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An elliptic nonlinear system of multiple functions with application

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Abstract. The purpose of this paper is to give a sufficient conditions for the existence and uniqueness of positive solutions to a rather general type of elliptic system of the Dirichlet problem on a bounded domain Ω in R^n. Also considered are the effects of perturbations on the coexistence state and uniqueness. The techniques used in this paper are super-sub solutions method, eigenvalues of operators, maximum principles, spectrum estimates, inverse function theory, and general elliptic theory. The arguments also rely on some detailed properties for the solution of logistic equations. These results yield an algebraically computable criterion for the positive coexistence of competing species of animals in many biological models.

Contents

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1. Introduction

One of the prominent subjects of study and analysis in mathematical biology concerns the competition of two or more species of animals in the same environment. Especially pertinent areas of investigation include the conditions under which the species can coexist, as well as the conditions under which any one of the species becomes extinct, that is, one of the species is excluded by the others. In this paper, we focus on the general competition model to better understand the competitive interactions between multiple species. Specifically, we investigate the conditions needed for the coexistence of the general N species when the factors affecting them are fixed or perturbed. In earlier literature, the models were concerned with studying those with homogeneous Neumann boundary conditions. Later on, the more important Dirichlet problems, which allow flux across the boundary, became the subject of study.

2. Literature Review

Within the academia of mathematical biology, extensive academic work has been devoted to investigation of the simple competition model, commonly known as the Lotka-Volterra competition model. This system describes the competitive interaction of two species residing in the same environment in the following manner:

Suppose two species of animals, rabbits and squirrels for instance, are competing in a bounded domain Ω. Let $u(x,t)$ and $v(x,t)$ be densities of the two habitats in the place x of Ω at time t. Then we have the dynamic competition model

$$\begin{cases} \begin{array}{l} u_t(x,t) = \Delta u(x,t) + au(x,t) - bu^2(x,t) - cu(x,t)v(x,t) \\ v_t(x,t) = \Delta v(x,t) + dv(x,t) - fv^2(x,t) - eu(x,t)v(x,t) \end{array} \quad \text{in } \Omega \times [0,\infty), \\ u(x,t) = v(x,t) = 0 \ \text{ for } \ x \in \partial\Omega, \end{cases}$$

where $a, d > 0$ are self-reproduction rates, $b, f > 0$ are self-limitation rates, and $c, e > 0$ are competition rates. Here we are interested in the time independent, positive solutions, i.e. the positive solutions $u(x), v(x)$ of

$$\begin{cases} \begin{array}{l} \Delta u(x) + u(x)(a - bu(x) - cv(x)) = 0 \\ \Delta v(x) + v(x)(d - fv(x) - eu(x)) = 0 \end{array} \quad \text{in } \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

which are called the coexistence state or the steady state. The coexistence state is the positive density solution depending only on the spatial variable x, not on the time variable t, and so its existence means that the two species of animals can live peacefully and forever.

The mathematical community has already established several results for the existence, uniqueness and stability of the positive steady state solution to (2.1) (see [**1**], [**2**], [**3**], [**4**], [**5**], [**6**], [**7**].).

One of the initial important results for the time-independent Lotka-Volterra model was obtained by Cosner and Lazer. In 1984, they published the following sufficient conditions for the existence and uniqueness of a positive steady state solution to (2.1):

Theorem 2.1. *(in* [**4**]*)*
(A) If $a > \lambda_1 + \frac{cd}{f}, d > \lambda_1 + \frac{ae}{b}$, *where* λ_1 *is the smallest eigenvalue of* $-\Delta$ *with*

homogeneous boundary conditions as in the Lemma 3.3, then there exist positive smooth functions u and v in Ω satisfying (2.1).
(B) Furthermore, if $4bf > \frac{fc^2}{b}\sup_{x\in\Omega}[\frac{\theta_a(x)}{\theta_{d-\frac{ae}{b}}(x)}] + 2ce + \frac{be^2}{f}\sup_{x\in\Omega}[\frac{\theta_d(x)}{\theta_{a-\frac{cd}{f}}(x)}]$, *where* $\theta_M(x)$ *for* $M > 0$ *is the unique positive solution to the logistic equation as mentioned in the Lemma 3.5, then the positive solution is unique.*

Biologically, the conditions in Theorem 2.1 implies that if the self-reproduction and self-limitation rates are relatively large, and the competition rates are relatively small, in other words, if members of each species interact strongly among themselves and weakly with members of the other species, then there is a unique positive steady state solution to (2.1), that is, the two species within the same domain will coexist indefinitely at unique population densities.

