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Traveling waves of a generalized nonlinear Beam equation

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Abstract. We consider the existence and stability of traveling waves of a nonlinear beam equation for a general class of non-homogeneous nonlinearities. We use variational methods to prove existence of ground state traveling wave solutions for this class and analyze their stability. We also present a numerical method based on the variational characterization of ground states and use it to determine intervals of wave speeds for which ground states are stable.

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1. Introduction

Consider the nonlinear beam equation

$$u_{tt} + \Delta^2 u - b\Delta u + au = f(u, Du, D^2u), \tag{1.1}$$

where f has the variational form

$$f(u, Du, D^2u) = F_u(u, Du) - \sum_{i=1}^{n} \left(F_{uu_{x_i}}(u, Du)u_{x_i} + \sum_{j=1}^{n} F_{u_{x_i}u_{x_j}}(u, Du)u_{x_ix_j} \right)$$

for some $F : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ that is C^2 and satisfies $F(0, 0) = 0$. A traveling wave with velocity c is a solution of the form $u(x, t) = \varphi(x - ct)$. Existence and stability of

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traveling waves was considered in [**12**] for homogeneous nonlinear terms of the form $f(u) = |u|^{p-1}u$. Recently, in [**6**], these results were improved upon and expanded to include standing wave solutions. In both prior works the homogeneity of the nonlinear term played a key role in both the existence and stability results. The goal of this paper is to extend these results to a general class of nonhomogeneous nonlinear terms that satisfy the conditions given in Assumption 2.2. Compared with [**12, 6**], a nonlinearity containing the derivative of u makes it difficult to show stability of the traveling waves. To overcome this problem, rather than following directly the classical approach laid down in [**2**], we will make the best of Nehari–Pankov manifold (see [**18**]) to prove our expected existence result (actually, nontrivial solutions with least possible energy) in Theorem 2.10 and then we get the stability result (Theorem 3.6) by adapting the definition of stability with this new variational problem. We also show, under Assumption 2.2, that the obtained traveling waves are in $H^4(\mathbb{R}^n)$ (see Theorem 2.13). To study the instability of traveling waves of (1.1), we give a modification of the methods of Angulo-Pava [**1**], which are based on those of Grillakis, Shatah and Strauss [**9**]. Indeed, we show by a geometric approach (see [**8**]) that for each traveling wave of (1.1) there exists an unstable direction. We note that Equation (1.1) may be written as the system

$$\begin{cases} u_t = v \\ v_t = -\Delta^2 u + b\Delta u - au + f(u, Du, D^2u) \end{cases} \tag{1.2}$$

and formally has the conserved quantities

$$\begin{aligned} E(u,v) &= \int_{\mathbb{R}^n} \frac{1}{2}|\Delta u|^2 + \frac{b}{2}|\nabla u|^2 + \frac{a}{2}|u|^2 + \frac{1}{2}|v|^2 - F(u, Du)\,\mathrm{d}x \\ Q(u,v) &= \int_{\mathbb{R}^n} v\nabla u\,\mathrm{d}x \end{aligned}$$

and thus the natural space in which to work is the energy space $X = H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. We also note that the system may be written in Hamiltonian form

$$w_t = JE'(w), \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{1.3}$$

Well-posedness of (1.2) in X was proven for the case $f(u) = |u|^{p-1}u$ in [**11**] and those results hold for nonlinearities that depend only on u and satisfy Assumption 2.2. Global existence for rough solutions of (1.2) with $f(u) = |u|^2 u$ in $H^s(\mathbb{R}^n)$ was obtained in [**25**]. The global existence and scattering theory in the energy space for (1.2) in the defocusing case was investigated in [**21, 22**]. These results were improved in [**7**] by the concentration-compactness argument. Esquivel-Avila in [**5**] considered (1.2) in a bounded domain with $f(u, Du) = |u|^p u + \tau \sum_{j=1}^n (g(u_{x_j}))_{x_j}$ and $g(s) = |s|^q s \pm s$. He studied the dynamic behavior of problem under Neumann and Dirichlet boundary conditions. Well-posedness for nonlinearities that also depend on the derivatives of u is beyond the scope of this paper, so we make the following assumption.

Assumption 1.1. For any $w_0 \in X$, there exists some $T > 0$ and a unique solution w of (1.2) in $C([0,T), X)$ satisfying $w(0) = w_0$.

By stability we mean the following.

Definition 1.2. Given a subset $\mathcal{D}$ of X, and $\epsilon > 0$ we denote by

$$V_\epsilon(\mathcal{D}) = \{v \in H^2(\mathbb{R}^n) : \|u - v\|_{H^2} < \epsilon \text{ for some } u \in \mathcal{D}\}$$

the ϵ-neighborhood of $\mathcal{D}$. A subset $\mathcal{D}$ of X is stable with respect to (1.2) if for every $\epsilon > 0$ there is some $\delta > 0$ such that for any $w_0 \in V_\delta(\mathcal{D})$, the solution $w(t)$ of (1.2) with $w(0) = w_0$ exists for all $t > 0$ and $w(t) \in V_\epsilon(\mathcal{D})$ for all $t > 0$.

We will use the following notation throughout the paper. Set

$$L^2(\mathbb{R}^n) = \left\{ u : \mathbb{R}^n \to \mathbb{R} : \|v\|_{L^2} \equiv \left(\int_{\mathbb{R}^n} |v|^2 \, \mathrm{d}x \right)^{1/2} < \infty \right\}$$

$$H^2(\mathbb{R}^n) = \left\{ u : \mathbb{R}^n \to \mathbb{R} : \|u\|_{H^2} \equiv \left(\int_{\mathbb{R}^n} |\Delta u|^2 + |u|^2 \, \mathrm{d}x \right)^{1/2} < \infty \right\}$$

$$X = H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n), \qquad X^* = H^{-2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$$

and denote by $\langle v_1, v_2 \rangle$ the standard inner product of $v_1, v_2 \in L^2(\mathbb{R}^n)$ and $\langle u_1, u_2 \rangle$ the pairing of $u_1 \in H^{-2}$ with $u_2 \in H^2$. For $w_1 = (u_1, v_1) \in H^{-2} \times L^2$ and $w_2 = (u_2, v_2) \in H^2 \times L^2$ we will denote by

$$\langle w_1, w_2 \rangle = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle \,,$$

the dual pairing of $H^{-2} \times L^2$ and $H^2 \times L^2$.

The paper is organized as follows. In Section 2, we prove the main existence result, Theorem 2.10. Sections 3 and 4 contain the proofs of the main stability and instability results, Theorem 3.6 and Theorem 4.1. Finally, in Section 5 we present a numerical method based for computing ground states based on their variational characterization, and apply it to determine regions of stability and instability for a particular mixed-power nonlinear term.

2. Existence of Solitary Waves

In this section we prove that traveling wave solutions of (1.1) exist for the class of nonlinearities that satisfy Assumption 2.2 given below. We first note that $u(x,t) = \varphi(x - ct)$, with $c \in \mathbb{R}^n$, is a solution of (1.1) if and only if φ satisfies

$$\Delta^2 \varphi - b\Delta\varphi + (c \cdot \nabla)^2 \varphi + a\varphi = f(\varphi, D\varphi, D^2\varphi), \tag{2.1}$$

and by the variational form of the nonlinear term, φ must be a critical point of

$$S(u) = \int_{\mathbb{R}^n} \frac{1}{2}(\Delta u)^2 + \frac{b}{2}|\nabla u|^2 - \frac{1}{2}(c \cdot \nabla u)^2 + \frac{a}{2}u^2 - F(u, Du) \, \mathrm{d}x$$

in $H^2(\mathbb{R}^n)$. Equivalently, $w(x,t) = \Phi(x - ct)$ is a solution of (1.2) if and only if $\Phi = (\varphi, -c \cdot \nabla\varphi)$ and φ satisfies (2.1). By (1.3) this implies $E'(\Phi) + c \cdot Q'(\Phi) = 0$ and thus Φ must be a critical point in X of the action functional $S_X : X \to \mathbb{R}$ defined by

$$S_X(w) = E(w) + c \cdot Q(w). \tag{2.2}$$

It is useful to note that the functionals S_X and S satisfy the relation

$$S_X(u, v) = S(u) + q(u, v) \tag{2.3}$$

where

$$q(u, v) = \frac{1}{2} \int_{\mathbb{R}^n} |v + c \cdot \nabla u|^2 \, \mathrm{d}x, \tag{2.4}$$

for all $(u,v) \in X$. The set of critical points of S_X in X is isomorphic to the set of critical points of S in $H^2(\mathbb{R}^n)$.

LEMMA 2.1. *$\Phi = (\varphi, \psi) \in X$ is a critical point of S_X if and only if $\varphi \in H^2(\mathbb{R}^n)$ is a critical point of S and $\psi = -c \cdot \nabla\varphi$.*

PROOF. First suppose Φ is a critical point of S_X. Then since

$$\langle q'(w_1), w_2\rangle = \int_{\mathbb{R}^n} (v_1 + c\cdot\nabla u_1)(v_2 + c\cdot\nabla u_2)\,\mathrm{d}x$$

for any $w_1, w_2 \in X$, we have

$$0 = \langle S_X'(\Phi), (0, \psi + c\cdot\nabla\varphi)\rangle = \int_{\mathbb{R}^n} |\psi + c\cdot\nabla\varphi|^2\,\mathrm{d}x,$$

so $\psi = -c\cdot\nabla\varphi$. Thus $q'(\Phi) = 0$, and $\langle S'(\varphi), u\rangle = \langle S_X'(\Phi) - q'(\Phi), w\rangle = 0$ for any $w = (u,v) \in X$, so φ is a critical point of S. Conversely, if $\varphi \in H^2$ is a critical point of S, then $\Phi = (\varphi, -c\cdot\nabla\varphi) \in X$ satisfies $q(\Phi) = 0$ and thus $q'(\Phi) = 0$. Hence $\langle S_X'(\Phi), w\rangle = \langle S'(\varphi), u\rangle + \langle q'(\Phi), w\rangle = 0$ for any $w = (u,v) \in X$ and thus Φ is a critical point of S_X. □

Next observe that any critical point Φ of S_X satisfies $P_X(\Phi) = 0$, where

$$P_X(w) = \langle S_X'(w), w\rangle \tag{2.5}$$

and any critical point φ of S in $H^2(\mathbb{R}^n)$ must satisfy $P(u) = 0$, where

$$P(u) = \langle S'(u), u\rangle = \int_{\mathbb{R}^n} |\Delta u|^2 + b|\nabla u|^2 - |c\cdot\nabla u|^2 + au^2 - uf(u, Du, D^2u)\,\mathrm{d}x.$$

Thus it is natural to look for solutions of (2.1) in the Nehari manifold

$$\mathcal{N} = \{u \in H^2(\mathbb{R}^n) : u \neq 0, P(u) = 0\}.$$

We will prove existence of solutions of (2.1) by showing that there exist minimizers of S on $\mathcal{N}$ under the assumptions on f given below. The functionals S and P may be expressed as $S(u) = I(u) - K(u)$ and $P(u) = 2I(u) - N(u)$, where

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^n} (\Delta u)^2 + b|\nabla u|^2 - |c\cdot\nabla u|^2 + au^2\,\mathrm{d}x \\ K(u) &= \int_{\mathbb{R}^n} F(u, Du)\,\mathrm{d}x \\ N(u) &= \int_{\mathbb{R}^n} uf(u, Du, D^2u)\,\mathrm{d}x \end{aligned}$$

We first note that the functional I is coercive over $H^2(\mathbb{R}^n)$ provided $a > 0$ and $b > |c|^2 - 2\sqrt{a}$, and under these conditions we have $I(u) \geq C(a,b,c)\|u\|^2_{H^2(\mathbb{R}^n)}$, where

$$C(a,b,c) = \begin{cases} \frac{4a - (b-|c|^2)^2}{4(a+1+\sqrt{(a-1)^2 + (b-|c|^2)^2})} & |c|^2 - 2\sqrt{a} < b \leq |c|^2, \\ \min\{1, a\} & b > |c|^2. \end{cases} \tag{2.6}$$

The relationship between the functionals K and N is central to the analysis. In the case that f is homogeneous of degree p, one has $N(u) = (p+1)K(u)$. Our assumptions on f given below imply that N is bounded below by a multiple of K.

ASSUMPTION 2.2. We assume $F(w) = F(u,v)$ is a real valued C^2 function on $\mathbb{R}\times\mathbb{R}^n$ such that $F(0) = 0$ and the following conditions hold.

(a) There exist $C > 0$ and $1 < p_k \le q_k$, $1 \le k \le 6$ such that

$$|F_{uu}(u,v)| \le C(|u|^{p_1-1} + |u|^{q_1-1} + |v|^{p_2-1} + |v|^{q_2-1})$$
$$|F_{uv_i}(u,v)| \le C(|u|^{p_3-1} + |u|^{q_3-1} + |v|^{p_4-1} + |v|^{q_4-1}) \text{ for } 1 \le i \le n$$
$$|F_{v_iv_j}(u,v)| \le C(|u|^{p_5-1} + |u|^{q_5-1} + |v|^{p_6-1} + |v|^{q_6-1}) \text{ for } 1 \le i,j \le n,$$

where $q_1 < \frac{n+4}{n-4}$, $q_3 < \frac{n+2}{n-4}$ and $q_5 < \frac{n}{n-4}$ when $n > 4$, and $q_2 < \min\left\{\frac{3n-2}{n-2}, \frac{n+6}{n-2}\right\}$, $q_4 < \min\left\{\frac{2n}{n-2}, \frac{n+4}{n-2}\right\}$ and $q_6 < \frac{n+2}{n-2}$ when $n > 2$.

(b) There exists $r > 1$ such that $w \cdot \nabla G(w) \ge (r+1)G(w)$ for all $w \in \mathbb{R}^{n+1}$, where $G(w) = w \cdot \nabla F(w)$.

(c) There exists $u \in H^2(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} F(u, Du)\,\mathrm{d}x > 0$.

Remark 2.3. Part (a) guarantees that the functionals K and N are bounded by a sum of powers of the H^2-norm. Part (b) is a generalization of the classical Abrosetti-Rabinowitz condition.

Examples. In addition to homogeneous nonlinear terms, the following nonlinearities satisfy Assumption 2.2.

(a) $F(u) = |u|^{p+1} + |u|^{q+1}$ where $1 < p < q$ and $q < \frac{n+4}{n-4}$ if $n > 4$.

(b) $F(Du) = |Du|^{p+1} + |Du|^{q+1}$, where $1 < p < q$ and $q < \frac{n+2}{n-2}$ if $n > 2$.

(c) More generally, $F(u, Du) = \displaystyle\sum_{i=1}^{l} |u|^{\alpha_i+1} + \sum_{j=1}^{m} |Du|^{\beta_j+1}$, where $\alpha_i > 1$, $\beta_j > 1$, $\alpha_i < \frac{n+4}{n-4}$ if $n > 4$ and $\beta_j < \frac{n+2}{n-2}$ if $n > 2$.

(d) $F(u) = |u|^{q+1} - |u|^{p+1}$ where $1 < p < q$ and $q < \frac{n+4}{n-4}$ if $n > 4$.

(e) $F(u) = |u|^{q+1} - |Du|^{p+1}$ where $1 < p < q$, $q < \frac{n+2}{n-2}$ if $n > 2$ and $q < \frac{n+4}{n-4}$ if $n > 4$.

Lemma 2.4. *Suppose $w \cdot \nabla G(w) \ge (r+1)G(w)$ for all $w \in \mathbb{R}^{n+1}$, where $G(w) = w \cdot \nabla F(w)$. Then*

(a) $G(w) \ge (r+1)F(w)$ for all $w \in \mathbb{R}^{n+1}$.

(b) For any $w \in \mathbb{R}^{n+1}$, $G(\alpha w) \ge \alpha^{r+1}G(w)$ for all $\alpha \ge 1$ and $G(\alpha w) \le \alpha^{r+1}G(w)$ for all $\alpha \le 1$.

(c) For any $w \in \mathbb{R}^{n+1}$, $F(\alpha w) \ge \alpha^{r+1}F(w)$ for all $\alpha \ge 1$ and $F(\alpha w) \le \alpha^{r+1}F(w)$ for all $\alpha \le 1$.

(d) $N(u) \ge (r+1)K(u)$ for all $u \in H^2(\mathbb{R}^n)$.

