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On the well-posedness of the incompressible Euler equations in a larger space of Besov-Morrey type

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Abstract. We obtain a local-in-time well-posedness result and blow-up criterion for the incompressible Euler equations in a new framework, namely Besov spaces based on modified weak-Morrey spaces, covering critical and supercritical cases of the regularity. In comparison with some previous results and considering the same level of regularity, we provide a larger initial-data class for the well-posedness of the Euler equations. For that matter, following the Chemin approach, we need to prove some properties and estimates in those spaces such as preduality, the action of volume preserving diffeomorphism, product and commutator-type estimates, logarithmic-type inequalities, among others.

Contents

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1. Introduction

We consider the initial value problem (IVP) for the incompressible Euler equations

$$\begin{cases} \frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla P = 0, & \text{in } \mathbb{R}^n \times (0,T), \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0, & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 2$, $u = (u_j)_{j=1}^n$ is the velocity field of the fluid and u_0 is the initial velocity that satisfies $\nabla \cdot u_0 = 0$. The scalar function P stands for the pressure and can be formally obtained from u via $\nabla P = -\pi(u,u)$ where

$$\pi(u,v) = \sum_{k,l=1}^{n} \nabla \Delta^{-1} \partial_k u_l \partial_l v_k = \nabla \Delta^{-1} \mathrm{div}\left((u,\nabla)\, v\right). \tag{1.2}$$

Over the years, the solvability of the Cauchy problem (1.1) has been considered by several authors who addressed (1.1) by means of both velocity and vorticity formulations and in different domains and functional spaces; see, e.g., [**9**], [**10**], [**11**], [**17**], [**27**], [**33**], [**34**] and references therein.

In what follows, we review some works of the relevant literature that are more directly related to our purposes. For $u_0 \in H^s(\mathbb{R}^3)$, $s \in \mathbb{N}$ and $s \geq 3$, Kato [**17**] proved the existence and uniqueness of local-in-time solutions for (1.1) in the class $C\left([0,T]; H^s(\mathbb{R}^3)\right) \cap C^1\left([0,T]; H^{s-1}(\mathbb{R}^3)\right)$ with $T = T(\|u_0\|_{H^s(\mathbb{R}^3)})$. In [**19**], Kato and Ponce obtained a unique global-in-time solution $u \in C\left([0,\infty); H_p^s(\mathbb{R}^2)\right)$ provided that $s > 2/p+1$, $1 < p < \infty$, and $u_0 \in H_p^s(\mathbb{R}^2)$. In [**18**], still in the context of Sobolev spaces and considering $n \geq 2$, $s > n/p+1$, $1 < p < \infty$ and $u_0 \in H_p^s(\mathbb{R}^n)$, they showed the existence of a unique local-in-time solution $u \in C\left([0,T]; H_p^s(\mathbb{R}^n)\right) \cap C^1\left([0,T]; H_p^{s-2}(\mathbb{R}^n)\right)$, where the existence-time depends on the initial-data norm $\|u_0\|_{H_p^s(\mathbb{R}^n)}$. Due to the relevance of the index $s_0 = n/p+1$ in the context of Euler equations, the case $s = n/p+1$ is called critical and the range $s > n/p+1$ is called supercritical.

Assuming $n \geq 2$, $1 < p, r < \infty$, and $s > n/p+1$ Chae [**8**] showed existence and uniqueness of solutions in the Triebel-Lizorkin space $F_{p,r}^s$. This result includes the ones of [**18**] and complements the ones of [**10**] because $H_p^s = F_{p,2}^s$ for $1 < p < \infty$ and $F_{\infty,\infty}^s = C^{[s],\gamma}$ for $\gamma = s - [s]$ and $s > 0$ non-integer. The critical case $p = 1$, $n = 2$ and $s = 3$ was addressed in [**7**] for $1 \leq r \leq \infty$.

Considering (1.1) in the framework of Besov spaces, Vishik [**32**] showed the existence of a unique global solution $u \in L_{loc}^\infty\left((0,\infty); B_{p,1}^{2/p+1}(\mathbb{R}^2)\right)$ $(1 < p < \infty)$ for the initial data $u_0 \in B_{p,1}^{2/p+1}(\mathbb{R}^2)$. Later, a local-in-time version of this result was proved in the space $B_{p,1}^{n/p+1}(\mathbb{R}^n)$ in [**6, 35**] for $1 < p < \infty$ and $n \geq 3$. Pak and Park [**28**] and [**29**] considered the endpoint cases $p = \infty$ and $p = 1$, respectively, by obtaining existence and uniqueness of local-in-time solutions. Moreover, in the case $n = 2$ and $p = 1$ they proved that the solutions can be extended to the infinity. See also [**14**] for the continuous dependence property in $B_{\infty,1}^1$ with respect to the initial data. For $n \geq 3$, $1 < p < \infty$, $s \geq n/p+1$ and $1 \leq r \leq \infty$ with $r = 1$ when $s = n/p+1$, Takada [**30**] obtained existence and uniqueness results for (1.1) in the weak-Besov space $w\text{-}B_{p,q}^s$. These are Besov spaces with underlying space weak-L^p instead of L^p and then it follows that $w\text{-}B^s_{p,q} \supset B^s_{p,q}$ because weak-$L^p \supset L^p$. More recently, Ferreira and Pérez-López [**13**] considered the Besov-Herz space $BK^{\alpha,s}_{p,q,r}$, which is a Besov-type space based on the Herz spaces $K^{\alpha}_{p,q}$, and showed well-posedness of solutions for $1 \leq p < \infty$, $1 \leq r, q \leq \infty$, $0 \leq \alpha < n(1-\frac{1}{p})$ with $\alpha = 0$ if $p = 1$, and $s \geq n/p + 1$ with $r = 1$ if $s = n/p + 1$.

Let us recall that well-posedness for (1.1) involves more properties than only existence of solutions, namely existence, uniqueness, persistence, and continuous dependence on initial data, which together characterize a good behavior of the Euler flow in the corresponding setting. The well-posedness in Sobolev or Besov spaces is subtle, even locally-in-time. In fact, Bourgain and Li [**4, 5**] proved that in the critical case $s = n/p + 1$ with $n = 2, 3$, the problem (1.1) is strong ill-posed in $H^s_p(\mathbb{R}^n)$ and $B^s_{p,r}(\mathbb{R}^n)$ for any $1 \leq p < \infty$ and $1 < r \leq \infty$. More precisely, they showed that for any smooth initial data there exists a perturbation with small norm such that the perturbed solution loses the regularity for any $t > 0$ and the persistence property is instantaneously broken; consequently, the continuous dependence is not true. This motivates to investigate the flow generated from (1.1) in frameworks different from H^s_p and $B^s_{p,r}$, specially about the persistence property, which is not trivial outside spaces of conservation laws, and the continuous dependence on the initial data.

In this paper we present a new functional class in which (1.1) is locally-in-time well-posed, namely modified Besov-weak-Morrey spaces that are Besov-type spaces based on the modified weak-Morrey spaces $W\tilde{\mathcal{M}}^l_p$. More precisely, they are Besov-E spaces BE^s_r with underlying space $E = W\tilde{\mathcal{M}}^l_p$ (see Definitions 2.1 and 3.3) and denoted here by $BW\tilde{\mathcal{M}}^{l,s}_{p,r}$, where s is the regularity index, $\{p, l\}$ are indexes that bring a sense of integrability linked to $W\tilde{\mathcal{M}}^l_p$, and the index r corresponds to the summation over frequency scales. Although the amount of indexes is a little uncomfortable, they are necessary for the breadth of our study. Our results include the so-called critical case for the regularity $s = n/p + 1$ if $r = 1$.

The space $W\tilde{\mathcal{M}}^l_p$ is larger than homogeneous Morrey $\mathcal{M}^l_p$ and weak-Morrey $W\mathcal{M}^l_p$, for all $p < l$. For example, $f = |x|^{-\frac{n}{p}} \in W\tilde{\mathcal{M}}^l_p$ while $f \notin W\mathcal{M}^l_p$ for all $p < l$. In the case $p = l$, $W\tilde{\mathcal{M}}^l_p = W\mathcal{M}^l_p$ becomes weak-L^p while $\mathcal{M}^l_p = L^p$. When $p > l$, those Morrey spaces are trivial (see Definition 2.1). So, the corresponding Besov space $BW\tilde{\mathcal{M}}^{l,s}_{p,r}$ is larger than the celebrated Besov-Morrey space $B\mathcal{M}^{l,s}_{p,r}$ ($E = \mathcal{M}^l_p$) introduced by Kozono and Yamazaki [**20**] (see also [**26**]) and the Besov-weak-Morrey space $BW\mathcal{M}^{l,s}_{p,r}$ ($E = W\mathcal{M}^l_p$) (see [**24**]).

Here we follow the Chemin approach (see [**9**]) in which a basic tool is the Littlewood-Paley decomposition. For that matter, we need to prove some properties and estimates in our framework such as convolution estimates, block-spaces properties, preduality, Bernstein inequalities, the action of volume preserving diffeomorphism, product and commutator-type estimates, logarithmic-type inequalities, among others. In [**24**], avoiding the use of that decomposition, Lemarié-Rieusset reobtained several existence and uniqueness results by developing a theory in a scale of functional spaces that covers a number of examples such as H^s_p, $F^s_{p,r}$, $B^s_{p,r}$, $B\mathcal{M}^{l,s}_{p,r}$, and $BW\mathcal{M}^{l,s}_{p,r}$, among other spaces. His approach is different from ours and relies on classical tools of real harmonic analysis such as transport equations with Lipschitz field, singular integrals, atomic decompositions and maximal functions.

Now we state our result of well-posedness for (1.1).

Theorem 1.1. *Let $1 < p < \infty$, $1 < l \leq \infty$ and $p \leq l$. Assume either $s > n/p+1$ with $1 \leq r \leq \infty$ or $s = n/p+1$ with $r = 1$.*

(i) *(Existence and uniqueness) For $u_0 \in BW\tilde{\mathcal{M}}^{l,s}_{p,r}$ satisfying $\nabla \cdot u_0 = 0$, there exists an existence-time $T > 0$ such that the IVP (1.1) has a unique solution $u \in L^\infty\big((0,T)\,; BW\tilde{\mathcal{M}}^{l,s}_{p,r}\big) \cap C\big([0,T]\,; BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}\big)$. Moreover, $u \in C\big([0,T]\,; BW\tilde{\mathcal{M}}^{l,s}_{p,r}\big)$ provided that $r < \infty$.*

(ii) *(Continuous dependence) Let $u_0 \in BW\tilde{\mathcal{M}}^{l,s}_{p,r}$ and let $(u_0^m)_{m\in\mathbb{N}}$ be a bounded sequence in $BW\tilde{\mathcal{M}}^{l,s}_{p,r}$ converging to u_0 in the topology of $BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}$. If u^m and u are the solutions with the initial data u_0^m and u_0, respectively, then there exists $T > 0$ such that $(u^m)_{m\in\mathbb{N}}$ is bounded in the space $L^\infty((0,T)\,; BW\tilde{\mathcal{M}}^{l,s}_{p,r})$ and $u^m \to u$ in $C([0,T]\,; BW\tilde{\mathcal{M}}^{l,s-1}_{p,r})$.*

Some comments on the family of spaces $BW\tilde{\mathcal{M}}^{l,s}_{p,r}$ are in order. These spaces increase with the indexes r, l and decrease with the indexes s, p. Note also that the growth with l is not verified by classical Morrey and weak-Morrey spaces. Recall that $H^s = B^s_{2,2}$ $(s \in \mathbb{R})$, $B^s_{p,p} \subset H^s_p \subset B^s_{p,2}$ $(s \in \mathbb{R}, 1 < p \leq 2)$, $B^s_{p,2} \subset H^s_p \subset B^s_{p,p}$ $(s \in \mathbb{R}, 2 \leq p < \infty)$ and $H^{s_1}_p \subset B^s_{p,1}(s_1 > s,\ 1 < p < \infty)$. Moreover, we have the continuous inclusions

$$\begin{aligned} &B^s_{p,r} \subset w-B^s_{p,r} \subset BW\tilde{\mathcal{M}}^{l,s}_{p,r} \subset BW\tilde{\mathcal{M}}^{\infty,s}_{p,r},\\ &B\mathcal{M}^{l,s}_{p,r} \subset BW\mathcal{M}^{l,s}_{p,r} \subset BW\tilde{\mathcal{M}}^{l,s}_{p,r} \subset BW\tilde{\mathcal{M}}^{\infty,s}_{p,r}, \end{aligned} \tag{1.3}$$

where $1 < p < \infty$, $1 \leq l, r \leq \infty$ and $s \in \mathbb{R}$. In light of (1.3), considering in Theorem 1.1 the cases $l = r = \infty$ with $s > n/p+1$, and $l = \infty$ with $r = 1$ and $s = n/p+1$, our result extends the previous ones in Sobolev and Besov-type spaces H^s, H^s_p, $B^s_{p,r}$, w-$B^s_{p,r}$, $B\mathcal{M}^{l,s}_{p,r}$ and $BW\mathcal{M}^{l,s}_{p,r}$ by providing (in a given level s of regularity) a larger initial-data class for the well-posedness of (1.1). As far as we know, considering the parameters range in Theorem 1.1 and the same index s, there is no inclusion relation between $BW\tilde{\mathcal{M}}^{l,s}_{p,r}$ and the Besov-Herz space used in [**13**]. Moreover, in the case $l = p$ Theorem 1.1 recovers the previous results by [**30**] because $BW\tilde{\mathcal{M}}^{p,s}_{p,r}$ coincides with w-$B^s_{p,r}$. With some effort and adaptation, we could also include the endpoint $p = l = \infty$ but we prefer not to do it, because $BW\tilde{\mathcal{M}}^{l,n/p+1}_{p,1} = B^{n/p+1}_{p,1} = B^1_{\infty,1}$ which has already been considered by [**28**].