In 1989, Cantrell and Cosner extended these results by proving that the reproduction rates may vary within bounds without losing the uniqueness result, given certain conditions. Biologically, Cantrell and Cosner's theorem suggests that two species can relax ecologically and maintain a coexistence state. Their primary result is given below:

THEOREM 2.2. *(in [**3**])*
If $a = d > \lambda_1, b = f = 1$, *and* $0 < c, e < 1$, *then there is a neighborhood* V *of* (a, a) *such that if* $(a_0, d_0) \in V$, *then (2.1) with* $(a, d) = (a_0, d_0)$ *has a unique positive solution.*

In Theorem 2.2, the condition $0 < c, e < 1$ biologically implies that the self-limitation rates of both species are relatively larger than competition rates. This condition plays an important role in the proof of Cantrell and Cosner's theorem by implying the invertibility of the Frechet derivative (linearization) of (2.1) at a fixed reproduction rate (a, a).

The work of Lazer, Cosner, and Cantrell provides insight into the competitive interactions of two species operating under the conditions described in the Lotka-Volterra model. However, their results are somewhat limited by a few key assumptions. In the Lotka-Volterra model that they studied, the rate of change of densities largely depends on constant self-reproduction, self-limitation, and competition rates. The model also assumes a linear relationship of the terms affecting the rate of change for both population densities.

However, in reality, the rates of change of population densities may vary in a more complicated and irregular manner than can be described by the simple competition model. Therefore, in the last decades, significant research has been focused on the existence and uniqueness of the positive steady state solution of the general competition model for general N species,

$$\begin{cases} (u_i)_t(x,t) = \Delta u_i(x,t) + g_i(u_1(x,t), ..., u_N(x,t)), & \text{in } \Omega \times R^+, \\ u_i(x,t)|_{\partial\Omega} = 0, i = 1, ..., N, \end{cases}$$

or, equivalently, the positive solution to

$$\begin{cases} \Delta u_i(x,t) + g_i(u_1(x,t), ..., u_N(x,t)) = 0, & \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1, ..., N, \end{cases} \tag{2.2}$$

where $g_i \in C^2$ designate reproduction, self-limitation and competition rates that satisfy certain growth conditions(see [**8**], [**9**], [**10**], [**11**].).

Because of its broader applicability, the general competition model has become a more popular subject of research within the mathematical community over the past few years.

The functions g_i's describe how species $u_1, ..., u_N$ interact among themselves and with each other.

The followings are questions raised in the general model with nonlinear growth rates.

Problem 1 : What are the sufficient conditions for existence of positive solutions?
Problem 2 : What are the sufficient conditions for uniqueness of positive solutions?
Problem 3 : What is the effect of perturbation for existence and uniqueness? In our analysis, we focus on the conditions required for the maintenance of the coexistence state of the model when bounded functions g_i's are slightly perturbed. Biologically, our conclusion implies that species may slightly relax ecologically and yet continue to coexist at unique densities.

In Section 4, we establish sufficient conditions for the existence and non-existence of positive solution of the system that generalizes the Theorem 2.1. We also achieve solution estimates in the Section 5 to prove the uniqueness and the invertibility of linearization in Sections 6, 7 and 8, where we investigate the effect of perturbation for existence and uniqueness that generalizes the Theorem 2.2.

An especially significant aspect of the global uniqueness result is the stability of the positive steady state solution, which has become an important subject of mathematical study. Indeed, researchers have obtained several stability results for the Lotka-Volterra model with constant rates(see [**3**], [**4**], [**5**], [**8**].). However, the stability of the steady state solution for the general model remains open to investigation. The research presented in this paper therefore begins the mathematical community's discussion on the stability of the steady state solution for the general competition model.

3. Preliminaries

Before entering into our primary arguments and results, we must first present a few preliminary items that we later employ throughout the proofs detailed in this paper. The following definition and lemmas are established and accepted throughout the literature on our topic.