(e) $\langle N'(u), u\rangle \ge (r+1)N(u)$ for all $u \in H^2(\mathbb{R}^n)$.

Proof. Fix $w \in \mathbb{R}^{n+1}$ and let $h(s) = G(sw) - (r+1)F(sw)$. Then $h(0) = 0$ and

$$\begin{aligned} h'(s) &= w \cdot \nabla G(sw) - (r+1)w \cdot \nabla F(sw) \\ &= s^{-1}(sw \cdot \nabla G(sw) - (r+1)G(sw)) \ge 0 \end{aligned}$$

for all $s > 0$, and thus $h(s) \ge 0$ for all $s > 0$, which proves part (a).

Next set $h(s) = G(sw)$. Then

$$h'(s) = w \cdot \nabla G(sw) \ge s^{-1}(r+1)G(sw) = (r+1)s^{-1}h(s)$$

for all $s > 0$. This implies $(s^{-(r+1)}h(s))' \ge 0$ for all $s > 0$, so integrating from $s = 1$ to $s = \alpha$ gives $\alpha^{-(r+1)}h(\alpha) \ge h(1)$, and thus $G(\alpha w) \ge \alpha^{r+1}G(w)$, for $\alpha \ge 1$. Likewise, integrating from α to 1 gives $G(\alpha w) \le \alpha^{r+1}G(w)$, for $0 < \alpha \le 1$.

Finally, setting $h(s) = F(sw)$, we have $h'(s) = w \cdot \nabla F(sw) = s^{-1}G(sw)$, so by part (a), $h'(s) \geq s^{-1}(r+1)h(s)$, and part (c) follows in the same way as part (b). By the relation between f and F, we have

$$\begin{aligned} N(u) &= \int_{\mathbb{R}^n} G(w)\,\mathrm{d}x, \\ \langle N'(u), u\rangle &= \int_{\mathbb{R}^n} w \cdot \nabla G(w)\,\mathrm{d}x. \end{aligned}$$

Thus part (d) follows immediately from part (a) and part (e) follows from Assumption 2.2(b). □

LEMMA 2.5. *If F satisfies Assumption 2.2, then there exists $C > 0$ such that*

$$\begin{aligned} |K(u)| &\leq C(\|u\|_{H^2}^{p+1} + \|u\|_{H^2}^{q+1}), \\ |N(u)| &\leq C(\|u\|_{H^2}^{p+1} + \|u\|_{H^2}^{q+1}), \\ |\langle N'(u), u\rangle| &\leq C(\|u\|_{H^2}^{p+1} + \|u\|_{H^2}^{q+1}) \end{aligned}$$

for all $u \in H^2(\mathbb{R}^n)$, where $p = \min\{p_i : 1 \leq i \leq 6\}$ and $q = \max\{q_i : 1 \leq i \leq 6\}$.

PROOF. By the relation between f and F, we have

$$\begin{aligned} N(u) &= \int_{\mathbb{R}^n} G(w)\,\mathrm{d}x, \\ \langle N'(u), u\rangle &= \int_{\mathbb{R}^n} w \cdot \nabla G(w)\,\mathrm{d}x, \end{aligned}$$

where $G(w) = w \cdot \nabla F(w)$ and $w = (u, Du)$. Define

$$\begin{aligned} M(w) = {} & |u|^{p_1+1} + |u|^{q_1+1} + |u|^2|v|^{p_2-1} + |u|^2|v|^{q_2-1} + |u|^{p_3}|v| + |u|^{q_3}|v| \\ & + |u||v|^{p_4} + |u||v|^{q_4} + |u|^{p_5-1}|v|^2 + |u|^{q_5-1}|v|^2 + |v|^{p_6+1} + |v|^{q_6+1}. \end{aligned}$$

We claim there exists a constant $C > 0$ such that $|G(w)| \leq CM(w)$, $|F(w)| \leq CM(w)$ and $|w \cdot \nabla G(w)| \leq CM(w)$ for all $w \in \mathbb{R}^{n+1}$. Fix $w = (u, v) \in \mathbb{R}^{n+1}$ and set $g(s) = F_u(su, sv)$. Then $g'(s) = uF_{uu}(su, sv) + \sum_{i=1}^n v_i F_{uv_i}(su, sv)$ so integrating from 0 to 1 and using the assumption we have

$$\begin{aligned} |F_u(u, v)| \lesssim {} & |u|^{p_1} + |u|^{q_1} + |u||v|^{p_2-1} + |u||v|^{q_2-1} + |u|^{p_3-1}|v| \\ & + |u|^{q_3-1}|v| + |v|^{p_4} + |v|^{q_4}. \end{aligned} \tag{2.7}$$

Likewise, for $g(s) = F_{v_j}(su, sv)$ we have $g'(s) = uF_{uv_j}(su, sv) + \sum_{i=1}^n v_i F_{v_i v_j}(su, sv)$ so

$$\begin{aligned} |F_{v_j}(u, v)| \lesssim {} & |u|^{p_3} + |u|^{q_3} + |u||v|^{p_4-1} + |u||v|^{q_4-1} + |u|^{p_5-1}|v| \\ & + |u|^{q_5-1}|v| + |v|^{p_6} + |v|^{q_6} \end{aligned} \tag{2.8}$$

for $1 \leq j \leq n$. For $g(s) = F(su, sv)$ we have $g'(s) = uF_u(su, sv) + \sum_{i=1}^n v_i F_{v_i}(su, sv)$ so

$$\begin{aligned} |F(w)| & \lesssim |u|^{p_1+1} + |u|^{q_1+1} + |u|^2|v|^{p_2-1} + |u|^2|v|^{q_2-1} + |u|^{p_3}|v| + |u|^{q_3}|v| \\ & \qquad + |u||v|^{p_4} + |u||v|^{q_4} + |u|^{p_5-1}|v|^2 + |u|^{q_5-1}|v|^2 + |v|^{p_6+1} + |v|^{q_6+1} \\ & = CM(w). \end{aligned}$$

Since $G(w) = uF_u(u,v) + \sum_{j=1}^n v_j F_{v_j}(u,v)$, it follows from (2.7) and (2.8) that $|G(w)| \le CM(w)$. Finally, since

$$w \cdot \nabla G(w) = G(u,v) + u^2 F_{uu}(u,v) + \sum_{i=1}^n \left(2uv_i F_{uv_i}(u,v) + \sum_{j=1}^n v_i v_j F_{v_i v_j}(u,v) \right),$$

it follows from the assumptions on F and the bound on G that $|w \cdot \nabla G(w)| \le CM(u,v)$.

Next let $u \in H^2(\mathbb{R}^n)$. By the Sobolev embeddings, we have $\int_{\mathbb{R}^n} |u|^{r+1}\,\mathrm{d}x \le C\|u\|_{H^2}^{r+1}$ for any $r > 1$ provided $r < \frac{n+4}{n-4}$ when $n > 4$, and $\int_{\mathbb{R}^n} |Du|^{r+1}\,\mathrm{d}x \le C\|u\|_{H^2}^{r+1}$ for any $r > 1$ such that $r < \frac{n+2}{n-2}$ when $n > 2$.

If $n \le 2$ then $\|Du\|_{L^\infty} \le C\|u\|_{H^2}$, so $\int_{\mathbb{R}^n} u^2|Du|^{r-1}\,\mathrm{d}x \le C\|u\|_{H^2}^{r+1}$. If $n > 2$, then $\frac{4n}{2n-(r-1)(n-2)} > 2$ for $r > 1$ and when $n > 4$ we also have $r - 1 < \frac{2n}{n-2}$ and $\frac{4n}{2n-(r-1)(n-2)} < \frac{2n}{n-4}$ provided $r < \min\left\{\frac{3n-2}{n-2}, \frac{n+6}{n-2}\right\}$, so by Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}^n} u^2|Du|^{r-1}\,\mathrm{d}x &\le \left(\int_{\mathbb{R}^n} |u|^{\frac{4n}{2n-(r-1)(n-2)}}\,\mathrm{d}x\right)^{1-\frac{(r-1)(n-2)}{2n}} \left(\int_{\mathbb{R}^n} |Du|^{\frac{2n}{n-2}}\right)^{\frac{(r-1)(n-2)}{2n}} \\ &\le C\|u\|_{H^2}^2\|u\|_{H^2}^{r-1} = C\|u\|_{H^2}^{r+1}.\end{aligned}$$

In the case $n \le 4$, we have $\|u\|_{L^{2r}} \le C\|u\|_{H^2}$ for any $r > 1$ and thus

$$\int_{\mathbb{R}^n} |u|^r|Du|\,\mathrm{d}x \le \|u\|_{L^{2r}}^r\|Du\|_{L^2} \le C\|u\|_{H^2}^{r+1}.$$

When $n > 4$ and $1 < r < \frac{n+2}{n-4}$, we have either $1 < r \le 1 + \frac{2}{n}$ or $1 + \frac{2}{n} < r < \frac{n+2}{n-4}$. If $1 < r \le 1 + \frac{2}{n}$, then $2 < \frac{2}{2-r} < \frac{2n}{n-2}$, so

$$\begin{aligned}\int_{\mathbb{R}^n} |u|^r|Du|\,\mathrm{d}x &\le \left(\int_{\mathbb{R}^n} |u|^2\,\mathrm{d}x\right)^{\frac{r}{2}} \left(\int_{\mathbb{R}^n} |Du|^{\frac{2}{2-r}}\,\mathrm{d}x\right)^{\frac{2-r}{2}} \\ &\le C\|u\|_{H^2}^r\|u\|_{H^2} = C\|u\|_{H^2}^{r+1};\end{aligned}$$

while if $1 + \frac{2}{n} < r < \frac{n+2}{n-4}$, then $2 < \frac{2nr}{n+2} < \frac{2n}{n-4}$, so

$$\begin{aligned}\int_{\mathbb{R}^n} |u|^r|Du|\,\mathrm{d}x &\le \left(\int_{\mathbb{R}^n} |u|^{\frac{2nr}{n+2}}\,\mathrm{d}x\right)^{\frac{n+2}{2n}} \left(\int_{\mathbb{R}^n} |Du|^{\frac{2n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{2n}} \\ &\le C\|u\|_{H^2}^r\|u\|_{H^2} = C\|u\|_{H^2}^{r+1}.\end{aligned}$$

By similar reasoning it follows that

$$\int_{\mathbb{R}^n} |u||Du|^r\,\mathrm{d}x \le C\|u\|_{H^2}^{r+1}$$

provided $r > 1$ and $r < \min\left\{\frac{2n}{n-2}, \frac{n+4}{n-2}\right\}$ when $n > 2$, and

$$\int_{\mathbb{R}^n} |u|^{r-1}|Du|^2\,\mathrm{d}x \le C\|u\|_{H^2}^{r+1}$$

provided $r > 1$ and $r < \frac{n}{n-4}$ when $n > 4$. By the assumptions on p_i and q_i, it therefore follows that

$$\int_{\mathbb{R}^n} M(u,Du)\,\mathrm{d}x \le C\sum_{i=1}^6 \|u\|_{H^2}^{p_i+1} + \|u\|_{H^2}^{q_i+1} \le C(\|u\|_{H^2}^{p+1} + \|u\|_{H^2}^{q+1})$$

for any $u \in H^2(\mathbb{R}^n)$. $\square$

We now return to the problem of minimizing S on $\mathcal{N}$. We first show that any minimizer is in fact a solution of (2.1).

Since $P(u) = 0$ for any $u \in \mathcal{N}$, minimizing S on $\mathcal{N}$ is equivalent to minimizing

$$S_\lambda(u) = S(u) - \frac{1}{1+\lambda} P(u)$$

on $\mathcal{N}$ for any $\lambda \in \mathbb{R}$, and we define

$$m = m_0 = \inf\{S(u) : u \in \mathcal{N}\} = \inf\{S_\lambda(u) : u \in \mathcal{N}\}. \tag{2.9}$$

Two choices of λ are of particular interest, $\lambda = r$ and $\lambda = 1$. We have from Lemma 2.4 that

$$S_r(u) = \frac{r-1}{r+1} I(u) + \frac{1}{r+1} N(u) - K(u)$$

is nonnegative for all $u \in H^2(\mathbb{R}^n)$, while

$$S_1(u) = \frac{1}{2} N(u) - K(u)$$

is independent of c.

Lemma 2.6. *If $u \in H^2(\mathbb{R}^n)$ satisfies $S(u) = m$ and $P(u) = 0$, then u is a weak solution of* (2.1).

Proof. By the Lagrange multiplier theorem, $S'(u) = \mu P'(u)$ for some $\mu \in \mathbb{R}$. Thus $0 = P(u) = \langle S'(u), u \rangle = \mu \langle P'(u), u \rangle$. By Lemma 2.4,

$$\langle P'(u), u \rangle = 4I(u) - \langle N'(u), u \rangle \leq 4I(u) - (r+1)N(u) = 2(1-r)I(u) < 0;$$

so $\mu = 0$ and thus $S'(u) = 0$. $\square$

We will now apply the concentrated-compactness method to minimizing sequences for the operator S_r. To do so, we consider the family of minimization problems

$$m_\sigma = \inf\{S_r(u) : u \in \mathcal{N}_\sigma\}, \qquad \mathcal{N}_\sigma = \{u \in H^2(\mathbb{R}^n) : u \neq 0, P(u) = \sigma\}.$$

Lemma 2.7. *Let $u \in H^2(\mathbb{R}^n)$ be such that $N(u) > 0$.*

(a) For each $\sigma \leq 0$, there exists a unique $\alpha > 0$ such that $P(\alpha u) = \sigma$.

(b) $S(\alpha u)$ attains its maximum for $\alpha > 0$ when $P(\alpha u) = 0$.

Proof. Set $g(\alpha) = \alpha^{-2} P(\alpha u) = 2I(u) - \alpha^{-2} N(\alpha u)$. Since $N(\alpha u) \leq \alpha^{r+1} N(u)$ for $0 < \alpha < 1$, $g(\alpha) > 0$ for small $\alpha > 0$ and since $N(\alpha u) \geq \alpha^{r+1} N(u)$ for $\alpha > 1$, $g(\alpha) < \sigma$ for large α, and thus $g(\alpha) = \sigma$ for some $\alpha > 0$. Since $I(u) > 0$, it follows that for any $\alpha > 0$ such that $P(\alpha u) \leq 0$ we have $N(\alpha u) > 0$ and thus

$$g'(\alpha) = -\frac{\langle N'(\alpha u), \alpha u \rangle - 2N(\alpha u)}{\alpha^3} \leq \frac{(r-1)N(\alpha u)}{\alpha^3} < 0.$$

Thus g is strictly decreasing when $g(\alpha) \leq 0$. Hence there is at most one $\alpha > 0$ for which $P(\alpha u) = \sigma$.

Next let $h(\alpha) = S(\alpha u)$. Part (b) then follows since $h'(\alpha) = \langle P'(\alpha u), u \rangle = \alpha^{-1} P(\alpha u) = \alpha g(\alpha)$. $\square$

Lemma 2.8. *Suppose $a > 0$ and $b > |c|^2 - 2\sqrt{a}$. There exists $\sigma_0 > 0$ such that $\mathcal{N}_\sigma$ is nonempty for any $\sigma \leq \sigma_0$. For $\sigma \leq 0$, $\mathcal{N}_\sigma$ is bounded away from zero in $H^2(\mathbb{R}^n)$ independent of σ.*

PROOF. By Assumption 2.2 and Lemma 2.4, there exists some $u \in H^2(\mathbb{R}^n)$ such that $N(u) > 0$, and thus by Lemma 2.7, $\mathcal{N}_\sigma$ is nonempty for any $\sigma \leq 0$. For $g(\alpha)$ as in the proof of Lemma 2.7 we have $g(\alpha) > 0$ for small $\alpha > 0$, so $\mathcal{N}_\sigma$ is nonempty for small $\sigma > 0$.

For $u \in \mathcal{N}_\sigma$ with $\sigma \leq 0$ we have by Lemma 2.5 that

$$0 \geq \sigma = P(u) \geq 2C(a,b,c)\|u\|_{H^2}^2 - C(\|u\|_{H^2}^{p+1} + \|u\|_{H^2}^{q+1})$$

for some $C > 0$, where $C(a,b,c)$ is given by (2.6). Since $u \neq 0$, we have

$$\|u\|_{H^2}^{p-1} + \|u\|_{H^2}^{q-1} \geq 2C^{-1}C(a,b,c)$$

and thus $\|u\|_{H^2} \geq \epsilon_0$ for some $\epsilon_0 > 0$. □

The following lemma establishes the strict subadditivity condition $m_\sigma + m_{-\sigma} > m_0$ needed to apply the concentrated-compactness method.

