For each $R > 0$, we denote the cube in $\mathbb{R}^3$ centered at the origin with radius R by

$$Q_R = [-R,R]^3.$$

Introduce $\phi \in C_0^\infty(\mathbb{R}^3)$ be a standard cut-off function such that

$$\phi(x) = \phi(|x|) = \begin{cases} 1, & |x| < 1/2, \\ 0, & |x| > 1, \end{cases}$$

and $0 \leq \phi \leq 1$ for $1/2 < |x| < 1$. Without of loss of generality, we may assume that $\phi(|x|)$ is a monotonic decreasing in $[0,+\infty)$. Then, for each $R > 0$, we define $\phi_R(x) = \phi(|x|/R)$, then the support of $\nabla\phi_R(x)$ is contained in $Q_R \backslash Q_{R/2}$ and satisfy

$$|\nabla\phi_R| \leq \frac{C}{R} \quad \text{and} \quad |\nabla^2\phi_R| \leq \frac{C}{R^2} \quad \text{in} \quad \mathbb{R}^3. \tag{3.3}$$

Into (3.2) we substitute $\xi = \phi_R$, we are led to

$$\begin{aligned}\int_{Q_{R/2}} |\nabla \otimes \mathbf{v}|^2 \mathrm{d}x &\leq \frac{1}{2}\int_{Q_R\setminus Q_{R/2}} |\mathbf{v}|^2 |\Delta\phi_R| \mathrm{d}x + \\ &+ \frac{1}{2}\int_{Q_R\setminus Q_{R/2}} |\mathbf{v}|^2 |\mathbf{T}||\nabla\phi_R| \mathrm{d}x \\ &+ \int_{Q_R\setminus Q_{R/2}} |p||\mathbf{v}||\nabla\phi_R| \mathrm{d}x \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{3.4}$$

The rest of proof is to control these terms. To control I_1, we choose $q_1, r_1 \in (1, \infty)$ that satisfy

$$\frac{2}{q} + \frac{1}{q_1} = 1 \quad \text{and} \quad \frac{2}{r} + \frac{1}{r_1} = 1. \tag{3.5}$$

Then, we use Hölder's inequality with the exponents $\frac{q}{2}$ and q_1 for the integration in $x' = (x_1, x_2)$-variable, and then use Hölder's inequality with the exponents $\frac{r}{2}$ and r_1 for the integration in x_3-variable. From these calculation, we deduce that

$$\begin{aligned} I_1 &\leq \frac{1}{2}\|\Delta\phi_R\|_{L_{q_1,r_1}(Q_R\setminus Q_{R/2})}\|\mathbf{v}\|^2_{L_{q,r}(Q_R\setminus Q_{R/2})} \leq CR^{\frac{2}{q_1}+\frac{1}{r_1}-2}\|\mathbf{v}\|^2_{L_{q,r}(Q_R\setminus Q_{R/2})} \\ &= CR^{1-2(\frac{2}{q}+\frac{1}{r})}\|\mathbf{v}\|^2_{L_{q,r}(Q_R\setminus Q_{R/2})}. \end{aligned}$$

Where in the second step in the above calculation, we used the second estimates in (3.3). By our assumption, we see that $1 - 2(\frac{2}{q} + \frac{1}{r}) < 0$. From this, and since $(\mathbf{u}, \mathbf{h}) \in [L_{q,r}(\mathbb{R}^3, w)]^3$, that is, $(\mathbf{v}, \mathbf{T}) \in [L_{q,r}(\mathbb{R}^3, w)]^3$ we can infer from the last estimate that

$$\lim_{R\to\infty} I_1 = 0.$$

Next, we control I_2. As $q, r \in [3, \infty)$, we can choose the numbers $q_2, r_2 \in [1, \infty)$ such that

$$\frac{2}{q} + \frac{1}{q} + \frac{1}{q_2} = 1 \quad \text{and} \quad \frac{2}{r} + \frac{1}{r} + \frac{1}{r_2} = 1. \tag{3.6}$$