After obtaining a local-in-time existence result, it is natural to investigate if the solution can be extended for all time $t > 0$. There are blow-up criterions for (1.1) in different spaces which are based on logarithmic-type inequalities and can be seen as extensions of the blow-up criterion in Sobolev spaces due to Beale, Kato and Majda [**1**] (see also [**22**],[**21**]). Examples of blow-up results in the spaces $C^{1,\gamma}$, $F^s_{p,q}$, $B^s_{p,q}$, w-$B^s_{p,q}$ and $BK^{\alpha,s}_{p,q,r}$ can be found in [**9**],[**8**],[**6**], [**30**] and [**13**], respectively. In this direction, we obtain a logarithmic-type inequality in the framework of modified Besov-weak-Morrey spaces $BW\tilde{\mathcal{M}}^{l,s}_{p,,r}$ (see Lemma 4.2) and then prove the following blow-up criterion:

Theorem 1.2. *Let $1 < p < \infty$, $p \leq l \leq \infty$, $0 < T^* < \infty$ and let $\omega = \nabla \times u$ stand for the fluid vorticity.*

(i) *(Critical case) Let $s = n/p+1$ and $r = 1$. Then, a local in time solution $u \in C\big([0,T^*); BW\tilde{\mathcal{M}}^{l,s}_{p,1}\big) \cap L^1((0,T^*)\,; L^\infty)$ satisfies*

$$\limsup_{t \nearrow T^*} \|u(t)\|_{BW\tilde{\mathcal{M}}^{l,n/p+1}_{p,1}} = \infty, \tag{1.4}$$

if and only if

$$\int_0^{T^*} \|\omega(t)\|_{\dot{B}^0_{\infty,1}} \, dt = \infty. \tag{1.5}$$

(ii) *(Supercritical case) Let* $s > n/p+1$ *and* $1 \le r \le \infty$. *Then, a local-in-time solution* $u \in C\big([0,T^*); BW\tilde{\mathcal{M}}^{l,s}_{p,r}\big) \cap L^1((0,T^*); L^\infty)$ *satisfies*

$$\limsup_{t \nearrow T^*} \|u(t)\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} = \infty, \tag{1.6}$$

if and only if

$$\int_0^{T^*} \|\omega(t)\|_{\dot{B}^0_{\infty,\infty}} \, dt = \infty. \tag{1.7}$$

This manuscript is organized as follows. In Section 2 we introduce the modified weak-Morrey spaces and prove some properties such as boundedness of the Riesz transform, convolution in modified block spaces, Bernstein-type inequalities, among others. In Section 3 we address the modified Besov-weak-Morrey spaces $BW\tilde{\mathcal{M}}^{l,s}_{p,r}$ and their properties. The subject of Section 4 consists in a logarithmic-type inequality and product and commutator-type estimates in the framework of $BW\tilde{\mathcal{M}}^{l,s}_{p,r}$-spaces, as well as estimates for the volume-preserving flow map. In subsections 4.3 and 4.4, using the estimates and properties obtained in the previous sections and subsections 4.1 and 4.2, we show Theorems 1.1 and 1.2.

2. Modified weak-Morrey spaces

In this section we introduce the modified weak-Morrey spaces $W\tilde{\mathcal{M}}^l_p$ and discuss some basic properties of them.

Definition 2.1. Let $1 < p \le l \le \infty$ and $D(x_0, R) = \{x \in \mathbb{R}^n; |x - x_0| < R\}$. The modified weak-Morrey space $W\tilde{\mathcal{M}}^l_p = W\tilde{\mathcal{M}}^l_p(\mathbb{R}^n)$ is defined as the set of all measurable functions such that

$$\|f\|_{W\tilde{\mathcal{M}}^l_p} := \sup_{x_0 \in \mathbb{R}^n} \sup_{R \ge 1} R^{\frac{n}{l} - \frac{n}{p}} \|f\|_{L^{p,\infty}(D(x_0,R))} < \infty. \tag{2.1}$$

The case $p > l$ is not included because $W\tilde{\mathcal{M}}^l_p$ would be trivial. For $1 < p \le l \le \infty$, the pair $(W\tilde{\mathcal{M}}^l_p, \|\cdot\|_{W\tilde{\mathcal{M}}^l_p})$ is a Banach space. It follows immediately from the definition that $W\tilde{\mathcal{M}}^p_p = L^{p,\infty} \supset L^p$ and $\mathcal{M}^l_p \hookrightarrow W\mathcal{M}^l_p \hookrightarrow W\tilde{\mathcal{M}}^l_p$ where $\mathcal{M}^l_p$ and $W\mathcal{M}^l_p$ stand for the classical Morrey and weak-Morrey spaces. Moreover, we also have the continuous inclusions $W\tilde{\mathcal{M}}^{l_0}_p \hookrightarrow W\tilde{\mathcal{M}}^{l_1}_p$ if $l_1 \ge l_0$, $L^\infty \hookrightarrow W\tilde{\mathcal{M}}^\infty_p$ and $W\tilde{\mathcal{M}}^l_{p_1} \hookrightarrow W\tilde{\mathcal{M}}^l_{p_0}$ if $p_1 \ge p_0$.

As in the case of Morrey spaces, we also have a version of the Hölder inequality in the framework of modified Morrey spaces. To be more specific, let $p \le l$ and $p_i \le l_i$ (for $i = 0, 1$) be such that $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$ and $\frac{1}{l} = \frac{1}{l_0} + \frac{1}{l_1}$. Then, there exists a universal constant $C > 0$ such that

$$\|fg\|_{W\tilde{\mathcal{M}}^l_p} \le C \|f\|_{W\tilde{\mathcal{M}}^{l_0}_{p_0}} \|g\|_{W\tilde{\mathcal{M}}^{l_1}_{p_1}}, \tag{2.2}$$

The following lemma consists in a convolution estimate in modified Morrey spaces. We omit its proof since it is similar to that of weak-Morrey spaces (see [**12**]).

Lemma 2.2. ***(Convolution in modified Morrey spaces)*** *Let $1 < p \leq l \leq \infty$ and $\theta \in L^1(\mathbb{R}^n)$. Then, there exists $C > 0$ (independent of θ) such that*

$$\|\theta * f\|_{W\tilde{\mathcal{M}}_p^l} \leq C\|\theta\|_{L^1}\|f\|_{W\tilde{\mathcal{M}}_p^l}, \text{ for all } f \in W\tilde{\mathcal{M}}_p^l. \tag{2.3}$$

Now we prove that the Riesz transform is bounded in the framework of modified Morrey spaces.

Lemma 2.3. *Let $1 < p < \infty$. The Riesz transform is bounded in $W\tilde{\mathcal{M}}_p^l$.*

Proof: Let $K(x) = \frac{x_i}{|x|^{n+1}}$ and $\rho \geq 1$ fixed. We have $K(x) = K_\rho(x) + \bar{K}_\rho(x)$ where $K_\rho(x) = K(x)\chi_{D(0,\rho)}(x)$ and $\bar{K}_\rho(x) = K(x) - K_\rho(x)$.

For $\varphi \in W\tilde{\mathcal{M}}_p^l$, define $\varphi_1 = K_\rho * \varphi$ and $\varphi_2 = \bar{K}_\rho * \varphi$. Then, we have

$$\begin{aligned}|\varphi_2(x)| = \left|\bar{K}_\rho * \varphi\right| &\leq \int_{|y|>\rho} \frac{1}{|y|^n}|\varphi(x-y)|\,dy = \int_\rho^\infty \frac{1}{r^n}d\mu_x(r)\\ &\leq -\int_\rho^\infty \partial_r\left(\frac{1}{r^n}\right)\mu_x(r)dr,\end{aligned}$$

where $\mu_x(r) = \int_{D(x,r)}|\tilde{\varphi}(y)|\,dy = \|\tilde{\varphi}\|_{L^1(D(x,r))}$. Using Hölder inequality in Lorentz spaces we can estimate $\|\mu_x(r)\|_{L^\infty} \leq Cr^{n-\frac{n}{p}}\|\tilde{\varphi}\|_{L^{p,\infty}(D(x,r))} \leq Cr^{n-\frac{n}{l}}\|\varphi\|_{W\tilde{\mathcal{M}}_p^l}$. Thus

$$|\varphi_2(x)| \leq C\|\varphi\|_{W\tilde{\mathcal{M}}_p^l}\int_\rho^\infty \frac{1}{r^{n+1}}r^{n-\frac{n}{l}}dr \leq C\|\varphi\|_{W\tilde{\mathcal{M}}_p^l}\rho^{-\frac{n}{l}},$$

which yields

$$\|\varphi_2\|_{L^{p,\infty}(D(x,\bar{\rho}))} \leq C\|\varphi\|_{W\tilde{\mathcal{M}}_p^l}\rho^{-\frac{n}{l}}\bar{\rho}^{\frac{n}{p}}.$$

For φ_1 we have

$$\begin{aligned}\|\varphi_1\|_{L^{p,\infty}(D(x,\bar{\rho}))} &= \left\|\int_{\mathbb{R}^n} K_\rho(y)\varphi(z-y)dy\right\|_{L^{p,\infty}(D(x,\bar{\rho}))}\\ &= \left\|\int_{\mathbb{R}^n} K_\rho(y)\varphi(z-y)\chi_{D(x,\rho+\bar{\rho})}(z-y)dy\right\|_{L^{p,\infty}(D(x,\bar{\rho}))}\\ &\leq \left\|\int_{\mathbb{R}^n} K_\rho(y)\varphi(z-y)\chi_{D(x,\rho+\bar{\rho})}(z-y)dy\right\|_{L^{p,\infty}(\mathbb{R}^n)}\\ &\leq C\left\|\varphi\chi_{D(x,\rho+\bar{\rho})}\right\|_{L^{p,\infty}(\mathbb{R}^n)} \leq \|\varphi\|_{L^{p,\infty}(D(x,\rho+\bar{\rho}))} \leq C\|\varphi\|_{W\tilde{\mathcal{M}}_p^l}(\rho+\bar{\rho})^{-\left(\frac{n}{l}-\frac{n}{p}\right)}.\end{aligned}$$

From the previous estimates, we conclude that

$$\|K * \varphi\|_{L^{p,\infty}(D(x,\bar{\rho}))} \leq C\|\varphi\|_{W\tilde{\mathcal{M}}_p^l}\left[\rho^{-\frac{n}{l}}\bar{\rho}^{\frac{n}{p}} + (\rho+\bar{\rho})^{-\left(\frac{n}{l}-\frac{n}{p}\right)}\right].$$

Taking in particular $\rho = 2\bar{\rho}$, we arrive at

$$\|K * \varphi\|_{L^{p,\infty}(D(x,\bar{\rho}))} \leq C\|\varphi\|_{W\tilde{\mathcal{M}}_p^l}\left[\bar{\rho}^{-\left(\frac{n}{l}-\frac{n}{p}\right)}\right],$$

which implies the desired boundedness.

◇

In what follows, we deal with the modified block spaces that play an important role in the characterization of the preduals of modified Morrey spaces.

Let $1 < l \leq p \leq \infty$. A measurable function b is called a modified p-block if $\mathrm{supp}(b) \subset D(a,\rho)$ for some $a \in \mathbb{R}^n$ and $\rho \geq 1$, and verifies

$$\rho^{\frac{n}{l}-\frac{n}{p}} \left\| b \right\|_{L^{p,1}(D(a,\rho))} \leq 1. \tag{2.4}$$

The space $\tilde{\mathcal{P}}\mathcal{D}_p^l$ is defined as the set of all functions f that can be expressed as

$$f(x) = \sum_{k=1}^{\infty} \alpha_k b_k(x), \text{ a.e } x \in \mathbb{R}^n, \tag{2.5}$$

where b_k is a modified p-block and $\sum_{k=1}^{\infty} |\alpha_k| < \infty$. The space $\tilde{\mathcal{P}}\mathcal{D}_p^l$ is a Banach space endowed with the norm

$$\|h\|_{\tilde{\mathcal{P}}\mathcal{D}_p^l} = \inf \left\{ \sum_{k=1}^{\infty} |\alpha_k| \ ; \ h = \sum_{k=1}^{\infty} \alpha_k b_k \text{ where the } b_k\text{'s are modified } p\text{-block} \right\}. \tag{2.6}$$

The following lemma shows the duality between modified block and modified Morrey spaces. We omit the proof because it follows the same ideas that the case of the duality between usual block and weak-Morrey spaces (see [**12**]).