LEMMA 2.9. *Suppose F satisfies Assumption 2.2, $a > 0$ and $b > |c|^2 - 2\sqrt{a}$. Then*

(a) $m_\sigma \geq 0$ for any σ for which $\mathcal{N}_\sigma$ is nonempty.
(b) $m_\sigma > 0$ for all $\sigma \leq 0$.
(c) m_σ is strictly decreasing for $\sigma \leq 0$.

PROOF. Suppose $u \in \mathcal{N}_\sigma$ for some σ. By part (d) of Lemma 2.4 $\frac{1}{r+1}N(u) - K(u) \geq 0$, so $S_r(u) \geq \frac{r-1}{r+1}I(u) > 0$ and thus $m_\sigma \geq 0$. For $\sigma \leq 0$, since $\mathcal{N}_\sigma$ is bounded away from zero, we have $\|u\|_{H^2} \geq C_0 > 0$ for all $u \in \mathcal{N}_\sigma$ and thus $m_\sigma \geq S_r(u) \geq \frac{1}{4}\left(\frac{r-1}{r+1}\right)C(a,b,c)C_0^2 > 0$. This proves (a) and (b).

Next suppose $\sigma_1 < \sigma_2 \leq 0$ and write $\mathcal{N}_{\sigma_1} = \mathcal{N}^+ \cup \mathcal{N}^-$, where $\mathcal{N}^+ = \mathcal{N}_{\sigma_1} \cap \{u \in H^2(\mathbb{R}^n) : S_r(u) > 2m_{\sigma_2}\}$ and $\mathcal{N}^- = \mathcal{N}_{\sigma_1} \cap \{u \in H^2(\mathbb{R}^n) : S_r(u) \leq 2m_{\sigma_2}\}$. If $\mathcal{N}^-$ is empty, then $m_{\sigma_1} \geq 2m_{\sigma_2} > m_{\sigma_2}$. Otherwise, notice that $\mathcal{N}^-$ is bounded in $H^2(\mathbb{R}^n)$ since S_r is coercive. For $u \in \mathcal{N}^-$, setting $g(\alpha) = P(\alpha u)$ we have $g(1) = \sigma_1$ and since $N(u) > 0$, Lemma 2.7 implies that there exists a unique $\alpha(u) < 1$ such that $P(\alpha(u)u) = \sigma_2$. Since $g'(\alpha) = 2\alpha I(u) - \alpha^{-1}\langle N'(\alpha u), \alpha u\rangle$, it follows from Lemma 2.5 that there exists some $C_1 > 0$ independent of u such that $|g'(\alpha)| \leq C_1$ for all $\alpha < 1$. Hence $\sigma_2 - \sigma_1 = g(\alpha(u)) - g(1) \leq C_0(1 - \alpha(u))$, so $\alpha(u) \leq 1 - \frac{\sigma_2 - \sigma_1}{C_0} \equiv \alpha_0$ for all $u \in \mathcal{N}^-$. Next for $h(\alpha) = S_r(\alpha u)$ we have

$$\begin{aligned} h'(\alpha) &= 2\alpha\left(\frac{r-1}{r+1}\right)I(u) + \frac{\langle N'(\alpha u), \alpha u\rangle - (r+1)N(\alpha u)}{(r+1)\alpha} \\ &\geq 2\alpha^{-1}\left(\frac{r-1}{r+1}\right)I(\alpha u) \end{aligned}$$

for any $\alpha > 0$. Since $P(\alpha u) \leq 0$ for $\alpha(u) \leq \alpha \leq 1$, we have by Lemma 2.8 that $\|\alpha u\|_{H^2} \geq \epsilon_0$ and thus $h'(\alpha) \geq 2\left(\frac{r-1}{r+1}\right)C(a,b,c)\epsilon_0^2 \equiv C_1 > 0$ for $\alpha(u) \leq \alpha \leq 1$. Thus

$$S_r(u) - S_r(\alpha(u)) = h(1) - h(\alpha(u)) \geq C_1(1 - \alpha(u)) \geq C_1(1 - \alpha_0)$$

so $S_r(u) \geq m_{\sigma_2} + C_1(1 - \alpha_0)$ for any $u \in \mathcal{N}^-$. This implies that

$$m_{\sigma_1} \geq \min\{m_{\sigma_2} + C_1(1 - \alpha_0), 2m_{\sigma_2}\} > m_{\sigma_2},$$

so m_σ is strictly decreasing for $\sigma \leq 0$. □

THEOREM 2.10. *Suppose F satisfies Assumption 2.2, $a>0$ and $b>|c|^2-2\sqrt{a}$. Let $\{u_k\}$ be a sequence in $H^2(\mathbb{R}^n)$ such that $P(u_k)\to 0$ and $S(u_k)\to m$. Then there exists a subsequence (renamed $\{u_k\}$), a sequence $\{y_k\}$ in $\mathbb{R}^n$ and $\varphi\in H^2(\mathbb{R}^n)$ such that $u_k(\cdot-y_k)\to\varphi$ strongly in $H^2(\mathbb{R}^n)$. Moreover, $S(\varphi)=m$ and $P(\varphi)=0$, so u is a minimizer of S on $\mathcal{N}$.*

PROOF. Since $S_r(u_k)\geq \frac{1}{4}\left(\frac{r-1}{r+1}\right)C(a,b,c)\|u_k\|^2_{H^2}$, the sequence $\{u_k\}$ is bounded in H^2. Since $S_r(u)\leq C(\|u\|^2_{H^2}+\|u\|^{p+1}_{H^2}+\|u\|^{q+1}_{H^2})$ and $m>0$, $\|u_k\|_{H^2}$ is bounded below. Thus if we set $\rho_k=|\Delta u_k|^2+|u_k|^2$, then there exists some $L>0$ and a subsequence such that $\int_{\mathbb{R}^n}\rho_k\,\mathrm{d}x\to L$, and be rescaling we may assume $\int_{\mathbb{R}^n}\rho_k\,\mathrm{d}x=L$ for all k. By the concentrated compactness lemma there are three possibilities:
Vanishing. For every $R>0$,

$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}^n}\int_{|x-y|\leq R}\rho_k\,\mathrm{d}x=0 \tag{2.10}$$

Dichotomy. There exists $\ell\in(0,L)$ such that for any $\epsilon>0$ there exist $R,R_k\to\infty$, $y_k\in\mathbb{R}^n$ and k_0 so that

$$\left|\int_{|x-y_k|\leq R}\rho_k\,\mathrm{d}x-\ell\right|<\epsilon \quad\text{and}\quad \left|\int_{R<|x-y_k|<R_k}\rho_k\,\mathrm{d}x\right|<\epsilon \tag{2.11}$$

for $k\geq k_0$.
Compactness. There exist $y_k\in\mathbb{R}^n$ such that for any $\epsilon>0$ there exists $R(\epsilon)$ so that for all k,

$$\int_{|x-y_k|\leq R(\epsilon)}\rho_k\,\mathrm{d}x\geq\int_{\mathbb{R}}\rho_k\,\mathrm{d}x-\epsilon.$$

By Lemma 1.1 in [**14**], vanishing would imply $\nabla u_k\to 0$ in $L^{r+1}(\mathbb{R}^n)$ for all $1<r<\frac{n+2}{n-2}$ and $u_k\to 0$ in $L^{r+1}(\mathbb{R}^n)$ for all $1<r<\frac{n+4}{n-4}$. This implies that $K(u_k),N(u_k)\to 0$ as$k\to\infty$ and therefore since $P(u_k)\to 0$ as $k\to\infty$, it follows that $I(u_k)\to 0$ as $k\to\infty$, which contradicts the fact that $\|u_k\|_{H^2}$ is bounded below by a positive constant. Hence vanishing cannot occur.

Next, dichotomy would imply there exist sequences $\{u_k^1\}$ and $\{u_k^2\}$ such that

$$\begin{aligned}\lim_{k\to\infty}I(u_k)-(I(u_k^1)+I(u_k^2))&=0,\\ \lim_{k\to\infty}K(u_k)-(K(u_k^1)+K(u_k^2))&=0,\\ \lim_{k\to\infty}N(u_k)-(N(u_k^1)+N(u_k^2))&=0,\end{aligned}$$

and therefore

$$\begin{aligned}\lim_{k\to\infty}P(u_k)-(P(u_k^1)+P(u_k^2))&=0,\\ \lim_{k\to\infty}S_r(u_k)-(S_r(u_k^1)+S_r(u_k^2))&=0.\end{aligned}$$

For some further subsequence, the limits of I, K and N evaluated at u_k^1, u_k^2 and u_k exist, so we may set $\sigma_1=\lim_k P(u_k^1)$ and $\sigma_2=\lim_k P(u_k^2)$. Then $\sigma_1+\sigma_2=0$, and either $\sigma_1\neq 0$ or $\sigma_1=0$.
Case 1: $\sigma_1\neq 0$. Without loss of generality, suppose $\sigma_1<0$. Then for all sufficiently large k, we have $P(u_k^1)<\frac{1}{2}\sigma_1$ and therefore $S_r(u_k^1)\geq m_{\frac{1}{2}\sigma_1}$. Since $S_r(u_k^2)\geq 0$ for all k, we then have by Lemma 2.9 that

$$m=\lim_{k\to\infty}S_r(u_k)=\lim_{k\to\infty}S_r(u_k^1)+S_r(u_k^2)\geq m_{\frac{1}{2}\sigma_1}>m_0=m;$$

which is a contradiction.
Case 2: $\sigma_1 = 0$. By coercivity of I, dichotomy implies that $I_1 = \lim_k I(u_k^1) > 0$ and $I_2 = \lim_k I(u_k^2) > 0$, and thus $\lim_k N(u_k^1) = 2I_1 > 0$ and $\lim_k N(u_k^2) = 2I_2 > 0$, so given $\epsilon > 0$ there exists k_0 such that $\frac{I(u_k^1)}{2N(u_k^1)} < (1+\epsilon)^{r-1}$ and $\frac{I(u_k^2)}{2N(u_k^2)} < (1+\epsilon)^{r-1}$ for $k \geq k_0$. By the assumptions on f and the boundedness of $\{u_k^1\}$ in $H^2(\mathbb{R}^n)$ there exists a constant $C > 0$ such that

$$|K(u_k^1) - K(\alpha u_k^1)| \leq C(\alpha - 1)$$
$$|N(u_k^1) - N(\alpha u_k^1)| \leq C(\alpha - 1)$$

for any k and $\alpha > 1$. If $P(u_k^1) > 0$, then since

$$P(\alpha u_k^1) \leq 2\alpha^2 I(u_k^1) - \alpha^{r+1} N(u_k^1)$$

for $\alpha > 1$, it follows that $P(\alpha u_k^1) = 0$ for some $\alpha \leq \left(\frac{2I(u_k^1)}{N(u_k^1)}\right)^{\frac{1}{r-1}}$. For this α, the inequalities above imply

$$\begin{aligned}
m_0 \leq S_r(\alpha u_k^1) &= \left(\frac{r-1}{r+1}\right) I(\alpha u_k^1) + \frac{1}{r+1} N(\alpha u_k) - K(\alpha u_k^1) \\
&= S_r(u_k^1) + (\alpha^2 - 1)\left(\frac{r-1}{r+1}\right) I(u_k^1) + \frac{1}{r+1}(N(\alpha u_k^1) \\
&\quad - N(u_k^1)) + K(u_k^1) - K(\alpha u_k^1) \\
&\leq S_r(u_k^1) + C_1(\alpha - 1) \\
&< S_r(u_k^1) + C_1\epsilon
\end{aligned}$$

for some constant C_1. Hence $S_r(u_k^1) \geq m_0 - C_1\epsilon$. On the other hand, if $P(u_k^1) \leq 0$, then $S_r(u_k^1) \geq m_0$, so in either case we have $S_r(u_k^1) \geq m_0 - C_1\epsilon$. Since the same inequality holds for $S_r(u_k^2)$, we have $S_r(u_k^1) + S_r(u_k^2) > 2m_0 - 2C_1\epsilon$ for all $k \geq k_0$, so

$$m_0 = \lim_{k\to\infty} S(u_k) = \lim_{k\to\infty} S_r(u_k) = \lim_{k\to\infty} S_r(u_k^1) + S_r(u_k^2) \geq 2m_0 - 2C_1\epsilon.$$

Since this holds for any $\epsilon > 0$, we obtain $m_0 \geq 2m_0$, a contradiction. Thus dichotomy does not occur.

Since the sequence $\{\rho_k\}$ is compact, it follows that there exists a sequence $\{y_k\}$ in $\mathbb{R}^n$ and some $\varphi \in H^2(\mathbb{R}^n)$ such that $v_k(x) = u_k(x - y_k)$ converges to φ weakly in $H^2(\mathbb{R}^n)$ and strongly in $L^{r+1}_{loc}(\mathbb{R}^n)$ for $1 < r < \frac{n+4}{n-4}$ and in $W^{1,r+1}_{loc}(\mathbb{R}^n)$ for $1 < r < \frac{n+2}{n-2}$. By a standard argument, compactness then implies strong convergence in $L^{r+1}(\mathbb{R}^n)$ for $1 < r < \frac{n+4}{n-4}$ and in $W^{1,r+1}(\mathbb{R}^n)$ for $1 < r < \frac{n+2}{n-2}$. Thus by the assumptions on f and the translation invariance of the functionals K and N, $\lim_k K(u_k) = \lim_k K(v_k) = K(\varphi)$ and $\lim_k N(u_k) = \lim_k N(v_k) = N(\varphi)$. The weak lower semicontinuity of I implies that

$$S(\varphi) + K(\varphi) = I(\varphi) \leq \liminf_{k\to\infty} I(u_k) = \liminf_{k\to\infty} S(u_k) + K(u_k) = m_0 + K(\varphi)$$

and

$$\begin{aligned}
P(\varphi) + N(\varphi) &= 2I(\varphi) \\
&\leq \liminf_{k\to\infty} 2I(u_k) \\
&= \liminf_{k\to\infty} P(u_k) + N(u_k) = N(\varphi),
\end{aligned}$$

so $S(\varphi) \le m_0$ and $P(\varphi) \le 0$. On the other hand, since

$$\begin{aligned} S_r(\varphi) + K(\varphi) &= \left(\frac{r-1}{r+1}\right) I(\varphi) + \frac{1}{r+1} N(\varphi) \\ &\le \liminf_{k\to\infty} \left(\frac{r-1}{r+1}\right) I(u_k) + \frac{1}{r+1} N(u_k) \\ &= \liminf_{k\to\infty} S_r(u_k) + K(u_k) \\ &= \lim_{k\to\infty} S_r(u_k) + K(u_k) = m_0 + K(\varphi) \\ &< m_\sigma + K(\varphi), \end{aligned}$$

we have $S_r(\varphi) < m_\sigma$ for all $\sigma < 0$, and thus we cannot have $P(\varphi) = \sigma$ for any $\sigma < 0$ since this would imply $m_\sigma \le S_r(\varphi) < m_\sigma$, a contradiction. Thus $P(\varphi) = 0$, and it then follows that $S(\varphi) = m_0$. Hence φ achieves the minimum m_0. Furthermore, since $\lim_k S(v_k) = m_0 = S(\varphi)$, we have

$$\lim_{k\to\infty} I(v_k) = \lim_{k\to\infty} S(v_k) + K(v_k) = S(\varphi) + K(\varphi) = I(\varphi)$$

so together with the fact that $v_k \to \varphi$ weakly in $H^2(\mathbb{R}^n)$, it follows that $I(v_k - \varphi) \to 0$, and thus since I is coercive, $v_k \to \varphi$ strongly in $H^2(\mathbb{R}^n)$. □

By Lemma 2.6, the minimizer φ in Theorem 2.10 is a weak solution of (2.1) and therefore minimizes S among all nontrivial solutions. We call such a solution a ground state. The set of all ground state solutions is denoted $\mathcal{G}$ and we have

$$\begin{aligned} \mathcal{G} &= \{\varphi \in H^2 : \varphi \ne 0, S'(\varphi) = 0, S(\varphi) \le S(u) \text{ for all } u \in H^2(\mathbb{R}^n) \\ &\qquad\qquad \text{such that } u \ne 0 \text{ and } S'(u) = 0\} \\ &= \{\varphi \in H^2(\mathbb{R}^n) : S(\varphi) = m, P(\varphi) = 0\}. \end{aligned}$$

Since m_σ is strictly decreasing for $\sigma \le 0$, it also follows that φ minimizes S_r among all nonzero u such that $P(u) \le 0$. That is,

$$m = S(\varphi) = S_r(\varphi) = \inf\{S_r(u) : u \in H^2(\mathbb{R}^n), u \ne 0, P(u) \le 0\} \tag{2.12}$$

for any $\varphi \in \mathcal{G}$.