If $q_2 < \infty$ and $r_2 < \infty$, as in the previous step of controlling I_1, once more applying Hölder's inequality, we are left with

$$\begin{aligned} I_2 &\leq C\|\nabla\phi_R\|_{L_{q_2,r_2}(Q_R\setminus Q_{R/2})}\|\mathbf{v}\|^2_{L_{q,r}(Q_R\setminus Q_{R/2})}\|\mathbf{T}\|_{L_{q,r}(Q_R\setminus Q_{R/2})} \\ &\leq CR^{\frac{2}{q_2}+\frac{1}{r_2}-1}\|\mathbf{v}\|^2_{L_{q,r}(Q_R\setminus Q_{R/2})}\|\mathbf{T}\|_{L_{q,r}(Q_R\setminus Q_{R/2})} \\ &= CR^{2-3(\frac{2}{q}+\frac{1}{r})}\|\mathbf{v}\|^2_{L_{q,r}(Q_R\setminus Q_{R/2})}\|\mathbf{T}\|_{L_{q,r}(Q_R\setminus Q_{R/2})} \end{aligned}$$

Observe also that when $q_2 = \infty$ or $r_2 = \infty$, the above estimates also holds. Now, from our assumption, we see that $2 - 3(\frac{2}{q} + \frac{1}{r}) \leq 0$ and $(\mathbf{v}, \mathbf{T}) \in [L_{q,r}(\mathbb{R}^3, w)]^3$. Therefore, we conclude that

$$\lim_{R\to\infty} I_2 = 0.$$

Finally, we estimate I_3. By repeating above arguments in the estimates of I_2, we use Lemma 2.1 and the fact $(\mathbf{u},\mathbf{h}) \in [L_{q,r}(\mathbb{R}^3)]^3$ to get

$$\begin{aligned} I_3 &\leq C\|\nabla\phi_R\|_{L_{q_2,r_2}(Q_R\backslash Q_{R/2})}\|\mathbf{v}\|_{L_{q,r}(Q_R\backslash Q_{R/2})}\|p\|_{L_{\frac{q}{2},\frac{r}{2}}(\mathbb{R}^3)} \\ &\leq CR^{\frac{2}{q_2}+\frac{1}{r_2}-1}\|\mathbf{v}\|_{L_{q,r}(Q_R\backslash Q_{R/2})}(\|\mathbf{u}\|^2_{L_{q,r}(\mathbb{R}^3)} + \|\mathbf{h}\|^2_{L_{q,r}(\mathbb{R}^3)}) \\ &\leq CR^{2-3(\frac{2}{q}+\frac{1}{r})}\|\mathbf{v}\|_{L_{q,r}(Q_R\backslash Q_{R/2})}(\|\mathbf{u}\|^2_{L_{q,r}(\mathbb{R}^3)} + \|\mathbf{h}\|^2_{L_{q,r}(\mathbb{R}^3)}) \to 0,\ R\to\infty. \end{aligned}$$

By collecting the estimates, we obtain

$$\int_{\mathbb{R}^3} |\nabla\otimes\mathbf{v}|^2 \mathrm{d}x = \lim_{R\to\infty}\int_{Q_{R/2}} |\nabla\otimes\mathbf{v}|^2\mathrm{d}x = 0.$$

Therefore, $\nabla\mathbf{v} = 0$, namely, $\mathbf{v}$ is a constant function in $\mathbb{R}^3$. Combining this with the condition $\mathbf{v} \in [L_{q,r}(\mathbb{R}^3)]^3$, and therefore $\mathbf{v} \equiv 0$, to wit, $\mathbf{u} \equiv -\mathbf{h}$. Substituting this into $(1.1)_3$ and $(1.1)_4$, we know that

$$\begin{cases} \Delta\mathbf{h} = 0, \\ div\ \mathbf{h} = 0. \end{cases} \tag{3.7}$$

Accordingly, the Liouville theorem for a harmonic function and the condition $(\mathbf{u},\mathbf{h}) \in [L_{q,r}(\mathbb{R}^3)]^3$ imply $\mathbf{u} = \mathbf{h} = 0$. □

4. PROOF OF THEOREM 1.4

Proof of Theorem 1.4. For each $R > 0$, let us denote B'_R the ball in $\mathbb{R}^2$ centered at the origin with radius R. Also, let $\phi \in C_0^\infty(\mathbb{R}^2)$ be a standard cut-off function with $0 \leq \phi \leq 1$ and

$$\phi = 1 \text{ on } B'_{1/2} \quad\text{and}\quad \phi = 0 \text{ on } \mathbb{R}^2\backslash B'_1.$$