Definition 3.1. (Super and Sub solutions)
The vector functions $(\bar{u}^1, ..., \bar{u}^N), (\underline{u}^1, ..., \underline{u}^N)$ form an super/sub solution pairs for the system

$$\begin{cases} \Delta u^i + g_i(u^1, ..., u^N) = 0 \text{ in } \Omega, \\ \qquad\qquad u^i = 0 \text{ on } \partial\Omega, \end{cases}$$

if for $i = 1, ..., N$

$$\begin{cases} \Delta \bar{u}^i + g_i(u^1, ..., u^{i-1}, \bar{u}^i, u^{i+1}, ..., u^N) \le 0, \\ \Delta \underline{u}^i + g_i(u^1, ..., u^{i-1}, \underline{u}^i, u^{i+1}, ..., u^N) \ge 0 \text{ in } \Omega, \\ \underline{u}^j \le u^j \le \bar{u}^j, j \ne i, \end{cases}$$

and

$$\begin{aligned} \underline{u}^i \le \bar{u}^i &\text{ on } \Omega, \\ \underline{u}^i \le 0 \le \bar{u}^i &\text{ on } \partial\Omega. \end{aligned}$$

LEMMA 3.2. *If g^i in the Definition 3.1 are in C^1 and the system admits an super/sub solution pairs $(\bar{u}^1, ..., \bar{u}^N), (\underline{u}^1, ..., \underline{u}^N)$, then there is a solution of the system in Definition 3.1 with $\underline{u}^i \le u^i \le \bar{u}^i$ in $\bar{\Omega}$. If*

$$\begin{aligned} \Delta \bar{u}^i + g_i(\bar{u}^1, ..., \bar{u}^N) \ne 0, \\ \Delta \underline{u}^i + g_i(\underline{u}^1, ..., \underline{u}^N) \ne 0 \end{aligned}$$

in Ω for $i = 1, ..., N$, then $\underline{u}^i < u^i < \bar{u}^i$ in Ω.

LEMMA 3.3. *(The first eigenvalue) Consider*

$$\begin{cases} -\Delta u + q(x)u = \lambda u \ \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{3.1}$$

where $q(x)$ is a smooth function from Ω to R and Ω is a bounded domain in R^n.
(A) The first eigenvalue $\lambda_1(q)$ of (3.1), denoted by simply λ_1 when $q \equiv 0$, is simple with a positive eigenfunction ϕ_q.
(B) If $q_1(x) < q_2(x)$ for all $x \in \Omega$, then $\lambda_1(q_1) < \lambda_1(q_2)$.
(C)(Variational Characterization of the first eigenvalue)

$$\lambda_1(q) = \min_{\phi \in W_0^1(\Omega), \phi \ne 0} \frac{\int_\Omega (|\nabla \phi|^2 + q\phi^2) dx}{\int_\Omega \phi^2 dx}.$$

LEMMA 3.4. *(Maximum Principles) Consider*

$$Lu = \sum_{i,j=1}^n a_{ij}(x) D_{ij} u + \sum_{i=1}^n a_i(x) D_i u + a(x) u = f(x) \ \text{ in } \Omega,$$

where Ω is a bounded domain in R^n with
(M1) $\partial\Omega \in C^{2,\alpha}(0 < \alpha < 1)$,
(M2) $|a_{ij}(x)|_\alpha, |a_i(x)|_\alpha, |a(x)|_\alpha \le M (i, j = 1, ..., n)$,
(M3) L is uniformly elliptic in $\bar{\Omega}$ with ellipticity constant $\gamma > 0$, i.e., for every $x \in \bar{\Omega}$ and every real vector $\xi = (\xi_1, ..., \xi_n)$,

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \ge \gamma \sum_{i=1}^n |\xi_i|^2.$$

Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of $Lu \ge 0 (Lu \le 0)$ in Ω.
(A) If $a(x) \equiv 0$, then $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u (\min_{\bar{\Omega}} u = \min_{\partial\Omega} u)$.
(B) If $a(x) \equiv 0$, Ω is connected and u attains its maximum (minimum) at an interior point of Ω, then u is identically a constant in Ω.
(C) If $a(x) \le 0$, then $\max_{\bar{\Omega}} u \le \max_{\partial\Omega} u^+ (\min_{\bar{\Omega}} u \ge -\max_{\partial\Omega} u^-)$, where $u^+ = \max(u, 0), u^- = -\min(u, 0)$. [Thus a nonnegative maximum(nonpositive minimum) must be attained on the boundary.] In particular, if $Lu = 0$ in Ω, then $\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|$.

(D) If $a(x) \leq 0$, Ω is connected and u attains a nonnegative maximum(nonpositive minimum) at an interior point of Ω, then u is identically a constant in Ω.

In our proof, we also employ accepted conclusions concerning the solutions of the following logistic equations.

LEMMA 3.5. *Consider*

$$\begin{cases} \Delta u + uf(u) = 0 \ \ in \ \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$

where f is a decreasing C^1 function such that there exists $c_0 > 0$ such that $f(u) \leq 0$ for $u \geq c_0$, and Ω is a bounded domain in R^n.