If we define

$$\mathscr{G} = \{\Phi = (\varphi, -c\nabla\varphi) : \varphi \in \mathcal{G}\}$$

to be the set of ground state pairs in X, then we have the following variational characterization of $\mathscr{G}$.

Lemma 2.11. *$\Phi \in \mathscr{G}$ if and only if $S_X(\Phi) = \inf\{S_X(w) : w \in X, w \ne 0, P_X(w) = 0\}$.*

Proof. We first recall that $S_X(u, v) = S(u) + q(u, v)$ and also note that $P_X(u, v) = P(u) + 2q(u, v)$, where q is defined by (2.4). Suppose $\Phi = (\varphi, -c\cdot\nabla\varphi) \in \mathscr{G}$, and suppose $w = (u, v) \ne 0$ satisfies $P_X(w) = 0$. Then $u \ne 0$, $P(u) \le 0$, and thus

$$\begin{aligned} S_X(\Phi) = S_r(\varphi) \le S_r(u) &= S(u) - \frac{1}{r+1} P(u) \\ &\le S(u) - \frac{1}{2} P(u) = S(u) + q(u, v) = S_X(w). \end{aligned}$$

Conversely, suppose $\Phi = (\varphi, \psi)$ minimizes S_X over all nonzero $w \in X$ such that $P_X(w) = 0$. Then $S'_X(\Phi) = \mu P'_X(\Phi)$ for some $\mu \in \mathbb{R}$, so $0 = P_X(\Phi) = \mu \langle P'_X(\Phi), \Phi\rangle$. But

$$\langle P'_X(\Phi), \Phi\rangle = 4I(\varphi) + 4q(\varphi, \psi) - \langle N'(\varphi), \varphi\rangle = 2N(\varphi) - \langle N'(\varphi), \varphi\rangle \le (1 - r)N(\varphi)$$

and since $N(\varphi) = 2I(\varphi)+2q(\varphi, \psi) > 0$ this implies $\langle P'_X(\Phi), \Phi\rangle < 0$ and thus $\mu = 0$. Hence $S'_X(\Phi) = 0$, which implies

$$0 = \langle S'_X(\varphi, \psi), (0, \psi + c \cdot \nabla\varphi)\rangle = \int_{\mathbb{R}^n} |\psi + c \cdot \nabla\varphi|^2 \, \mathrm{d}x$$

and thus $\psi = -c\cdot\nabla\varphi$. Thus φ is nonzero and $q(\Phi) = 0$, so $P(\varphi) = 0$. Now suppose $u \in H^2(\mathbb{R}^n)$ is nonzero and $P(u) = 0$. Then $w = (u, -c \cdot \nabla u)$ satisfies $P_X(w) = 0$, so

$$S(\varphi) = S_X(\Phi) \le S_X(w) = S(u);$$

and thereby φ minimizes S over all such u and is therefore in $\mathcal{G}$, which implies $\Phi \in \mathscr{G}$. $\square$

In the case that $F(u) = |u|^{p+1}$ with $1 < p < \frac{n+4}{n-4}$, existence of traveling waves was proven in [**11**] by showing that there exist minimizers of $I(u)$ subject to the constraint $K(u) = 1$ and then using homogeneity to scale away the Lagrange multiplier. The same method proves existence for any $F(u, Du)$ that is homogeneous of degree $p + 1$, where $1 < p < \frac{n+2}{n-2}$ if F depends on Du, and there exist $u \in H^2(\mathbb{R}^n)$ such that $K(u) > 0$. By homogeneity, minimizers achieve the minimum

$$m^* = \inf \left\{ \frac{I(u)}{K(u)^{\frac{2}{p+1}}} : u \in H^2(\mathbb{R}^n), K(u) > 0 \right\}.$$

If we denote by $\mathcal{G}^*$ the set of all such minimizers, then we have the following.

Theorem 2.12. *For $|c|^2 < b + 2\sqrt{a}$, $m^* = m$ and $\mathcal{G}^* = \mathcal{G}$.*

Proof. By homogeneity of F, $N(u) = (p + 1)K(u)$, and thus $P(u) = 2I(u) - (p + 1)K(u)$ for all $u \in H^2(\mathbb{R}^n)$.

First suppose $u \in \mathcal{G}$, and let $v \in H^2(\mathbb{R}^n)$ satisfy $K(v) > 0$. Then by Lemma 2.4 $N(v) > 0$ and

$$P(\alpha v) = 2\alpha^2 I(v) - \alpha^{p+1}N(v) = 0$$

when $\alpha^{p-1} = \frac{2I(v)}{N(v)}$. Thus $S(u) \le S(\alpha v)$ by definition of $\mathcal{G}$. Since $P(u) = P(\alpha v) = 0$, we have $I(u) = \frac{1}{2}N(u) = \frac{p+1}{2}K(u) = \frac{p+1}{p-1}S(u)$ and $I(\alpha v) = \frac{p+1}{p-1}S(\alpha v)$. This implies

$$\begin{aligned} \frac{I(u)}{K(u)^{\frac{2}{p+1}}} &= \frac{p+1}{2}K(u)^{\frac{p-1}{p+1}} = \frac{p+1}{2}\left(\frac{2}{p-1}S(u)\right)^{\frac{p-1}{p+1}} \\ &\le \frac{p+1}{2}\left(\frac{2}{p-1}S(\alpha v)\right)^{\frac{p-1}{p+1}} = \frac{I(\alpha v)}{K(\alpha v)^{\frac{2}{p+1}}} = \frac{I(v)}{K(v)^{\frac{2}{p+1}}} \end{aligned}$$

and thus $u \in \mathcal{G}^*$.

Next suppose $u \in \mathcal{G}^*$. Then since $2I(u) = (p + 1)K(u)$, we have $P(u) = 0$ and $S(u) = \frac{p-1}{p+1}I(u) = \frac{p-1}{2}K(u)$. Now let v be any nonzero element of $H^2(\mathbb{R})$ such that

$P(v) = 0$. Then $S(v) = \frac{p-1}{p+1} I(v) = \frac{p-1}{2} K(v)$ and

$$\frac{I(u)}{K(u)^{\frac{2}{p+1}}} \leq \frac{I(v)}{K(v)^{\frac{2}{p+1}}};$$

so

$$S(v) \geq \frac{p-1}{p+1} \frac{K(v)^{\frac{2}{p+1}}}{K(u)^{\frac{2}{p+1}}} I(u) = \frac{S(v)^{\frac{2}{p+1}}}{S(u)^{\frac{2}{p+1}}} S(u)$$

and it follows that $S(v) \geq S(u)$, and thus $u \in \mathcal{G}$. □

THEOREM 2.13. *Suppose $\varphi \in H^2(\mathbb{R}^n)$ is a solution of* (2.1)*. Then $\varphi \in H^4(\mathbb{R}^n)$.*

PROOF. Set $g = f(\varphi, D\varphi, D^2\varphi)$, and note that equation (2.1) may be written $L\varphi = g$ where $L = \Delta^2 + (c \cdot \nabla)^2 + Id$. Since L^{-1} maps $H^s(\mathbb{R}^n)$ into $H^{s+4}(\mathbb{R}^n)$ for any $s \in \mathbb{R}$, if $g \in H^s(\mathbb{R}^n)$, then $u \in H^{s+4}(\mathbb{R}^n)$. Thus it suffices to show that $g \in L^2(\mathbb{R}^n)$. We will make use of the embedding $H^s(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$ for $2 \leq r \leq \frac{2n}{n-2s}$ and $0 \leq s < n/2$ and the dual embeddings $L^r(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ for $-\frac{n}{2} < s \leq \frac{n}{2} - \frac{n}{r}$ and $1 < r \leq 2$.

By Assumption 2.2 and (2.7), we have

$$\begin{aligned} |g| \lesssim\ & |\varphi|^{p_1} + |\varphi|^{q_1} + |\varphi||D\varphi|^{p_2-1} + |\varphi||D\varphi|^{q_2-1} + |\varphi|^{p_3-1}|D\varphi| \\ & + |\varphi|^{q_3-1}|D\varphi| + |D\varphi|^{p_4} + |D\varphi|^{q_4} + |\varphi|^{p_5-1}|D^2\varphi| + |\varphi|^{q_5-1}|D^2\varphi| \\ & + |D\varphi|^{p_6-1}|D^2\varphi| + |D\varphi|^{q_6-1}|D^2\varphi|. \end{aligned}$$

We will denote

$$\begin{aligned} g_1 &= |\varphi|^{p_1} + |\varphi|^{q_1}, & g_2 &= |\varphi| \left(|D\varphi|^{p_2-1} + |D\varphi|^{q_2-1}\right), \\ g_3 &= \left(|\varphi|^{p_3-1} + |\varphi|^{q_3-1}\right) |D\varphi|, & g_4 &= |D\varphi|^{p_4} + |D\varphi|^{q_4}, \\ g_5 &= \left(|\varphi|^{p_5-1} + |\varphi|^{q_5-1}\right) |D^2\varphi|, & g_6 &= \left(|D\varphi|^{p_6-1} + |D\varphi|^{q_6-1}\right) |D^2\varphi|. \end{aligned}$$

Case 1: $n = 1$**.** In this case, $\varphi \in W^{1,\infty}(\mathbb{R})$ and thus $g \in L^2(\mathbb{R})$.
Case 2: $n = 2$**.** In this case, $\varphi \in L^\infty(\mathbb{R}^2)$ and $D\varphi \in L^q(\mathbb{R}^2)$ for $2 \leq q < \infty$, so g_1 through g_5 are in $L^2(\mathbb{R}^2)$, and $g_6 \in L^r(\mathbb{R}^2)$ for $\max\{1, \frac{2}{p_6}\} \leq r < 2$. In particular, there is some $1 < r < 2$ such that $g_6 \in L^r(\mathbb{R}^2)$, and thus $g_6 \in H^s(\mathbb{R}^2)$ for $s = 1 - \frac{2}{r}$. Since $s < 0$, we therefore have $g \in H^s(\mathbb{R}^2)$ and $\varphi \in H^{s+4}(\mathbb{R}^2)$. Since $s > -1$ this implies $\varphi \in H^3(\mathbb{R}^2)$, so $D\varphi \in L^\infty(\mathbb{R}^2)$, and thus $g \in L^2(\mathbb{R}^2)$.
Case 3: $n = 3$**.** In this case we write $g = F_u(\varphi, D\varphi) - \sum_{i=1}^3 (F_{v_i}(\varphi, D\varphi))_{x_i}$ and note that by (2.7) and (2.8) we have

$$\begin{aligned} |F_u(\varphi, D\varphi)| &\lesssim g_1 + g_2 + g_3 + g_4, \\ |F_{v_i}(\varphi, D\varphi)| &\lesssim \tilde{g}_3 + \tilde{g}_4 + \tilde{g}_5 + \tilde{g}_6, \end{aligned}$$

where $\tilde{g}_3 = |\varphi|^{p_3} + |\varphi|^{q_3}$, $\tilde{g}_4 = |\varphi||D\varphi|^{p_4-1} + |\varphi||D\varphi|^{q_4-1}$, $\tilde{g}_5 = |\varphi|^{p_5-1}|D\varphi| + |\varphi|^{q_6-1}|D\varphi|$ and $\tilde{g}_6 = |D\varphi|^{p_6} + |D\varphi|^{q_6}$. Since $\varphi \in L^\infty(\mathbb{R}^3)$, g_1, g_3, $\tilde{g}_3$ and $\tilde{g}_5$ are in $L^2(\mathbb{R}^3)$.

We proceed inductively, supposing that $\varphi \in H^s(\mathbb{R}^3)$ for $s \geq 2$. Since $\varphi \in L^\infty(\mathbb{R}^3)$, g_1, g_3, $\tilde{g}_3$ and $\tilde{g}_5$ are in $L^2(\mathbb{R}^3)$. Since $|D\varphi| \in L^r(\mathbb{R}^3)$ for $2 \leq r \leq \frac{6}{3-2(s-1)}$, it follows that

$$|\varphi||D\varphi|^{q-1} \in L^r(\mathbb{R}^3) \quad \text{if} \quad \max\left\{1, \frac{2}{q}\right\} \leq r \leq \frac{6}{(q-1)(3-2(s-1))}$$

for $1 < q < \frac{6}{3-2(s-1)} + 1$, and

$$|D\varphi|^q \in L^r(\mathbb{R}^3) \quad \text{if} \quad \max\left\{1, \frac{2}{q}\right\} \leq r \leq \frac{6}{q(3-2(s-1))}$$

for $1 < q < \frac{6}{3-2(s-1)}$.

We use the first of these inclusions to consider g_2. If $s > 1 + \frac{3}{2} - \frac{3}{2(q_2-1)}$ then $\frac{6}{(p_2-1)(3-2(s-1))} \geq \frac{6}{(q_2-1)(3-2(s-1))} \geq 2$ and thus $g_2 \in L^2(\mathbb{R}^3)$. Otherwise $|\varphi||D\varphi|^{q_2-1} \in L^{r_2}(\mathbb{R}^3)$, where $r_2 = \frac{6}{(q_2-1)(3-2(s-1))} < 2$ and thus $|\varphi||D\varphi|^{q_2-1} \in H^{t_2}(\mathbb{R}^3)$, where $t_2 = \frac{3}{2} - \frac{3}{r_2} = \frac{1}{2}(3-(q_2-1)(3-2(s-1))) < 0$. Since $p_2 \leq q_2$, it follows that $|\varphi||D\varphi|^{p_2-1} \in H^{t_2}(\mathbb{R}^3)$ as well. Thus $g_2 \in H^{t_2(s)}(\mathbb{R}^3)$, where

$$t_2(s) = \begin{cases} (q_2-1)s + \frac{3}{2} - \frac{5(q_2-1)}{2} & s < 1 + \frac{3}{2} - \frac{3}{2(q_2-1)} \\ 0 & s \geq 1 + \frac{3}{2} - \frac{3}{2(q_2-1)} \end{cases}.$$

We now consider g_4. If $s \geq 1 + \frac{3}{2} - \frac{3}{2q_4}$, then $\frac{6}{p_4(3-2(s-1))} \geq \frac{6}{q_4(3-2(s-1))} \geq 2$, so $g_4 \in L^2(\mathbb{R}^3)$. Otherwise $|D\varphi|^{q_4} \in L^{r_4}(\mathbb{R}^3)$, where $r_4 = \frac{6}{q_4(3-2(s-1))} < 2$ and thus $|D\varphi|^{q_4} \in H^{t_4}(\mathbb{R}^n)$ with $t_4 = \frac{3}{2} - \frac{3}{r_4} = \frac{1}{2}(3-q_4(3-2(s-1))) < 0$. Since $1 < p_4 \leq q_4$, it follows that $|D\varphi|^{p_4} \in H^{t_4}(\mathbb{R}^3)$ as well, so $g_4 \in H^{t_4(s)}(\mathbb{R}^3)$, where

$$t_4(s) = \begin{cases} q_4 s + \frac{3}{2} - \frac{5q_4}{2} & s < 1 + \frac{3}{2} - \frac{3}{2q_4} \\ 0 & s \geq 1 + \frac{3}{2} - \frac{3}{2q_4} \end{cases}.$$

It then follows that $F_u(\varphi, D\varphi) \in H^{t(s)}(\mathbb{R}^3)$, where $t(s) = \min\{t_2(s), t_4(s)\}$.