For each $R > 2$, we defined $\phi_R(x') = \phi(\frac{x'}{R})$ for $x' \in \mathbb{R}^2$. Then, it follows that

$$\phi_R = 1 \text{ on } B'_{R/2} \quad\text{and}\quad \phi_R = 0 \text{ on } \mathbb{R}^2\backslash B'_R.$$

Moreover, there is a universal constant C independent on R such that

$$|\nabla\phi_R| \leq \frac{C}{R} \quad\text{and}\quad |\nabla^2\phi_R| \leq \frac{C}{R^2} \quad \forall R > 0. \tag{4.1}$$

Let us also denote

$$D_R = (B'_R\backslash B'_{R/2}) \times [-R_0, R_0].$$

From (3.2) with $\phi_R(x')$ in place of ξ and the fact supp$(\mathbf{u},\mathbf{h}) \subset \mathbb{R}^2 \times [-R_0, R_0]$, we easily see that

$$\begin{aligned} \int_{B'_R\times[-R_0,R_0]} |\nabla\otimes\mathbf{v}|^2\mathrm{d}x &= \frac{1}{2}\int_{D_R} |\mathbf{v}|^2\Delta\phi_R\mathrm{d}x + \\ &+ \frac{1}{2}\int_{D_R} |\mathbf{v}|^2(\mathbf{T}_1,\mathbf{T}_2)\nabla\phi_R\mathrm{d}x \\ &+ \int_{D_R} p(\mathbf{v}_1,\mathbf{v}_2)\nabla\phi_R\mathrm{d}x. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned}\int_{B'_{R/2}\times[-R_0,R_0]}|\nabla\otimes\mathbf{v}|^2\mathrm{d}x &\le \frac{1}{2}\int_{D_R}|\mathbf{v}|^2|\Delta\phi_R|\mathrm{d}x+\\ &+\frac{1}{2}\int_{D_R}|\mathbf{v}|^2|\mathbf{T}||\nabla\phi_R|\mathrm{d}x\\ &+\int_{D_R}|p||\mathbf{v}||\nabla\phi_R|\mathrm{d}x\\ &= J_1+J_2+J_3. \end{aligned}\tag{4.2}$$

Note that for $R>2$,

$$w_1(x')\sim R^{-\alpha},\quad \forall x'=(x_1,x_2)\in B'_R\backslash B'_{R/2}.$$

Therefore, J_1 is bounded from above by

$$\begin{aligned}J_1 &\le CR^{\frac{2\alpha}{q}}\int_{-R_0}^{R_0}\int_{B'_R\backslash B'_{R/2}}(|\mathbf{v}|^2w_1(x')^{\frac{2}{q}})|\Delta\phi_R(x')|\mathrm{d}x'\mathrm{d}x_3\\ &\le CR^{\frac{2\alpha}{q}}\|\mathbf{v}\|^2_{L_{q,r}(D_R,w)}\|\Delta\phi_R\|_{L_{q_1,r_1}(D_R)}\\ &\le CR^{\frac{2\alpha}{q}+\frac{2}{q_1}-2}R_0^{1/r_1}\|\mathbf{v}\|^2_{L_{q,r}(D_R,w)}\\ &= CR^{\frac{2(\alpha-2)}{q}}R_0^{1-\frac{2}{r}}\|\mathbf{v}\|^2_{L_{q,r}(D_R,w)},\end{aligned}$$

where q_1,r_1 are defined in (3.5). This last estimate along with $\mathbf{v}\in[L_{q,r}(\mathbb{R}^3,w)]^3$ particularly implies that

$$\lim_{R\to 0}J_1=0.$$

Now, in a similar way, we can also have

$$J_2\le CR^{\frac{3\alpha}{q}}\int_{-R_0}^{R_0}\int_{B'_R\backslash B'_{R/2}}(|\mathbf{v}|^2w_1(x')^{\frac{2}{q}})(|\mathbf{T}|w_1(x')^{\frac{1}{q}})|\nabla\phi_R(x')|\mathrm{d}x'\mathrm{d}x_3.$$

