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Global wellposedness for 2D quasilinear wave without Lorentz

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Abstract. We consider the two-dimensional quasilinear wave equations with standard null-form type quadratic nonlinearities. We introduce a new streamlined framework and prove global wellposedness without using the Lorentz boost vector fields.

Contents

1. Introduction

Denote $\Box = \partial_{tt} - \Delta$ as the usual wave operator. We consider the Cauchy problem for the following two-dimensional quasilinear wave equation:

$$\begin{cases} \Box u = g^{kij}\partial_k u \partial_{ij} u, \quad t > 2, \quad x \in \mathbb{R}^2, \\ u|_{t=2} = \varepsilon f_1, \quad \partial_t u|_{t=2} = \varepsilon f_2. \end{cases} \tag{1.1}$$

Here and throughout this note we adopt the Einstein summation convention with $\partial_0 = \partial_t$ and $\partial_l = \partial_{x_l}$ for $l = 1, 2$. For simplicity we assume g^{kij} are constant coefficients, $g^{kij} = g^{kji}$ for any i, j, and satisfy the standard null condition:

$$g^{kij}\omega_k\omega_i\omega_j = 0, \quad \text{for any null } \omega, \text{ i.e. } \omega = (-1, \cos\theta, \sin\theta),\ \theta \in [0, 2\pi]. \tag{1.2}$$

In [**1**] Alinhac showed that under the general null condition (1.2) the system (1.1) has small data global wellposedness, and the highest norm of the solution grows

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at most polynomially in time. Alinhac's proof relies on the construction of an approximation solution, combined with a judiciously chosen time-dependent weighted energy estimate known since then as the ghost weight method. The energy estimate used therein involves a collection of vector fields which are well adapted to the d'Alembertian operator. In particular, in order to harness sufficient time-decay of the solution, the Lorentz boost vector fields were used heavily in conjunction with the scaling operator. Heuristically, the benefit of the Lorentz boost can be seen from the following identities (below $\Omega_{i0} = t\partial_i + x_i\partial_t$ denotes the usual Lorentz boost):

$$L_0 = t\partial_t + r\partial_r, \qquad \frac{x_i}{r}\Omega_{i0} = r\partial_t + t\partial_r. \tag{1.3}$$

Clearly away from the light cone (i.e. $r \leq t/2$ or $r \geq 2t$), we have ∂_t and $\partial_r \approx \frac{1}{t}(O(L_0) + O(\Omega_{i0}))$ which readily leads to time-decay estimates. Whilst the Lorentz boost can produce strong decay estimates, they are not suitable for general wave systems which are not Lorentz invariant. Such systems include non-relativistic wave systems with multiple wave speeds (cf. [**3, 15**]), nonlinear wave equations on non-flat space-time (cf. [**18**]) and exterior domains (cf. [**13**]). From this perspective it is of fundamental importance to remove the Lorentz boost operator and develop a new strategy for the general non-Lorentz-invariant systems. In [**2**], Hoshiga considered a quasilinear system with multiple speeds of propagation, and proved global wellposedness under some suitable null conditions. A notable novelty in [**2**] is an L^∞-L^∞ estimate which relies on the fundamental solution of the wave equation. In [**19**] (see also [**14**]), Zha considered (1.1)–(1.2) with the following additional symmetry condition:

$$g^{kij} = g^{ikj} = g^{jik}, \qquad \forall\, i, j, k. \tag{1.4}$$

For this case Zha proved the global wellposedness without using the Lorentz boost vector fields. Note that the condition (1.4) is a bit too restrictive. For example, it does not include the standard nonlinearity $\partial(|\partial_t u|^2 - |\nabla u|^2)$. In recent [**5**], the first three authors introduced a novel strong null form which includes several prototypical strong null forms such as $\partial(|\partial_t u|^2 - |\nabla u|^2)$ in the literature as special cases. Moreover a new normal-form type strategy was developed in [**5**] to prove uniform boundedness of highest norm of the solution. Other related developments with different strategies can be found in the papers [**4, 6, 7, 8, 9, 10, 11, 12, 16, 17**].

The purpose of this note is to develop further the program initiated in [**5, 14, 19**], and obtain a full wellposedness result under the standard null condition (1.2) without employing the Lorentz boost vector fields. Thanks to the aforementioned developments, it is now possible to build a robust and streamlined Lorentz-free framework for general quasilinear equations. Our main result reads as follows.

Theorem 1.1. Consider (1.1) with g^{kij} satisfying the standard null condition (1.2). Let $m \geq 5$ and assume $f_1 \in H^{m+1}(\mathbb{R}^2)$, $f_2 \in H^m(\mathbb{R}^2)$ are compactly supported in the disk $\{|x| \leq 1\}$. There exists $\varepsilon_0 > 0$ depending on g^{kij} and $\|f_1\|_{H^{m+1}} + \|f_2\|_{H^m}$ such that for all $0 \leq \varepsilon < \varepsilon_0$, the system (1.1) has a unique global solution. Furthermore, the highest norm of the solution is polynomially bounded in time, and the second highest norm of the solution remains uniformly bounded, namely

$$\sup_{t\geq 2} E_{m-1}(u(t,\cdot)) = \sup_{t\geq 2} \sum_{|\alpha|\leq m-1} \|(\partial\Gamma^\alpha u)(t,\cdot)\|^2_{L^2_x(\mathbb{R}^2)} < \infty. \tag{1.5}$$

Here $\Gamma = \{\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_\theta, t\partial_t + r\partial_r\}$ does not include the Lorentz boost (see (2.2) for notation).

We now outline the key steps of the proof of Theorem 1.1 (see section 2 for the relevant notation). To elucidate the idea, we fix any multi-index α and denote $v = \Gamma^\alpha u$ (we suppress the dependence on α for simplicity of notation). By Lemma 2.2, we have

$$\Box v = \sum_{\alpha_1+\alpha_2 \le \alpha} g^{kij}_{\alpha;\alpha_1,\alpha_2} \partial_k \Gamma^{\alpha_1} u \partial_{ij} \Gamma^{\alpha_2} u. \tag{1.6}$$

In the forthcoming energy estimates, we shall sum over $|\alpha| \le m_0$, where m_0 is a running parameter. If $m_0 = m-1$, we seek to show the uniform-in-time boundedness of $E_{m-1}(u(t,\cdot))$. If $m_0 = m$, we show $t^{\tilde{\epsilon}}$ ($\tilde{\epsilon}$ is a small exponent) growth of $E_m(u(t,\cdot))$.

Step 1. Weighted energy estimates: LHS of (1.6). We choose $p(r,t) = q(r-t)$ with $q'(s)$ nearly scales as $\langle s \rangle^{-1}$ to derive

$$\int \Box v \partial_t v e^p dx = \frac{1}{2}\frac{d}{dt}(\|e^{\frac{p}{2}} \partial v\|_2^2) + \frac{1}{2}\int e^p q' |Tv|^2 dx. \tag{1.7}$$

In more detail, we have

$$\sum_{|\alpha| \le m_0} \|e^{\frac{p}{2}} \partial v\|_2^2 \sim E_{m_0} = \sum_{|\alpha| \le m_0} \|\partial \Gamma^\alpha u(t,\cdot)\|_2^2; \tag{1.8}$$

$$\sum_{|\alpha| \le m_0} \int e^p q' |Tv|^2 dx = \sum_{|\alpha| \le m_0} \int e^p q' |T\Gamma^\alpha u|^2 dx. \tag{1.9}$$

When carrying out the energy estimates, we shall use the convention (3.5).

Step 2. Refined decay estimates. To remedy the lack of Lorentz boost vector fields, one has to employ L^∞ and L^2 estimates involving the weight-factor $\langle r-t \rangle$. At the expense of certain smallness of $E_{\lfloor \frac{m}{2} \rfloor + 3}$ and using in an essential way the nonlinear null form (see Lemma 2.3), we obtain

$$\|\langle r-t \rangle (\partial^2 \Gamma^{\le l_0} u)(t,\cdot)\|_2 \lesssim \|(\partial \Gamma^{\le l_0+1} u)(t,\cdot)\|_2, \qquad \forall\, l_0 \le m-1; \tag{1.10}$$

$$|\langle r-t \rangle (\partial^2 \Gamma^{\le l_0} u)(t,x)| \lesssim |(\partial \Gamma^{l_0+1})(t,x)|, \qquad \forall\, r \ge t/10,\, l_0 \le m-1; \tag{1.11}$$

$$\|(\partial \Delta \Gamma^{\le m-3} u)(t,\cdot)\|_{L^2_x(|x| \le \frac{2}{3}t)} \lesssim t^{-2} \|(\partial \Gamma^{\le m-1} u)(t,\cdot)\|_2. \tag{1.12}$$

These in turn lead to a handful of strong decay estimates (see Lemma 2.5):

$$t^{\frac{1}{2}} \|\partial \Gamma^{\le m-3} u\|_\infty + t^{\frac{3}{2}} \|\frac{T\Gamma^{\le m-3} u}{\langle r-t \rangle}\|_\infty + t \|\frac{T\Gamma^{\le m-2} u}{\langle r-t \rangle}\|_2 \lesssim E_{m-1}^{\frac{1}{2}}; \tag{1.13}$$

$$t^{\frac{1}{2}} \|\langle r-t \rangle \partial^2 \Gamma^{\le m-4} u\|_\infty + t^{\frac{3}{2}} \|T\partial \Gamma^{\le m-4} u\|_\infty + t \|T\partial \Gamma^{\le m-4} u\|_2 \lesssim E_{m-1}^{\frac{1}{2}}. \tag{1.14}$$

These decay estimates play an important role in the nonlinear energy estimates.