By the same reasoning as above, it follows that $\tilde{g}_4 \in H^{\tilde{t}_4(s)}(\mathbb{R}^3)$, where

$$\tilde{t}_4(s) = \begin{cases} (q_4-1)s + \frac{3}{2} - \frac{5(q_4-1)}{2} & s < 1 + \frac{3}{2} - \frac{3}{2(q_4-1)} \\ 0 & s \geq 1 + \frac{3}{2} - \frac{3}{2(q_4-1)} \end{cases}$$

and $\tilde{g}_6 \in H^{\tilde{t}_6(s)}(\mathbb{R}^3)$, where

$$\tilde{t}_6(s) = \begin{cases} q_6 s + \frac{3}{2} - \frac{5q_6}{2} & s < 1 + \frac{3}{2} - \frac{3}{2q_6} \\ 0 & s \geq 1 + \frac{3}{2} - \frac{3}{2q_6} \end{cases},$$

and thus $(F_{v_i}(\varphi, D\varphi))_{x_i} \in H^{\tilde{t}(s)}(\mathbb{R}^3)$, where $\tilde{t}(s) = \min\{\tilde{t}_4(s), \tilde{t}_6(s)\} - 1$. Altogether this implies $\varphi \in H^{h(s)}(\mathbb{R}^3)$ whenever $\varphi \in H^s(\mathbb{R}^3)$, where $h(s) = 4 + \min\{t(s), \tilde{t}(s)\}$. Since $1 < q_2 < 7$, $1 < q_4 < 6$ and $1 < q_6 < 5$, it follows that the iteration $s_{k+1} = h(s_k)$ with $s_0 = 2$ converges to 3 in finitely many steps. Therefore $\varphi \in H^3(\mathbb{R}^3)$. It then follows that $\varphi \in W^{1,\infty}(\mathbb{R}^3)$, so $g \in L^2(\mathbb{R}^3)$.

Case 4: $n \geq 4$**.** We proceed inductively by supposing $\varphi \in H^s(\mathbb{R}^n)$ for some $s \geq 2$. First suppose $s \leq n/2$. Then since $|\varphi| \in L^r(\mathbb{R}^n)$ for all $2 \leq r \leq \frac{2n}{n-2s}$ when $s < n/2$ and for $2 \leq r < \infty$ when $s = n/2$, $|D\varphi| \in L^r(\mathbb{R}^n)$ for $2 \leq r \leq \frac{2n}{n-2(s-1)}$ and $|D^2\varphi| \in L^r(\mathbb{R}^n)$ for $2 \leq r \leq \frac{2n}{n-2(s-2)}$, it follows in the case $s < n/2$ that

$$|\varphi|^q \in L^r(\mathbb{R}^n) \text{ if } \max\left\{1, \frac{2}{q}\right\} \leq r \leq \frac{2n}{q(n-2s)}$$

for $1 < q < \frac{2n}{n-2s}$,

$$|\varphi||D\varphi|^{q-1} \in L^r(\mathbb{R}^n) \text{ if } \max\left\{1, \frac{2}{q}\right\} \leq r \leq \frac{2n}{q(n-2(s-1))-2}$$

for $1 < q < \frac{2(n+1)}{n-2(s-1)}$,

$$|\varphi|^{q-1}|D\varphi| \in L^r(\mathbb{R}^n) \text{ if } \max\left\{1, \frac{2}{q}\right\} \leq r \leq \frac{2n}{q(n-2s)+2}$$

for $1 < q < \frac{2(n-1)}{n-2s}$

$$|\varphi|^{q-1}|D^2\varphi| \in L^r(\mathbb{R}^n) \text{ if } \max\left\{1, \frac{2}{q}\right\} \leq r \leq \frac{2n}{q(n-2s)+4}$$

for $1 < q < \frac{2(n-2)}{n-2s}$, and while the case $s = n/2$,

$$|\varphi|^q \in L^r(\mathbb{R}^n) \text{ if } \max\left\{1, \frac{2}{q}\right\} \leq r < \infty$$

for $1 < q < \infty$. We first consider g_1. If $s = \frac{n}{2}$, then clearly $g_1 \in L^2(\mathbb{R}^n)$. If $\frac{n}{2} > s \geq \frac{n}{2} - \frac{n}{2q_1}$, then $\frac{2n}{p_1(n-2s)} \geq \frac{2n}{q_1(n-2s)} \geq 2$ and thus $g_1 \in L^2(\mathbb{R}^n)$. Otherwise $|\varphi|^{q_1} \in L^{r_1}(\mathbb{R}^n)$, where $r_1 = \frac{2n}{q_1(n-2s)}$, and thus $|\varphi|^{q_1} \in H^{t_1}(\mathbb{R}^n)$ with $t_1 = \frac{n}{2} - \frac{n}{r_1}$. Hence, since $p_1 \leq q_1$, $|\varphi|^{p_1} \in H^{t_1}(\mathbb{R}^n)$ as well, and thus $g_1 \in H^{t_1(s)}(\mathbb{R}^n)$, where

$$t_1(s) = \begin{cases} q_1 s + \frac{1}{2}n - \frac{1}{2}nq_1 & s < \frac{n}{2} - \frac{n}{2q_1} \\ 0 & s \geq \frac{n}{2} - \frac{n}{2q_1} \end{cases}.$$

We next consider g_2. If $\frac{n}{2} \geq s \geq 1 + \frac{n}{2} - \frac{n+2}{2q_2}$, then

$$\frac{2n}{p_2(n-2(s-1))-2} \geq \frac{2n}{q_2(n-2(s-1))-2} \geq 2$$

and thus $g_2 \in L^2(\mathbb{R}^n)$. Otherwise $|\varphi||D\varphi|^{q_2-1} \in L^{r_2}(\mathbb{R}^n)$, where

$$r_2 = \frac{2n}{q_2(n-2(s-1))-2},$$

so $|\varphi||D\varphi|^{q_2-1} \in H^{t_2}(\mathbb{R}^n)$, where $t_2 = \frac{n}{2} - \frac{n}{r_2}$. Since $p_2 \leq q_2$, $|\varphi||D\varphi|^{p_2-1} \in H^{t_2}(\mathbb{R}^n)$ and thus $g_2 \in H^{t_2(s)}(\mathbb{R}^n)$, where

$$t_2(s) = \begin{cases} q_2 s - \frac{1}{2}(q_2-1)(n+2) & s < 1 + \frac{n}{2} - \frac{n+2}{2q_2} \\ 0 & s \geq 1 + \frac{n}{2} - \frac{n+2}{2q_2}. \end{cases}$$

It follows similarly that $g_i \in H^{t_i(s)}(\mathbb{R}^n)$, where

$$t_3(s) = \begin{cases} q_3 s - 1 + \frac{n}{2} - \frac{nq_3}{2} & s < \frac{n}{2} - \frac{n-2}{2q_3} \\ 0 & s \geq \frac{n}{2} - \frac{n-2}{2q_3} \end{cases},$$

$$t_4(s) = \begin{cases} q_4 s + \frac{n}{2} - \frac{q_4(n+2)}{2} & s < 1 + \frac{n}{2} - \frac{n}{2q_4} \\ 0 & s \geq 1 + \frac{n}{2} - \frac{n}{2q_4} \end{cases},$$

$$t_5(s) = \begin{cases} q_5 s - 2 + \frac{n}{2} - \frac{q_5 n}{2} & s < \frac{n}{2} - \frac{n-4}{2q_5} \\ 0 & s \geq \frac{n}{2} - \frac{n-4}{2q_5} \end{cases},$$

$$t_6(s) = \begin{cases} q_6 s - 1 + \frac{n}{2} - \frac{q_6(n+2)}{2} & s < 1 + \frac{n}{2} - \frac{n-2}{2q_6} \\ 0 & s \geq 1 + \frac{n}{2} - \frac{n-2}{2q_6} \end{cases}.$$

Thus $g \in H^{t(s)}(\mathbb{R}^n)$ with $t(s) = \min\{t_i(s) : 1 \leq i \leq 6\}$ and $\varphi \in H^{h(s)}(\mathbb{R}^n)$, where $h(s) = t(s) + 4$, provided $2 \leq s \leq \frac{n}{2}$. It is easily verified using the assumptions on p_i and q_i that the sequence defined by $s_0 = 2$ and $s_{k+1} = h(s_k)$ is increasing and converges to 4 in finitely many steps. If $n \geq 8$, we have $s_k \leq \frac{n}{2}$ for all k and the iteration terminates with $\varphi \in H^4(\mathbb{R}^n)$. If $4 \le n \le 7$, the iteration terminates with either $\varphi \in H^4(\mathbb{R}^n)$, or $\varphi \in H^s(\mathbb{R}^n)$ for some $s < 4$ such that $s > \frac{n}{2}$. If $s > 1 + \frac{n}{2}$, then $\varphi \in W^{1,\infty}(\mathbb{R}^n)$ and thus $g \in L^2(\mathbb{R}^n)$. So it remains to consider the case $\varphi \in H^s(\mathbb{R}^n)$, where $\frac{n}{2} < s \le 1 + \frac{n}{2}$ and $4 \le n \le 7$. In this case $\varphi \in L^\infty(\mathbb{R}^n)$ and it follows that g_1, g_3 and g_5 are in $L^2(\mathbb{R}^n)$. Since $D\varphi \in L^r(\mathbb{R}^n)$ for $2 \le r < \infty$ if $s = 1 + \frac{n}{2}$ and for $2 \le r \le \frac{2n}{n-2(s-1)}$ if $s < 1 + \frac{n}{2}$, we have

$$|\varphi||D\varphi|^{q-1} \in L^r(\mathbb{R}^n) \text{ if } \max\left\{1, \frac{2}{q}\right\} \le r \le \frac{2n}{(q-1)(n-2(s-1))}$$

for $1 \le q < \frac{2n}{n-2(s-1)} + 1$ if $s < 1 + \frac{n}{2}$, and

$$|\varphi||D\varphi|^{q-1} \in L^r(\mathbb{R}^n) \text{ if } \max\left\{1, \frac{2}{q}\right\} \le r < \infty$$

for $1 \le q < \infty$ if $s = 1 + \frac{n}{2}$. It then follows that $g_2 \in H^{t_2(s)}(\mathbb{R}^n)$, where

$$t_2(s) = \begin{cases} (q_2 - 1)s + \frac{n}{2} - \frac{(q_2-1)(n+2)}{2} & s < 1 + \frac{n}{2} - \frac{n}{2(q_2-1)} \\ 0 & s \ge 1 + \frac{n}{2} - \frac{n}{2(q_2-1)} \end{cases}$$

and $g_4 \in H^{t_4(s)}(\mathbb{R}^n)$ and $g_6 \in H^{t_6(s)}(\mathbb{R}^n)$, where t_4 and t_6 are as above. Thus $\varphi \in H^{h(s)}(\mathbb{R}^n)$ provided $\varphi \in H^s(\mathbb{R}^n)$ for some $\frac{n}{2} < s \le 1 + \frac{n}{2}$, where $h(s) = \min\{t_2(s), t_4(s), t_6(s)\} + 4$. Again the assumptions on p_i and q_i imply that the sequence $s_0 = 2$, $s_{k+1} = h(s_k)$ converges to 4 in finitely many steps. Thus the iteration terminates with $\varphi \in H^4(\mathbb{R}^n)$ or $\varphi \in H^s(\mathbb{R}^n)$ for some $s > 1 + \frac{n}{2}$, which implies $g \in L^2(\mathbb{R}^n)$ and thus $\varphi \in H^4(\mathbb{R}^n)$. □

3. Stability

In this section we prove the main stability result. The proof makes use of arguments similar to those in [**12**], with modifications to account for the difference in variational characterization of the set of ground states. For the remainder of this section, we make explicit the dependence on the wave velocity c by writing $\mathcal{G}(c)$, $\mathscr{G}(c)$, $I(u;c)$, $S(u;c)$, $S_r(u;c)$, $P(u;c)$, $S_X(w;c)$ and $P_X(w;c)$.

We now define

$$d(c) = S_X(\Phi; c) = E(\Phi) + c \cdot Q(\Phi) \tag{3.1}$$

for $\Phi \in \mathscr{G}(c)$ and observe that

$$d(c) = S(\varphi; c) = m = \inf\{S(u;c) : u \in H^2(\mathbb{R}^n), u \ne 0, P(u;c) = 0\}$$

so d is independent of the choice of $\Phi \in \mathscr{G}(c)$ and therefore well-defined.

Lemma 3.1. *For each fixed unit vector $\nu \in \mathbb{R}^n$, $d_\nu(s) = d(s\nu)$ is continuous and strictly decreasing in s for $s \in [0, \sqrt{b + 2\sqrt{a}})$. At all but countable many points in $(0, \sqrt{b+2\sqrt{a}})$, d_ν is differentiable with $d_\nu'(s) = -s \int_{\mathbb{R}^n} |\nabla\varphi|^2 \,\mathrm{d}x$ for any $\varphi \in \mathcal{G}(s\nu)$.*

Proof. Define

$$\beta^+(s) = \sup\left\{\int_{\mathbb{R}^n} |\nu \cdot \nabla\varphi|^2 \,\mathrm{d}x : \varphi \in \mathcal{G}(s\nu)\right\},$$

$$\beta^-(s) = \inf\left\{\int_{\mathbb{R}^n} |\nu \cdot \nabla\varphi|^2 \,\mathrm{d}x : \varphi \in \mathcal{G}(s\nu)\right\}.$$

Since $d_\nu(s) = S_r(\varphi) \geq \left(\frac{r-1}{r+1}\right) I(\varphi; s\nu) \geq \left(\frac{(r-1)(2-s^2)}{2(r+1)}\right) \int_{\mathbb{R}^n} |\nu \cdot \nabla\varphi|^2 \, \mathrm{d}x$ for any $\varphi \in \mathcal{G}(s\nu)$, we have

$$0 \leq \beta^-(s) \leq \beta^+(s) \leq \frac{2(r+1)d_\nu(s)}{(r-1)(2-s^2)}$$

for $s \in (0, \sqrt{b+2\sqrt{a}})$. Suppose $0 \leq s_1 < s_2 < s_0 < \sqrt{b+2\sqrt{a}}$ and let $\varphi_j \in \mathcal{G}(c_j)$, where $c_j = s_j\nu$ for $j = 1, 2$. Since $P(\varphi_1; c_1) = 0$, we have

$$P(\varphi_1; c_2) = P(\varphi_1; c_1) + (s_1^2 - s_2^2) \int_{\mathbb{R}^n} |\nu \cdot \nabla\varphi_1|^2 \, \mathrm{d}x < 0.$$

Since $d(c_2)$ is the minimum of $S_r(u; c_2)$ subject to $P(u; c_2) \leq 0$, we have

$$\begin{aligned} d(c_2) \leq S_r(\varphi_1; c_2) &= S_r(\varphi_1; c_1) + \frac{1}{2}(s_1^2 - s_2^2)\frac{r-1}{r+1} \int_{\mathbb{R}^n} |\nu \cdot \nabla\varphi_1|^2 \, \mathrm{d}x \\ &= d(c_1) + \frac{1}{2}(s_1^2 - s_2^2)\frac{r-1}{r+1} \int_{\mathbb{R}^n} |\nu \cdot \nabla\varphi_1|^2 \, \mathrm{d}x < d(c_1) \end{aligned}$$

so $d_\nu(s)$ is strictly decreasing in s and hence differentiable at all but countably many points in $(0, \sqrt{b+2\sqrt{a}})$.