Lemma 2.4. *Let $1 < p \leq l \leq \infty$. Then*

$$\left(\tilde{\mathcal{P}}\mathcal{D}_{p'}^{l'}\right)' = W\tilde{\mathcal{M}}_p^l.$$

Using duality we now prove a convolution estimate in the modified block spaces.

Lemma 2.5. ***(Convolution in modified block spaces)*** *Let $1 \leq p \leq l < \infty$ and $\theta \in L^1$. Then, there exists $C > 0$ such that*

$$\|\theta * f\|_{\tilde{\mathcal{P}}\mathcal{D}_p^l} \leq C \|\theta\|_{L^1} \|f\|_{\tilde{\mathcal{P}}\mathcal{D}_p^l}, \text{ for all } f \in \tilde{\mathcal{P}}\mathcal{D}_p^l. \tag{2.7}$$

Proof. For $f \in \tilde{\mathcal{P}}\mathcal{D}_p^l$, we can estimate

$$\begin{aligned}
\|\theta * f\|_{f\in\tilde{\mathcal{P}}\mathcal{D}_p^l} &= \sup_{\|h\|_{W\tilde{\mathcal{M}}_{p'}^{l'}} \leq 1} |\langle \theta * f, h\rangle| = \sup_{\|h\|_{W\tilde{\mathcal{M}}_{p'}^{l'}} \leq 1} \left|\left\langle f, h * \tilde{\theta}\right\rangle\right| \\
&\leq \|f\|_{\tilde{\mathcal{P}}\mathcal{D}_p^l} \sup_{\|h\|_{W\tilde{\mathcal{M}}_{p'}^{l'}} \leq 1} \left\|h * \tilde{\theta}\right\|_{W\tilde{\mathcal{M}}_{p'}^{l'}} \\
&\leq C \left\|\tilde{\theta}\right\|_{L^1} \|f\|_{\tilde{\mathcal{P}}\mathcal{D}_p^l} \sup_{\|h\|_{W\tilde{\mathcal{M}}_{p'}^{l'}} \leq 1} \|h\|_{W\tilde{\mathcal{M}}_{p'}^{l'}} \\
&\leq C \|\theta\|_{L^1} \|f\|_{\tilde{\mathcal{P}}\mathcal{D}_p^l}.
\end{aligned}$$

◇

The lemma below contains Bernstein-type inequalities in modified Morrey spaces.

Lemma 2.6. ***(Bernstein-type inequalities in modified Morrey spaces)*** *Let $1 < p \leq l \leq \infty$.*

i) *Given $C_1 > 0$, there exists $C > 0$ such that*

$$\|D^\gamma f\|_{W\tilde{\mathcal{M}}_p^l} \leq C 2^{j|\gamma|} \|f\|_{W\tilde{\mathcal{M}}_p^l}, \tag{2.8}$$

for all $j \in \mathbb{Z}$ and $f \in W\tilde{\mathcal{M}}_p^l$ such that supp $\left(\hat{f}\right) \subset D(0, C_1 2^j)$.

ii) *For $j \geq -1$, we have that*

$$\|f\|_{L^\infty} \leq C2^{j\frac{n}{p}} \|f\|_{W\tilde{\mathcal{M}}_p^l}, \tag{2.9}$$

for all $f \in W\tilde{\mathcal{M}}_p^l$ such that supp $\left(\hat{f}\right) \subset D(0, C_1 2^j)$. Moreover, if $j \leq -2$ we have

$$\|f\|_{L^\infty} \leq C2^{j\frac{n}{l}} \|f\|_{W\tilde{\mathcal{M}}_p^l}, \tag{2.10}$$

for all $f \in W\tilde{\mathcal{M}}_p^l$ such that supp $\left(\hat{f}\right) \subset D(0, C_1 2^j)$.

iii) *Let $C_2 > C_1 > 0$, $j \in \mathbb{Z}$ and $f \in W\tilde{\mathcal{M}}_p^l$ be such that supp $\left(\hat{f}\right) \subset \left\{\xi \in \mathbb{R}^n; C_1 2^j \leq |\xi| < C_2 2^j\right\}$. Then, there exists a constant $C > 0$ depending on γ, n, C_1, C_2 such that*

$$\|f\|_{W\tilde{\mathcal{M}}_p^l} \leq C2^{-j|\gamma|} \|D^\gamma f\|_{W\tilde{\mathcal{M}}_p^l}. \tag{2.11}$$

Proof. Taking $\theta \in C_c^\infty(\mathbb{R}^n)$ such that $\theta \equiv 1$ on $D(0, C_1)$, and defining $\theta_j(\xi) := \theta\left(2^{-j}\xi\right)$, it follows that $\theta_j \equiv 1$ on $D(0, C_1 2^j)$ and $\hat{f} = \theta_j \hat{f}$. Therefore, we have

$$\widehat{D^\gamma f}(\xi) = 2\pi i^{|\gamma|} \xi^\gamma \hat{f}(\xi) = 2\pi i^{|\gamma|} \xi^\gamma \theta_j(\xi) \hat{f}(\xi) = \widehat{D^\gamma \check{\theta}_j}(\xi) \hat{f}(\xi),$$

which implies that

$$D^\gamma f = 2^{j|\gamma|} 2^{jn} \left(D^\gamma \check{\theta}\right)(2^j \cdot) * f.$$

Now item **(i)** follows from Lemma 2.2.

Next we deal with item **(ii)**. Let $x \in \mathbb{R}^n$, $\rho \in \mathbb{R}$ and $\mu(\rho) := \int\limits_{D(0,\rho)} \left|\tau_x \tilde{f}(y)\right| dy$, where $\tau_x \tilde{f}(y) = f(x-y)$. We have that

$$|\mu(\rho)| \leq C\rho^{n-\frac{n}{p}} \|f\|_{L^{p,\infty}(D(-x,\rho))} \leq C\rho^{n-\frac{n}{l}} \|f\|_{W\tilde{\mathcal{M}}_p^l},$$

and

$$\begin{aligned} |f(x)| &= |((\theta_j)\check{} * f)(x)| \leq \left| \int_{\mathbb{R}^n} (\theta_j)\check{}(y) f(x-y)\, dy \right| \\ &\leq \left| \int_{D(0,1)} (\theta_j)\check{}(y) f(x-y)\, dy \right| + \left| \int_{D(0,1)^c} (\theta_j)\check{}(y) f(x-y)\, dy \right| = I + II. \end{aligned}$$

Assume further that θ is radially symmetric and abusively denote $\check{\theta}(y) = \check{\theta}(\rho)$ with $\rho = |y|$. To estimate II, fixed j we denote by P and N the subsets of $[1, \infty)$ where the function $\check{\theta}\left(2^j \cdot\right)$ is positive and negative, respectively. Note that P and N are unions of open intervals and the number 1 is the inferior endpoint of one of such intervals. So, proceeding as in the proof of Lemma 2.3 we have

$$\begin{aligned}
&\left|\int_{D(0,1)^c} (\theta_j)\check{}(y)\, f(x-y)\, dy\right| = \left|\int_{D(0,1)^c} 2^{jn}\check{\theta}\left(2^j y\right) f(x-y)\, dy\right| \\
&\leq \int_{D(0,1)^c} 2^{jn}\left|\check{\theta}\left(2^j y\right)\right| |f(x-y)|\, dy = \int_1^\infty 2^{jn}\left|\check{\theta}\left(2^j\rho\right)\right| d\mu(\rho) \\
&= \int_P 2^{jn}\check{\theta}\left(2^j\rho\right) d\mu(\rho) + \int_N 2^{jn}\left(-\check{\theta}\left(2^j\rho\right)\right) d\mu(\rho) \\
&\leq -\int_P 2^j 2^{jn}\left(\left(\check{\theta}\right)'\left(2^j\rho\right)\right)\mu(\rho)\, d\rho + \int_N 2^j 2^{jn}\left(\left(\check{\theta}\right)'\left(2^j\rho\right)\right)\mu(\rho)\, d\rho \\
&\leq C\int_1^\infty 2^j 2^{jn}\left|\left(\check{\theta}\right)'\left(2^j\rho\right)\right|\mu(\rho)\, d\rho \\
&\leq C\|f\|_{W\tilde{\mathcal{M}}_p^l}\int_1^\infty 2^j 2^{jn}\left|\left(\check{\theta}\right)'\left(2^j\rho\right)\right|\rho^{n-\frac{n}{l}}\, d\rho \\
&= C2^{j\frac{n}{l}}\|f\|_{W\tilde{\mathcal{M}}_p^l}\int_1^\infty 2^j\left|\left(\check{\theta}\right)'\left(2^j\rho\right)\right|\left(2^j\rho\right)^{n-\frac{n}{l}}\, d\rho \\
&= C2^{j\frac{n}{l}}\|f\|_{W\tilde{\mathcal{M}}_p^l}\int_{2^j}^\infty \left|\left(\check{\theta}\right)'(z)\right| z^{n-\frac{n}{l}}\, dz \\
&\leq C2^{j\frac{n}{l}}\|f\|_{W\tilde{\mathcal{M}}_p^l}\int_0^\infty \left|\left(\check{\theta}\right)'(z)\right| z^{n-\frac{n}{l}}\, dz \\
&\leq C2^{j\frac{n}{l}}\|f\|_{W\tilde{\mathcal{M}}_p^l}. \qquad (2.12)
\end{aligned}$$

For the parcel I we get

$$\begin{aligned}
I &\leq C\left\|\check{\theta}_j\right\|_{L^{p',1}(D(0,1))}\left\|\tau_x\tilde{f}\right\|_{L^{p,\infty}(D(0,1))} \leq C\left\|\check{\theta}_j\right\|_{L^{p',1}}\left\|\tilde{f}\right\|_{L^{p,\infty}(D(x,1))} \\
&\leq C2^{j\frac{n}{p}}\left\|\check{\theta}\right\|_{L^{p',1}}\|f\|_{W\tilde{\mathcal{M}}_p^l}. \qquad (2.13)
\end{aligned}$$

Since $2^{j\frac{n}{l}} \leq C2^{j\frac{n}{p}}$ for $j \geq -1$, we obtain (2.9) from (2.12) and (2.13). In the case $j \leq -2$ we have that $2^{j\frac{n}{p}} \leq C2^{j\frac{n}{l}}$, and (2.10) follows by using again (2.12) and (2.13).

Finally, we turn to item **(iii)**. Considering $\phi_j(\xi) = \phi\left(2^{-j}\xi\right)$ with $\phi \in C_c^\infty(\mathbb{R}^n)$ and $\phi \equiv 1$ on $\{\xi;\, C_1 \leq |\xi| < C_2\}$, we have that $\phi_j \equiv 1$ on $\left\{\xi;\, C_1 2^j \leq |\xi| < C_2 2^j\right\}$ and

$$\widehat{f}(\xi) = \phi_j(\xi)\widehat{f}(\xi) = i^{-|\gamma|}2^{-j|\gamma|}\phi_j(\xi)\left(\frac{\xi}{2^j}\right)^{-\gamma} i^{|\gamma|}\xi^\gamma\widehat{f}(\xi).$$

Choosing $\theta(\xi) = \phi(\xi)\xi^{-\gamma}$, we can write f as

$$f = i^{-|\gamma|}2^{-j|\gamma|}2^{jn}\check{\theta}(2^j\cdot) * D^\gamma f.$$

Then, in view of Lemma 2.2, the $W\tilde{\mathcal{M}}_p^l$-norm of f can be estimated as

$$\|f\|_{W\tilde{\mathcal{M}}_p^l} \leq 2^{-j|\gamma|}\left\|2^{jn}(\check{\theta}(2^j\cdot)) * D^\gamma f\right\|_{W\tilde{\mathcal{M}}_p^l} \leq C2^{-j|\gamma|}\left\|D^\gamma f\right\|_{W\tilde{\mathcal{M}}_p^l},$$

which gives (2.11).

◇

3. Modified Besov-weak-Morrey spaces

This section is devoted to the modified Besov-weak-Morrey spaces and their properties.

We begin by recalling some facts about sequence spaces. For $1 \leq r \leq \infty$, $s \in \mathbb{R}$ and a Banach space E, we define $l_r^s(E)$ as the space of all sequences $a = (a_k)_{k \geq -1}$ with $a_k \in E$ for all k and such that

$$\|a\|_{l_r^s(E)} = \left(\sum_{k=-1}^{\infty} 2^{ksr} \|a_k\|_E^r \right)^{1/r} < \infty.$$

As we will show later, the spaces in which we are interested in behave like sequence spaces. In what follows, we recall two classical lemmas about real interpolation in this kind of spaces (see, e.g., [**2**]).

Lemma 3.1. *Assume that* $1 \leq r_0, r_1, r \leq \infty$ *and* $s_0 \neq s_1$. *Then, we have that*

$$\left(l_{r_0}^{s_0}(E), l_{r_1}^{s_1}(E)\right)_{\theta,r} = l_r^s(E),$$

where $0 < \theta < 1$ *and* $s = (1-\theta)s_0 + \theta s_1$. *If* $s = s_0 = s_1$ *and* $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$, *then*

$$\left(l_{r_0}^{s}(E), l_{r_1}^{s}(E)\right)_{\theta,r} = l_r^s(E)$$

The subject of the next lemma is a basic duality property of sequence spaces (see [**16**]).