Step 3. Weighted energy estimates: nonlinear terms. We discuss several cases.

Case 1: $\alpha_1 < \alpha$ and $\alpha_2 < \alpha$. Since $g^{kij}_{\alpha;\alpha_1,\alpha_2}$ still satisfies the null condition, by Lemma 2.2 we rewrite

$$\begin{aligned} &\sum_{\substack{\alpha_1<\alpha,\alpha_2<\alpha\\ \alpha_1+\alpha_2\le\alpha}} g^{kij}_{\alpha;\alpha_1,\alpha_2}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u\\ =&\sum_{\substack{\alpha_1<\alpha,\alpha_2<\alpha\\ \alpha_1+\alpha_2\le\alpha}} g^{kij}_{\alpha;\alpha_1,\alpha_2}(T_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u-\omega_k\partial_t\Gamma^{\alpha_1}uT_i\partial_j\Gamma^{\alpha_2}u\\ &+\omega_k\omega_i\partial_t\Gamma^{\alpha_1}uT_j\partial_t\Gamma^{\alpha_2}u). \end{aligned} \tag{1.15}$$

By using the decay estimates proved in Step 2, we show that

$$\sup_{|\alpha|\le m_0}\|\sum_{\substack{\alpha_1<\alpha,\alpha_2<\alpha\\ \alpha_1+\alpha_2\le\alpha}} g^{kij}_{\alpha;\alpha_1,\alpha_2}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u\|_2\lesssim t^{-\frac32}E^{\frac12}_{\lfloor\frac{m_0}{2}\rfloor+3}E^{\frac12}_{m_0}. \tag{1.16}$$

Case 2: The quasilinear piece $\alpha_1=0$, $\alpha_2=\alpha$. By using successive integration by parts, we have

$$\int g^{kij}\partial_k u\partial_{ij}v\partial_t ve^p\lesssim \text{OK}+t^{-\frac32}E_3^{\frac12}E_{m_0}, \tag{1.17}$$

where OK is in the sense of (3.5). Here a crucial observation is the algebraic identity (see (3.16))

$$\begin{aligned} &-\partial_j\varphi\partial_i v\partial_t v+\partial_t\varphi\partial_i v\partial_j v-\partial_i\varphi\partial_t v\partial_j v\\ =&-T_j\varphi\partial_i v\partial_t v+\partial_t\varphi T_i vT_j v-T_i\varphi\partial_t v\partial_j v-\omega_i\omega_j\partial_t\varphi(\partial_t v)^2, \end{aligned} \tag{1.18}$$

where we take $\varphi=\partial_k u$ or $\varphi=e^p$. The standard null form condition amounts to the annihilation of the term $\omega_i\omega_j\omega_k$ when $\varphi=\partial_k u$ and $\partial_t\varphi$ is replaced by $T_k\partial_t u-\omega_k\partial_{tt}u$.

Case 3: the main piece $\alpha_1=\alpha$, $\alpha_2=0$. By using Lemma 2.2 with the decay estimates, we derive

$$\int g^{kij}\partial_k v\partial_{ij}u\partial_t ve^p\lesssim \text{OK}+t^{-\frac32}E_3^{\frac12}E_{m_0}+\underbrace{\int g^{kij}T_k v\partial_{ij}u\partial_t ve^p}_{=:Y_1}. \tag{1.19}$$

We then discuss two sub-cases. If $m_0=m-1$, we show

$$|Y_1|\lesssim t^{-\frac32}E_3^{\frac12}E_m^{\frac12}E_{m_0}^{\frac12}. \tag{1.20}$$

If $m_0=m$, we use Cauchy-Schwartz to bound Y_1 as

$$|Y_1|\le \text{OK}+\text{const}\cdot\int\frac{1}{q'}|\partial_{ij}u|^2|\partial_t v|^2dx\lesssim \text{OK}+\frac1t E_3E_m. \tag{1.21}$$

Collecting all the estimates and assuming smallness of the initial data, we finally obtain

$$\sup_{t\ge2}E_{m-1}(u(t,\cdot))\le\epsilon_3\ll1,\qquad \sup_{t\ge2}\frac{E_m(u(t,\cdot))}{t^{\epsilon_4}}\le1, \tag{1.22}$$

where $\epsilon_3>0$, $\epsilon_4>0$ are small constants. This concludes the proof of Theorem 1.1.

Remark 1.1. At this point, it is worthwhile pin-pointing exactly where the symmetry condition $g^{kij}=g^{ikj}=g^{jik}$ for all k, i, j was used in [**14**]. In our notation, this comes from bounding the quasilinear piece $\alpha_2 = \alpha$. Namely

$$\begin{aligned}\int g^{kij}\partial_k u\partial_{ij}v\partial_t v e^p &= \int g^{kij}\partial_j(\partial_k u\partial_i v\partial_t v e^p) - \int g^{kij}\partial_{kj}u\partial_i v\partial_t v e^p \\ &\quad - \int g^{kij}\partial_k u\partial_i v\partial_{tj}v e^p - \int g^{kij}\partial_k u\partial_i v\partial_t v\partial_j(e^p). \end{aligned} \tag{1.23}$$

By using the symmetry $g^{kij} = g^{kji}$ which is harmless, we have

$$\begin{aligned} & - \int g^{kij}\partial_k u\partial_i v\partial_{tj}v e^p \\ = & -\frac{1}{2}\int g^{kij}\partial_t(\partial_k u\partial_i v\partial_j v e^p) \\ & + \frac{1}{2}\int g^{kij}\partial_{tk}u\partial_i v\partial_j v e^p + \frac{1}{2}\int g^{kij}\partial_i v\partial_j v\partial_t(e^p). \end{aligned} \tag{1.24}$$

It follows that

$$\begin{aligned} & \int g^{kij}\partial_k u\partial_{ij}v\partial_t v e^p dx \\ = & -\int g^{kij}\partial_{kj}u\partial_i v\partial_t v e^p \\ & + \frac{1}{2}\int g^{kij}\partial_{tk}u\partial_i v\partial_j v e^p + \frac{1}{2}\int g^{kij}\partial_k u\partial_i v\partial_j v\partial_t(e^p) + \cdots, \end{aligned} \tag{1.25}$$

where $\cdots$ denotes harmless terms. The second term on the RHS of (1.25) is not a problem thanks to the good decay of $\partial_{tk}u$. On the other hand, in [**14**] the time-decay of $\partial^2 u$ in the regime $r \le t/2$ was not sufficient to treat the first term on the RHS of (1.25). For this reason (see (3.11) in [**14**]), Peng and Zha made use of the other piece corresponding to $\alpha_1 = \alpha$ and the symmetry $g^{kij} = g^{ikj}$ to eliminate the above term, i.e.:

$$\int g^{kij}\partial_k v\partial_{ij}u\partial_t v e^p = \int g^{kij}\partial_i v\partial_{kj}u\partial_t v e^p. \tag{1.26}$$

One of the main novelty of this work is that we obtained $t^{-\frac{3}{2}}$ decay of $\partial^2 u$ in the regime $r \le t/2$. This and several other new estimates can have useful applications in many other problems.

The rest of this note is organized as follows. In Section 2 we collect some preliminaries and useful lemmas. In Section 3 we give the proof of Theorem 1.1.

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2. Preliminaries

Notation. We shall use the Japanese bracket notation: $\langle x\rangle = \sqrt{1+|x|^2}$, for $x \in \mathbb{R}^d$. We denote $\partial_0 = \partial_t$, $\partial_i = \partial_{x_i}$, $i = 1, 2$ and (below ∂_θ and ∂_r correspond to the usual polar coordinates)

$$\partial = (\partial_i)_{i=0}^2,\ \partial_\theta = x_1\partial_2 - x_2\partial_1,\ L_0 = t\partial_t + r\partial_r, \tag{2.1}$$

$$\Gamma = (\Gamma_i)_{i=1}^5, \quad \text{where } \Gamma_1 = \partial_t, \Gamma_2 = \partial_1, \Gamma_3 = \partial_2, \Gamma_4 = \partial_\theta, \Gamma_5 = L_0; \tag{2.2}$$

$$\Gamma^\alpha = \Gamma_1^{\alpha_1}\Gamma_2^{\alpha_2}\Gamma_3^{\alpha_3}\Gamma_4^{\alpha_4}\Gamma_5^{\alpha_5}, \qquad \alpha = (\alpha_1, \cdots, \alpha_5) \text{ is a multi-index;} \tag{2.3}$$

$$\partial_+ = \partial_t + \partial_r, \qquad \partial_- = \partial_t - \partial_r; \tag{2.4}$$

$$T_i = \omega_i\partial_t + \partial_i,\ \omega_0 = -1,\ \omega_i = x_i/r,\ i = 1, 2. \tag{2.5}$$

Note that in (2.2) we do not include the Lorentz boosts. Note that $T_0 = 0$. For simplicity of notation, we define for any integer $k \ge 1$, $\Gamma^k = (\Gamma^\alpha)_{|\alpha|=k}$, $\Gamma^{\le k} = (\Gamma^\alpha)_{|\alpha|\le k}$. In particular

$$|\Gamma^{\le k} u| = \left(\sum_{|\alpha|\le k} |\Gamma^\alpha u|^2\right)^{\frac{1}{2}}. \tag{2.6}$$

Informally speaking, it is useful to think of $\Gamma^{\le k}$ as any one of the vector fields Γ^α with $|\alpha| \le k$. For integer $J \ge 3$, we shall denote

$$E_J = E_J(u(t,\cdot)) = \|(\partial\Gamma^{\le J}u)(t,\cdot)\|^2_{L^2_x(\mathbb{R}^2)}. \tag{2.7}$$

We shall need the following convention for multi-indices: for $\beta = (\beta_1, \cdots, \beta_5)$ and $\alpha = (\alpha_1, \cdots, \alpha_5)$, we denote $\beta < \alpha$ if $\beta_i \le \alpha_i$ for $i = 1, \cdots, 5$ and $|\beta| < |\alpha|$ (Here $|\alpha| = \sum_{i=1}^5 \alpha_i$). Similarly we denote $\beta \le \alpha$ if $\beta_i \le \alpha_i$ for $i = 1, \cdots, 5$.