(1) *If $f(0) > \lambda_1$, then the above equation has a unique positive solution, where λ_1 is the first eigenvalue of $-\Delta$ with homogeneous boundary condition as in the Lemma 3.3. We denote this unique positive solution as θ_f.*
(2) *If $f(0) \leq \lambda_1$, then $u \equiv 0$ is the only nonnegative solution to the above equation.*

The main property about this positive solution is that θ_f is increasing as f is increasing.

Especially, for $a > \lambda_1$, we denote θ_a as the unique positive solution of

$$\begin{cases} \Delta u + u(a - u) = 0 \ \ \text{in} \ \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0. \end{cases}$$

Hence, θ_a is increasing as $a > 0$ is increasing.

We need the following lemmas for the perturbation of uniqueness in the Sections 7 and 8.

LEMMA 3.6. *(Implicit Function Theorem)*
Let X, Y, Z be Banach spaces. For a given $(u_0, v_0) \in X \times Y$ and $a, b > 0$, let $S = \{(u, v) :\| u - u_0 \|\leq a, \| v - v_0 \|\leq b\}$. Suppose $A : S \to Z$ satisfies the followings:

(1) A is continuous.
(2) $A_v(\cdot, \cdot)$ exists and is continuous in S(in the operator norm).
(3) $A(u_0, v_0) = 0$.
(4) $[A_v(u_0, v_0)]^{-1}$ exists and belongs to $B(Z, Y)$.

Then there are neighborhoods U of u_0 and V of v_0 such that the equation $A(u, v) = 0$ has exactly one solution $v \in V$ for every $u \in U$. The solution v depends continuously on u.

LEMMA 3.7. *(Schauder's Boundary Estimate)*
We consider

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u + \sum_{i=1}^{n} a_i(x) D_i u + a(x) u = f(x), x \in \Omega$$

with (M1), (M2), and (M3) in the Lemma 3.4. If $u \in C^{2,\alpha}(\bar{\Omega})$ and $u|_{\partial\Omega} = \phi \in C^{2,\alpha}(\partial\Omega)$, then

$$|u|_{2,\alpha} \leq c(|Lu|_\alpha + |u|_0 + |\phi|^{\partial\Omega}_{2,\alpha})$$

for some $c > 0$ that is independent of u.

LEMMA 3.8. *(Fredholm Alternative)*
Let $T : X \to X$ be a compact linear operator on the normed space X and let $S : X \to X$ be a linear operator such that $S^{-1} \in B(X)$, where $B(X)$ is the set of bounded linear operators on X. Then either one of the followings is true:
(a) The homogeneous equation $T(x) - S(x) = 0$ has a nontrivial solution $x \in X$.
(b) For each $y \in X$ the inhomogeneous equation $T(x) - S(x) = y$ has a unique solution $x \in X$.
Furthermore, in case (b), the operator $(T - S)^{-1} \in B(X)$.

Having established these preliminaries, we now commence our investigation of the general competition model.

4. Existence, Nonexistence

The most general type of elliptic interacting system of N functions with homogeneous boundary condition is

$$(4.1) \qquad \begin{cases} \Delta u_i + g_i(u_1, u_2, ..., u_i, u_{i+1}, ..., u_N) = 0, & \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1, ..., N, \end{cases}$$

where we assume that the C^2 functions g_i's, $i = 1, 2, ..., N$, are relative growth rates satisfying the following so-called growth rate conditions:

(EN_1) $(g_i)_{u_j} < 0, (g_i)_{u_i u_i} < 0, (g_i)_{u_i u_j} < 0$ for $i, j = 1, 2, ..., N, i \neq j$, where in general, $(g_i)_{u_j}$ is the first partial derivative of g_i with respect to the jth variable and $(g_i)_{u_k u_j}$ is the second partial derivative of g_i with respect to the kth and jth variables.
(EN_2) $g_i(u_1, ..., u_i, ..., u_N) \leq u_i (g_i)_{u_i}(u_1, ..., u_{i-1}, \frac{u_i}{2}, u_{i+1}, ..., u_N), i = 1, ..., N$.
(EN_3) $g_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) = 0, i = 1, ..., N$.
(EN_4) There exists a constant $c_0 > 0$ such that $(g_i)_{u_i}(0, ..., 0, u_i, 0, ..., 0) < 0$ for $u_i > c_0$ and $i = 1, ..., N$.

The following inequalities will be useful for later use which are true for any C^2 functions satisfying only (EN_3).