Lemma 3.2. *For* $1 \leq r < \infty$ *and* $s \in \mathbb{R}$, *we have that*

$$\left(l_r^s(E)\right)' = \left(l_{r'}^{-s}(E')\right).$$

In the remainder of this paper, we consider a nonnegative radial function $\varphi \in C_c^\infty(\mathbb{R}^n \backslash \{0\})$ satisfying $\operatorname{supp}(\varphi) \subset \{\xi \,;\, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ and

$$\sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1, \ \ \forall \xi \in \mathbb{R}^n \backslash \{0\}, \text{ where } \varphi_j(\xi) := \varphi\left(2^{-j}\xi\right).$$

Moreover, define ψ as $\psi(\xi) = \sum_{j \leq -1} \varphi_j(\xi)$ if $\xi \neq 0$ and $\psi(\xi) = 1$ if $\xi = 0$. So, $\psi \in C_c^\infty(\mathbb{R}^n)$, $\operatorname{supp}(\psi) \subset \{\xi \,;\, |\xi| \leq \frac{4}{3}\}$ and

$$\psi(\xi) + \sum_{j \geq 0} \varphi_j(\xi) = 1, \, \forall \xi \in \mathbb{R}^n.$$

For simplicity in the calculations, we also define the functions $\tilde{\varphi}_j = \sum_{|k-j| \leq 1} \varphi_k$, $\tilde{\psi} = \psi + \varphi$ and the set $\tilde{D}_j = D_{j-1} \cup D_j \cup D_{j+1}$, where $D_j = \left\{x \,;\, \frac{3}{4}2^j \leq |x| \leq \frac{8}{3}2^j\right\}$ and $j \in \mathbb{Z}$. So, we have $\tilde{\varphi}_j \equiv 1$ on D_j, $\varphi_j = \tilde{\varphi}_j \varphi_j$ $(\forall j \in \mathbb{Z})$ and $\psi = \tilde{\psi}\psi$.

The localization operators Δ_j, $\bar{\Delta}_j$ and S_j are defined as

$$\begin{aligned}
\Delta_j f = \varphi_j(D) f &= \mathcal{F}^{-1}\left(\varphi_j \hat{f}\right) && \text{for all } j \in \mathbb{Z}, \\
\bar{\Delta}_j f &= \Delta_j f && \text{if}\, j \geq 0, \\
\bar{\Delta}_{-1} f = \psi(D) f &= \mathcal{F}^{-1}\left(\psi \hat{f}\right), && \\
\bar{\Delta}_j f &= 0 && \text{if}\, j \leq -2, \\
S_j f = \psi_j(D) f &= \mathcal{F}^{-1}\left(\psi_j \hat{f}\right) && \text{for all}\, j \in \mathbb{Z},
\end{aligned}$$

where $\psi_j(\xi) = \psi\left(2^{-j}\xi\right)$. In view of the supports of φ_j and ψ_j, we can see that

$$\Delta_j \Delta_k f = 0 \text{ if } |j-k| \geq 2 \text{ and } \Delta_j\left(S_{k-2}\Delta_k f\right) = 0 \text{ if } |j-k| \geq 3. \tag{3.1}$$

Using the operators Δ_j and S_j, we have the Littlewood-Paley decomposition $f = S_k f + \Sigma_{j\geq k}\Delta_j f$, for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $k \in \mathbb{Z}$. Moreover, if $\lim\limits_{k\to-\infty} S_k f = 0$ in $\mathcal{S}'$ (as is the case for $f \in L^\infty\backslash\{\text{constants}\}$), then the equality $f = \sum\limits_{j\in\mathbb{Z}} \Delta_j f$ is called the homogeneous Littlewood-Paley decomposition of f (see [**23**]). On the other hand, using Bony's paraproduct (see [**3**]), it follows that for $f, g \in \mathcal{S}'$ we can define the product fg as

$$fg = T_f g + T_g f + R(f,g), \tag{3.2}$$

where

$$T_f g = \sum_{j\geq 2} S_{j-2} f \bar{\Delta}_j g \text{ and } R(f,g) = \sum_{j\geq -1} \bar{\Delta}_j f \tilde{\bar{\Delta}}_j g \text{ with } \tilde{\bar{\Delta}}_j g = \sum_{|j-j'|\leq 2} \bar{\Delta}_{j'} g.$$

Note that if $f = \sum\limits_{j\in\mathbb{Z}} \Delta_j f$, then, for any $N \in \mathbb{Z}$, we have that $S_N f = \sum\limits_{j\leq N-1} \Delta_j f$. Thus, if $f = \sum\limits_{j\in\mathbb{Z}} \Delta_j f$ and $g = \sum\limits_{j\in\mathbb{Z}} \Delta_j g$, then their product can be written as

$$fg = \dot{T}_f g + \dot{T}_g f + \dot{R}(f,g),$$

where

$$\dot{T}_f g = \sum_{j\in\mathbb{Z}} S_{j-2} f \Delta_j g \text{ and } \dot{R}(f,g) = \sum_{j\in\mathbb{Z}} \Delta_j f \tilde{\Delta}_j g \text{ with } \tilde{\Delta}_j g = \sum_{|j-j'|\leq 2} \Delta_{j'} g.$$

Below we recall a general definition of Besov-type spaces based on a Banach space E (see, e.g., [**26, 23**]).

Definition 3.3. Let $E \subset \mathcal{S}'$ be a Banach space, $1 \leq r \leq \infty$ and $s \in \mathbb{R}$. The Besov-E space BE_r^s is defined as

$$BE_r^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^n);\ \|f\|_{BE_r^s} < \infty \right\},$$

where

$$\|f\|_{BE_r^s} := \begin{cases} \left(\sum\limits_{j\geq -1} 2^{jsr} \left\|\bar{\Delta}_j f\right\|_E^r \right)^{1/r} & \text{if } r < \infty, \\ \sup\limits_{j\geq -1} 2^{js} \left\|\bar{\Delta}_j f\right\|_E & \text{if } r = \infty. \end{cases} \tag{3.3}$$

Similarly, the homogeneous Besov-E space $\dot{B}E_r^s$ is defined as

$$\dot{B}E_r^s = \left\{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P};\ \|f\|_{\dot{B}E_r^s} < \infty \right\},$$

where

$$\|f\|_{\dot{B}E_r^s} := \begin{cases} \left(\sum\limits_{j\in\mathbb{Z}} 2^{jsr} \left\|\Delta_j f\right\|_E^r \right)^{1/r} & \text{if } r < \infty, \\ \sup\limits_{j\in\mathbb{Z}} 2^{js} \left\|\Delta_j f\right\|_E & \text{if } r = \infty. \end{cases} \tag{3.4}$$

Let us point out that there are other notations for the Besov-E space $\dot{B}E^s_r$, for example, $\dot{B}^{s,r}_E$ or $\dot{B}^s_{E,r}$ (see [**23**]). If we take $E = L^p$ then we obtain the classical non-homogeneous Besov space $B^s_{p,r}$. In the case $E = M^l_p$ we have the Besov-Morrey space $N^s_{p,l,r}$ introduced by Kozono and Yamazaki [**20**]. Throughout this paper we consider $E = W\tilde{\mathcal{M}}^l_p$ and $E = \tilde{\mathcal{P}}\mathcal{D}^l_p$ in order to obtain the modified Besov-weak-Morrey space $BW\tilde{\mathcal{M}}^{l,s}_{p,r}$ and modified Besov-block space $B\tilde{\mathcal{P}}\mathcal{D}^{l,s}_{p,r}$, respectively.

Lemma 3.4. *Let $1 < p \leq l \leq \infty$, $1 \leq r \leq \infty$ and $s \in \mathbb{R}$. Then, the space $BW\tilde{\mathcal{M}}^{l,s}_{p,r}$ is a retract of $l^s_r(W\tilde{\mathcal{M}}^l_p)$.*

Proof. Let the operator $\mathcal{I}$ be defined as $(\mathcal{I}(f))_j := \bar{\Delta}_j f$ for $j \geq -1$, for each $f \in \mathcal{S}'$. It is not difficult to see that

$$\mathcal{I} : BW\tilde{\mathcal{M}}^{l,s}_{p,r} \longrightarrow l^s_r\left(W\tilde{\mathcal{M}}^l_p\right).$$

Consider also the operator $\mathcal{L}(\gamma) = \sum_{j=-1}^{\infty} \tilde{\bar{\Delta}}_j \gamma_j$, where $\gamma = (\gamma_j) \in l^s_r\left(W\tilde{\mathcal{M}}^l_p\right)$. We have that $\bar{\Delta}_{-1}\mathcal{L}(\gamma) = \bar{\Delta}_{-1}\left(\tilde{\bar{\Delta}}_{-1}\gamma_{-1} + \tilde{\bar{\Delta}}_0\psi_0\right)$, $\bar{\Delta}_0\mathcal{L}(\gamma) = \bar{\Delta}_0\left(\tilde{\bar{\Delta}}_{-1}\gamma_{-1} + \sum_{j=0}^{1}\tilde{\bar{\Delta}}_j\gamma_j\right)$ and $\bar{\Delta}_j\mathcal{L}(\gamma) = \bar{\Delta}_j \sum_{k=j-1}^{j+1} \tilde{\bar{\Delta}}_k\gamma_k$ for $j \geq 1$. Next, we can employ Lemma 2.2 in order to obtain

$$\begin{aligned}
\|\mathcal{L}(\gamma)\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} &= \left(\sum_{j\geq -1} 2^{jsr}\left\|\bar{\Delta}_j\mathcal{L}(\gamma)\right\|^r_{W\tilde{\mathcal{M}}^l_p}\right)^{1/r} \\
&\leq C\left\|\bar{\Delta}_{-1}\left(\tilde{\bar{\Delta}}_{-1}\gamma_{-1} + \tilde{\bar{\Delta}}_0\gamma_0\right)\right\|_{W\tilde{\mathcal{M}}^l_p} + C\left\|\bar{\Delta}_0\left(\tilde{\bar{\Delta}}_{-1}\gamma_{-1} + \sum_{j=0}^{1}\tilde{\bar{\Delta}}_j\gamma_j\right)\right\|_{W\tilde{\mathcal{M}}^l_p} \\
&\quad + C\left(\sum_{j\geq 1} 2^{jsr}\left\|\bar{\Delta}_j\sum_{k=j-1}^{j+1}\tilde{\bar{\Delta}}_k\gamma_k\right\|^r_{W\tilde{\mathcal{M}}^l_p}\right)^{1/r} \\
&\leq C\left(\sum_{j\geq -1} 2^{jsr}\left\|\gamma_j\right\|^r_{W\tilde{\mathcal{M}}^l_p}\right)^{1/r} = C\left\|\gamma\right\|_{l^s_r(W\tilde{\mathcal{M}}^l_p)}.
\end{aligned}$$

Therefore $\mathcal{L} : l^s_r\left(W\tilde{\mathcal{M}}^l_p\right) \longrightarrow BW\tilde{\mathcal{M}}^{l,s}_{p,r}$. Moreover, since $\bar{\Delta}_j = \tilde{\bar{\Delta}}_j\bar{\Delta}_j$ it follows that $\mathcal{L}\circ\mathcal{I}$ is the identity in $BW\tilde{\mathcal{M}}^{l,s}_{p,r}$, and we are done.

◇

The next lemma is similar to Lemma 3.4 but now in the framework of modified Besov-block spaces. The proof follows by using Lemma 2.5 and proceeding as in the previous lemma. The details are left to the reader.

Lemma 3.5. *Let $1 \leq l \leq p < \infty$, $1 \leq r \leq \infty$ and $s \in \mathbb{R}$. Then, the space $B\tilde{\mathcal{P}}\mathcal{D}^{l,s}_{p,r}$ is a retract of $l^s_r(\tilde{\mathcal{P}}\mathcal{D}^l_p)$.*

Finally, we present the duality relation between modified Besov-block spaces and modified Besov-weak-Morrey spaces.

LEMMA 3.6. *Let* $1 < p \leq l \leq \infty$, $1 < r \leq \infty$ *and* $s \in \mathbb{R}$. *Then*

$$\left(B\tilde{\mathcal{P}}\mathcal{D}_{p',r'}^{l',-s} \right)' = BW\tilde{\mathcal{M}}_{p,r}^{l,s}. \tag{3.5}$$

Proof. In view of Lemma 3.2, we have that

$$\left(l_{r'}^{-s} \left(\tilde{\mathcal{P}}\mathcal{D}_{p'}^{l'} \right) \right)' = l_r^s \left(W\tilde{\mathcal{M}}_p^l \right).$$

Now we can conclude (3.5) by using Lemmas 3.4 and 3.5.

◇

4. Key estimates and proof of the theorems

In order to prove our well-posedness result for (1.1), we need estimates for volume-preserving maps, product and commutator operators in the context of modified Besov-weak-Morrey spaces. This is the subject of the present section.