For any two quantities A, $B \ge 0$, we write $A \lesssim B$ if $A \le CB$ for some unimportant constant $C > 0$. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We write $A \ll B$ if $A \le cB$ and $c > 0$ is a sufficiently small constant. The needed smallness is clear from the context.

Lemma 2.1 (Sobolev and Hardy). For $v \in C_c^\infty(\mathbb{R}^2)$, we have

$$|v(x)| \lesssim \begin{cases} \|v\|_2 + \|\Delta v\|_2, & |x| \le 1, \\ \langle x\rangle^{-\frac{1}{2}} \|\partial_\theta^{\le 1}\partial_r^{\le 1} v\|_2, & |x| > 1. \end{cases} \tag{2.8}$$

Suppose $u = u(t,x)$ $(t \ge 0)$ is smooth and compactly supported in the space-time slab $\{(t,x) : |x| \le 1 + t\}$, then

$$\|\langle |x| - t\rangle^{-1} u\|_{L^2_x(\mathbb{R}^2)} \lesssim \|\partial_r u\|_{L^2_x(\mathbb{R}^2)}, \tag{2.9}$$

$$\langle |x_0| - t\rangle^{-1} |u(t,x_0)| \lesssim \langle x_0\rangle^{-\frac{1}{2}} \|\partial\Gamma^{\le 1} u\|_{L^2_x(\mathbb{R}^2)}, \qquad \forall\ x_0 \in \mathbb{R}^2. \tag{2.10}$$

Proof. This is essentially Lemma 2.1 and 2.2 in [**5**]. For the sake of completeness we sketch the details here. For a one-variable function $h \in C_c^\infty([0,\infty))$, we have

$$\rho|h(\rho)|^2 \le \int_0^\infty |h|^2 r dr + \int_0^\infty |\partial_r h|^2 r dr, \quad \forall\, \rho > 0. \tag{2.11}$$

For $x = (\rho\cos\theta, \rho\sin\theta)$, by considering the average of $v(\rho,\theta)$ over θ we obtain

$$\rho|\overline{v}(\rho)|^2 = \rho\left(\frac{1}{2\pi}\int_0^{2\pi} v(\rho,\theta)d\theta\right)^2 \lesssim \|v\|^2_{L^2(\mathbb{R}^2)} + \|\partial_r v\|^2_{L^2(\mathbb{R}^2)}.$$

Similarly

$$\rho\|\partial_\theta v(\rho,\theta)\|_{L^2_\theta}^2 = \rho\int_0^{2\pi} |\partial_\theta v(\rho,\theta)|^2 d\theta \lesssim \|\partial_\theta v\|_{L^2(\mathbb{R}^2)}^2 + \|\partial_r\partial_\theta v\|_{L^2(\mathbb{R}^2)}^2.$$

Noting that $|v(\rho,\theta)-\overline{v}(\rho)|^2 \lesssim \|\partial_\theta v\|_{L^2_\theta}^2$, we obtain (2.8). To see (2.9), we observe that for any real-valued $h \in C_c^\infty([0, M+1))$ with $M > 0$, we have

$$\int_0^{M+1} \frac{h(\rho)^2}{(2+M-\rho)^2}\rho d\rho \le 4\int_0^\infty (h'(\rho))^2 \rho d\rho. \tag{2.12}$$

Then (2.9) follows from (2.12) and the fact that $\langle |x|-t\rangle^{-2} \sim (2+t-|x|)^{-2}$ for $|x| \le 1+t$. For (2.10), we first consider the case $|x_0| > 1$. Clearly by (2.8) we have

$$\begin{aligned}\langle |x_0|-t\rangle^{-1}|u(t,x_0)| &\lesssim \langle x_0\rangle^{-\frac12}\|\partial_r^{\le 1}\partial_\theta^{\le 1}(\langle r-t\rangle^{-1}u)\|_2 \\ &\sim \langle x_0\rangle^{-\frac12}\|\partial_r^{\le 1}(\langle r-t\rangle^{-1}\partial_\theta^{\le 1}u)\|_2 \\ &\lesssim \langle x_0\rangle^{-\frac12}\|\partial_r\partial_\theta^{\le 1}u\|_2 \qquad \text{(by (2.9))} \\ &\lesssim \langle x_0\rangle^{-\frac12}\|\partial\Gamma^{\le 1}u\|_2.\end{aligned}$$

For $|x_0| \le 1$, we have

$$\begin{aligned}\langle |x_0|-t\rangle^{-1}|u(t,x_0)| &\lesssim \langle t\rangle^{-1}(\|u\|_{L^2_x(|x|\le 10)} + \|\nabla^2 u\|_{L^2_x(|x|\le 10)}) \\ &\lesssim \|\langle |\cdot|-t\rangle^{-1}u\|_{L^2_x(\mathbb{R}^2)} + \|\Delta u\|_{L^2_x(\mathbb{R}^2)} \lesssim \|\nabla u\|_2 + \|\Delta u\|_2.\end{aligned}$$

$\square$

Lemma 2.2. If g^{kij} satisfies the null condition, then for $t > 0$ we have

$$g^{kij}\partial_k f\partial_{ij}h = g^{kij}(T_k f\partial_{ij}h - \omega_k\partial_t f T_i\partial_j h + \omega_k\omega_i\partial_t f T_j\partial_t h), \tag{2.13}$$

where $T = (T_1, T_2)$ is defined in (2.5). It follows that

$$|g^{kij}\partial_k f\partial_{ij}h| \lesssim |Tf||\partial^2 h| + |\partial f||T\partial h| \tag{2.14}$$

$$\lesssim \frac{1}{\langle r+t\rangle}(|\Gamma f||\partial^2 h| + |\partial f||\Gamma\partial h| + |\partial f|\cdot|\partial^2 h|\cdot|r-t|). \tag{2.15}$$

Suppose g^{kij} satisfies the null condition and $\Box u = g^{kij}\partial_k u\partial_{ij}u$. Then for any multi-index α, we have

$$\Box\Gamma^\alpha u = \sum_{\alpha_1+\alpha_2\le\alpha} g^{kij}_{\alpha;\alpha_1,\alpha_2}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u, \tag{2.16}$$

where for each (α_1, α_2), $g^{kij}_{\alpha;\alpha_1,\alpha_2}$ also satisfies the null condition. In addition, we have $g^{kij}_{\alpha;\alpha,0} = g^{kij}_{\alpha;0,\alpha} = g^{kij}$.

Proof. We sketch the details. The identity (2.13) follows from appealing to the definition $\partial_l = T_l - \omega_l\partial_t$ and the null condition. The inequality (2.15) is obvious if $r \le \frac{t}{2}$ or $r \ge 2t$, or $r \sim t \lesssim 1$ since $\langle r+t\rangle \sim \langle r-t\rangle$ in these regimes. On the other hand, if $r \sim t \gtrsim 1$, then one can use the identities

$$T_1 = \omega_1\partial_+ - \frac{\omega_2}{r}\partial_\theta, \quad T_2 = \omega_2\partial_+ + \frac{\omega_1}{r}\partial_\theta; \quad \partial_+ = \frac{1}{t+r}(2L_0 - (t-r)\partial_-). \tag{2.17}$$

We will prove (2.16) by induction. Clearly, the result holds for $|\alpha| = 0$. Assume that it is true for $|\alpha| = l$. We consider $\Gamma_j\Gamma^\alpha$ for $j = 1, \cdots, 5$. It suffices for us to consider $\widetilde{\Gamma} = \partial_\theta$ or L_0. To alleviate the notation, we use the characterization (below we denote $z = (t, x_1, x_2)^\top$)

$$(\partial_\theta h)(z) = \frac{d}{d\tau}\Big|_{\tau=0} h(e^{\tau A} z), \qquad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \tag{2.18}$$

$$(L_0 h)(z) = \frac{d}{d\tau}\Big|_{\tau=0} h(e^{\tau A} z), \qquad A = \mathrm{I}. \tag{2.19}$$