Then, with q_2,r_2 as in (3.6), our using Hölder inequality and $(\mathbf{v},\mathbf{T})\in[L_{q,r}(\mathbb{R}^3,w)]^3$ yields

$$\begin{aligned}J_2 &\le CR^{\frac{3\alpha}{q}}\|\mathbf{v}\|^2_{L_{q,r}(D_R,w)}\|\mathbf{T}\|_{L_{q,r}(D_R,w)}\|\nabla\phi_R\|_{L_{q_2,r_2}(D_R)}\\ &\le CR^{\frac{3\alpha}{q}+\frac{2}{q_2}-1}R_0^{1/r_2}\|\mathbf{v}\|^2_{L_{q,r}(D_R,w)}\|\mathbf{T}\|_{L_{q,r}(D_R,w)}\\ &= CR^{1+\frac{3(\alpha-2)}{q}}R_0^{1-\frac{3}{r}}\|\mathbf{v}\|^2_{L_{q,r}(D_R,w)}\|\mathbf{T}\|_{L_{q,r}(D_R,w)}\to 0,\ R\to\infty.\end{aligned}$$

Finally, we control J_3 in the same fashion. Again, we have

$$J_3\le CR^{\frac{3\alpha}{q}}\int_{-R_0}^{R_0}\int_{B'_R\backslash B'_{R/2}}(pw_1(x')^{\frac{2}{q}})(|\mathbf{v}|w_1(x')^{\frac{1}{q}})|\nabla\phi_R(x')|\mathrm{d}x'\mathrm{d}x_3.$$

By inspection of $q\in[3,6],\alpha=\frac{6-q}{\alpha}\in[0,1]\subset[0,n)$, it follows from Proposition 2.2 that $w_1\in A_{\frac{q}{2}}(\mathbb{R}^2)$, from which we are able to apply Lemma 2.1 and the fact $(\mathbf{u},\mathbf{h})\in[L_{q,r}(\mathbb{R}^3,w)]^3$ again to derive that

$$\begin{aligned}J_3 &\le CR^{\frac{3\alpha}{q}}\|p\|_{L_{\frac{q}{2},\frac{r}{2}}(\mathbb{R}^3,w)}\|\mathbf{v}\|_{L_{q,r}(D_R,w)}\|\nabla\phi_R\|_{L_{q_2,r_2}(D_R)}\\ &\le CR^{\frac{3\alpha}{q}+\frac{2}{q_2}-1}R_0^{1/r_2}(\|\mathbf{u}\|^2_{L_{q,r}(\mathbb{R}^3,w)}+\|\mathbf{h}\|^2_{L_{q,r}(\mathbb{R}^3,w)})\|\mathbf{v}\|_{L_{q,r}(D_R,w)}\\ &= CR^{1+\frac{3(\alpha-2)}{q}}R_0^{1-\frac{3}{r}}(\|\mathbf{u}\|^2_{L_{q,r}(\mathbb{R}^3,w)}+\|\mathbf{h}\|^2_{L_{q,r}(\mathbb{R}^3,w)})\|\mathbf{v}\|_{L_{q,r}(D_R,w)}\to 0,\ R\to\infty.\end{aligned}$$

Accordingly, we conclude from estimates of J_k and (4.2) that

$$\int_{\mathbb{R}^3} |\nabla \otimes \mathbf{v}|^2 \mathrm{d}x = \lim_{R\to\infty} \int_{B'_R \times [-R_0, R_0]} |\nabla \otimes \mathbf{v}|^2 \mathrm{d}x = 0,$$

from which our conclusion follows by repeating above arguments in the proof of Theorem 1.2. □

Remark 4.1. *By combining the proofs of Theorem 1.2 and Theorem 1.4 and using (2.1), it can be easily verified that the assertion of Theorem 1.2 also holds if* $(\mathbf{u}, \mathbf{h}) \in L_q(\mathbb{R}^3, w)$ *where*

$$\|f\|_{L_q(\mathbb{R}^3,w)} = \left(\int_{\mathbb{R}^3} |f(x)|^q w(x) \mathrm{d}x \right)^{\frac{1}{q}}$$

for $q \in [3, \frac{9}{2}]$ *and*

$$w(x) = (1 + |x|)^{-(\frac{9}{q}-2)}, \forall x \in \mathbb{R}^3.$$

5. PROOF OF THEOREM 1.6

The proofs of Theorem 1.6 can be completed by the method analogous to that used above, the major change being the substitution of test function, and important details in the calculation needed to be adjusted.