4.1. Volume-preserving maps and product estimates. We start with a result that provides a control in $W\tilde{\mathcal{M}}_p^l$-spaces for the action of a volume preserving diffeomorphism.

LEMMA 4.1. *Let* $1 < p, l \leq \infty$ *and assume that* $X : \mathbb{R}^n \to \mathbb{R}^n$ *is a volume-preserving diffeomorphism such that*

$$|X(x_0) - X(y_0)| \leq \gamma |x_0 - y_0|, \ \forall x_0, y_0 \in \mathbb{R}^n, \tag{4.1}$$

for some fixed $\gamma \geq 1$, *where* $|\cdot|$ *stands for the Euclidean norm in* $\mathbb{R}^n$. *Then, there exists a positive constant* $C = C(n, p, l, \gamma)$ *such that*

$$C^{-1} \|f\|_{W\tilde{\mathcal{M}}_p^l} \leq \|f \circ X\|_{W\tilde{\mathcal{M}}_p^l} \leq C \|f\|_{W\tilde{\mathcal{M}}_p^l}. \tag{4.2}$$

Proof. Let $R \geq 1$. In view of (4.1), we have that

$$X^{\pm 1}(D(x_0, R)) \subset D\left(X^{\pm 1}(x_0), R\gamma\right),$$

and then

$$\begin{aligned} R^{\frac{n}{l} - \frac{n}{p}} \|f\|_{L^{p,\infty}(D(x_0,R))} &= R^{\frac{n}{l} - \frac{n}{p}} \|f \circ X\|_{L^{p,\infty}(X^{-1}(D(x_0,R)))} \\ &\leq R^{\frac{n}{l} - \frac{n}{p}} \|f \circ X\|_{L^{p,\infty}(D(X^{-1}(x_0),R\gamma))} \\ &\leq C\gamma^{\frac{n}{p} - \frac{n}{l}} (\gamma R)^{\frac{n}{l} - \frac{n}{p}} \|f \circ X\|_{L^{p,\infty}(D(X^{-1}(x_0),R\gamma))} \\ &\leq C \|f \circ X\|_{W\tilde{\mathcal{M}}_p^l}. \end{aligned}$$

Taking the supremum over $R \geq 1$ and $x_0 \in \mathbb{R}^n$, it follows that

$$\|f\|_{W\tilde{\mathcal{M}}_p^l} \leq C \|f \circ X\|_{W\tilde{\mathcal{M}}_p^l}.$$

Replacing X by X^{-1}, we arrive at

$$\|f\|_{W\tilde{\mathcal{M}}_p^l} \leq C \left\|f \circ X^{-1}\right\|_{W\tilde{\mathcal{M}}_p^l}, \tag{4.3}$$

and afterwards obtain the second inequality in (4.2) by taking $g = f \circ X$ in (4.3).

◇

For further references, we recall that if $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is a continuous vector field that is Lipschitzian in the first variable for each fixed $t \in [0,T]$, and $X : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is the flow defined by

$$\begin{cases} \frac{\partial X(x_0,t)}{\partial t} = u\left(X\left(x_0,t\right),t\right), \\ X\left(x_0,0\right) = x_0, \end{cases} \tag{4.4}$$

then

$$\left|X\left(x_0,t\right) - X\left(y_0,t\right)\right| \leq e^{\int_0^t b(t')dt'} \left|x_0 - y_0\right|, \tag{4.5}$$

where $b(t')$ is the Lipschitz constant.

In the proof of our blow-up theorem, a key ingredient is a logarithmic-type inequality in $BW\tilde{\mathcal{M}}^{l,s}_{p,r}$ spaces. This is the subject of the lemma below.

Lemma 4.2. ***(Logarithmic inequality)*** *For* $1 < p < \infty$, $\rho > \frac{n}{p}$ *and* $1 \leq r \leq \infty$, *we have the* log-*estimate*

$$\|f\|_{L^\infty} \leq C\left\{1 + \|f\|_{\dot{B}^0_{\infty,\infty}}\left(\log^+ \|f\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}} + 1\right)\right\}, \tag{4.6}$$

where $C > 0$ *is a universal constant.*

Proof. For each $N \in \mathbb{N}$, we can write f as

$$f = \sum_{j \leq -N} \Delta_j f + \sum_{|j|<N} \Delta_j f + \sum_{j \geq N} \Delta_j f = I_1 + I_2 + I_3. \tag{4.7}$$

Using Lemmas 2.6 and 2.2, we can estimate the parcels in (4.7) as

$$|I_1| \leq \sum_{j \leq -N} \|\Delta_j f\|_{L^\infty} \leq C \sum_{j \leq -N} 2^{j\frac{n}{l}} \|\Delta_j f\|_{W\tilde{\mathcal{M}}^l_p} \leq C 2^{-N\frac{n}{l}} \|f\|_{W\tilde{\mathcal{M}}^l_p},$$

$$|I_2| \leq \sum_{|j|<N} \|\Delta_j f\|_{L^\infty} \leq CN \|f\|_{\dot{B}^0_{\infty,\infty}},$$

and

$$\begin{aligned} |I_3| &\leq \sum_{j \geq N} \|\Delta_j f\|_{L^\infty} \leq C \sum_{j \geq N} 2^{j\frac{n}{p}} \|\Delta_j f\|_{W\tilde{\mathcal{M}}^l_p} = C \sum_{j \geq N} 2^{j\left(\frac{n}{p}-\rho\right)} 2^{j\rho} \|\Delta_j f\|_{W\tilde{\mathcal{M}}^l_p} \\ &= C\|f\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,\infty}} \sum_{j \geq N} 2^{-j\left(\rho - \frac{n}{p}\right)} \leq C 2^{-N\left(\rho-\frac{n}{p}\right)} \|f\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}}. \end{aligned}$$

Considering the last three estimates in (4.7), it follows that

$$\begin{aligned} \|f\|_{L^\infty} &\leq C\left(2^{-N\frac{n}{l}} \|f\|_{W\tilde{\mathcal{M}}^l_p} + N\|f\|_{\dot{B}^0_{\infty,\infty}} + 2^{-N\left(\rho-\frac{n}{p}\right)} \|f\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}}\right) \\ &\leq C\left(2^{-N\frac{n}{l}} \|f\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}} + N\|f\|_{\dot{B}^0_{\infty,\infty}} + 2^{-N\left(\rho-\frac{n}{p}\right)} \|f\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}}\right) \\ &\leq C\left(2^{-Na} \|f\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}} + N\|f\|_{\dot{B}^0_{\infty,\infty}}\right), \end{aligned} \tag{4.8}$$

where $a = \min\left\{\rho - \frac{n}{p}, \frac{n}{l}, \frac{1}{\log 2}\right\}$. Since the previous inequality is valid for an arbitrary N, we can choose $N = 1 + (a\log 2)^{-1}\log^+ \|f\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}}$ in order to obtain (4.6).

◇

In order to handle with the nonlinearity in (1.1), it is important to have product estimates in our setting, i.e., Leibniz-like rules. In the next lemma we obtain such a kind of estimate.

LEMMA 4.3. ***(Product estimate)*** *Let $1 < p \leq l \leq \infty$, $1 \leq r \leq \infty$ and $\rho > 0$. Then*

$$\|uv\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}} \leq C\left(\|u\|_{L^\infty} \|v\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}} + \|u\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}} \|v\|_{L^\infty}\right), \tag{4.9}$$

for all $u, v \in BW\tilde{\mathcal{M}}^{l,\rho}_{p,r} \cap L^\infty$. Moreover, considering $p < \infty$ and assuming either $s > n/p+1$ with $1 \leq r \leq \infty$ or $s = n/p+1$ with $r = 1$, we have that

$$\|u \cdot \nabla v\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}} \leq C \|u\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}} \|v\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}, \tag{4.10}$$

for all $u \in BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}$ and $v \in BW\tilde{\mathcal{M}}^{l,s}_{p,r}$.

Proof. It follows from the Bony decomposition (3.2) that

$$uv = \sum_{k\geq 2} S_{k-2}u\bar{\Delta}_k v + \sum_{k\geq 2} S_{k-2}v\bar{\Delta}_k u + \sum_{k\geq -1} \bar{\Delta}_k u \left(\sum_{|k-l|\leq 2} \bar{\Delta}_l v\right).$$

Using the cancellation relations (3.1), we can write

$$\bar{\Delta}_{-1}(uv) = \bar{\Delta}_{-1}\left(\sum_{k\geq -1} \bar{\Delta}_k u \left(\sum_{|k-l|\leq 2} \bar{\Delta}_l v\right)\right).$$

Now we can use Lemma 2.2 and Hölder-type inequality in order to estimate

$$\left\|\bar{\Delta}_{-1}(uv)\right\|_{W\tilde{\mathcal{M}}^l_p} \leq \sum_{k\geq -1} \left\|\bar{\Delta}_k u \left(\sum_{|k-l|\leq 2} \bar{\Delta}_l v\right)\right\|_{W\tilde{\mathcal{M}}^l_p} \leq C\|v\|_{L^\infty} \sum_{k\geq -1} \left\|\bar{\Delta}_k u\right\|_{W\tilde{\mathcal{M}}^l_p}$$

$$\leq C\|v\|_{L^\infty} \sum_{k\geq -1} 2^{-\rho k} 2^{\rho k} \left\|\bar{\Delta}_k u\right\|_{W\tilde{\mathcal{M}}^l_p} \leq \|v\|_{L^\infty} \|u\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,\infty}}.$$

For $j \geq 0$ we have

$$\begin{aligned}\bar{\Delta}_j(uv) &= \sum_{k\geq 2, |k-j|\leq 4} \bar{\Delta}_j\left(S_{k-2}u\bar{\Delta}_k v\right) + \sum_{k\geq 2, |k-j|\leq 4} \bar{\Delta}_j\left(S_{k-2}v\bar{\Delta}_k u\right) \\ &\quad + \sum_{k\geq j-2} \bar{\Delta}_j\left(\bar{\Delta}_k v \tilde{\bar{\Delta}}_k u\right) \\ &:= I_1^j + I_2^j + I_3^j. \end{aligned} \tag{4.11}$$

The parcel I_1^j can be estimated as

$$\begin{aligned} 2^{j\rho}\left\|I_1^j\right\|_{W\tilde{\mathcal{M}}^l_p} &\leq C2^{j\rho} \sum_{|k-j|\leq 2} \left\|S_{k-2}u\bar{\Delta}_k v\right\|_{W\tilde{\mathcal{M}}^l_p} \\ &\leq C2^{j\rho} \sum_{|k-j|\leq 2} \|S_{k-2}u\|_{L^\infty} \left\|\bar{\Delta}_k v\right\|_{W\tilde{\mathcal{M}}^l_p} \leq C \sum_{|k-j|\leq 2} \|u\|_{L^\infty} 2^{j\rho} \left\|\bar{\Delta}_k v\right\|_{W\tilde{\mathcal{M}}^l_p} \\ &\leq C\|u\|_{L^\infty} \sum_{|k-j|\leq 2} 2^{k\rho} \left\|\bar{\Delta}_k v\right\|_{W\tilde{\mathcal{M}}^l_p}. \end{aligned}$$

Since $k \sim j$, it follows that

$$\begin{aligned}\left(\sum_{j\geq 0} 2^{j\rho r}\left\|I_1^j\right\|^r_{W\tilde{\mathcal{M}}^l_p}\right)^{1/r} &\leq C\left\|u\right\|_{L^\infty}\left(\sum_{j\geq 0} 2^{j\rho r}\left\|\bar{\Delta}_j v\right\|^r_{W\tilde{\mathcal{M}}^l_p}\right)^{1/r} \\ &\leq C\left\|u\right\|_{L^\infty}\left\|v\right\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}}, \qquad (4.12)\end{aligned}$$

with the usual modification in the case $r = \infty$. By symmetry, it is easy to see that estimate (4.12) also holds true for I_2^j.