It follows that

$$\begin{aligned} \widetilde{\Gamma}\Gamma^\alpha (g^{kij}\partial_k u \partial_{ij} u) =& \widetilde{\Gamma}(\sum_{\alpha_1+\alpha_2\le\alpha} g^{kij}_{\alpha;\alpha_1,\alpha_2} \partial_k \Gamma^{\alpha_1} u \partial_{ij}\Gamma^{\alpha_2} u) \\ =& \sum_{\alpha_1+\alpha_2\le\alpha} \frac{d}{d\tau}\Big|_{\tau=0} \left(g^{kij}_{\alpha;\alpha_1,\alpha_2} (\partial_k \Gamma^{\alpha_1} u \partial_{ij}\Gamma^{\alpha_2} u)(e^{\tau A} z)\right). \end{aligned}$$

For simplicity of notation, we work with

$$\frac{d}{d\tau}\Big|_{\tau=0} \Big(g^{kij} (\partial_k \Phi \partial_{ij} \Psi)(Mz)\Big), \qquad M = e^{\tau A}. \tag{2.20}$$

Observe that (one can tacitly write $(Mz)_k = M_{kb} z_b$)

$$\begin{aligned} &\partial_{z_b}\Big(f(Mz)\Big) = M_{kb}(\partial_k f)(Mz) \\ \Rightarrow& (\partial_k f)(Mz) = (M^{-1})_{bk}\partial_{z_b}\Big(f(Mz)\Big) = W_{kb}\partial_{z_b}\Big(f(Mz)\Big), \qquad W = (M^\top)^{-1}. \end{aligned} \tag{2.21}$$

Consequently we have

$$\begin{aligned} & g^{kij}(\partial_k \Phi \partial_{ij}\Psi)(Mz) \\ =& g^{kij} W_{kb} W_{ia} W_{jc} \partial_{z_b}\Big(\Phi(Mz)\Big)\partial_{z_a z_c}\Big(\Psi(Mz)\Big). \end{aligned} \tag{2.22}$$

Observe that for any null vector $\vec{\omega}$, we have

$$g^{kij} W_{kb} W_{ia} W_{jc} \omega_b \omega_a \omega_c = 0. \tag{2.23}$$

Thus

$$\frac{d}{d\tau}\Big|_{\tau=0} g^{kij} W_{kb} W_{ia} W_{jc} \omega_b \omega_a \omega_c = 0. \tag{2.24}$$

The desired conclusion easily follows.

□

LEMMA 2.3. Suppose $\tilde{u} = \tilde{u}(t,x)$ has continuous second order derivatives. Then

$$\begin{aligned} & |\langle r-t\rangle \partial_{tt}\tilde{u}(t,x)| + |\langle r-t\rangle \partial_t \nabla \tilde{u}(t,x)| + |\langle r-t\rangle \Delta\tilde{u}(t,x)| \\ \lesssim& |(\partial\Gamma^{\le 1}\tilde{u})(t,x)| + (r+t)|(\Box\tilde{u})(t,x)|, \quad r = |x|,\ t\ge 0; \end{aligned} \tag{2.25}$$

$$\begin{aligned} & |\langle r-t\rangle \partial^2\tilde{u}(t,x)| \lesssim |(\partial\Gamma^{\le 1}\tilde{u})(t,x)| + (r+t)|(\Box\tilde{u})(t,x)|, \\ & \forall\, r \ge t/10,\ t\ge 1. \end{aligned} \tag{2.26}$$

Suppose $T_0 \ge 1$ and $u \in C^\infty([1,T_0]\times\mathbb{R}^2)$ solves (1.1) with support in $|x| \le 1+t$, $1 \le t \le T_0$. For any integer $l_0 \ge 2$, there exists $\epsilon_1 > 0$ depending only on l_0, such that if at some $1 \le t \le T_0$,

$$\|(\partial\Gamma^{\le \lceil \frac{l_0}{2}\rceil + 2} u)(t,\cdot)\|_{L^2_x(\mathbb{R}^2)} \le \epsilon_1, \qquad (\text{ here } \lceil z\rceil = \min\{n\in\mathbb{N}:\ n\ge z\}\ ) \tag{2.27}$$

then for the same t, we have the L^2 estimate:

$$\|(\langle r-t\rangle\partial^2\Gamma^{\le l_0}u)(t,\cdot)\|_{L^2_x(\mathbb{R}^2)} \lesssim \|(\partial\Gamma^{\le l_0+1}u)(t,\cdot)\|_{L^2_x(\mathbb{R}^2)}. \tag{2.28}$$

For any integer $l_1 \ge 2$, there exists $\epsilon_2 > 0$ depending only on l_1, such that if at some $1 \le t \le T_0$,

$$\|(\partial\Gamma^{\le l_1+2}u)(t,\cdot)\|_{L^2_x(\mathbb{R}^2)} \le \epsilon_2, \tag{2.29}$$

then for the same t, we have the point-wise estimate:

$$|(\langle r-t\rangle\partial^2\Gamma^{\le l_1}u)(t,x)| \lesssim |(\partial\Gamma^{\le l_1+1}u)(t,x)|, \qquad \forall\, r \ge t/10. \tag{2.30}$$

Moreover, we have

$$\|\partial\Delta\Gamma^{\le l_1-1}u\|_{L^2_x(|x|\le\frac{2}{3}t)} \lesssim t^{-2}\|(\partial\Gamma^{\le l_1+1}u)(t,\cdot)\|_{L^2_x(\mathbb{R}^2)}. \tag{2.31}$$

REMARK 2.1. It also holds that

$$\|\partial^3\Gamma^{\le l_1-1}u\|_{L^2_x(|x|\le\frac{2}{3}t)} \lesssim t^{-2}\|(\partial\Gamma^{\le l_1+1}u)(t,\cdot)\|_{L^2_x(\mathbb{R}^2)}.$$

PROOF. All estimates except (2.31) were proved in Lemma 2.4 of [**5**]. For the sake of completeness we sketch the details.

Denote $Q = |(\partial\Gamma^{\le 1}\tilde{u})(t,x)| + (r+t)|(\Box\tilde{u})(t,x)|$. By using $L_0 = t\partial_t + r\partial_r$ and $\Delta = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}$, we have

$$\begin{cases} L_0\partial_t\tilde{u} = t\partial_{tt}\tilde{u} + r\partial_r\partial_t\tilde{u} = t\Box\tilde{u} + t\Delta\tilde{u} + r\partial_r\partial_t\tilde{u}, \\ L_0\partial_r\tilde{u} = t\partial_t\partial_r\tilde{u} + r\partial_{rr}\tilde{u} = t\partial_t\partial_r\tilde{u} + r\Delta\tilde{u} - \partial_r\tilde{u} - \frac{1}{r}\partial_{\theta\theta}\tilde{u}, \end{cases}$$

$$\Longrightarrow \begin{cases} |t\Delta\tilde{u} + r\partial_r\partial_t\tilde{u}| \lesssim Q, \\ |r\Delta\tilde{u} + t\partial_r\partial_t\tilde{u}| \lesssim Q, \end{cases} \quad \Longrightarrow \langle r-t\rangle(|\Delta\tilde{u}| + |\partial_t\partial_r\tilde{u}|) \lesssim Q.$$

This implies $\langle r-t\rangle(|\partial_{tt}\tilde{u}| + |\Delta\tilde{u}| + |\partial_t\partial_r\tilde{u}|) \lesssim Q$.

To deal with $|r-t|\cdot|\nabla\partial_t\tilde{u}|$, we observe

$$\begin{aligned} |r-t|\cdot|\nabla\partial_t\tilde{u}| &\lesssim \frac{1}{r}|t-r||\partial_\theta\partial_t\tilde{u}| + |t-r||\partial_r\partial_t\tilde{u}| \\ &\lesssim \frac{1}{r}|t-r||\partial_\theta\partial_t\tilde{u}| + Q \\ &\lesssim Q \qquad (\text{since } \partial_\theta L_0\tilde{u} = (t-r)\partial_\theta\partial_t\tilde{u} + r(\partial_t+\partial_r)\partial_\theta\tilde{u}). \end{aligned} \tag{2.32}$$

This proves (2.25).

The estimate (2.26) can be conducted similarly. Observe that for $r \ge t/10$ we have (note that for ∇^2 includes $\partial_i\partial_j$, $1 \le i,j \le 2$, $\partial_i = \omega_i\partial_r + \frac{\omega_i^\perp}{r}\partial_\theta$):

$$\begin{aligned} |r-t||\partial_{ij}\tilde{u}(t,x)| &\lesssim |r-t||\partial_r\partial_j\tilde{u}(t,x)| + |\partial_\theta\partial_j\tilde{u}(t,x)| \\ &\lesssim |r-t||\partial_r(\omega_j\partial_r\tilde{u} + \frac{\omega_j^\perp}{r}\partial_\theta\tilde{u})| + |\partial_\theta(\omega_j\partial_r\tilde{u} + \frac{\omega_j^\perp}{r}\partial_\theta\tilde{u})| \\ &\lesssim |r-t||\partial_{rr}\tilde{u}| + |\partial_r\partial_\theta\tilde{u}| + \frac{1}{r}|\partial_\theta\tilde{u}| + |\partial_r\tilde{u}| + \frac{1}{r}|\partial_{\theta\theta}\tilde{u}|. \end{aligned} \tag{2.33}$$

Using

$$\left|(r-t)(\partial_{rr}\tilde{u} + \frac{1}{r}\partial_r\tilde{u} + \frac{1}{r^2}\partial_{\theta\theta}\tilde{u})\right| \lesssim Q \Rightarrow |r-t||\partial_{rr}\tilde{u}| \lesssim Q, \qquad r \ge \frac{t}{10}, \tag{2.34}$$

we arrive at

$$\sum_{i,j=1}^{2} |r-t||\partial_{ij}\tilde{u}| \lesssim Q. \tag{2.35}$$

For (2.28), using integration by parts one has (below $k_0 \geq 0$ is a running parameter)

$$\sum_{i,j=1}^{2} \|\langle r-t\rangle \partial_i\partial_j \Gamma^{\leq k_0} u\|_2 \lesssim \|\partial \Gamma^{\leq k_0} u\|_2 + \|\langle r-t\rangle \Delta \Gamma^{\leq k_0} u\|_2.$$

Thus it suffices for us to bound $\|\langle r-t\rangle \partial_{tt}\Gamma^{\leq k_0}u\|_2 + \|\langle r-t\rangle \partial_t \nabla \Gamma^{\leq k_0} u\|_2 + \|\langle r-t\rangle \Delta \Gamma^{\leq k_0} u\|_2$.