For $0 < \alpha < 1$ we have by Lemma 2.4 that

$$\begin{aligned} P(\alpha\varphi_1; c_2) &= 2\alpha^2 I(\varphi_1; c_2) - N(\alpha\varphi_1) \\ &\geq 2\alpha^2 I(\varphi_1; c_2) - \alpha^{r+1} N(\varphi_1) \\ &= 2\alpha^2 I(\varphi_1; c_2) - \alpha^{r+1} 2I(\varphi_1; c_1) \end{aligned}$$

so $P(\alpha\varphi_1; c_2) > 0$ for $\alpha < \left(\frac{I(\varphi_1;c_2)}{I(\varphi_1;c_1)}\right)^{\frac{1}{r-1}}$. Thus there exists $\left(\frac{I(\varphi_1;c_2)}{I(\varphi_1;c_1)}\right)^{\frac{1}{r-1}} < \alpha_1 < 1$ such that $P(\alpha\varphi_1; c_2) = 0$, and

$$\begin{aligned} d(c_2) \leq S(\alpha_1\varphi_1; c_2) &= \alpha_1^2 I(\varphi_1; c_2) - K(\alpha_1\varphi_1) \\ &= \alpha_1^2 I(\varphi_1; c_1) - \alpha_1^2 \frac{1}{2}(s_2^2 - s_1^2) \int_{\mathbb{R}^n} |\nu \cdot \nabla\varphi_1|^2 \, \mathrm{d}x - K(\alpha_1\varphi_1) \\ &= S(\alpha_1\varphi_1; c_1) - \alpha_1^2 \frac{1}{2}(s_2^2 - s_1^2) \int_{\mathbb{R}^n} |\nu \cdot \nabla\varphi_1|^2 \, \mathrm{d}x \end{aligned}$$

By Lemma 2.7, since $P(\varphi_1; c_1) = 0$, $S(\alpha\varphi_1; c_1)$ attains its maximum at $\alpha = 1$, and thus $S(\alpha_1\varphi_1; c_1) \leq S(\varphi_1; c_1) = d(c_1)$, and we have

$$d(c_2) \leq d(c_1) - \alpha_1^2 \frac{1}{2}(s_2^2 - s_1^2) \int_{\mathbb{R}^n} |\nu \cdot \nabla\varphi_1|^2 \, \mathrm{d}x.$$

Since $I(u; s\nu) \geq \frac{1}{2}(b + 2\sqrt{a} - s^2) \int_{\mathbb{R}^n} |\nu \cdot \nabla u|^2 \, \mathrm{d}x$ for all $u \in H^2(\mathbb{R}^n)$, we have

$$\begin{aligned} 1 \geq \frac{I(\varphi_1; c_2)}{I(\varphi_1; c_1)} &= 1 - \frac{\frac{1}{2}(s_2^2 - s_1^2) \int_{\mathbb{R}^n} |\nu \cdot \nabla\varphi_1|^2 \, \mathrm{d}x}{I(\varphi_1; c_1)} \\ &\geq 1 - \frac{s_2^2 - s_1^2}{b + 2\sqrt{a} - s_2^1} \geq 1 + \frac{s_2^2 - s_1^2}{b + 2\sqrt{a} - s_0^2} \end{aligned}$$

for $s_1 < s_2 < s_0 < \sqrt{b+2\sqrt{a}}$. Thus $\alpha_1 \geq \left(1 - \frac{s_2^2 - s_1^2}{2 - s_0^2}\right)^{\frac{1}{r-1}}$, and

$$d(c_2) \leq d(c_1) - \frac{1}{2}(s_2^2 - s_1^2) \left(1 - \frac{s_2^2 - s_1^2}{2 - s_0^2}\right)^{\frac{2}{r-1}} \int_{\mathbb{R}^n} |\nu \cdot \nabla\varphi_1|^2 \, \mathrm{d}x.$$

Since $\varphi_2 \in \mathcal{G}(c_2)$ is arbitrary, this implies

$$d_\nu(s_2) \leq d_\nu(s_1) - \frac{1}{2}(s_2^2 - s_1^2)\left(1 - \frac{s_2^2 - s_1^2}{2 - s_0^2}\right)^{\frac{2}{r-1}} \beta^+(s_1).$$

It follows similarly that

$$d_\nu(s_1) \leq d_\nu(s_2) + \frac{1}{2}(s_2^2 - s_1^2)\left(1 + \frac{s_2^2 - s_1^2}{2 - s_0^2}\right)^{\frac{2}{r-1}} \beta^-(s_2);$$

and thus

$$\begin{aligned} -\frac{1}{2}(s_2 + s_1)\left(1 + \frac{s_2^2 - s_1^2}{2 - s_0^2}\right)^{\frac{2}{r-1}} \beta^-(s_2) \leq \frac{d_\nu(s_2) - d_\nu(s_1)}{s_2 - s_1} \\ \leq -\frac{1}{2}(s_1 + s_2)\left(1 - \frac{s_2^2 - s_1^2}{2 - s_0^2}\right)^{\frac{2}{r-1}} \beta^+(s_1) \end{aligned}$$

for $s_1 < s_2 < s_0$. This proves d_ν is locally Lipschitz and hence continuous on $(0, \sqrt{2})$. By Theorem 2.10 and the same reasoning as in the proof of Lemma 4.2 in [**12**], $\limsup_{s_2 \to s_1} \beta^-(s_2) \leq \beta^+(s_1)$ and $\liminf_{s_1 \to s_2} \beta^+(s_1) \geq \beta^-(s_2)$ so

$$\lim_{s_2 \to s_1^+} \frac{d_\nu(s_2) - d_\nu(s_1)}{s_2 - s_1} = -s_1 \beta^+(s_1),$$
$$\lim_{s_1 \to s_2^-} \frac{d_\nu(s_2) - d_\nu(s_1)}{s_2 - s_1} = -s_2 \beta^-(s_2).$$

Thus $d_\nu'(s^+) = -s\beta^+(s)$ and $d_\nu'(s^-) = -s\beta^-(s)$, so d_ν is differentiable at s if and only if $\beta^+(s) = \beta^-(s)$, and in this case $d_\nu'(s) = -s\int_{\mathbb{R}^n} |\nu \cdot \nabla\varphi|^2 \, dx$ for any $\varphi \in \mathcal{G}(s\nu)$. ☐

We recall that the functional $S_1 = S - \frac{1}{2}P = \frac{1}{2}N - K$ is independent of c, and $d(c) = S_1(\varphi)$ for any $\varphi \in \mathcal{G}(c)$. We use this to associate a wave speed to functions in a neighborhood of $\mathcal{G}(c)$.

LEMMA 3.2. *Fix a unit vector $\nu \in \mathbb{R}^n$ and let $c = a\nu$ for some $a \in (0, \sqrt{2})$. There exists $\epsilon > 0$ and a continuous map $c_\nu : V_\epsilon(\mathcal{G}(c)) \to \mathbb{R}^n$ such that $d(c_\nu(u)) = S_1(u)$ for all $u \in V_\epsilon(\mathcal{G}(c))$.*

PROOF. By Lemma 3.1 $d_\nu(s) = d(s\nu)$ has a continuous inverse d_ν^{-1} defined on the range of d_ν. Since d_ν is strictly decreasing and $d(c) = S_1(\varphi)$ for any $\varphi \in \mathcal{G}(c)$, it follows that there is some $\epsilon > 0$ such that $S_1(u)$ is in the range of d_ν for all $u \in V_\epsilon(\mathcal{G}(c))$. We may therefore define

$$c_\nu(u) = d_\nu^{-1}(S_1(u))\nu \tag{3.2}$$

for $u \in V_\epsilon(\mathcal{G}(c))$. Then $d(c_\nu(u)) = d_\nu(d_\nu^{-1}(S_1(u))) = S_1(u)$, and the continuity of S_1 and d_ν^{-1} imply the continuity of c. ☐

REMARK 3.3. Equation (3.2) agrees with, and thus generalizes, equation (5.3) in [**12**] in the case of a homogeneous nonlinearity.

We will denote $s_\nu(u) = d_\nu^{-1}(S_1(u))$, so $c_\nu(u) = s_\nu(u)\nu$.

LEMMA 3.4. *Let $\nu \in \mathbb{R}^n$ be a unit vector, and suppose u is in the domain of c_ν. Then $P(u; c_\nu(u)) \geq 0$ and $S(u; c_\nu(u)) \geq d(c_\nu(u))$.*

Proof. By definition we have

$$d(c_\nu(u)) = S_1(u) = S(u;c_\nu(u)) - \frac{1}{2}P(u;c_\nu(u)). \tag{3.3}$$

Since $d(c_\nu(u))$ is the minimum of $S_r(v;c_\nu(u))$ over all v such that $P(v;c_\nu(u)) \leq 0$ it follows that

$$d(c_\nu(u)) \leq S_r(u;c_\nu(u)) = S(u;c_\nu(u)) - \frac{1}{r+1}P(u;c_\nu(u)). \tag{3.4}$$

Combining these proves that $P(u;c_\nu(u)) \geq 0$, and thus $S(u;c_\nu(u)) \geq d(c_\nu(u))$. □

Lemma 3.5. *Fix a unit vector $\nu \in \mathbb{R}^n$, let $c = |c|\nu$ for some $|c| < \sqrt{2}$ and suppose $d_\nu''(|c|) > 0$. There exists $\epsilon > 0$ such that for any $\Phi \in \mathscr{G}(c)$ and any $(u,v) \in V_\epsilon(\mathscr{G}(c))$ we have*

$$E(u,v) - E(\Phi) + s_\nu(u)\nu \cdot (Q(u,v) - Q(\Phi)) \geq \frac{1}{4}d_\nu''(|c|)(s_\nu(u) - |c|)^2.$$

Proof. By Lemma 3.1, $d_\nu'(|c|) = \nu \cdot Q(\Phi)$ and thus for some $\epsilon_0 > 0$

$$\begin{aligned} d_\nu(s) &\geq d_\nu(|c|) + \nu \cdot Q(\Phi)(s - |c|) + \frac{1}{4}d_\nu''(|c|)(s-|c|)^2 \\ &= E(\Phi) + \nu s \cdot Q(\Phi) + \frac{1}{4}d_\nu''(|c|)(s-|c|)^2 \end{aligned}$$

for $|s - |c|| < \epsilon_0$. Let $(u,v) \in V_\epsilon(\mathscr{G}(c))$. Then $u \in V_\epsilon(\mathcal{G}(c))$, so u is in the domain of c_ν and $|s_\nu(s) - |c|| < \epsilon_0$ for ϵ sufficiently small and thus

$$d_\nu(s_\nu(u)) \geq E(\Phi) + \nu s_\nu(u) \cdot Q(\Phi) + \frac{1}{4}d_\nu''(|c|)(s_\nu(u) - |c|)^2.$$

By Lemma 3.4 and (2.3)

$$d_\nu(s_\nu(u)) = d(c_\nu(u)) \leq S(u;c\nu(u)) \leq E(u,v) + c_\nu(u) \cdot Q(u,v)$$

so combining the two inequalities proves the lemma. □

We are now ready to prove the main stability result.

Theorem 3.6. *Let $c \in \mathbb{R}^n$ satisfy $0 < |c| < \sqrt{2}$ and let $\nu = c/|c|$. If $d_\nu''(|c|) > 0$, then $\mathscr{G}(c)$ is stable.*

Proof. If $\mathscr{G}(c)$ is unstable, then there exist $\epsilon_1 > 0$, $g_k \in V_{1/k}(\mathscr{G}(c))$ and $t_k > 0$ such that $\|w_k(t_k) - \Phi\|_X \geq \epsilon_1$ for every $\Phi \in \mathscr{G}(c)$, where $w_k(t) = (u_k(t), v_k(t))$ is the solution of (1.2) with $w_k(0) = g_k$. We may also assume that $\epsilon_1 < \epsilon$ and therefore by continuity of $w_k(t)$ that $w_k(t_k) \in V_\epsilon(\mathscr{G}(c))$ for the ϵ in Lemma 3.5. Thus there exist $\Phi_k \in \mathscr{G}(c)$ such that $\|g_k - \Phi_k\| < \frac{1}{k}$ for each k and therefore

$$\|E(\Phi_k) - E(w_k(t_k))\| = \|E(\Phi_k) - Q(g_k)\| \leq \frac{C}{k}$$

and

$$\|Q(\Phi_k) - Q(w_k(t_k))\| = \|Q(\Phi_k) - Q(g_k)\| \leq \frac{C}{k}$$

for some constant $C > 0$. By Lemma 3.5 we then have

$$E(w_k(t_k)) - E(\Phi_k) + s_\nu(u_k(t_k))\nu \cdot (Q(w_k(t_k)) - Q(\Phi_k)) \geq \frac{1}{4}d_\nu''(|c|)(s_\nu(u_k(t_k)) - |c|)^2.$$

for all k. Since $s_\nu(u_k(t_k))$ is bounded, it then follows that $\lim_k s_\nu(u_k(t_k)) = |c|$. Thus $\lim_k c_\nu(u_k(t_k)) = c$, so

$$\lim_{k\to\infty} S_1(u_k(t_k)) = \lim_{k\to\infty} d(c_\nu(u_k(t_k)) = d(c).$$

Since

$$\lim_{k\to\infty} E(w_k(t_k)) + c \cdot Q(w_k(t_k)) = \lim_{k\to\infty} E(\Phi_k) + c \cdot Q(\Phi_k) = d(c),$$

we have by (2.3) that

$$\limsup_{k\to\infty} S(u_k(t_k); c) \le \lim_{k\to\infty} E(w_k(t_k)) + c \cdot Q(w_k(t_k)) = d(c),$$

from which we obtain

$$\limsup_{k\to\infty} P(u_k(t_k); c) = \limsup_{k\to\infty} 2S(u_k(t_k), c) - 2S_1(u_k(t_k)) \le 0.$$

By Lemma 3.4, $P(u_k(t_k); c_\nu(u_k(t_k))) \ge 0$ and

$$S(u_k(t_k); c_\nu(u_k(t_k))) \ge d(c_\nu(u_k(t_k)))$$

for all k, so since

$$P(u_k(t_k); c) = P(u_k(t_k); c_\nu(u_k(t_k))) + (s_\nu(u_k(t_k))^2 - |c|^2) \int_{\mathbb{R}^n} |\nu \cdot \nabla u_k(t_k)|^2 \,\mathrm{d}x,$$

$$S(u_k(t_k); c) = S(u_k(t_k); c_\nu(u_k(t_k))) + \frac{1}{2}(s_\nu(u_k(t_k))^2 - |c|^2) \int_{\mathbb{R}^n} |\nu \cdot \nabla u_k(t_k)|^2 \,\mathrm{d}x,$$

we have $\liminf_{k\to\infty} P(u_k(t_k); c) \ge 0$ and $\liminf_{k\to\infty} S(u_k(t_k); c) \ge d(c)$ and therefore

$$\lim_{k\to\infty} P(u_k(t_k); c) = 0$$

and

$$\lim_{k\to\infty} S(u_k(t_k); c) = d(c).$$

Thus $\{u_k(t_k)\}$ is a minimizing sequence, so by Theorem 2.10, there exists a subsequence (renamed $\{u_k(t_k)\}$) and a sequence $\{\psi_k\}$ in $\mathcal{G}(c)$ such that $\|u_k(t_k) - \psi_k\|_{H^2} \to 0$. Furthermore, by (2.3) we have

$$\begin{aligned}\lim_{k\to\infty} \frac{1}{2} \int_{\mathbb{R}^n} |v_k(t_k) + c \cdot \nabla u_k(t_k)|^2 \,\mathrm{d}x &= \lim_{k\to\infty} (E(w_k(t_k)) + c \cdot Q(w_k(t_k)) \\ &\qquad - S(u_k(t_k); c)) \\ &= d(c) - d(c) = 0;\end{aligned}$$

so

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |v_k(t_k) + c \cdot \nabla \psi_k|^2 \,\mathrm{d}x = 0$$

and we have

$$\lim_{k\to\infty} \|w_k(t_k) - \Psi_k\|_X = 0,$$

where $\Psi_k = (\psi_k, -c \cdot \nabla \psi_k) \in \mathscr{G}(c)$. This contradicts the fact that $\|w_k(t_k) - \Phi\|_X \ge \epsilon_1 > 0$ for all k and all $\Phi \in \mathscr{G}(c)$. Hence $\mathscr{G}(c)$ is stable. $\square$

4. Instability

In this section we prove the main instability result. It follows from a modification of the methods of Angulo-Pava [**1**], which are based on those of Grillakis, Shatah and Strauss [**9**]. Fix a unit vector $\nu \in \mathbb{R}^n$ and define

$$Q_\nu(w) = \nu \cdot Q(w) = \int_{\mathbb{R}^n} v(\nu \cdot \nabla u)\,\mathrm{d}x$$

for $w = (u, v) \in X$. Notice that $JQ'_\nu(w) = \nu \cdot \nabla w$. Given a ground state $\Phi \in \mathscr{G}(c)$ with $c/|c| = \nu$, we define the orbit of Φ by

$$\mathcal{O}(\Phi) = \{T(r)\Phi : r \in \mathbb{R}\},$$

where $T(r)w(x) = w(x + r\nu)$.