For I_3^j, we proceed as follows

$$\left\|I_3^j\right\|_{W\tilde{\mathcal{M}}^l_p} \leq C\sum_{k\geq j-2}\left\|\bar{\Delta}_k u\right\|_{W\tilde{\mathcal{M}}^l_p}\left\|\tilde{\Delta}_k v\right\|_{L^\infty} \leq C\left\|v\right\|_{L^\infty}\sum_{k\geq j-2}\left\|\bar{\Delta}_k u\right\|_{W\tilde{\mathcal{M}}^l_p}.$$

Thus, for $1 \leq r < \infty$ we obtain that

$$\left(\sum_{j\geq 0} 2^{j\rho r}\left\|I_3^j\right\|^r_{W\tilde{\mathcal{M}}^l_p}\right)^{1/r} \leq C\left\|v\right\|_{L^\infty}\left(\sum_{j\geq 0}\left(\sum_{k\geq j-2} 2^{j\rho}\left\|\bar{\Delta}_k v\right\|_{W\tilde{\mathcal{M}}^l_p}\right)^r\right)^{1/r}.$$

Changing of variable $m = j - k$ and using Minkowski's inequality, we arrive at

$$\begin{aligned}\left(\sum_{j\geq 0} 2^{j\rho r}\left\|I_3^j\right\|^r_{W\tilde{\mathcal{M}}^l_p}\right)^{1/r} &\leq C\left\|v\right\|_{L^\infty}\sum_{m\leq 2}\left(\sum_{j\geq 0}\left(2^{m\rho}2^{(j-m)\rho}\left\|\bar{\Delta}_{j-m} u\right\|_{W\tilde{\mathcal{M}}^l_p}\right)^r\right)^{1/r} \\ &\leq C\left\|v\right\|_{L^\infty}\sum_{m\leq 2} 2^{m\rho}\left(\sum_{j\geq 0}\left(2^{(j-m)\rho}\left\|\bar{\Delta}_{j-m} u\right\|_{W\tilde{\mathcal{M}}^l_p}\right)^r\right)^{1/r} \\ &\leq C\left\|v\right\|_{L^\infty}\left\|u\right\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}}\sum_{m\leq 2} 2^{m\rho} \leq C\left\|v\right\|_{L^\infty}\left\|u\right\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}}.\end{aligned}$$

For the case $r = \infty$, we use the change of variable $m = j - k$ and obtain the corresponding estimate directly.

Recalling the expression of the norm in (3.3) and considering the estimates for I_1^j, I_2^j and I_3^j in (4.11), we obtain (4.9). Inequality (4.10) can be proved by employing (4.9), the decomposition $f = \Sigma_{j\geq -1}\bar{\Delta}_j f$, and estimate (2.9).

◇

4.2. Commutator estimates in modified Besov-weak-Morrey spaces. In this part we present our commutator-type estimates. Let us begin with some decompositions and preliminary estimates in the framework of $W\tilde{\mathcal{M}}^l_p$-spaces linked to the bilinear term $u \cdot \nabla v$. Proceeding as in Vishik [**32**] and Chemin [**9**], we define

$$R_j(u, v) = \bar{\Delta}_j(u \cdot \nabla v) - S_{j-2}u \cdot \nabla\bar{\Delta}_j v, \qquad (4.13)$$

for $u = (u_1, ..., u_n)$ and $v = (v_1, ..., v_n)$ such that $\nabla \cdot u = \nabla \cdot v = 0$. It follows that $R_j(u, v)$ can be decomposed as

$$R_j(u, v) = \sum_{i=1}^{4} R_j^i(u, v), \qquad (4.14)$$

where

$$R_j^1(u,v) = \sum_{k=1}^{n} \bar{\Delta}_j T_{\partial_k v} u_k,$$
$$R_j^2(u,v) = -\sum_{k=1}^{n} \left[T_{u_k}\partial_k, \bar{\Delta}_j\right] v,$$
$$R_j^3(u,v) = \sum_{k=1}^{n} T_{u_k - S_{j-2}u_k}\partial_k \bar{\Delta}_j v,$$
$$R_j^4(u,v) = \sum_{k=1}^{n} \left\{\bar{\Delta}_j R\left(u_k, \partial_k v\right) - R\left(S_{j-2}u_k, \bar{\Delta}_j \partial_k v\right)\right\},$$

with $\left[T_{u_k}\partial_k, \bar{\Delta}_j\right] v := T_{u_k}\partial_k \bar{\Delta}_j v - \bar{\Delta}_j T_{u_k}\partial_k v$ and $R(u,v)$ as in (3.2).

The first parcel in (4.14) can be written as

$$R_j^1(u,v) = \sum_{k=1}^{n} \bar{\Delta}_j \sum_{j'\geq 2} S_{j'-2}\partial_k v \bar{\Delta}_{j'} u_k = \sum_{k=1}^{n} \bar{\Delta}_j \sum_{|j-j'|\leq 4} S_{j'-2}\partial_k v \bar{\Delta}_{j'} u_k.$$

Thus, we can estimate it as follows

$$\left\|R_j^1(u,v)\right\|_{W\tilde{\mathcal{M}}_p^l} \leq C\sum_{k=1}^{n} \sum_{j'\geq 2, |j-j'|\leq 3} \left\|S_{j'-2}\partial_k v\right\|_{L^\infty} \left\|\bar{\Delta}_{j'} u_k\right\|_{W\tilde{\mathcal{M}}_p^l}. \tag{4.15}$$

Now we consider $R_j^2(u,v)$. For $j \geq 0$, it follows that

$$R_j^2(u,v) = -\sum_{j'\geq 2, |j-j'|\leq 3jh} \sum_{k=1}^{n} 2^{j(n+1)} \int_{\mathbb{R}^n} \left(\partial_k \check{\varphi}\right)\left(2^i(x-y)\right)$$
$$\cdot \left(\sum_{m=1}^{n} \int_0^1 S_{j'-2}\partial_m u_k(x+\tau(y-x)) \cdot (x_m - y_m) d\tau\right) \cdot \bar{\Delta}_{j'} v(y)\, dy,$$

and then

$$\left|R_j^2(u,v)\right| \leq \sum_{j'\geq 2, |j-j'|\leq 3} \sum_{k,m=1}^{n} \left\|S_{j'-2}\partial_m u_k\right\|_{L^\infty} 2^{jn} \left|\left(\partial_k \check{\varphi}\right)\left(2^i(\cdot)\right)\left(2^j(\cdot)_m\right)\right| * \left|\bar{\Delta}_{j'} v\right|.$$

Similarly, we have that

$$\left|R_{-1}^2(u,v)\right| \leq \sum_{j'\geq 2, |-1-j'|\leq 3} \sum_{k,m=1}^{n} \left\|S_{j'-2}\partial_m u_k\right\|_{L^\infty} \left|\left(\partial_k \check{\psi}\right)(\cdot)(\cdot)_m\right| * \left|\bar{\Delta}_{j'} v\right|.$$

So, using Lemma 2.2, we obtain

$$\left\|R_j^2(u,v)\right\|_{W\tilde{\mathcal{M}}_p^l} \leq C \sum_{j'\geq 2, |j-j'|\leq 4} \sum_{k,m=1}^{n} \left\|S_{j'-2}\partial_m u_k\right\|_{L^\infty} \left\|\bar{\Delta}_{j'} v\right\|_{W\tilde{\mathcal{M}}_p^l}. \tag{4.16}$$

In order to handle the term $R_j^3(u,v)$, first note that

$$R_j^3(u,v) = \sum_{k=1}^{n} \sum_{j'\geq 2, |j-j'|\leq 1} \left(\sum_{m=j-2}^{j-1} \bar{\Delta}_m S_{j'-2} u_k\right) \partial_k \bar{\Delta}_{j'} \bar{\Delta}_j v$$

and then we can estimate

$$\begin{aligned} & \left\|R_j^3(u,v)\right\|_{W\tilde{\mathcal{M}}_p^l} \\ & \leq \sum_{k=1}^{n} \sum_{j'\geq 2,|j-j'|\leq 1} \left(\sum_{m=j-2}^{j-1} \left\|\bar{\Delta}_m S_{j'-2} u_k\right\|_{L^\infty} \right) 2^j \left\|\bar{\Delta}_j v\right\|_{W\tilde{\mathcal{M}}_p^l} \end{aligned} \tag{4.17}$$

and

$$\left\|R_j^3(u,v)\right\|_{W\tilde{\mathcal{M}}_p^l} \leq C \left\|\nabla v\right\|_{L^\infty} \sum_{k=1}^{n} \sum_{j'\geq 2,|j-j'|\leq 1} \sum_{m=j-2}^{j-1} \left\|\bar{\Delta}_m S_{j'-2} u_k\right\|_{W\tilde{\mathcal{M}}_p^l}. \tag{4.18}$$

For $R_j^4(u,v)$, we can split $R_j^4(u,v) = R_j^{4,1}(u,v) + R_j^{4,2}(u,v)$ with

$$\begin{aligned} R_j^{4,1}(u,v) &= \sum_{k=1}^{n} \partial_k \bar{\Delta}_j \sum_{l\geq j-3} \left(\bar{\Delta}_l \left(u_k - S_{j-2}u_k\right)\right) \left(\sum_{|l-l'|\leq 2} \bar{\Delta}_{l'} v \right), \\ R_j^{4,2}(u,v) &= \sum_{k=1}^{n} \sum_{|l-j|\leq 5} \left[\bar{\Delta}_j, \bar{\Delta}_l S_{j-2} u_k\right] \left(\sum_{|l-l'|\leq 2} \bar{\Delta}_{l'} \partial_k v \right). \end{aligned}$$

For the first parcel, we have the following estimates

$$\left\|R_j^{4,1}(u,v)\right\|_{W\tilde{\mathcal{M}}_p^l} \leq C \sum_{k=1}^{n} \sum_{l\geq j-3} 2^j \left\|\bar{\Delta}_l u_k\right\|_{L^\infty} \left\| \sum_{|l-l'|\leq 2} \bar{\Delta}_{l'} v \right\|_{W\tilde{\mathcal{M}}_p^l}, \tag{4.19}$$

and

$$\begin{aligned} \left\|R_j^{4,1}(u,v)\right\|_{W\tilde{\mathcal{M}}_p^l} &\leq C \sum_{k=1}^{n} \sum_{l\geq j-3} \left\| \partial_k \left[\left(\bar{\Delta}_l \left(u_k - S_{j-2}u_k\right)\right) \sum_{|l-l'|\leq 2} \bar{\Delta}_{l'} v \right] \right\|_{W\tilde{\mathcal{M}}_p^l} \\ &\leq C \sum_{k=1}^{n} \sum_{l\geq j-3} \left\|\partial_k \left(\Delta_l \left(u_k - S_{j-2}u_k\right)\right)\right\|_{L^\infty} \left\| \sum_{|l-l'|\leq 2} \bar{\Delta}_{l'} v \right\|_{W\tilde{\mathcal{M}}_p^l} \\ &\quad + C \sum_{k=1}^{n} \sum_{l\geq j-3} \left\|\Delta_l \left(u_k - S_{j-2}u_k\right)\right\|_{W\tilde{\mathcal{M}}_p^l} \left\| \partial_k \sum_{|l-l'|\leq 2} \bar{\Delta}_{l'} v \right\|_{L^\infty} \\ &\leq C \left\|\nabla u\right\|_{L^\infty} \sum_{l\geq j-3} \left\| \sum_{|l-l'|\leq 2} \bar{\Delta}_{l'} v \right\|_{W\tilde{\mathcal{M}}_p^l} + C \left\|\nabla v\right\|_{L^\infty} \sum_{l\geq j-3} \left\|\bar{\Delta}_l u\right\|_{W\tilde{\mathcal{M}}_p^l}. \end{aligned} \tag{4.20}$$

In the case of the second parcel, computing the $W\tilde{\mathcal{M}}_p^l$-norm and using the previous estimates, we arrive at (for all $j \geq -1$)

$$\left\|R_j^{4,2}(u,v)\right\|_{W\tilde{\mathcal{M}}_p^l} \leq C \sum_{k,m=1}^{n} \sum_{|j-l|\leq 5} \left\|\bar{\Delta}_l S_{j-2} \partial_m u_k\right\|_{L^\infty} \left\| \sum_{|l-l'|\leq 2} \bar{\Delta}_{l'} v \right\|_{W\tilde{\mathcal{M}}_p^l}. \tag{4.21}$$

In the next lemma we state our commutator estimates. Using the above estimates (4.15) to (4.21), the harmonic analysis tools developed in Sections 2 and 3, and arguments similar to those in the proof of Lemma 4.3, we can handle each

$R_j^i(u,v)$, $i=1,...,4$, and then, in view of decomposition (4.14), obtain commutator estimates in the framework of modified Besov-weak-Morrey spaces. The proof is omitted in order to avoid extensive computations and repetition of similar arguments.

LEMMA 4.4. ***(Commutator estimates)*** *Let* $1<p<\infty$, $1\leq l,r\leq\infty$, $p\leq l$ *and let* u,v *be divergence-free vector fields.*

(i) *Assume that* $\rho>0$. *Then, there exists a universal constant* $C>0$ *such that*

$$\left(\sum_{j\geq -1} 2^{j\rho r}\left\|R_j(u,v)\right\|^r_{W\tilde{\mathcal{M}}^l_p}\right)^{1/r} \leq C\left(\|\nabla u\|_{L^\infty}\|v\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}}+\|\nabla v\|_{L^\infty}\|u\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}}\right), \tag{4.22}$$

where we consider the usual modification for $r=\infty$.

(ii) *Assume that* $s\geq \frac{n}{p}+1$ *and that* $r=1$ *in the case* $s=\frac{n}{p}+1$. *Then, there exists a universal constant* $C>0$ *such that*

$$\left(\sum_{j\geq -1} 2^{jsr}\left\|R_j(u,v)\right\|^r_{W\tilde{\mathcal{M}}^l_p}\right)^{1/r} \leq C\|u\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}\|v\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}, \tag{4.23}$$

$$\left(\sum_{j\geq -1} 2^{j(s-1)r}\left\|R_j(u,v)\right\|^r_{W\tilde{\mathcal{M}}^l_p}\right)^{1/r} \leq C\|u\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}}\|v\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}, \tag{4.24}$$

$$\left(\sum_{j\geq -1} 2^{j(s-1)r}\left\|R_j(u,v)\right\|^r_{W\tilde{\mathcal{M}}^l_p}\right)^{1/r} \leq \|u\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}\|v\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}}, \tag{4.25}$$

where we consider the usual modification for $r=\infty$.