Proof of Theorem 1.2. For each $R > 0$, let $\phi \in C_0^\infty(\mathbb{R})$ be a standard cut-off function with $0 \le \phi \le 1$ and

$$\phi = 1 \text{ on } [-\frac{1}{2}, \frac{1}{2}], \text{ and } \phi = 0 \text{ on } \mathbb{R} \backslash [-1, 1].$$

For each $R > 2$, we define $\phi_R(x_3) = \phi(\frac{x_3}{R})$ with $x_3 \in \mathbb{R}$ such that

$$|\phi'_R| \le \frac{C}{R} \quad \text{and} \quad |\phi''_R| \le \frac{C}{R^2} \quad \forall R > 0. \tag{5.1}$$

We also define

$$E_R = B'_{R_0} \times ([-R, R] \backslash [-R/2, R/2]).$$

Likewise, substituting ξ for $\phi_R(x_3)$ in (3.2) and the fact $\mathrm{supp}(\mathbf{u}, \mathbf{h}) \subset C_{R_0} = B'_{R_0} \times \mathbb{R}$, it is easily checked that

$$\begin{aligned} \int_{B'_{R_0} \times [-R,R]} |\nabla \otimes \mathbf{v}|^2 \mathrm{d}x &= \frac{1}{2} \int_{E_R} |\mathbf{v}|^2 \phi''_R(x_3) \mathrm{d}x + \\ &+ \frac{1}{2} \int_{E_R} |\mathbf{v}|^2 \mathbf{T}_3 \phi'_R \mathrm{d}x \\ &+ \int_{E_R} p \mathbf{v}_3 \phi'_R \mathrm{d}x, \end{aligned}$$

from which we arrive at

$$\begin{aligned}\int_{B'_{R_0}\times[-R/2,R/2]}|\nabla\otimes\mathbf{v}|^2\mathrm{d}x &\le \frac{1}{2}\int_{E_R}|\mathbf{v}|^2|\phi''_R(x_3)|\mathrm{d}x+\\ &+\frac{1}{2}\int_{E_R}|\mathbf{v}|^2|\mathbf{T}||\phi'_R|\mathrm{d}x\\ &+\int_{E_R}|p||\mathbf{v}||\phi'_R|\mathrm{d}x\\ &= K_1+K_2+K_3. \end{aligned} \tag{5.2}$$

Similarly, as $R>2$, we have

$$w_2(x_3)\sim R^{-\alpha}\qquad \forall x_3\in([-R,R]\backslash[-R/2,R/2]).$$

Therefore, for q_1,r_1 are defined in (3.5), we can perform the Hölder inequality along with the fact $(\mathbf{v},\mathbf{T})\in[L_{q,r}(\mathbb{R}^3,w)]^3$ to get

$$\begin{aligned} K_1 &\le CR^{\frac{2\alpha}{r}}\int_{E_R}(|\mathbf{v}|^2w_2(x_3)^{\frac{2}{r}})|\phi''_R(x')|\mathrm{d}x\\ &\le CR^{\frac{2\alpha}{r}}\|\mathbf{v}\|^2_{L_{q,r}(D_R,w)}\|\phi''_R\|_{L_{q_1,r_1}(D_R)}\\ &\le CR^{\frac{2\alpha}{r}+\frac{1}{r_1}-2}R_0^{2/q_1}\|\mathbf{v}\|^2_{L_{q,r}(D_R,w)}\\ &= CR^{\frac{2(\alpha-1)}{r}-1}R_0^{2-\frac{4}{q}}\|\mathbf{v}\|^2_{L_{q,r}(D_R,w)}\to 0,\ R\to\infty. \end{aligned}$$