By (2.25) we have

$$\begin{aligned}&|(\langle r-t\rangle \partial_{tt}\Gamma^{\leq k_0}u)(t,x)| + |(\langle r-t\rangle \partial_t \nabla \Gamma^{\leq k_0}u)(t,x)| + |(\langle r-t\rangle \Delta \Gamma^{\leq k_0}u)(t,x)| \\ \lesssim& |\partial \Gamma^{\leq k_0+1}u| + (r+t)|\Box \Gamma^{\leq k_0}u|.\end{aligned}$$

Using (2.15) (recall $\langle r+t\rangle |g^{kij}\partial_k f \partial_{ij} h| \lesssim |\Gamma f||\partial^2 h| + |\partial f||\Gamma \partial h| + |\partial f| \cdot |\partial^2 h| \cdot |r-t|$), we get

$$\begin{aligned}(r+t)|\Box \Gamma^{\leq k_0} u| \lesssim& \sum_{m+l\leq k_0} (|\Gamma^{\leq m+1}u||\partial^2\Gamma^{\leq l}u| + |\partial \Gamma^{\leq m}u||\Gamma^{\leq l+1}\partial u|) \\ &+ \sum_{m+l\leq k_0} |\partial \Gamma^{\leq m}u||\partial^2 \Gamma^{\leq l}u||r-t|.\end{aligned}$$

Thus we obtain

$$\begin{aligned}\|\langle r-t\rangle \partial^2 \Gamma^{\leq k_0}u\|_2 \lesssim& \sum_{m+l\leq k_0} (\||\Gamma^{\leq m+1}u||\partial^2\Gamma^{\leq l}u|\|_2 + \||\partial\Gamma^{\leq m}u||\Gamma^{\leq l+1}\partial u|\|_2) \\ &+ \sum_{m+l\leq k_0} \||\partial\Gamma^{\leq m}u||\partial^2\Gamma^{\leq l}u||r-t|\|_2 + \|\partial \Gamma^{\leq k_0+1}u\|_2.\end{aligned}$$

If $m \leq l+1$, then we use the estimates (note that $m+2 \leq \lfloor \frac{k_0+1}{2} \rfloor + 2 \leq \lceil \frac{k_0}{2} \rceil + 2$)

$$\langle r-t\rangle^{-1}|(\Gamma^{\leq m+1}u)(t,x)| \lesssim \|\partial \Gamma^{\leq m+2}u\|_2, \quad \|\partial \Gamma^{\leq m}u\|_\infty \lesssim \|\partial\Gamma^{\leq m+2}u\|_2. \tag{2.36}$$

If $m \geq l+2$, then $l \leq \frac{k_0-2}{2}$ and we use the estimates (see (2.49) for the second estimate)

$$\|\frac{\Gamma^{\leq m+1}u}{\langle |\cdot| - t\rangle}\|_2 \lesssim \|\partial \Gamma^{\leq m+1}u\|_2, \quad |\langle r-t\rangle \partial^2\Gamma^{\leq l}u(t,x)| \lesssim \|\langle |\cdot|-t\rangle \partial^2 \Gamma^{\leq l+2}u\|_2. \tag{2.37}$$

Thus if $\|\partial\Gamma^{\leq k_0+1}u\|_2 \ll 1$, we obtain

$$\|\langle r-t\rangle \partial^2 \Gamma^{\leq k_0}u(t,\cdot)\|_2 \lesssim \|\partial \Gamma^{\leq k_0+1}u\|_2 \ll 1. \tag{2.38}$$

To prove (2.28) under the assumption (2.27) we first take $k_0 = \lceil \frac{l_0}{2} \rceil + 1$ and show that

$$\|\langle r-t\rangle \partial^2 \Gamma^{\leq \lceil \frac{l_0}{2}\rceil + 1}u)(t,\cdot)\|_2 \lesssim \|(\partial \Gamma^{\leq \lceil \frac{l_0}{2}\rceil+2}u)(t,\cdot)\|_2 \ll 1. \tag{2.39}$$

We then use this smallness in (2.37) and obtain the desired result for $k_0 = l_0$ (Note that $\lceil \frac{l_0-2}{2} \rceil + 2 \le \lceil \frac{l_0}{2} \rceil + 1$). The estimate of (2.30) follows from (2.26):

$$
\begin{aligned}
&|\langle r-t\rangle \partial^2 \Gamma^{\le l_1} u| \\
\lesssim& |\partial \Gamma^{\le l_1+1} u| + (r+t)|\Box \Gamma^{\le l_1} u| \qquad \text{(by (2.26))} \\
\lesssim& |\partial \Gamma^{\le l_1+1} u| + \sum_{m+n\le l_1} (|\Gamma^{\le m+1} u||\partial^2 \Gamma^{\le n} u| + |\partial \Gamma^{\le m} u||\Gamma \partial \Gamma^{\le n} u|) \\
&+ \sum_{m+n\le l_1} |\partial \Gamma^{\le m} u||\partial^2 \Gamma^{\le n} u||r-t| \qquad \text{(by (2.15))} \\
\lesssim& |\partial \Gamma^{\le l_1+1} u| + (|\frac{\Gamma^{\le l_1+1} u}{\langle r-t\rangle}| + |\partial \Gamma^{\le l_1} u|)|\langle r-t\rangle \partial^2 \Gamma^{\le l_1} u| \\
\lesssim& |\partial \Gamma^{\le l_1+1} u| + \underbrace{\|\partial \Gamma^{\le l_1+2} u\|_2}_{\text{Here we need (2.29)}} |\langle r-t\rangle \partial^2 \Gamma^{\le l_1} u| \qquad \text{(by (2.8), (2.10))}.
\end{aligned}
$$

We now sketch the proof of (2.31). Applying (2.25) to $\tilde{u} = \partial \Gamma^{\le l_1-1} u$ with $r \le \frac{2}{3}t$, we get

$$
|\Delta \partial \Gamma^{\le l_1-1} u| \lesssim \frac{1}{t} |\partial^2 \Gamma^{\le l_1} u| + |\partial \Box \Gamma^{\le l_1-1} u|. \tag{2.40}
$$

By Lemma 2.2, we have

$$
\begin{aligned}
|\partial \Box \Gamma^{\le l_1-1} u| &\lesssim \sum_{a+b\le l_1-1} |\partial(\partial \Gamma^a u \partial^2 \Gamma^b u)| \\
&\lesssim |\partial^2 \Gamma^{\le l_1-1} u||\partial^2 \Gamma^{\le l_1-1} u| + |\partial \Gamma^{\le l_1-1} u||\partial^3 \Gamma^{\le l_1-1} u|.
\end{aligned} \tag{2.41}
$$

Note that

$$
\begin{aligned}
|\partial^3 \Gamma^{\le l_1-1} u| &\lesssim |\underbrace{\partial_{tt} \partial \Gamma^{\le l_1-1} u}_{\partial_t \text{ appears twice or more}}| + \sum_{1\le i_1, i_2 \le 2} |\underbrace{\partial \partial_{i_1} \partial_{i_2} \Gamma^{\le l_1-1} u}_{\partial_t \text{ appears at most once}}| \\
&\lesssim |\partial \Box \Gamma^{\le l_1-1} u| + |\partial \tilde{\partial}^2 \Gamma^{\le l_1-1} u|,
\end{aligned} \tag{2.42}
$$

where we have denoted $\tilde{\partial} = (\partial_1, \partial_2)$. By using the smallness of the pre-factor $\|\partial \Gamma^{\le l_1-1} u\|_\infty$ and (2.42), we then derive from (2.41)

$$
|\partial \Box \Gamma^{\le l_1-1} u| \lesssim |\partial^2 \Gamma^{\le l_1-1} u||\partial^2 \Gamma^{\le l_1-1} u| + |\partial \Gamma^{\le l_1-1} u||\partial \tilde{\partial}^2 \Gamma^{\le l_1-1} u|. \tag{2.43}
$$