Now we present estimates that will be useful to handle the bilinear operator (1.2) and the pressure term ∇P in $BW\tilde{\mathcal{M}}^{l,s}_{p,r}$-spaces. Their proofs are also omitted because they follow by using estimates for the product operator and the Riesz transform proved in Lemmas 4.3 and 2.3, respectively, and proceeding as in [**32**] and [**28**] but adapting the arguments to our setting.

LEMMA 4.5. *Let* $1<p<\infty$, $1\leq l,r\leq\infty$ *and* $p\leq l$. *Assume that* u,v *are divergence-free vector fields.*

(i) *Let* $\rho>1$. *Then, there exists a universal constant* $C>0$ *such that*

$$\|\pi(u,v)\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}} \leq C\left(\|\nabla u\|_{L^\infty}\|v\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}}+\|\nabla v\|_{L^\infty}\|u\|_{BW\tilde{\mathcal{M}}^{l,\rho}_{p,r}}\right). \tag{4.26}$$

(ii) *Assume that* $s\geq \frac{n}{p}+1$ *and that* $r=1$ *in the case* $s=\frac{n}{p}+1$. *Then, there exists a universal constant* $C>0$ *such that*

$$\|\pi(u,v)\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} \leq C\|u\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}\|v\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}, \tag{4.27}$$

$$\|\pi(u,v)\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}} \leq C\|u\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}}\|v\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}, \tag{4.28}$$

$$\|\pi(u,v)\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}} \leq C\|u\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}\|v\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}}. \tag{4.29}$$

4.3. Proof of Theorem 1.1. Existence part: Define $u^1(t) = S_1 u_0$ and for $m \geq 1$ let u^{m+1} be the solution of the system

$$\begin{cases} \frac{\partial u^{m+1}}{\partial t} + u^m \cdot \nabla u^{m+1} + \nabla P^{m+1} = 0, & \text{in } \mathbb{R}^n \times (0,T), \\ \nabla \cdot u^{m+1} = 0, & \text{in } \mathbb{R}^n \times (0,T), \\ u^{m+1}(0) = S_{m+1} u_0, & \text{in } \mathbb{R}^n. \end{cases} \tag{4.30}$$

Applying the operator $\bar{\Delta}_j$ in (4.30), adding and subtracting the term $S_{j-2}u^m \cdot \nabla \bar{\Delta}_j u^{m+1}$, and afterwards recalling the definition of the commutator $R_j(u,f)$, we arrive at the system $(x \in \mathbb{R}^n, t \in (0,T))$

$$\begin{cases} \frac{\partial \bar{\Delta}_j u^{m+1}}{\partial t} + S_{j-2}u^m \cdot \nabla \bar{\Delta}_j u^{m+1} = -R_j(u^m, u^{m+1}) - \bar{\Delta}_j \nabla P^{m+1}, \\ \nabla \cdot \bar{\Delta}_j u^{m+1} = 0, \\ \bar{\Delta}_j u^{m+1}(0) = \bar{\Delta}_j S_{m+1} u_0. \end{cases} \tag{4.31}$$

Next consider the flow $X_j^m : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ associated with the problem

$$\begin{cases} \frac{\partial X_j^m(x_0,t)}{\partial t} = (S_{j-2}u^m)\left(X_j^m(x_0,t), t\right) \\ X_j^m(x_0,0) = x_0. \end{cases} \tag{4.32}$$

Since $\nabla \cdot S_{j-2}u^m = 0$, we have that $X_j^m(\cdot,t)$ is a volume-preserving diffeomorphism for each $t \geq 0$ (see, e.g., [**15**]). For now, let us assume that there exists a constant $\gamma \geq 1$ such that

$$\left| \left(X_j^m\right)^{\pm 1}(x_0,t) - \left(X_j^m\right)^{\pm 1}(y_0,t) \right| \leq \gamma \left| x_0 - y_0 \right|, \tag{4.33}$$

for all $x_0, y_0 \in \mathbb{R}^n,\ j \geq -1,\ m \geq 1$ and $t \in [0,T]$.

Using the flow $X_j^m(\cdot,t)$, we can solve (4.31) and obtain that

$$\begin{aligned} \bar{\Delta}_j u^{m+1}\left(X_j^m(x_0,t), t\right) = \bar{\Delta}_j u^{m+1}(x_0,0) - \int_0^t R_j(u^m, u^{m+1})\left(X_j^m(x_0,\tau),\tau\right) d\tau \\ - \int_0^t \bar{\Delta}_j \nabla P^{m+1}\left(X_j^m(x_0,\tau),\tau\right) d\tau. \end{aligned}$$

By means of Lemma 4.1, we can estimate

$$\begin{aligned} \left\|\bar{\Delta}_j u^{m+1}(t)\right\|_{W\tilde{\mathcal{M}}_p^l} &\leq C \left\|\bar{\Delta}_j u^{m+1}\left(X_j^m(\cdot,t),t\right)\right\|_{W\tilde{\mathcal{M}}_p^l} \\ &\leq C \left\|\bar{\Delta}_j u^{m+1}(0)\right\|_{W\tilde{\mathcal{M}}_p^l} + C\int_0^t \left\|R_j(u^m,u^{m+1})\left(X_j^m(\cdot,\tau),\tau\right)\right\|_{W\tilde{\mathcal{M}}_p^l} d\tau \\ &\qquad + \int_0^t \left\|\bar{\Delta}_j \nabla P^{m+1}\left(X_j^m(\cdot,\tau),\tau\right)\right\|_{W\tilde{\mathcal{M}}_p^l} d\tau \\ &\leq C\left(\left\|\bar{\Delta}_j u_0\right\|_{W\tilde{\mathcal{M}}_p^l} + \int_0^t \left\|R_j(u^m,u^{m+1})\right\|_{W\tilde{\mathcal{M}}_p^l} d\tau \right. \\ &\qquad \left. + \int_0^t \left\|\bar{\Delta}_j \nabla P^{m+1}\right\|_{W\tilde{\mathcal{M}}_p^l} d\tau \right). \end{aligned} \tag{4.34}$$

Since $\nabla P^{m+1} = -\pi\left(u^m, u^{m+1}\right)$, using Lemmas 4.4 and 4.5, we obtain

$$\begin{aligned}\left\|u^{m+1}(t)\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} &\leq C_0 \left\|u_0\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} + C_1 \int_0^t \left\|u^m\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} \left\|u^{m+1}\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} d\tau \\ &\leq C_0 \left\|u_0\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} + C_1 \left\|u^m\right\|_{L^\infty_T(BW\tilde{\mathcal{M}}^{l,s}_{p,r})} \int_0^t \left\|u^{m+1}\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} d\tau. \end{aligned} \tag{4.35}$$

Now, Gronwall's inequality in (4.35) leads us to

$$\begin{aligned}\left\|u^{m+1}(t)\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} &\leq C_0 \left\|u_0\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} \exp\left(C_1 \left\|u^m\right\|_{L^\infty_T(BW\tilde{\mathcal{M}}^{l,s}_{p,r})} T\right) \\ &\leq L \left\|u_0\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} \exp\left(L \left\|u^m\right\|_{L^\infty_T(BW\tilde{\mathcal{M}}^{l,s}_{p,r})} T\right). \end{aligned} \tag{4.36}$$

Up to now, note that we have showed that (4.36) holds if we have (4.33). To prove the general case, first note that $S_{j-2}u^m$ is a Lipschitz vector field provided that $\nabla u^m \in L^\infty_T\left(L^\infty\left(\mathbb{R}^n\right)\right)$. Also, by (4.5) we have that

$$\begin{aligned}\left|\left(X^m_j\right)^{\pm 1}(x_0,t) - \left(X^m_j\right)^{\pm 1}(y_0,t)\right| &\leq e^{T\left\|\nabla S_{j-2}u^m\right\|_{L^\infty_T(L^\infty(\mathbb{R}^n))}} \left|x_0 - y_0\right| \\ &\leq e^{TC\left\|\nabla u^m\right\|_{L^\infty_T(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r})}} \left|x_0 - y_0\right|, \\ &\leq e^{CT\left\|u^m\right\|_{L^\infty_T(BW\tilde{\mathcal{M}}^{l,s}_{p,r})}} \left|x_0 - y_0\right|. \end{aligned} \tag{4.37}$$

Choosing a suitable C_0, for u^1 we obtain that

$$\left\|u^1(t)\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} = \left\|S_1 u_0\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} \leq C_0 \left\|u_0\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} \leq L \left\|u_0\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}. \tag{4.38}$$

Let $\tilde{C} > 0$, $T_2 > 0$ and γ be such that $L\left\|u_0\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} \leq \tilde{C}/2$, $e^{CT\tilde{C}} \leq \gamma$ and

$$\frac{\tilde{C}}{2}\exp\left(L\tilde{C}T_2\right) \leq \tilde{C}.$$

We claim that for $T = T_2$ it holds

$$\left\|u^m\right\|_{L^\infty_{T_2}(BW\tilde{\mathcal{M}}^{l,s}_{p,r})} \leq \tilde{C}, \text{ for } m \geq 1. \tag{4.39}$$

In order to see that, we proceed by induction. In view of (4.38), note that u^1 satisfies (4.39). Next, suppose that u^m also satisfies (4.39). Using (4.37) we obtain

$$\left|\left(X^m_j\right)^{\pm 1}(x_0,t) - \left(X^m_j\right)^{\pm 1}(y_0,t)\right| \leq e^{CT\tilde{C}} \left|x_0 - y_0\right| \leq \lambda \left|x_0 - y_0\right|, \tag{4.40}$$

for all $j \geq -1$ and $t \in [0, T_2]$. So, it follows that (4.36) holds true, and then

$$\left\|u^{m+1}\right\|_{L^\infty_{T_2}(BW\tilde{\mathcal{M}}^{l,s}_{p,r})} \leq L\left(\left\|u_0\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}\right) \exp\left(L\tilde{C}T_2\right) \leq \frac{\tilde{C}}{2}\exp\left(L\tilde{C}T_2\right) \leq \tilde{C},$$

which, by induction, gives the desired claim.

In what follows, we prove that (u^m) is Cauchy in the space $L^\infty_{T_3}\left(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}\right)$, for some $T_3 \in (0, T_2]$. In fact, considering the difference $z^{m+1} = u^{m+1} - u^m$ for $m \geq 2$, subtracting $(4.30)_m$ from $(4.30)_{m+1}$ and applying $\bar{\Delta}_j$, we arrive at the system $(x \in \mathbb{R}^n, t \in (0,T))$

$$
(4.41)\quad \begin{cases} \frac{\partial \bar{\Delta}_j z^{m+1}}{\partial t} + S_{j-2}u^m \cdot \nabla \bar{\Delta}_j z^{m+1} = \\ \qquad\qquad = -R_j\left(u^m, z^{m+1}\right) + \bar{\Delta}_j \pi\left(z^m, u^{m+1}\right) + \bar{\Delta}_j \pi\left(u^{m-1}, z^{m+1}\right), \\ \nabla \cdot \bar{\Delta}_j\left(z^{m+1}\right) = 0, \\ \bar{\Delta}_j\left(z^{m+1}\right)(0) = \bar{\Delta}_j \bar{\Delta}_{m+1} u_0. \end{cases}
$$

After solving (4.41), we can use Lemma 4.1, (4.10), and Lemmas 4.4 and 4.5 in order to estimate

$$
\begin{aligned}
(4.42)\quad & \left\|\left(z^{m+1}\right)(t)\right\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}} \\
&\leq C_0 \left\|\bar{\Delta}_{m+1} u_0\right\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}} + C_1 \int_0^t \left\|u^m\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} \left\|z^{m+1}\right\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}} d\tau \\
&\quad + C_2 \int_0^t \left\|u^{m+1}\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} \left\|z^{m}\right\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}} d\tau \\
&\quad + C_2 \int_0^t \left\|u^{m-1}\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} \left\|z^{m+1}\right\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}} d\tau \\
&\leq C 2^{-m} \left\|u_0\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} + C_3 \tilde{C} \left\|z^{m+1}\right\|_{L^\infty_T(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r})} T \\
&\quad + C_2 \tilde{C} \left\|z^{m}\right\|_{L^\infty_T(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r})} T.
\end{aligned}
$$

Considering $0 < T \leq T_2$ such that $C_3\tilde{C}T \leq 1/2$ and $2CT < 1$, we obtain

$$
\left\|z^{m+1}\right\|_{L^\infty_T(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r})} \leq \frac{C}{1 - C_3\tilde{C}T}\left(2^{-m}\left\|u_0\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} + T\left\|z^m\right\|_{L^\infty_T(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r})}\right).
$$

Therefore, we can choose $T_3 \in (0, T_2]$ in such a way that the estimate

$$
\left\|z^{m+1}\right\|_{L^\infty_{T_3}(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r})} \leq C 2^{-(m-1)}\left\|u_0\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} + \frac{1}{2}\left\|z^m\right\|_{L^\infty_{T_3}(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r})}
$$

holds true, for $m \geq 2$. It follows that

$$
\left\|z^{m+1}\right\|_{L^\infty_{T_3}(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r})} \leq C m 2^{-(m-1)}\left\|u_0\right\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} + 2^{-(m-1)}\left\|z^2\right\|_{L^\infty_{T_3}(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r})},
$$

and then

$$
\sum_{m\geq 1}\left\|\left(u^{m+1} - u^m\right)\right\|_{L^\infty_{T_3}(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r})} = \sum_{m\geq 1}\left\|z^{m+1}\right\|_{L^\infty_{T_3}(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r})} < \infty.
$$

This proves that (u^m) is a Cauchy sequence in $L^\infty_{T_3}\big(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}\big)$. Therefore, there exists $v \in L^\infty_{T_3}\big(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}\big)$ such that $u^m \to v$ in $L^\infty_{T_3}\big(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}\big)$. Moreover, it follows from (4.39) that (u^m) is bounded in $L^\infty_{T_3}\big(BW\tilde{\mathcal{M}}^{l,s}_{p,r}\big)$. By Lemma 3.6, we have that $BW\tilde{\mathcal{M}}^{l,s}_{p,r}$ has a predual, and then there exists a subsequence (u^{m_k}) and $u \in L^\infty_{T_3}\big(BW\tilde{\mathcal{M}}^{l,s}_{p,r}\big)$ such that $u^{m_k} \overset{*}{\rightharpoonup} u$ in $L^\infty_{T_3}\big(BW\tilde{\mathcal{M}}^{l,s}_{p,r}\big)$; but this implies that $u^{m_k} \overset{*}{\rightharpoonup} u$ in $u \in L^\infty_{T_3}\big(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}\big)$. By the uniqueness of the weak*-limit, we get $u = v$.