By Theorem 2.13, $\varphi \in H^4(\mathbb{R}^n)$, so $\Phi \in H^4(\mathbb{R}^n) \times H^3(\mathbb{R}^n)$ and thus $Q'_\nu(\Phi) = -J(\nu \cdot \nabla\Phi) \in H^2(\mathbb{R}^n) \times H^3(\mathbb{R}^n) \subset X$, and $(\nu \cdot \nabla)^2\Phi \subset H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \subset X$. The main result of this section is the following.

Theorem 4.1. *Let $c = s\nu$ for some $s \in (0, \sqrt{2})$ and suppose $\Phi \in \mathscr{G}(c)$. If there exist $\Psi \in X$ such that $\langle Q'_\nu(\Phi), \Psi\rangle = 0$ and $\langle S''_X(\Phi)\Psi, \Psi\rangle < 0$, then $\mathcal{O}(\Phi)$ is unstable.*

Lemma 4.2. *Suppose $\langle P'_X(\Phi), w\rangle = 0$. Then $\langle S''_X(\Phi)w, w\rangle \geq 0$.*

Proof. Set $h(t, z) = P_X(\Phi + tw + zL^{-1}P'_X(\Phi))$, where $L(u, v) = (\Delta^2 u + u, v)$ is the natural isomorphism from X to X^*. Then $h(0, 0) = 0$ and $\frac{\partial h}{\partial z}(0, 0) = \left\langle P'_X(\Phi), L^{-1}P'_X(\Phi)\right\rangle$. As shown in the proof of Lemma 2.11, $\langle P'_X(\Phi), \Phi\rangle < 0$, so $P'_X(\Phi) \neq 0$ and thus $\frac{\partial h}{\partial z}(0, 0) \neq 0$. The implicit function theorem therefore implies there exists a C^1 function $z(t)$ defined in a neighborhood of $t = 0$ such that $z(0) = 0$ and $P_X(\Phi + tw + z(t)L^{-1}P'_X(\Phi)) = 0$. The curve $\gamma(t) = \Phi + tw + z(t)L^{-1}P'_X(\Phi)$ then satisfies $\gamma(0) = \Phi$ and $\gamma'(0) = w + z'(0)L^{-1}P'_X(\Phi)$. But $P_X(\gamma(t)) = 0$ implies $\langle P'_X(\gamma(t)), \gamma'(t)\rangle = 0$, so when $t = 0$ we have $0 = \left\langle P'_X(\Phi), w + z'(0)L^{-1}P'_X(\Phi)\right\rangle = z'(0)\left\langle P'_X(\Phi), L^{-1}P'_X(\Phi)\right\rangle$, and thus $z'(0) = 0$, so $\gamma'(0) = w$. Since Φ minimizes S_X subject to the constraint $P_X = 0$, the function $g(t) = S_X(\gamma(t))$ has a local minimum at $t = 0$, and thus $g''(0) \geq 0$. The lemma follows since $g''(0) = \langle S''_X(\Phi)w, w\rangle$. □

Lemma 4.3. *There exists a C^1 map $\rho : V_\epsilon(\mathcal{O}(\Phi)) \to \mathbb{R}$ such that*

$$\langle JQ'_\nu(\Phi), T(\rho(w))w\rangle = 0$$

for every $w \in V_\epsilon(\mathcal{O}(\Phi))$, and

$$\rho'(w) = \frac{T(-\rho(w))JQ'_\nu(\Phi)}{\langle T(-\rho(w))(\nu \cdot \nabla)^2\Phi, w\rangle}.$$

Proof. First note that since $\Phi \in H^4 \times H^3$, $JQ'_\nu(\Phi) = \nu \cdot \nabla\Phi \in H^3 \times H^2 \subset X^*$. Thus the map $g : \mathbb{R} \times X \to \mathbb{R}$ defined by $g(\rho, w) = \langle \nu \cdot \nabla\Phi, T(\rho)w\rangle = \langle JQ'_\nu(\Phi), T(\rho)w\rangle$ is C^1 with

$$g_\rho(\rho, w) = -\left\langle T(-\rho)((\nu \cdot \nabla)^2\Phi), w\right\rangle \qquad g_w(\rho, w) = T(-\rho)(\nu \cdot \nabla\Phi).$$

Since $g(0, \Phi) = \langle \nu \cdot \nabla\Phi, \Phi\rangle = 0$ and

$$g_\rho(0, \Phi) = \langle JQ'_\nu(\Phi), \nu \cdot \nabla\Phi\rangle = \int_{\mathbb{R}^n} |\nu \cdot \nabla\varphi|^2 + |c|^2|(\nu \cdot \nabla)^2\varphi|^2\,\mathrm{d}x > 0,$$

the implicit function theorem implies there exists a neighborhood $V_\epsilon(\Phi)$ and a unique C^1 map $\rho : U \to \mathbb{R}$ such that $g(\rho(w), w) = 0$ for all $w \in V_\epsilon(\Phi)$. Since

$g(\rho, T(r)w) = g(\rho+r, w)$ for any $w \in X$ and ρ and r in $\mathbb{R}$, it follows that if $w_1, w_2 \in V_\epsilon(\Phi)$ satisfy $w_2 = T(r)w_1$ for some r, then $0 = g(\rho(w_2), w_2) = g(\rho(T(r)w_1)+r, w_1)$, so by uniqueness $\rho(T(r)w_1) + r = \rho(w_1)$. For any $w \in V_\epsilon(\mathcal{O}(\Phi))$ there exists $r \in \mathbb{R}$ such that $T_\nu(r) \in V_\epsilon(\Phi)$, and we may define $\rho(w) = \rho(T_\nu(r)) + r$. This extends ρ to $V_\epsilon(\mathcal{O}(\Phi))$ in a well-defined manner, such that $g(\rho(w), w) = 0$ for all $w \in V_\epsilon(\mathcal{O}(\Phi))$. This may be rewritten $\langle T(-\rho(w))(\nu \cdot \nabla\Phi), w\rangle = 0$, so differentiating with respect to w gives

$$\langle T(-\rho(w))(\nu \cdot \nabla\Phi), z\rangle - \langle \rho'(w), z\rangle \left\langle T(-\rho(w))(\nu \cdot \nabla)^2\Phi, w\right\rangle$$

and thus

$$\langle \rho'(w), z\rangle = \frac{\langle T(-\rho(w))(\nu \cdot \nabla\Phi), z\rangle}{\langle T(-\rho(w))(\nu \cdot \nabla)^2\Phi, w\rangle} = \frac{\langle T(-\rho(w))JQ'_\nu(\Phi), z\rangle}{\langle T(-\rho(w))(\nu \cdot \nabla)^2\Phi, w\rangle}$$

for $z \in X$. □

LEMMA 4.4. *Define*

$$B_\Psi(w) = T(-\rho(w))\Psi + \left(\frac{\langle Q'_\nu(w), T(-\rho(w))\Psi\rangle}{\langle T(-\rho(w))(\nu \cdot \nabla)^2\Phi, w\rangle}\right) T(-\rho(w))Q'_\nu(\Phi).$$

Then $B : X \to X$ is C^1 and satisfies

$$\begin{aligned}\langle JB_\Psi(w), z\rangle &= \langle JT(-\rho(w))\Psi, z\rangle + \left(\frac{\langle Q'_\nu(w), T(-\rho(w))\Psi\rangle}{\langle T(-\rho(w))(\nu \cdot \nabla)^2\Phi, w\rangle}\right)\langle JT(-\rho(w))Q'_\nu(\Phi), z\rangle \\ &= \langle JT(-\rho(w))\Psi, z\rangle + \langle \rho'(w), z\rangle \langle Q'_\nu(w), T(-\rho(w))\Psi\rangle\end{aligned}$$

and

$$\langle Q'_\nu(w), B_\Psi(w)\rangle = 0$$

for any $w \in X$.

The following lemma is a direct consequence of the definition of B_Ψ (see [**6**]).

LEMMA 4.5. *Let $U(s, w_0)$ denote the solution of the equation $\frac{dw}{ds} = B_\Psi(w)$ with initial data $w(0) = w_0 \in X$. Then we have*

(i) *U is C^1 for $|s| < s_0(w_0)$ for any $w_0 \in V_\epsilon(\mathcal{O}(\Phi))$,*
(ii) *$U(s, T(\alpha)w_0) = T(\alpha)U(s, w_0)$ for all $w_0 \in V_\epsilon(\mathcal{O}(\Phi))$, $r \in \mathbb{R}$,*
(iii) *$Q(U(s, w_0)) = Q(w_0)$,*
(iv) $\dfrac{dU}{ds}(0, \Phi) = \Psi$.

LEMMA 4.6. *Define $Z(w) = \langle S'_X(w), B_\Psi(w)\rangle$. There exist $\epsilon > 0$ and $\sigma > 0$ such that*

(i) $S_X(U(s, w_0)) \leq S_X(w_0) + Z(w_0)s$ for all $w_0 \in V_\epsilon(\mathcal{O}(\Phi))$ and all $s \in (-\sigma, \sigma)$,
(ii) $S_X(\Phi) \leq S_X(U(\delta, \Phi)) - Z(U(\delta, \Phi))\delta$ for all $\delta \in (-\sigma, \sigma)$,
(iii) $S_X(U(\delta, \Phi)) < S_X(\Phi)$ for all nonzero $\delta \in (-\sigma, \sigma)$,
(iv) $Z(U(\delta, \Phi)) < 0$ for all $\delta \in (0, \sigma)$,
(v) for all $w_0 \in V_\epsilon(\mathcal{O}(\Phi))$, there exists $s = s(w_0) \in (-\sigma, \sigma)$ such that $S_X(\Phi) \leq S_X(w_0) + Z(w_0)s$.

PROOF. Taylor's theorem implies

$$S_X(U(s, w_0)) = S_X(w_0) + Z(w_0)s + \frac{1}{2}R(U(\eta s, w_0))$$

for some $\eta \in (0, 1)$, where $R(w) = \langle S''_X(w)B(w), B(w)\rangle + \langle S'_X(w), B'(w)(B(w))\rangle$. Part (i) then follows since $R(\Phi) = \langle S''_X(\Phi)\Psi, \Psi\rangle < 0$. Part (ii) follows by applying part (i) with $w_0 = U(\delta, \Phi)$ and setting $s = -\delta$. Part (iii) again follows from Taylor's theorem and the fact that $Z(\Phi) = 0$ and $\langle S''_X(\Phi)\Psi, \Psi\rangle < 0$. Combining parts (ii) and (iii) proves part (iv). Since Φ minimizes S_X subject to the constraint $P_X = 0$, part (v) will follow from part (i) once it is shown that for each w_0 there exists s such that $P_X(U(s, w_0)) = 0$. So define $g(s, w) = P_X(U(s, w_0))$ and note that $g(0, \Phi) = 0$. The desired result then follows since $\frac{\partial g}{\partial s}(0, \Phi) = \langle P'_X(\Phi), \Psi\rangle$ is nonzero by Lemma 4.2 and the assumption that $\langle S''_X(\Phi)\Psi, \Psi\rangle < 0$. □

Next define

$$A(w) = \langle T(-\rho(w))J\Psi, w\rangle \, . \tag{4.1}$$

LEMMA 4.7. *A is a C^1 map from $V_\epsilon(\mathcal{O}(\Phi))$ to $\mathbb{R}$, and $A'(w) = JB(w)$.*

PROOF. The fact that A is C^1 follows from Lemma 4.3, and by direct calculation we have

$$\begin{aligned} A'(w) &= T(-\rho(w))J\Psi - \rho'(w)\,\langle T(-\rho(w))\nu\cdot\nabla J\Psi, w\rangle \\ &= T(-\rho(w))J\Psi + \rho'(w)\,\langle T(-\rho(w))\Psi, Q'_\nu(w)\rangle \\ &= JB(w). \end{aligned}$$

□

Proof of Theorem 4.1. Let $\sigma > 0$ be as in Lemma 4.6, and choose any sequence $\delta_j \in (0, \sigma)$ such that $\delta_j \to 0$ as $j \to \infty$, and define $w_j = U(\delta_j, \Phi)$. By continuity of U it follows that $w_j \to \Phi$ in X. To establish instability of $\mathcal{O}(\Phi)$ it suffices to show that for every j the solution $w_j(t)$ of (1.2) exits $V_\epsilon(\mathcal{O}(\Phi))$ in finite time. Denote

$$T_j = \sup\{t : w_j(t) \in V_\epsilon(\mathcal{O}(\Phi))\}.$$

By Lemma 4.6, we have $S_X(w_j) < S_X(\Phi)$ and $Z(w_j) < 0$ for all j. Since S_X is a conserved quantity of (1.2), we then have $S_X(w_j(t)) < S_X(\Phi)$ for all $t \in [0, T_j]$. Furthermore, we must also have $Z(w_j(t)) < 0$ for all $t \in [0, T_j]$, for otherwise we would have $Z(w_j(t)) = 0$ for some t, and by part (v) of Lemma (4.6) we would have $S_X(\Phi) \le S_X(w_j(t))$, a contradiction. Thus the region

$$\mathcal{D} = \{w \in X : S_X(w) < S_X(\Phi), Z(w) < 0\}$$

is invariant under the flow of (1.2). By Lemma 4.6 it follows that for each $t \in [0, T_j]$ there exists $s = s(w_j(t)) \in (-\sigma, \sigma)$ such that $S_X(\Phi) \le S_X(w_j(t)) + Z(w_j(t))s$. Since $w_j(t) \in \mathcal{D}$, it follows that $s \in (-\sigma, 0)$, and since $S_X(w_j(t)) = S_X(w_j)$ for all $t \in [0, T_j]$, we have

$$-Z(w_j(t)) \ge \frac{S_X(\Phi) - S_X(w_j)}{\sigma} \equiv \eta_j > 0 \tag{4.2}$$

for all $t \in [0, T_j]$. For all $t \in [0, T_j]$ we have $w_j(t) \in V_\epsilon(\mathcal{O}(\Phi))$ and thus $A(w_j(t))$ is bounded on $[0, T_j]$. But by Lemma 4.7 and Lemma 4.4 we have

$$\begin{aligned}\frac{d}{dt}A(w_j(t)) &= \left\langle A'(w_j(t)), w_j'(t)\right\rangle \\ &= \langle JB_\Psi(w_j(t)), JE'(w_j(t))\rangle \\ &= \langle B_\Psi(w_j(t)), E'(w_j(t))\rangle \\ &= \langle B_\Psi(w_j(t)), S_X'(w_j(t)) - c\cdot Q'(w_j(t))\rangle \\ &= \langle B_\Psi(w_j(t)), S_X'(w_j(t))\rangle \\ &= Z(w_j(t)).\end{aligned}$$

By (4.2) it then follows that $A(w_j(t)) \to -\infty$ as $t \to \infty$ and therefore T_j must be finite, which implies $w_j(t)$ exits $V_\epsilon(\mathcal{O}(\Phi))$ in finite time. □

We will make use of the following formula frequently.

LEMMA 4.8. *Let $\Phi \in \mathscr{G}(c)$. Then*

$$\langle S_X''(\Phi)w, w\rangle = 2I(u) - \langle K''(\varphi)u, u\rangle + 2q(u, v) \tag{4.3}$$

for any $w = (u, v) \in X$.

PROOF. This follows from the fact that $S_X(u, v) = I(u) - K(u) + q(u, v)$ and both I and q are quadratic. □

Next note that, since $N(u) = \langle K'(u), u\rangle$, we have $\langle N'(u), u\rangle = \langle K''(u)u, u\rangle + N(u)$ for any $u \in H^2(\mathbb{R}^n)$, and thus

$$\begin{aligned}\langle S_X''(\Phi)\Phi, \Phi\rangle &= 2I(\varphi) + N(\varphi) - \langle N'(\varphi), \varphi\rangle \\ &= 2N(\varphi) - \langle N'(\varphi), \varphi\rangle < (1 - r)N(\varphi) < 0\end{aligned} \tag{4.4}$$

for any $\Phi = (\varphi, -c\cdot\nabla\varphi) \in \mathscr{G}(c)$, since $q(\Phi) = 0$ and $2I(\varphi) = N(\varphi)$.

COROLLARY 4.1. Let $c = s\nu$ with $s^2 < 2$, and $\Phi \in \mathscr{G}(c)$. Suppose there exists a C^2 map $\Phi_\nu : (s - \epsilon, s + \epsilon) \to X$ such that $\Phi_\nu(r) \in \mathscr{G}(r\nu)$ for each r and $\Phi_\nu(s) = \Phi$. If $d_\nu''(s) < 0$, then $\mathcal{O}(\Phi)$ is unstable.