Then, let q_2,r_2 as in (3.6). In a similar way as we just did, we could derive

$$\begin{aligned} K_2 &\le CR^{\frac{3\alpha}{r}}\int_{E_R}(|\mathbf{v}|^2w_2(x_3)^{\frac{2}{r}})(|\mathbf{T}|w_2(x_3)^{\frac{1}{r}})|\phi'_R(x')|\mathrm{d}x.\\ &\le CR^{\frac{3\alpha}{r}}\|\mathbf{v}\|^2_{L_{q,r}(E_R,w)}\|\mathbf{T}\|_{L_{q,r}(E_R,w)}\|\phi'_R\|_{L_{q_2,r_2}(E_R)}\\ &\le CR^{\frac{3\alpha}{r}+\frac{1}{r_2}-1}R_0^{2/q_2}\|\mathbf{v}\|^2_{L_{q,r}(E_R,w)}\|\mathbf{T}\|_{L_{q,r}(E_R,w)}\\ &= CR^{\frac{3(\alpha-1)}{r}}R_0^{2(1-\frac{3}{q})}\|\mathbf{v}\|^2_{L_{q,r}(E_R,w)}\|\mathbf{T}\|_{L_{q,r}(E_R,w)}\to 0,\ R\to\infty. \end{aligned}$$

It remains to estimate K_3. This can be done exactly the same as the estimates of the other terms. Again, it is clear that

$$K_3\le CR^{\frac{3\alpha}{q}}\int_{E_R}(|p|w_2(x_3)^{\frac{2}{r}})(|\mathbf{v}|w_2(x_3)^{\frac{1}{r}})|\phi'_R(x')|\mathrm{d}x.$$

Now, we note that as $\alpha\in[0,1)$, it follows from Proposition 2.2 that $w_2\in A_{\frac{r}{2}}(\mathbb{R})$. Therefore, we are able to apply Lemma 2.1. Then, we can perform the calculation using Lemma 2.1 together with the fact $(\mathbf{u},\mathbf{h})\in[L_{q,r}(\mathbb{R}^3,w)]^3$ to obtain

$$\begin{aligned} K_3 &\le CR^{\frac{3\alpha}{r}}\|\mathbf{v}\|_{L_{q,r}(E_R,w)}\|p\|_{L_{\frac{q}{2},\frac{r}{2}}(\mathbb{R}^3,w)}\|\phi'_R\|_{L_{q_2,r_2}(E_R)}\\ &\le CR^{\frac{3\alpha}{r}+\frac{1}{r_2}-1}R_0^{2/q_2}\|\mathbf{v}\|_{L_{q,r}(E_R,w)}(\|\mathbf{u}\|^2_{L_{q,r}(\mathbb{R}^3,w)}+\|\mathbf{h}\|^2_{L_{q,r}(\mathbb{R}^3,w)})\\ &= CR^{\frac{3(\alpha-1)}{r}}R_0^{2(1-\frac{3}{q})}\|\mathbf{v}\|_{L_{q,r}(E_R,w)}(\|\mathbf{u}\|^2_{L_{q,r}(\mathbb{R}^3,w)}+\|\mathbf{h}\|^2_{L_{q,r}(\mathbb{R}^3,w)})\to 0, R\to\infty. \end{aligned}$$

Consequently, we infer that

$$\int_{\mathbb{R}^3}|\nabla\otimes\mathbf{v}|^2\mathrm{d}x=\lim_{R\to\infty}\int_{B'_R\times[-R_0,R_0]}|\nabla\otimes\mathbf{v}|^2\mathrm{d}x=0.$$

Arguing as in the proof of Theorem 1.2, we complete the proof. □

References

[1] M. Acheritogaray, P. Degond, A. Frouvelle and J. G. Liu, Kinetic formulation and global existence for the Hall-Magneto-hydrodynamics system, *Kinet. Relat. Models* **4, no. 4** (2011), 901-918.

[2] D. Chae, Note on the Liouville type problem for the stationary Navier-Stokes equations in $\mathbb{R}^3$, *J. Differential Equations* **268, no. 3** (2020), 1043-1049.

[3] D. Chae, P. Degond and J. Liu, Well-posedness for Hall-magnetohydrodynamics, *Ann. Inst. H. Poincaré Anal. Non Linaire* **31, no. 3** (2014), 555-565.