Clearly by Sobolev embedding $H^1(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$, we get (below denote $X = \|(\partial \Gamma^{\le l_1+1} u)(t,\cdot)\|_2$)

$$
\|\langle r-t\rangle \partial^2 \Gamma^{\le l_1-1} u\|_4 \lesssim \|\partial^{\le 1}(\langle r-t\rangle \partial^2 \Gamma^{\le l_1-1} u)\|_2 \lesssim X; \tag{2.44}
$$

$$
\|\langle r-t\rangle^{\frac{1}{2}} \partial \Gamma^{\le l_1-1} u\|_\infty \lesssim X, \qquad \text{(by Lemma 2.4)}. \tag{2.45}
$$

By using a smooth cut-off function localized to $|x| \le \frac{2}{3}t$, we then derive

$$
\|\Delta \partial \Gamma^{\le l_1-1} u\|_{L_x^2(|x|\le \frac{2}{3}t)} \lesssim t^{-\frac{3}{2}} X. \tag{2.46}
$$

It follows that (recall $\tilde{\partial} = (\partial_1, \partial_2)$)

$$
\|\tilde{\partial}^2 \partial \Gamma^{\le l_1-1} u\|_{L_x^2(|x|\le \frac{2}{3}t)} \lesssim t^{-\frac{3}{2}} X. \tag{2.47}
$$

Plugging this estimate into (2.43), we then obtain the estimate (2.31). $\square$

LEMMA 2.4. For any $f \in C_c^\infty(\mathbb{R}^2)$, we have

$$\begin{aligned} &\langle |x_0|-t\rangle^{\frac12}|f(x_0)| \lesssim \|f\|_2 + \|\langle |x|-t\rangle \nabla f\|_2 \\ &+ \|\langle |x|-t\rangle \partial_1\partial_2 f\|_2, \quad \forall\, x_0\in\mathbb{R}^2,\ t\ge 0; \qquad (2.48) \\ &\|\langle |x|-t\rangle \partial f\|_\infty \lesssim \|\langle |x|-t\rangle \partial f\|_2 + \|\langle |x|-t\rangle \partial^2 f\|_2 \\ &+ \|\langle |x|-t\rangle \partial^3 f\|_2, \quad \forall\, t\ge 0. \qquad (2.49) \end{aligned}$$

It follows that

$$\|f\|_{L_x^\infty(\mathbb{R}^2)} \lesssim \langle t\rangle^{-\frac12}(\|\widetilde{\Gamma}^{\le 1} f\|_2 + \|\langle |x|-t\rangle \nabla \tilde{\Gamma}^{\le 1} f\|_2), \qquad \forall\, t\ge 0, \tag{2.50}$$

where $\tilde{\Gamma} = (\partial_1, \partial_2, \partial_\theta)$.

PROOF. This is similar to Lemma 2.5 of [**5**]. We sketch the details here for the sake of completeness. The case $||x_0|-t| \le 2$ follows from the inequality $|f(x_0)|^2 \le \int |\partial_1\partial_2(f(x)^2)| dx_1 dx_2$. For $||x_0|-t| > 2$, we note that $\langle |x_0|-t\rangle \lesssim \frac{\langle |x_0|^2-t^2\rangle}{\langle x_0\rangle + t}$. Note that $\frac{\langle |x|^2-t^2\rangle}{\langle x\rangle+t} \lesssim \langle |x|-t\rangle$ for all $x\in\mathbb{R}^2$, $t\ge 0$. Let $W(x) = \frac{\langle |x|^2-t^2\rangle}{\langle x\rangle+t}$ and observe that

$$\sum_{1\le i\le 2} \|\partial_i W\|_\infty + \sum_{1\le i,j\le 2} \|\partial_i\partial_j W\|_\infty \lesssim 1. \tag{2.51}$$

Since $f\in C_c^\infty$, by the fundamental theorem of calculus we have

$$\begin{aligned} &\langle |x_0|-t\rangle |f(x_0)|^2 \\ \lesssim& \frac{\langle |x_0|^2-t^2\rangle}{\langle x_0\rangle + t}|f(x_0)|^2 \lesssim \int_{\mathbb{R}^2} \left|\partial_1\partial_2\Big(\frac{\langle |x|^2-t^2\rangle}{\langle x\rangle+t} f(x)^2\Big)\right| dx_1 dx_2 \\ \lesssim& \|f\|_2^2 + \|\nabla f\|_2\|f\|_2 + \|\langle |x|-t\rangle \nabla f\|_2 \|\nabla f\|_2 + \|\langle |x|-t\rangle \partial_1\partial_2 f\|_2 \|f\|_2 \\ \lesssim& \|f\|_2^2 + \|\langle |x|-t\rangle \nabla f\|_2^2 + \|\langle |x|-t\rangle \partial_1\partial_2 f\|_2^2 \qquad \text{(by (2.51))}. \end{aligned} \tag{2.52}$$

Thus (2.48) follows. The proof of (2.49) is similar by working with the expression $W(x_0)^2|\partial f(x_0)|^2$ for the case $|x_0 - t| > 2$ and deriving the result. For (2.50) we may assume $t\ge 2$. The case $|x_0|\le t/2$ follows from (2.48). The case $|x_0| > t/2$ follows from (2.8).

□

LEMMA 2.5 (Decay estimates). Suppose $T_0 \ge 2$ and $u \in C^\infty([2,T_0]\times\mathbb{R}^2)$ solves (1.1) with support in $|x|\le t+1$, $2\le t\le T_0$. Suppose $J\ge 3$ and

$$E_J = E_J(u(t,\cdot)) = \|(\partial\Gamma^{\le J} u)(t,\cdot)\|_2^2 \le \tilde{\epsilon}, \tag{2.53}$$

where $\tilde{\epsilon} > 0$ is sufficiently small. Then we have the following decay estimates:

$$t^{\frac{1}{2}}\|\partial\Gamma^{\leq J-2}u\|_{L_x^\infty} + t^{\frac{1}{2}}\|\langle|x|-t\rangle\partial^2\Gamma^{\leq J-3}u\|_{L_x^\infty(|x|>\frac{t}{10})} + \|\langle|x|-t\rangle\partial^2\Gamma^{\leq J-3}u\|_{L_x^\infty} \lesssim E_J^{\frac{1}{2}}; \tag{2.54}$$

$$\|\partial^2\Gamma^{\leq J-3}u\|_{L_x^\infty(|x|<\frac{t}{2})} \lesssim t^{-\frac{3}{2}}E_J^{\frac{1}{2}}; \tag{2.55}$$

$$\|\langle|x|-t\rangle\partial^2\Gamma^{\leq J-3}u\|_{L_x^\infty} \lesssim t^{-\frac{1}{2}}E_J^{\frac{1}{2}}; \tag{2.56}$$

$$\|\frac{T\Gamma^{\leq J-2}u}{\langle|x|-t\rangle}\|_{L_x^\infty} + \|T\partial\Gamma^{\leq J-3}u\|_{L_x^\infty} \lesssim t^{-\frac{3}{2}}E_J^{\frac{1}{2}}; \tag{2.57}$$

$$\|\frac{T\Gamma^{\leq J-1}u}{\langle|x|-t\rangle}\|_{L_x^2} + \|T\partial\Gamma^{\leq J-1}u\|_{L_x^2} \lesssim t^{-1}E_J^{\frac{1}{2}}. \tag{2.58}$$

More generally, for any integer $J_1 \geq 1$, we have

$$\|\frac{T\Gamma^{\leq J_1}u}{\langle|x|-t\rangle}\|_{L_x^2} \lesssim t^{-1}\|\partial\Gamma^{\leq J_1+1}u\|_{L_x^2}. \tag{2.59}$$

Proof. We shall take $\tilde{\epsilon}$ sufficiently small so that Lemma 2.3 can be applied. The estimate (2.54) follows from Lemma 2.3 and Lemma 2.4. To derive the estimate (2.55), we choose $\psi \in C_c^\infty(\mathbb{R}^2)$ such that $\psi(z) \equiv 1$ for $|z| \leq 0.5$ and $\psi(z) \equiv 0$ for $|z| \geq 0.52$. Applying the interpolation inequality $\|\tilde{v}\|_\infty \lesssim \|\tilde{v}\|_2^{\frac{1}{2}}\|\Delta\tilde{v}\|_2^{\frac{1}{2}}$ with $\tilde{v}(x) = \psi(\frac{x}{t})\partial^2\Gamma^{\leq J-3}u$, we obtain

$$\|\psi(\frac{x}{t})\partial^2\Gamma^{\leq J-3}u\|_\infty \lesssim \|\psi(\frac{x}{t})\partial^2\Gamma^{\leq J-3}u\|_2^{\frac{1}{2}}\|\Delta(\psi(\frac{x}{t})\partial^2\Gamma^{\leq J-3}u)\|_2^{\frac{1}{2}}. \tag{2.60}$$

By Lemma 2.3, it is not difficult to check that

$$\|\Delta(\psi(\frac{x}{t})\partial^2\Gamma^{\leq J-3}u)\|_2 \lesssim t^{-2}E_J^{\frac{1}{2}}, \quad \|\psi(\frac{x}{t})\partial^2\Gamma^{\leq J-3}u\|_2 \lesssim t^{-1}E_J^{\frac{1}{2}}. \tag{2.61}$$

The estimate (2.55) then follows. For the estimate (2.56) we only need to examine the regime $|x| \geq t/2$. Clearly

$$\begin{aligned}
&|\langle|x|-t\rangle\partial^2\Gamma^{\leq J-3}u| \\
\lesssim&\langle t\rangle^{-\frac{1}{2}}(\|\partial_r^{\leq 1}\partial_\theta^{\leq 1}(\langle|\cdot|-t\rangle\partial^2\Gamma^{\leq J-3}u)\|_2 \qquad \text{(by (2.8))} \\
\lesssim&\langle t\rangle^{-\frac{1}{2}}E_J^{\frac{1}{2}}. \qquad \text{(by (2.28))}.
\end{aligned}$$