PROOF. Define $\Psi = \Phi - \frac{2d_\nu'(s)}{d_\nu''(s)}\Phi_\nu'(s)$. By assumption $\psi \in X$ and $\Phi_\nu(r) \in \mathscr{G}(r\nu)$ and thus $d_\nu(r) = E(\Phi_\nu(r)) + rQ_\nu(\Phi_\nu(r))$ for $r \in (s - \epsilon, s + \epsilon)$. It then follows that

$$d_\nu'(r) = \langle E'(\Phi_\nu(r)) + rQ\nu'(\Phi_\nu(r)), \Phi_\nu'(r)\rangle + Q_\nu(\Phi_\nu(r)) = Q_\nu(\Phi_\nu(r))$$

and thus

$$d_\nu''(r) = \langle Q_\nu'(\Phi_\nu(r)), \Phi_\nu'(r)\rangle\,.$$

It then follows that

$$\langle Q_\nu'(\Phi), \Psi\rangle = \langle Q_\nu'(\Phi), \Phi\rangle - \frac{2d_\nu'(s)}{d_\nu''(s)}\langle Q_\nu'(\Phi_\nu(s)), \Phi_\nu'(s)\rangle = 2Q_\nu(\Phi(s)) - 2d_\nu'(s) = 0.$$

Since $E'(\Phi_\nu(r)) + rQ_\nu'(\Phi_\nu(r)) = 0$ for all $r \in (s - \epsilon, s + \epsilon)$, we have

$$\langle (E''(\Phi_\nu(r) + rQ_\nu''(\Phi_\nu(r)))\,\Phi_\nu'(r), w\rangle + \langle Q_\nu'(\Phi_\nu(r)), w\rangle = 0$$

for any $w \in X$. Applying this with $r = s$ and $w = \Phi$ and $w = \Phi_\nu'(s)$ gives

$$\begin{aligned}\langle S_X''(\Phi)\Phi_\nu'(s), \Phi\rangle &= -\langle Q_\nu'(\Phi), \Phi\rangle = -2d_\nu'(s), \\ \langle S_X''(\Phi)\Phi_\nu'(s), \Phi_\nu'(s)\rangle &= -\langle Q_\nu'(\Phi), \Phi_\nu'(s)\rangle = -d_\nu''(s).\end{aligned}$$

Thus

$$\begin{aligned}\langle S''_X(\Phi)\Psi,\Psi\rangle &= \langle S''_X(\Phi)\Phi,\Phi\rangle - \frac{4d'_\nu(s)}{d''_\nu(s)}\langle S''_X(\Phi)\Phi'_\nu(s),\Phi\rangle \\ &\quad + \left(\frac{2d'_\nu(s)}{d''_\nu(s)}\right)^2 \langle S''_X(\Phi)\Phi'_\nu(s),\Phi'_\nu(s)\rangle \\ &= \langle S''_X(\Phi)\Phi,\Phi\rangle + \frac{4(d'_\nu(s))^2}{d''_\nu(s)} < 0,\end{aligned}$$

since $d''_\nu(s) < 0$ and $\langle S''_X(\Phi)\Phi,\Phi\rangle < 0$. □

We next consider the choice $\Psi = (\varphi, c\cdot\nabla\varphi)$. It is clear that $\langle Q'_\nu(\Phi),\Psi\rangle = 0$, so $\mathcal{O}(\Phi)$ is unstable if $\langle S''_X(\Phi)\Psi,\Psi\rangle < 0$. By (4.3) and (4.4) we have

$$\langle S''_X(\Phi)\Psi,\Psi\rangle = 2N(\varphi) - \langle N'(\varphi),\varphi\rangle + 4\int_{\mathbb{R}^n} |c\cdot\nabla\varphi|^2\,\mathrm{d}x. \tag{4.5}$$

Corollary 4.2. Let $\Phi \in \mathscr{G}(c)$, and suppose $|c| < \sqrt{\frac{(b+2\sqrt{a})(r-1)}{r+3}}$. Then $\mathcal{O}(\Phi)$ is unstable.

Proof. Since $N(\varphi) = 2I(\varphi) \geq (b + 2\sqrt{a} - |c|^2)\int_{\mathbb{R}^n}|\nabla\varphi|^2\,\mathrm{d}x$, it follows from (4.5) and Lemma 2.4 that

$$\langle S''_X(\Phi)\Psi,\Psi\rangle \leq \left(1 - r + \frac{4|c|^2}{b+2\sqrt{a}-|c|^2}\right)N(\varphi) < 0$$

provided $|c|^2 < \frac{(b+2\sqrt{a})(r-1)}{r+3}$. □

5. Numerical Method

The result in Corollary 4.2 only provides a sufficient condition for instability. In general, the region of parameters for which instability occurs is larger than what this result guarantees. On the other hand, while the sign of d'' provides necessary and sufficient conditions for stability, exact formulas for d are generally not possible to obtain. However, one can use numerical approximations of traveling waves together with equation (3.1) and Lemma 3.1 to approximate d and d', and therefore d'' by doing so for varying wave speeds.

The following numerical method is motivated by the fact that solitary waves are minimizers of S subject to the constraint $P = 0$. It consists of a steepest descent algorithm applied to S where at each stage a scaling is performed to preserve the constraint $P = 0$.

(1) Choose some initial guess $\varphi_0 \in H^2(\mathbb{R}^n)$.
(2) For $k \geq 0$, define $\tilde{\varphi}_k = \alpha_k\varphi_k$, where α_k is the unique solution of $P(\alpha_k\varphi_k) = 0$.
(3) Set
$$\varphi_{k+1} = \tilde{\varphi}_k - \gamma_k L^{-1}S'(\tilde{\varphi}_k),$$
where L is the invertible linear operator $L(v) = \Delta^2 v - b\Delta v + (c\cdot\nabla)^2 v + av$ and $\gamma_k > 0$ is a control parameter to be chosen. Since $S'(\tilde{\varphi}_k) = L -$

$f(\tilde{\varphi}_k, D\tilde{\varphi}_k, D^2\tilde{\varphi}_k)$ this may be rewritten

$$\varphi_{k+1} = \tilde{\varphi}_k - \gamma_k(\tilde{\varphi}_k - L^{-1}f(\tilde{\varphi}_k, D\tilde{\varphi}_k, D^2\tilde{\varphi}_k)).$$

(4) Repeat Steps 2 and 3 until the L^2 norm of $\tilde{\varphi}_k - L^{-1}f(\tilde{\varphi}_k, D\tilde{\varphi}_k, D^2\tilde{\varphi}_k)$ is smaller than a desired tolerance.

REMARK 5.1. We note that in the case of a degree p homogeneous nonlinearity this method generalizes the well-known Petviashvili method, since setting $\gamma_k = 1$ results in

$$\varphi_{k+1} = M_k^{\frac{p}{p-1}} L^{-1}f(\varphi_k, D\varphi_k, D^2\varphi_k), \qquad M_k = \frac{2I(\varphi_k)}{N(\varphi_k)}.$$

Multiplication by the stabilizing factor M_k with the optimal exponent $\frac{p}{p-1}$ ensures that the constraint $P(\varphi_k) = 0$ is satisfied.

To demonstrate the convergence of this method we consider the one-dimensional beam equation with mixed power nonlinearity $f(u) = |u|^{p-1}u + |u|^{2p-2}u$, which admits a family of exact solutions, as stated in the following lemma.

LEMMA 5.2. *Let $f(u) = |u|^{p-1}u + |u|^{2p-2}u$ for some $p > 1$. Then for any $a > 0$ and any $c \in \mathbb{R}$ there exists $b > 0$, $A > 0$ and $s > 0$ such that $\varphi(x) = A\operatorname{sech}^r(sx)$ is a solution of* (2.1)*, where $r = \frac{2}{p-1}$.*

PROOF. Substituting $\varphi(x) = A\operatorname{sech}^r(sx)$ and equating coefficients leads to the system of equations

$$\begin{aligned} r(r+1)(r+2)(r+3)s^4 &= |A|^{2p-2}, \\ r(r+1)(b-c^2)s^2 - 2r(r+1)(r^2+2r+2)s^4 &= |A|^{p-1}, \\ r^4s^4 + (c^2-b)s^2s^2 + a &= 0. \end{aligned}$$

Solving the first equation for $|A|^{p-1}$ and the third equation for $b - c^2$, the middle equation becomes

$$r^2(r+1)(r+2)^2s^4 + r\sqrt{r(r+1)(r+2)(r+3)}s^2 - a(r+1) = 0.$$

This quadratic in s^2 has a unique positive solution for any $a > 0$. The first and third equations in the system then determine A and b. □

Figures 1 and 2 show the exact and numerically computed solutions in the case $p = 2$, $a = c = 1$ and $b \approx 4.43$, as well as the error between them after 25 iterations of the numerical method.

We illustrate the use of the numerical method to determine regions of stability and instability for the nonlinearity $f(u) = |u|u + |u|^2u$. The same approach could be used for any of the nonlinearities for which the existence and stability results of the previous sections apply. To reduce the scope of calculations we fix $b = 0$ and let a and c vary over the domain $0 < a < 1$, $|c|^2 < 2\sqrt{a}$. These computations were performed in dimensions one and two. Figure 3 shows two numerically computed traveling waves in two dimensions. We note that the first with wave speed $c = 0$ is radially symmetric, while the second with wave speed $c = 1.35$ is symmetric in both x and y, but not radially symmetric. Figures 4 and 5 show the regions of stability and instability within this domain, with $n = 1$ and $n = 2$, respectively. The dashed curves indicate the boundary of the regions of instability guaranteed by Corollary 4.2, while the solid curves bound the numerically computed regions of stability and instability.

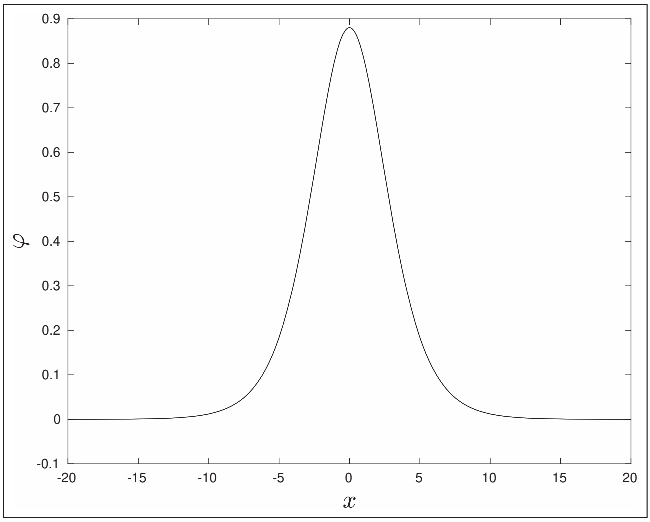

Figure 1. Numerically computed solution with exact solution for $f(u) = |u|u + u^3$, $a = c = 1$ and $b \approx 4.43$.

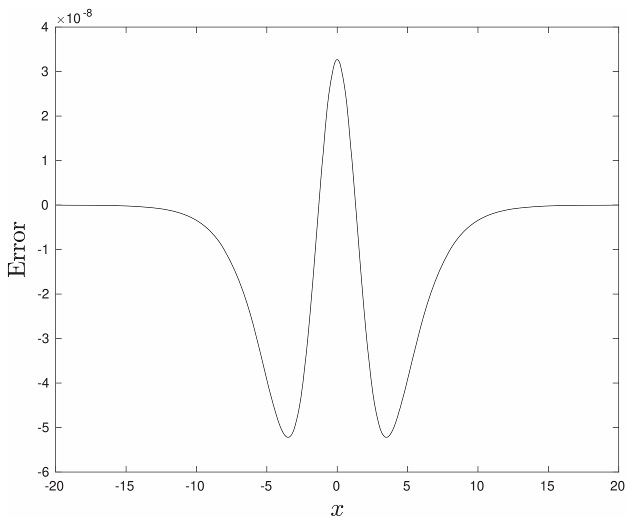

Figure 2. Error between exact solution and numerically computed solution for $f(u) = |u|u + u^3$, $a = c = 1$ and $b \approx 4.43$.

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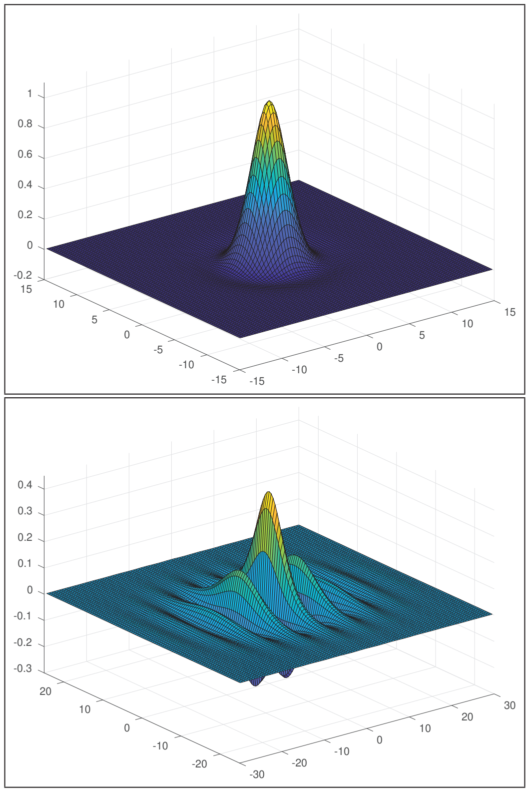

FIGURE 3. Numerically computed traveling wave profiles in two dimensions with $f(u) = |u|u + u^3$, $a = 1$, $b = 0$, $c = 0$ and $c = 1.35$.

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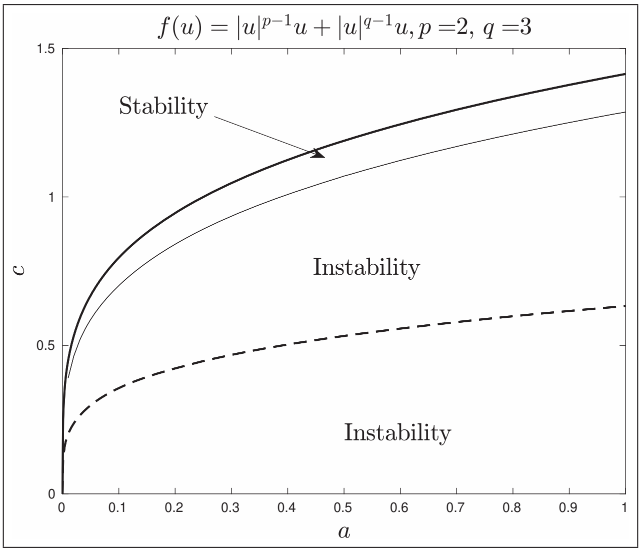

FIGURE 4. Regions of stability and instability for $f(u) = |u|u + u^3$ with $n = 1$ and $b = 0$.

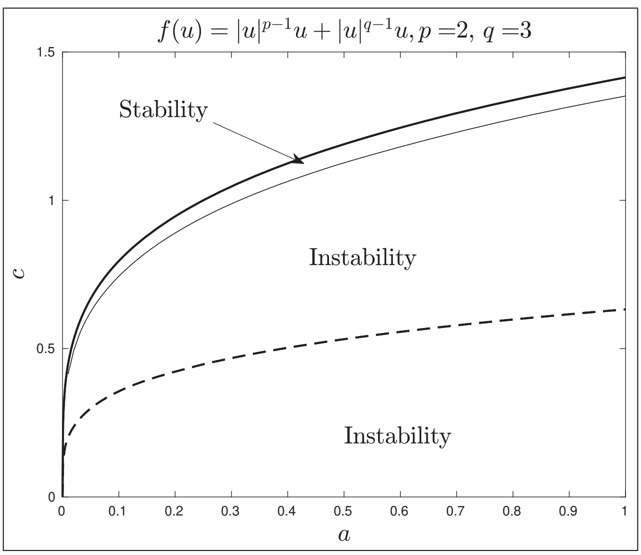

FIGURE 5. Regions of stability and instability for $f(u) = |u|u + u^3$ with $n = 2$ and $b = 0$.

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