LEMMA 4.1. *For all $u_1, ..., u_N$,*

$$g_i(u_1, ..., u_N) \geq u_i (g_i)_{u_i}(u_1, ..., u_N), i = 1, ...N.$$

PROOF. By (EN_3) and the Mean Value Theorem, there is $\bar{u}_i$ such that $0 \leq \bar{u}_i \leq u_i$ and

$$\begin{aligned} & g_i(u_1, ..., u_{i-1}, u_i, u_{i+1}, ..., u_N) \\ = \; & g_i(u_1, ..., u_{i-1}, u_i, u_{i+1}, ..., u_N) - g_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) \\ = \; & u_i (g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N). \end{aligned}$$

But, by the monotonicity of $(g_i)_{u_i}$, we have

$$\begin{aligned} g_i(u_1, ..., u_N) &= u_i (g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N) \\ &\geq u_i (g_i)_{u_i}(u_1, ..., u_{i-1}, u_i, u_{i+1}, ..., u_N). \end{aligned}$$

The following lemma says that if there is no competition between the species, then we can easily conclude their peaceful coexistence.

Lemma 4.2. *If we consider*

$$\begin{cases} \Delta u_i + g_i(0,...,0,u_i,0,...,0) = 0, & \text{in } \Omega, \\ u_i = 0 \ \text{ on } \partial\Omega, i = 1,...,N, \end{cases}$$

then the conditions $(g_i)_{u_i}(0,...,0) > \lambda_1, i = 1,...,N$ *(i.e, reproductions are relatively large) are sufficient to guarantee the existence of a positive density solution.*

Proof. Let $\underline{u}_i = \theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}$ and $\bar{u}_i = M > 0$ be a constant such that $M > 2c_0$.
Then by $(EN_2),(EN_4)$, the monotonicity of $(g_i)_{u_i}$ and the Lemmas 3.5, 4.1,

$$\begin{aligned} \Delta\underline{u}_i + g_i(0,...,0,\underline{u}_i,0,...,0) &\geq \Delta\underline{u}_i + \underline{u}_i(g_i)_{u_i}(0,...,0,\underline{u}_i,0,...,0) = 0, \\ \Delta\bar{u}_i + g_i(0,...,0,\bar{u}_i,0,...,0) &\leq M(g_i)_{u_i}(0,...,0,\tfrac{M}{2},0,...,0) \\ &\leq M(g_i)_{u_i}(0,...,0,c_0,0,...0) \\ &\leq 0, \end{aligned}$$

and so, $\underline{u}_i$ and $\bar{u}_i$ are subsolution and supersolution to

$$\begin{cases} \Delta u_i + g_i(0,...,0,u_i,0,...,0) &= 0, \\ u_i|_{\partial\Omega} &= 0, \end{cases}$$

respectively. So, by the Lemma 3.2, there is a positive solution u_i to

$$\begin{cases} \Delta u_i + g_i(0,...,0,u_i,0,...,0) &= 0, \\ u_i|_{\partial\Omega} &= 0. \end{cases}$$

But, if there is some competition between the species, then as we notice in the following Theorem, we should have much larger reproduction rates, i.e, we need stronger conditions to guarantee their coexistence.

We establish the following two existence results:

Theorem 4.3. *(A) If* $(g_i)_{u_i}(2c_0,2c_0,...,2c_0,0,2c_0,...,2c_0) > \lambda_1, i = 1,...,N,$ *then (4.1) has a positive solution* $(u_1,...,u_N)$.
(B) If $(g_i)_{u_i}[2\sup(\theta_{(g_1)_{u_1}(\cdot,0,...,0)}), 2\sup(\theta_{(g_2)_{u_2}(0,\cdot,0,...,0)}),...,$
$2\sup(\theta_{(g_{i-1})_{u_{i-1}}(0,...,0,\cdot,0,...,0)}), 0, 2\sup(\theta_{(g_{i+1})_{u_{i+1}}(0,...,0,\cdot,0,...,0)}),...,$
$2\sup(\theta_{(g_N)_{u_N}(0,...,0,\cdot)})] > \lambda_1, i = 1,...,N,$ *then (4.1) has a positive solution* $(u_1,...,u_N)$.

Biologically, we can interpret the conditions in Theorem 4.3 as follows. The functions (g_i)'s, $i = 1,...,N$, and their partial derivatives describe how species interact among themselves and with each other. Hence, the conditions in both (A) and (B) imply that each species must have large enough reproduction capacity for their peaceful coexistence.