[4] D. Chae and S. Weng, Liouville type theorems for the steady axially symmetric Navier-Stokes and magnetohydrodynamic equations, *Discrete Contin. Dyn. Syst.* **36, no. 10** (2016), 5267-5285.

[5] D. Chae and J. Wolf, On Liouville type theorems for the stationary MHD and Hall-MHD systems,*Submitted* (2018). arXiv:1812.04495.

[6] D. Chae and J. Wolf, On Liouville type theorems for the steady Navier-Stokes equations in $\mathbb{R}^3$, *J. Differential Equations* **261, no. 10** (2016), 5541-5560.

[7] D. Chae and J. Wolf, On Liouville type theorem for the stationary Navier-Stokes equations, *Calc. Var. Partial Differential Equations* **58, no. 13** (2019), Paper No. 111, 11 pp.

[8] H. Dong and D. Kim, On L^p-estimates for elliptic and parabolic equations with A_p weights. (English summary) *Trans. Amer. Math. Soc.* **370, no. 7** (2018), 5081-5130.

[9] H. Dong and T. Phan, Mixed-norm Lp-estimates for non-stationary Stokes systems with singular VMO coefficients and applications. (English summary) *J. Differential Equations* **276** (2018), 342-367.

[10] S. Gala, A note on the Liouville type theorem for the smooth solutions of the stationary Hall-MHD system, *AIMS Math.* **1, no. 3** (2016), 282-287.

[11] G. P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations. In: Steady-state problems, Second edition. Springer Monographs in Mathematics. Springer, New York, 2011.

[12] J. García-Cuerva and J. L. Rubio de Francia, Weighted norm inequalities and related topics. In: Steady-state problems, North-Holland Mathematics Studies, 116. Notas de Matematica [Mathematical Notes], 104. North-Holland Publishing Co., Amsterdam, 1985.

[13] L. Grafakos, Classical Fourier analysis. Second edition. Graduate Texts in Mathematics, 249. Springer, New York, 2008.

[14] N. V. Krylov, Parabolic equations with VMO coefficients in Sobolev spaces with mixed norms. (English summary) *J. Funct. Anal.* **250, no. 2** (2007), 521-558.

[15] Z. Li, P. Liu and P. Niu, Remarks on Liouville type theorems for the 3D stationary MHD equations. (English summary), *Bull. Korean Math. Soc.* **57, no. 5** (2020), 1151-1164.

[16] Z. Li and P. Niu, Liouville type theorems for the 3D stationary Hall-MHD equations, *ZAMM Z. Angew. Math. Mech.* **100, no. 5** (2020), e201900200, 9 pp.

[17] Z. Li and X. Pan, On the vanishing of some D-solutions to the stationary magnetohydrodynamics system. (English summary), *J. Math. Fluid Mech.* **21, no. 4** (2019), Art. 52, 13 pp.

[18] T. Phan, Liouville type theorems for 3D stationary Navier-Stokes equations in weighted mixed-norm Lebesgue spaces. (English summary), *Dyn. Partial Differ. Equ.* **17, no. 3** (2020), 229-243.

[19] S. Schulz, Liouville Type Theorem for the Stationary Equations of Magneto-Hydrodynamics, *Acta Mathematica Scientia* 39, no. 2 (2019), 491-497.

[20] G. Seregin and W. Wang, Sufficient conditions on Liouville type theorems for the 3D steady Navier-Stokes equations, *Algebra i Analiz* **31, no. 2** (2019), 269-278; *reprinted in St. Petersburg Math. J.* **31, no. 2** (2020), 387-393.

[21] B. Yuan and Y. Xiao, Liouville-type theorems for the 3D stationary Navier-Stokes, MHD and Hall-MHD equations. (English summary), *J. Math. Anal. Appl.* **491, no. 2** (2020), 124343, 10 pp.

[22] Z. Zhang, X. Yang and S. Qiu, Remarks on Liouville type result for the 3D Hall-MHD system. (English summary) *J. Partial Differ. Equ.* **28, no. 3** (2015), 286-290.

School of Mathematical Science, Zhejiang University, Hangzhou 310027, CHINA
Email address: hyfaner@zju.edu.cn

School of Mathematical Science, Zhejiang University, Hangzhou 310027, CHINA
Email address: mathdreamcn@zju.edu.cn