For (2.57), we note that the case $|x| \leq \frac{t}{2}$ follows from (2.54) and (2.55). On the other hand, for $|x| > \frac{t}{2}$ we denote $\tilde{u} = \Gamma^{\leq J-2}u$ and estimate $\|\frac{T_1\tilde{u}}{\langle|x|-t\rangle}\|_{L_x^\infty(|x|>\frac{t}{2})}$ (the estimate for T_2 is similar). Recall that

$$\begin{aligned}
T_1\tilde{u} &= \omega_1\partial_t\tilde{u} + \partial_1\tilde{u} = \omega_1(\partial_t+\partial_r)\tilde{u} - \frac{\omega_2}{r}\partial_\theta\tilde{u} \\
&= \omega_1\frac{1}{t+r}(2L_0\tilde{u} - (t-r)\partial_-\tilde{u}) - \frac{\omega_2}{r}\partial_\theta\tilde{u}.
\end{aligned} \tag{2.62}$$

Clearly for $r = |x| \geq \frac{t}{2}$,

$$\begin{aligned}
\left|\frac{T_1\tilde{u}}{\langle r-t\rangle}\right| &\lesssim \frac{1}{t}\Big(\left|\frac{L_0\tilde{u}}{\langle r-t\rangle}\right| + |\partial\tilde{u}|\Big) + \left|\frac{\partial_\theta\tilde{u}}{r\langle r-t\rangle}\right| \\
&\lesssim t^{-\frac{3}{2}}\|\partial\Gamma^{\leq 2}\tilde{u}\|_2 + t^{-\frac{3}{2}}\|\partial\Gamma^{\leq 1}\partial_\theta\tilde{u}\|_2 \qquad \text{(by (2.10), (2.54))} \\
&\lesssim t^{-\frac{3}{2}}E_J^{\frac{1}{2}}.
\end{aligned} \tag{2.63}$$

The estimates for (2.58)–(2.59) can be derived similarly. □

3. Proof of Theorem 1.1

In this section we carry out the proof of Theorem 1.1. Write $v = \Gamma^\alpha u$. By Lemma 2.2 we have

$$\square v = \sum_{\alpha_1+\alpha_2\le\alpha} g^{kij}_{\alpha;\alpha_1,\alpha_2}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u \tag{3.1}$$

$$= g^{kij}\partial_k v\partial_{ij}u + g^{kij}\partial_k u\partial_{ij}v + \sum_{\substack{\alpha_1<\alpha,\alpha_2<\alpha;\\ \alpha_1+\alpha_2\le\alpha}} g^{kij}_{\alpha;\alpha_1,\alpha_2}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u. \tag{3.2}$$

Choose $p(t,r) = q(r-t)$, where

$$q(s) = \int_0^s \langle\tau\rangle^{-1}\big(\log(2+\tau^2)\big)^{-2}d\tau, \quad s\in\mathbb{R}; \tag{3.3}$$

$$-\partial_t p = \partial_r p = q'(r-t) = \langle r-t\rangle^{-1}\big(\log(2+(r-t)^2)\big)^{-2}. \tag{3.4}$$

Multiplying both sides of (3.1) by $e^p\partial_t v$, we obtain

$$\begin{aligned}
\mathrm{LHS} &= \int e^p\partial_{tt}v\partial_t v - \int e^p\Delta v\partial_t v\\
&= \int e^p\partial_{tt}v\partial_t v + \int e^p\nabla v\cdot\nabla\partial_t v + \int e^p\nabla v\cdot\nabla p\partial_t v\\
&= \frac12\frac{d}{dt}\int e^p(\partial v)^2 - \frac12\int e^p|\partial v|^2 p_t + \int e^p\nabla v\cdot\nabla p\partial_t v\\
&= \frac12\frac{d}{dt}\|e^{\frac p2}\partial v\|_{L^2}^2 + \frac12\int e^p q'\cdot\Big(|\partial_+ v|^2 + \frac{|\partial_\theta v|^2}{r^2}\Big)\\
&= \frac12\frac{d}{dt}\|e^{\frac p2}\partial v\|_{L^2}^2 + \frac12\int e^p q'|Tv|^2.
\end{aligned}$$

We shall sum over $|\alpha|\le m_0$, where $m_0 = m-1$ or m is a running parameter. To simplify the notation in the subsequent nonlinear estimates, we introduce the following terminology.

Notation. For a quantity $X(t)$, we shall write $X(t) = \mathrm{OK}$ if $X(t)$ can be written as

$$X(t) = \frac{d}{dt}X_1(t) + X_2(t) + X_3(t), \tag{3.5}$$

where (below $\alpha_0>0$ is some constant)

$$\begin{aligned}
&|X_1(t)| \ll \|(\partial\Gamma^{\le m_0}u)(t,\cdot)\|^2_{L^2_x(\mathbb{R}^2)}, \quad |X_2(t)| \lesssim t^{-1-\alpha_0},\\
&|X_3(t)| \ll \sum_{|\alpha|\le m_0}\int e^p q'|(T\Gamma^\alpha u)(t,x)|^2dx.
\end{aligned} \tag{3.6}$$

In yet other words, the quantity X will be controllable if either it can be absorbed into the energy, or can be controlled by the weighted L^2-norm of the good unknowns from the Alinhac weight, or it is integrable in time.

We now proceed with the nonlinear estimates. We shall discuss several cases.

3.1. The case $\alpha_1 < \alpha$ and $\alpha_2 < \alpha$. Since $g^{kij}_{\alpha;\alpha_1,\alpha_2}$ still satisfies the null condition, by (2.13) we have

$$\begin{aligned}
&\sum_{\substack{\alpha_1<\alpha,\alpha_2<\alpha\\ \alpha_1+\alpha_2\le\alpha}} g^{kij}_{\alpha;\alpha_1,\alpha_2}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u\\
&=\sum_{\substack{\alpha_1<\alpha,\alpha_2<\alpha\\ \alpha_1+\alpha_2\le\alpha}} g^{kij}_{\alpha;\alpha_1,\alpha_2}(T_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u-\omega_k\partial_t\Gamma^{\alpha_1}uT_i\partial_j\Gamma^{\alpha_2}u\\
&\quad+\omega_k\omega_i\partial_t\Gamma^{\alpha_1}uT_j\partial_t\Gamma^{\alpha_2}u).
\end{aligned}\tag{3.7}$$

`Estimate of` $\|T_k\Gamma^{\alpha_1}u\partial^2\Gamma^{\alpha_2}u\|_2$**.** If $|\alpha_1|\le|\alpha_2|$, then by Lemma 2.5 we have

$$\|T_k\Gamma^{\alpha_1}u\partial^2\Gamma^{\alpha_2}u\|_2\lesssim\|\frac{T_k\Gamma^{\alpha_1}u}{\langle r-t\rangle}\|_\infty\cdot\|\langle r-t\rangle\partial^2\Gamma^{\alpha_2}u\|_2\lesssim t^{-\frac{3}{2}}E^{\frac12}_{\lfloor\frac{m_0}{2}\rfloor+3}E^{\frac12}_{m_0}.\tag{3.8}$$

If $|\alpha_1|>|\alpha_2|$, then we have

$$\|T_k\Gamma^{\alpha_1}u\partial^2\Gamma^{\alpha_2}u\|_2\lesssim\|\frac{T_k\Gamma^{\alpha_1}u}{\langle r-t\rangle}\|_2\cdot\|\langle r-t\rangle\partial^2\Gamma^{\alpha_2}u\|_\infty\lesssim t^{-\frac{3}{2}}E^{\frac12}_{\lfloor\frac{m_0}{2}\rfloor+3}E^{\frac12}_{m_0}.\tag{3.9}$$

`Estimate of` $\|\partial\Gamma^{\alpha_1}uT\partial\Gamma^{\alpha_2}u\|_2$. If $|\alpha_1|\le|\alpha_2|$, we have

$$\|\partial\Gamma^{\alpha_1}uT\partial\Gamma^{\alpha_2}u\|_2\lesssim\|\partial\Gamma^{\alpha_1}u\|_\infty\cdot\|T\partial\Gamma^{\alpha_2}u\|_2\lesssim t^{-\frac{3}{2}}E^{\frac12}_{\lfloor\frac{m_0}{2}\rfloor+3}E^{\frac12}_{m_0}.\tag{3.10}$$

If $|\alpha_1|>|\alpha_2|$, we have

$$\|\partial\Gamma^{\alpha_1}uT\partial\Gamma^{\alpha_2}u\|_2\lesssim\|\partial\Gamma^{\alpha_1}u\|_2\cdot\|T\partial\Gamma^{\alpha_2}u\|_\infty\lesssim t^{-\frac{3}{2}}E^{\frac12}_{\lfloor\frac{m_0}{2}\rfloor+3}E^{\frac12}_{m_0}.\tag{3.11}$$

Collecting the estimates, we have proved

$$\|\sum_{\substack{\alpha_1<\alpha,\alpha_2<\alpha\\ \alpha_1+\alpha_2\le\alpha}} g^{kij}_{\alpha;\alpha_1,\alpha_2}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u\|_2\lesssim t^{-\frac{3}{2}}E^{\frac12}_{\lfloor\frac{m_0}{2}\rfloor+3}E^{\frac12}_{m_0}.\tag{3.12}$$

3.2. The case $\alpha_2=\alpha$. Noting that $g^{kij}_{\alpha;0,\alpha}=g^{kij}$, we have

$$\begin{aligned}
&\int g^{kij}\partial_k u\partial_{ij}v\partial_t ve^p\\
&=\mathrm{OK}\underbrace{-\int g^{kij}\partial_{jk}u\partial_i v\partial_t ve^p}_{I_1}\underbrace{-\int g^{kij}\partial_k u\partial_i v\partial_t v\partial_j(e^p)}_{I_2}\\
&\quad-\int g^{kij}\partial_k u\partial_i v\partial_{tj}ve^p.
\end{aligned}\tag{3.13}$$

Here in the above, the term "OK" is zero if $\partial_j=\partial_1$ or ∂_2. This term is nonzero when $\partial_j=\partial_t$, i.e. we should absorb it into the energy when integrating by parts in the time variable.