Proof. (A) Let $(\underline{u}_1,\underline{u}_2,...,\underline{u}_N) = (\theta_{(g_1)_{u_1}(\cdot,2c_0,2c_0,...,2c_0)},$
$\theta_{(g_2)_{u_2}(2c_0,\cdot,2c_0,2c_0,...,2c_0)},...,\theta_{(g_N)_{u_N}(2c_0,2c_0,...,2c_0,\cdot)})$ and $(\bar{u}_1,\bar{u}_2,...,\bar{u}_N) =$
$(2\theta_{(g_1)_{u_1}(\cdot,0,...,0)}, 2\theta_{(g_2)_{u_2}(0,\cdot,0,...,0)},...,2\theta_{(g_N)_{u_N}(0,...,0,\cdot)})$. Then by the Mean Value Theorem, there is $\tilde{u}_i$ with $0 \leq \tilde{u}_i \leq \underline{u}_i$ such that

$$\begin{aligned} & g_i(\bar{u}_1,...,\bar{u}_{i-1},\underline{u}_i,\bar{u}_{i+1},...,\bar{u}_N) - g_i(\bar{u}_1,...,\bar{u}_{i-1},0,\bar{u}_{i+1},...,\bar{u}_N) \\ = \ & \underline{u}_i(g_i)_{u_i}(\bar{u}_1,...,\bar{u}_{i-1},\tilde{u}_i,\bar{u}_{i+1},...,\bar{u}_N), i = 1,...,N, \end{aligned}$$

and so, by the Lemma 4.1,

$$\begin{aligned}
& \Delta\underline{u}_i + g_i(\bar{u}_1, ..., \bar{u}_{i-1}, \underline{u}_i, \bar{u}_{i+1}, ..., \bar{u}_N) \\
\geq\ & \Delta\underline{u}_i + \underline{u}_i (g_i)_{u_i}(\bar{u}_1, ..., \bar{u}_{i-1}, \underline{u}_i, \bar{u}_{i+1}, ..., \bar{u}_N) \\
=\ & \Delta\underline{u}_i + \underline{u}_i[(g_i)_{u_i}(2c_0, ..., 2c_0, \underline{u}_i, 2c_0, ..., 2c_0) \\
& +(g_i)_{u_i}(\bar{u}_1, ..., \bar{u}_{i-1}, \underline{u}_i, \bar{u}_{i+1}, ..., \bar{u}_N) - (g_i)_{u_i}(2c_0, ..., 2c_0, \underline{u}_i, 2c_0, ..., 2c_0)] \\
=\ & (g_i)_{u_i}(\bar{u}_1, ..., \bar{u}_{i-1}, \underline{u}_i, \bar{u}_{i+1}, ..., \bar{u}_N) - (g_i)_{u_i}(2c_0, ..., 2c_0, \underline{u}_i, 2c_0, ..., 2c_0)]
\end{aligned}$$

for $i = 1, ..., N$.
We claim that $\bar{u}_i \leq c_0$. Suppose $\bar{u}_i(x) > c_0$ for some $x \in \Omega$. Then by the continuity of $\bar{u}_i$, there is an open set $\Omega' \subseteq \Omega$ containing x such that $\bar{u}_i(y) > c_0$ for all $y \in \Omega'$ and $\bar{u}_i|_{\partial\Omega'} = c_0$, so

$$\Delta\bar{u}_i = -\bar{u}_i(g_i)_{u_i}(0, ..., 0, \bar{u}_i, 0, ..., 0) > 0 \ \text{ in } \ \Omega',$$

which is a contradiction to the Maximum Principles Lemma 3.4.
Therefore, by the monotonicity of $(g_i)_{u_i}$, we conclude

$$\begin{aligned}
& \Delta\underline{u}_i + g_i(\bar{u}_1, ..., \bar{u}_{i-1}, \underline{u}_i, \bar{u}_{i+1}, ..., \bar{u}_N) \\
\geq\ & (g_i)_{u_i}(\bar{u}_1, ..., \bar{u}_{i-1}, \underline{u}_i, \bar{u}_{i+1}, ..., \bar{u}_N) - (g_i)_{u_i}(2c_0, ..., 2c_0, \underline{u}_i, 2c_0, ..., 2c_0) \\
\geq\ & 0
\end{aligned}$$

for $i = 1, ..., N$.
Furthermore, by (EN_2) and the monotonicity of $(g_i)_{u_i}$,

$$\begin{aligned}
& \Delta\bar{u}_i + g_i(\underline{u}_1, ..., \underline{u}_{i-1}, \bar{u}_i, \underline{u}_{i+1}, ..., \underline{u}_N) \\
\leq\ & \Delta\bar{u}_i + \bar{u}_i(g_i)_{u_i}(\underline{u}_1, ..., \underline{u}_{i-1}, \tfrac{\bar{u}_i}{2}, \underline{u}_{i+1}, ..., \underline{u}_N) \\
\leq\ & \Delta\bar{u}_i + \bar{u}_i(g_i)_{u_i}(0, ..., 0, \tfrac{\bar{u}_i}{2}, 0, ..., 0) \\
=\ & 2[\Delta\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)} \\
& +\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}(g_i)_{u_i}(0, ..., 0, \theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...0)}, 0, ..., 0)] \\
=\ & 0.
\end{aligned}$$