Next we check that u is a solution for (1.1). For each $m \in \mathbb{N}$, it follows from (4.30) that

$$
(4.43)\qquad u^{m+1}(\beta, t) = u^{m+1}(\beta, 0) - \int_0^t u^m \cdot \nabla u^{m+1} d\tau + \int_0^t \pi(u^m, u^{m+1}) d\tau.
$$

Since $(u^m) \subset C\big([0,T_3]\,;BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}\big)$ and (u^m) converges to u in $L^\infty_{T_3}\big(BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}\big)$, we have that $u \in C\big([0,T_3]\,;BK^{\alpha,s-1}_{p,q,r}\big)$. Taking $m\to\infty$ in (4.43), we obtain from Lemmas 4.3 and 4.5 that u satisfies

$$u(t) = u_0 - \int_0^t u\cdot\nabla u d\tau + \int_0^t \pi(u,u)d\tau, \tag{4.44}$$

as desired. It remains to verify that u is continuous in time with values in $BW\tilde{\mathcal{M}}^{l,s}_{p,r}$ for $r<\infty$. Considering $w_k = S_k u$ with $k\in\mathbb{N}$, we have that w_k converges to u in the space $L^\infty_T\left(BW\tilde{\mathcal{M}}^{l,s}_{p,r}\right)$ and

$$\begin{aligned}
\|w_k(t)-w_k(s)\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} &= \|S_k\left(u(t)-u(s)\right)\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}\\
&\le C\left(\sum_{j=-1}^{k+1} 2^{jsr}\left\|\bar{\Delta}_j\left(u(t)-u(s)\right)\right\|^r_{W\tilde{\mathcal{M}}^l_p}\right)^{1/r}\\
&\le C2^k\left(\sum_{j=-1}^{k+1} 2^{j(s-1)r}\left\|\bar{\Delta}_j\left(u(t)-u(s)\right)\right\|^r_{W\tilde{\mathcal{M}}^l_p}\right)^{1/r}\\
&\le C2^k\|u(t)-u(s)\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}}.
\end{aligned}$$

Since $u\in C\big([0,T_3]\,;BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}\big)$, we get $w_k\in C\big([0,T_3]\,;BW\tilde{\mathcal{M}}^{l,s}_{p,r}\big)$ for each k, and then $u\in C\big([0,T_3]\,;BW\tilde{\mathcal{M}}^{l,s}_{p,r}\big)$.

Continuous dependence on the data: In view of the proof of item (i), note that the existence-time T and the norm of the solution u in $L^\infty_T\left(BW\tilde{\mathcal{M}}^{l,s}_{p,r}\right)$ can be controlled by the $BW\tilde{\mathcal{M}}^{l,s}_{p,r}$-norm of u_0. Thus, since $(u_0^m)_{m\in\mathbb{N}}$ is bounded in the space $BW\tilde{\mathcal{M}}^{l,s}_{p,r}$, there exists $T>0$ such that the sequence of solutions $(u^m)_{m\in\mathbb{N}}$ is also bounded in $L^\infty_T\left(BW\tilde{\mathcal{M}}^{l,s}_{p,r}\right)$.

Next we have that $z^m = u^m - u$ satisfies the system ($x\in\mathbb{R}^n, t\in(0,T)$)

$$\begin{cases}
\frac{\partial\bar{\Delta}_j z^m}{\partial t} + \bar{\Delta}_j\left(u\cdot\nabla z^m\right) + \bar{\Delta}_j\left(z^m\cdot\nabla u^m\right) - \bar{\Delta}_j\left(\pi\left(z^m,u^m\right)+\pi\left(u,z^m\right)\right) = 0,\\
\nabla\cdot\bar{\Delta}_j z^m = 0,\\
\bar{\Delta}_j z^m(0) = \bar{\Delta}_j z_0^m,
\end{cases}$$

where $z_0^m = u_0^m - u_0$. Now we can proceed as in the proof of (4.42) in order to obtain

$$\begin{aligned}
&\|z^m(t)\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}}\\
&\le C_0\|z_0^m\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}} + C_1\int_0^t\left(2\|u^m\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} + \|u\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}\right)\|z^m(\tau)\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}}d\tau.
\end{aligned}$$

From Gronwall's inequality and the boundedness of $(u^m)_{m\in\mathbb{N}}$, we can estimate

$$\begin{aligned}
&\|z^m(t)\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}}\\
&\leq C_0\,\|z_0^m\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}}\exp\left(C_1\int_0^t\left(2\,\|u^m\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}+\|u\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}\right)d\tau\right)\\
&\leq C_0\,\|z_0^m\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}}\exp\left(2C_1\left(\|u^m\|_{L^\infty_T(BW\tilde{\mathcal{M}}^{l,s}_{p,r})}+\|u\|_{L^\infty_T(BW\tilde{\mathcal{M}}^{l,s}_{p,r})}\right)T\right)\\
&\leq C_0\,\|z_0^m\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}}\exp\left(CT\right)\leq C\,\|z_0^m\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}},
\end{aligned}$$

where $C>0$ is independent of m, and then the continuous dependence follows.

Uniqueness: Suppose that u and v are solutions of (4.44) with initial data u_0. Considering $z=u-v$, we have that $z_0=u(0)-v(0)=0$. Proceeding as above, it follows that

(4.45)

$$\|z(t)\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}}\leq C\int_0^t\left(2\,\|u\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}+\|v\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}\right)\|z(\tau)\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}}\,d\tau.$$

Since $u,v\in L^\infty_T\left(BW\tilde{\mathcal{M}}^{l,s}_{p,r}\right)$, Gronwall's inequality and the last estimate imply that $z=0$, and we are done.

$\diamond$

4.4. Proof of Theorem 1.2. We start with the supercritical case. Note that for each $j\geq -1$, we have that u satisfies

$$\begin{cases}\frac{\partial\bar{\Delta}_j u}{\partial t}+S_{j-2}u\cdot\nabla\bar{\Delta}_j u=R_j(u,u)+\bar{\Delta}_j\pi(u,u) & \text{in}\,\mathbb{R}^n\times(0,T^*),\\ \nabla\cdot\bar{\Delta}_j u=0 & \text{in}\,\mathbb{R}^n\times(0,T^*),\\ \bar{\Delta}_j u(0)=\bar{\Delta}_j u_0 & \text{in}\,\mathbb{R}^n.\end{cases}$$

Moreover, from the hypothesis we have that

$$\operatorname{div}(S_{j-2}u)=0\text{ and }S_{j-2}u\in L^1((0,T^*)\,;L^\infty).$$

Therefore, the flow associated to the vector field $S_{j-2}u$ is a volume preserving diffeomorphism and verifies the condition (4.1) where the constant γ is independent of j and $t\in(0,T^*)$. Now, we can proceed as in the proof of the existence theorem in order to get

$$\|u(t)\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}\leq C_0\,\|u_0\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}+C_1\int_0^t\|\nabla u\|_{L^\infty}\,\|u\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}\,d\tau,\ \text{for all } t\in(0,T^*).$$

Using Lemma 4.2, we can estimate

$$\begin{aligned}
\|u(t)\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}&\leq C_0\,\|u_0\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}\\
&\quad+C\int_0^t\left(1+\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}\left(\log^+\|\nabla u\|_{BW\tilde{\mathcal{M}}^{l,s-1}_{p,r}}+1\right)\right)\|u\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}\,d\tau.
\end{aligned}$$

Next let us recall that $\nabla u=\mathcal{C}(w)+Aw$ where $(w)_{i,j}=\partial_i u_j-\partial_j u_i$, $\mathcal{C}$ is an homogeneous singular integral operator of degree $-n$ and A is a constant matrix (see, e.g., [**25**]). Since $\mathcal{C}$ is bounded in $\dot{B}^0_{\infty,\infty}$ (see, e.g., [**31**]), we arrive at

$$\begin{aligned}\|u(t)\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} &\leq C_0 \|u_0\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} \\ &\quad + C\int_0^t \left(1+\|w\|_{\dot{B}^0_{\infty,\infty}}\left(\log^+\|u\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}+1\right)\right)\|u\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}\,d\tau \\ &\leq C_0 \|u_0\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} \\ &\quad + C\int_0^t \left(1+\|w\|_{\dot{B}^0_{\infty,\infty}}\right)\left(\log^+\|u\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}+1\right)\|u\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}\,d\tau,\end{aligned} \tag{4.46}$$

which together with Gronwall's inequality (twice) yield

$$\|u(t)\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}} \leq C_1 \exp\left[C_2 \exp\left\{C_3\int_0^t \left(1+\|w\|_{\dot{B}^0_{\infty,\infty}}\right)d\tau\right\}\right]. \tag{4.47}$$

Using (4.47), it is not difficult to see that (1.6) implies (1.7). The reverse implication follows directly from the estimate

$$\int_0^{T^*} \|\omega(t)\|_{\dot{B}^0_{\infty,\infty}}\,dt \leq \sup_{t\in(0,T^*)}\|\omega(t)\|_{\dot{B}^0_{\infty,\infty}}\,T^* \leq \sup_{t\in(0,T^*)}\|u(t)\|_{BW\tilde{\mathcal{M}}^{l,s}_{p,r}}\,T^*.$$

Now we turn to the critical case. Using the inclusion $\dot{B}^0_{\infty,1}\hookrightarrow L^\infty$ and the boundedness of $\mathcal{P}$ in $\dot{B}^0_{\infty,1}$, we obtain $\|\nabla u\|_{L^\infty} \leq C\|\nabla u\|_{\dot{B}^0_{\infty,1}} \leq C\|w\|_{\dot{B}^0_{\infty,1}}$. It follows that

$$\begin{aligned}\|u(t)\|_{BW\tilde{\mathcal{M}}^{l,n/p+1}_{p,1}} &\leq C_0\|u_0\|_{BW\tilde{\mathcal{M}}^{l,n/p+1}_{p,1}} + C_1\int_0^t \|\nabla u\|_{L^\infty}\|u\|_{BW\tilde{\mathcal{M}}^{l,n/p+1}_{p,1}}\,d\tau \\ &\leq C_0\|u_0\|_{BW\tilde{\mathcal{M}}^{l,n/p+1}_{p,1}} + C\int_0^t \|w\|_{\dot{B}^0_{\infty,1}}\|u\|_{BW\tilde{\mathcal{M}}^{l,n/p+1}_{p,1}}\,d\tau.\end{aligned}$$

An application of Gronwall's inequality leads us to

$$\|u(t)\|_{BW\tilde{\mathcal{M}}^{l,n/p+1}_{p,1}} \leq C\|u_0\|_{BW\tilde{\mathcal{M}}^{l,n/p+1}_{p,1}} \exp\left\{C\int_0^t \|w\|_{\dot{B}^0_{\infty,1}}\,d\tau\right\},$$

which shows that (1.4) implies (1.5). The converse follows from the estimate

$$\int_0^{T^*} \|\omega(t)\|_{\dot{B}^0_{\infty,1}}\,dt \leq \sup_{t\in(0,T)}\|\omega(t)\|_{\dot{B}^0_{\infty,1}}\,T^* \leq \sup_{t\in(0,T)}\|u(t)\|_{BW\tilde{\mathcal{M}}^{l,n/p+1}_{p,1}}\,T^*.$$

◇

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