Further integration by parts gives

$$
\begin{aligned}
&-\int g^{kij}\partial_k u\partial_i v\partial_{tj} v e^p \\
&= \mathrm{OK} + \underbrace{\int g^{kij}\partial_{tk} u\partial_i v\partial_j v e^p}_{I_3} + \underbrace{\int g^{kij}\partial_k u\partial_i v\partial_j v\partial_t (e^p)}_{I_4} \\
&\quad + \int g^{kij}\partial_k u\partial_{it} v\partial_j v e^p. \qquad (3.14)
\end{aligned}
$$

$$
\begin{aligned}
&\int g^{kij}\partial_k u\partial_{it} v\partial_j v e^p \\
&= \mathrm{OK} \underbrace{-\int g^{kij}\partial_{ik} u\partial_t v\partial_j v e^p}_{I_5} \underbrace{-\int g^{kij}\partial_k u\partial_t v\partial_j v\partial_i (e^p)}_{I_6} \\
&\quad - \int g^{kij}\partial_k u\partial_t v\partial_{ij} v e^p. \qquad (3.15)
\end{aligned}
$$

It follows that

$$
2\int g^{kij}\partial_k u\partial_{ij} v\partial_t v e^p = (I_1+I_3+I_5)+(I_2+I_4+I_6)+\mathrm{OK}\,.
$$

Observe that if $\varphi=\partial_k u$ or $\varphi=e^p$, then

$$
\begin{aligned}
&-\partial_j\varphi\partial_i v\partial_t v+\partial_t\varphi\partial_i v\partial_j v-\partial_i\varphi\partial_t v\partial_j v \\
=&-T_j\varphi\partial_i v\partial_t v+\omega_j\partial_t\varphi\partial_i v\partial_t v+\partial_t\varphi\partial_i v\partial_j v-T_i\varphi\partial_t v\partial_j v+\omega_i\partial_t\varphi\partial_t v T_j v \\
&-\omega_i\omega_j\partial_t\varphi(\partial_t v)^2 \\
=&-T_j\varphi\partial_i v\partial_t v+\partial_t\varphi\partial_i v T_j v-T_i\varphi\partial_t v\partial_j v+\omega_i\partial_t\varphi\partial_t v T_j v-\omega_i\omega_j\partial_t\varphi(\partial_t v)^2 \\
=&-T_j\varphi\partial_i v\partial_t v+\partial_t\varphi T_i v T_j v-T_i\varphi\partial_t v\partial_j v-\omega_i\omega_j\partial_t\varphi(\partial_t v)^2. \qquad (3.16)
\end{aligned}
$$

By (3.16) and rewriting $\partial_t\varphi=\partial_k\partial_t u=T_k\partial_t u-\omega_k\partial_{tt}u$, we have

$$
\begin{aligned}
&I_1+I_3+I_5 \\
=&\int g^{kij}(-T_j\partial_k u\partial_i v\partial_t v+\partial_t\partial_k u T_i v T_j v-T_i\partial_k u\partial_t v\partial_j v \\
&-\omega_i\omega_j T_k\partial_t u(\partial_t v)^2)e^p dx. \qquad (3.17)
\end{aligned}
$$

By Lemma 2.5, we have $\|T\partial u\|_\infty \lesssim t^{-\frac{3}{2}}E_3^{\frac{1}{2}}$ and $\|\langle r-t\rangle\partial^2 u\|_\infty \lesssim t^{-\frac{1}{2}}E_3^{\frac{1}{2}}$. Clearly then

$$
\begin{aligned}
&\int_{r<\frac{t}{2}\text{ or }r>2t}|\partial^2 u||Tv|^2dx \lesssim t^{-\frac{3}{2}}E_3^{\frac{1}{2}}E_{m_0}, \\
&\int_{r\sim t}|\partial^2 u||Tv|^2dx \ll \int e^p q'|Tv|^2dx. \qquad (3.18)
\end{aligned}
$$

It follows that

$$
I_1+I_3+I_5 \lesssim \mathrm{OK}+t^{-\frac{3}{2}}E_3^{\frac{1}{2}}E_{m_0}. \qquad (3.19)
$$

Plugging $\varphi = e^p$ in (3.16) and noting that $T_j(e^p) = 0$, we have

$$\begin{aligned} & I_2 + I_4 + I_6 \\ = & \int g^{kij} \partial_k u \Big(- T_j(e^p)\partial_i v \partial_t v - T_i(e^p)\partial_t v \partial_j v - \omega_i\omega_j(\partial_t v)^2 \partial_t(e^p) + \partial_t(e^p) T_i v T_j v \Big) \\ = & \int g^{kij} \left(-T_k u \cdot \omega_i \omega_j (\partial_t v)^2 \partial_t(e^p) + \partial_k u \partial_t(e^p) T_i v T_j v \right). \end{aligned}$$

By Lemma 2.5 we have $||Tu||\partial_t(e^p)| \lesssim t^{-\frac{3}{2}} E_3^{\frac{1}{2}}$. Clearly

(3.20)
$$\|\partial u \partial_t(e^p)\|_{L_x^\infty(r<\frac{t}{2},\text{ or } r>2t)} \lesssim t^{-\frac{3}{2}} E_3^{\frac{1}{2}}, \qquad \int_{r\sim t} |\partial u \partial_t (e^p)||Tv|^2 dx \ll \int e^p q' |Tv|^2 dx.$$

Thus

$$I_2 + I_4 + I_6 \lesssim \text{OK} + t^{-\frac{3}{2}} E_3^{\frac{1}{2}} E_{m_0}.$$

This concludes the case $\alpha_2 = \alpha$.

3.3. The case $\alpha_1 = \alpha$, $\alpha_2 = 0$. By (2.13), we have

$$\int g^{kij} \partial_k v \partial_{ij} u \partial_t v e^p = \int g^{kij} (T_k v \partial_{ij} u - \omega_k \partial_t v T_i \partial_j u + \omega_k \omega_i \partial_t v T_j \partial_t u) \partial_t v e^p$$

By Lemma 2.5, all terms containing $T\partial u$ decay as $O(t^{-\frac{3}{2}} E_3^{\frac{1}{2}} E_{m_0})$. Thus

$$\int g^{kij} \partial_k v \partial_{ij} u \partial_t v e^p \lesssim \text{OK} + t^{-\frac{3}{2}} E_3^{\frac{1}{2}} E_{m_0} + \underbrace{\int g^{kij} T_k v \partial_{ij} u \partial_t v e^p}_{=:Y_1}. \tag{3.21}$$

We now discuss two cases.

Case 1: $m_0 = m - 1$. By Lemma 2.5, we have

$$\|\frac{Tv}{\langle |x| - t\rangle}\|_{L_x^2} \lesssim t^{-1} E_{m_0+1}^{\frac{1}{2}} = t^{-1} E_m^{\frac{1}{2}}, \qquad \|\langle |x| - t\rangle \partial_{ij} u\|_{L_x^\infty} \lesssim t^{-\frac{1}{2}} E_3^{\frac{1}{2}}. \tag{3.22}$$

Thus

$$|Y_1| \lesssim t^{-\frac{3}{2}} E_3^{\frac{1}{2}} E_m^{\frac{1}{2}} E_{m_0}^{\frac{1}{2}}. \tag{3.23}$$

Case 2: $m_0 = m$. By using Cauchy-Schwartz, we have

$$\begin{aligned} |Y_1| &\le \text{OK} + \text{const} \cdot \int \frac{1}{q'} |\partial_{ij} u|^2 |\partial_t v|^2 dx \\ &\le \text{OK} + \frac{1}{t} E_3 E_m \qquad \text{(by (2.56))}. \end{aligned} \tag{3.24}$$

Collecting all the estimates and assuming the norm of the initial data is sufficiently small, we then obtain for some small constants $\epsilon_3 > 0$, $\epsilon_4 > 0$,

$$\sup_{t\ge 2} E_{m-1}(u(t,\cdot)) \le \epsilon_3 \ll 1, \qquad \sup_{t\ge 2} \frac{E_m(u(t,\cdot))}{t^{\epsilon_4}} \le 1. \tag{3.25}$$

This concludes the proof of Theorem 1.1.

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