Therefore, $(\underline{u}_1, ..., \underline{u}_N)$ is a subsolution and $(\bar{u}_1, ..., \bar{u}_N)$ is a supersolution to (4.1), so by the Lemma 3.2, there is a positive solution $(u_1, ..., u_N)$ to (4.1).

(B) Let $(\underline{u}_1, ..., \underline{u}_N) = (\epsilon\phi_0, ..., \epsilon\phi_0)$
and $(\bar{u}_1, ..., \bar{u}_N) = (2\theta_{(g_1)_{u_1}(\cdot,0,...,0)}, 2\theta_{(g_2)_{u_2}(0,\cdot,0,...,0)}, ..., 2\theta_{(g_N)_{u_N}(0,...,0,\cdot)})$, where $\phi_0 > 0$ is the eigenfuntion of $-\Delta$ with homogeneous boundary condition corresponding to the smallest eigenvalue λ_1.
Then by (EN_2) and the monotonicity of $(g_i)_{u_i}$, $i = 1, ..., N$,

$$\begin{aligned}
& \Delta(\bar{u}_i) + g_i(\underline{u}_1, ..., \underline{u}_{i-1}, \bar{u}_i, \underline{u}_{i+1}, ..., \underline{u}_N) \\
\leq\ & 2[\Delta(\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}) \\
& +\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}(g_i)_{u_i}(\underline{u}_1, ..., \underline{u}_{i-1}, \theta_{(g_i)_{u_i}(0,...,0,\cdot 0,...,0)}, \\
& \underline{u}_{i+1}, ..., \underline{u}_N)] \\
\leq\ & 2[\Delta(\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}) \\
& +\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}(g_i)_{u_i}(0, ..., 0, \theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}, 0, ..., 0)] \\
=\ & 0.
\end{aligned}$$

Since $(g_i)_{u_i}[2\sup(\theta_{(g_1)_{u_1}(\cdot,0,...,0)}), 2\sup(\theta_{(g_2)_{u_2}(0,\cdot,0,...,0)}), ...,$
$2\sup(\theta_{(g_{i-1})_{u_{i-1}}(0,...,0,\cdot,0,...,0)}), 0, 2\sup(\theta_{(g_{i+1})_{u_{i+1}}(0,...,0,\cdot,0,...,0)}), ...,$
$2\sup(\theta_{(g_N)_{u_N}(0,...,0,\cdot)})] > \lambda_1$, by the continuity of $(g_i)_{u_i}$, for sufficiently small $\epsilon > 0$,
$(g_i)_{u_i}[2\sup(\theta_{(g_1)_{u_1}(\cdot,0,...,0)}), 2\sup(\theta_{(g_2)_{u_2}(0,\cdot,0,...,0)}), ...,$
$2\sup(\theta_{(g_{i-1})_{u_{i-1}}(0,...,0,\cdot,0,...,0)}), \epsilon\phi_0, 2\sup(\theta_{(g_{i+1})_{u_{i+1}}(0,...,0,\cdot,0,...,0)}), ...,$
$2\sup(\theta_{(g_N)_{u_N}(0,...,0,\cdot)})] > \lambda_1$, so, by the Lemma 4.1 and monotonicity of $(g_i)_{u_i}$,

$$\begin{aligned}
& \Delta(\underline{u}_i) + g_i(\bar{u}_1, ..., \bar{u}_{i-1}, \underline{u}_i, \bar{u}_{i+1}, ..., \bar{u}_N) \\
\geq\ & -\epsilon\lambda_1\phi_0 + \epsilon\phi_0 (g_i)_{u_i}[2\theta_{(g_1)_{u_1}(\cdot,0,...,0)}, 2\theta_{(g_2)_{u_2}(0,\cdot,0,...,0)}, ..., \\
& 2\theta_{(g_{i-1})_{u_{i-1}}(0,...,0,\cdot,0,...,0)} \\
& , \epsilon\phi_0, 2\theta_{(g_{i+1})_{u_{i+1}}(0,...,0,\cdot,0,...,0)}, ..., 2\theta_{(g_N)_{u_N}(0,...,0,\cdot)}] \\
\geq\ & \epsilon\phi_0(-\lambda_1 + (g_i)_{u_i}[2\sup(\theta_{(g_1)_{u_1}(\cdot,0,...,0)}), 2\sup(\theta_{(g_2)_{u_2}(0,\cdot,0,...,0)}), ... \\
& , 2\sup(\theta_{(g_{i-1})_{u_{i-1}}(0,...,0,\cdot,0,...,0)}), \epsilon\phi_0, 2\sup(\theta_{(g_{i+1})_{u_{i+1}}(0,...,0,\cdot,0,...,0)}), ..., \\
& 2\sup(\theta_{(g_N)_{u_N}(0,...,0,\cdot)})]) \\
>\ & 0.
\end{aligned}$$

Hence, $(\underline{u}_1, ..., \underline{u}_N)$ is a subsolution and $(\bar{u}_1, ..., \bar{u}_N)$ is a supersolution to (4.1), so by the Lemma 3.2, there is a positive solution $(u_1, ..., u_N)$ to (4.1).

We also have the following nonexistence result.

THEOREM 4.4. *If $(g_i)_{u_i}(0, ..., 0) \leq \lambda_1$ for some $i = 1, ..., N$, then (4.1) does not have any positive solution.*

It says that if one of the species has small reproduction, then it may be extinct, which means that the species can not coexist.

PROOF. Suppose $(g_i)_{u_i}(0, ..., 0) \leq \lambda_1$ and $(u_1, ..., u_N)$ is a nonnegative solution to (4.1). Then by (EN_2) and the monotonicity of g_i,

$$\begin{aligned}
& \Delta \tfrac{u_i}{2} + \tfrac{u_i}{2}(g_i)_{u_i}(0, ..., 0, \tfrac{u_i}{2}, 0, ..., 0) \\
=\ & \tfrac{1}{2}[\Delta(u_i) + u_i (g_i)_{u_i}(0, ..., 0, \tfrac{u_i}{2}, 0, ..., 0)] \\
\geq\ & \tfrac{1}{2}[\Delta(u_i) + g_i(0, ..., 0, u_i, 0, ..., 0)] \\
\geq\ & \tfrac{1}{2}[\Delta(u_i) + g_i(u_1, ..., u_{i-1}, u_i, u_{i+1}, ..., u_N)] \\
=\ & 0,
\end{aligned}$$

and so, $\frac{u_i}{2}$ is a subsolution to

$$\begin{aligned}
& \Delta(\omega) + \omega(g_i)_{u_i}(0, ..., 0, \omega, 0, ..., 0) = 0 \text{ in } \Omega, \\
& \omega|_{\partial\Omega} = 0.
\end{aligned}$$

Any large constant $M > c_0$ is a supersolution to

$$\begin{aligned}
& \Delta(\omega) + \omega(g_i)_{u_i}(0, ..., 0, \omega, 0, ..., 0) = 0 \text{ in } \Omega, \\
& \omega|_{\partial\Omega} = 0.
\end{aligned}$$

Hence, by the Lemma 3.2, there is a positive solution $\bar{u}_i$ with $0 \leq \frac{u_i}{2} \leq \bar{u}_i$ to

$$\begin{aligned}
& \Delta(\omega) + \omega(g_i)_{u_i}(0, ..., 0, \omega, 0, ..., 0) = 0 \text{ in } \Omega, \\
& \omega|_{\partial\Omega} = 0.
\end{aligned}$$

But, since $(g_i)_{u_i}(0, ..., 0) \leq \lambda_1$, by the Lemma 3.5, $\bar{u}_i \equiv 0$, so $u_i \equiv 0$.

5. Solution estimates

In this section, we build up upper and lower bounds of solutions under certain conditions to establish uniqueness results in the next sections.

We have the following solution estimates.

THEOREM 5.1. *If* $(g_i)_{u_i}(2c_0, 2c_0, ..., 2c_0, 0, 2c_0, ..., 2c_0) > \lambda_1, i = 1, ..., N,$ *then for any positive solution* $(u_1, ..., u_N)$ *to (4.1),*

$$\theta_{(g_i)_{u_i}(2c_0,2c_0,...,2c_0,\cdot,2c_0,2c_0,...,2c_0)} \leq u_i \leq 2\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}.$$

PROOF. Suppose $(u_1, ..., u_N)$ is a positive solution to (4.1). Then by (EN_2) and the monotonicity of $g_i, i = 1, ..., N$,

$$\begin{array}{rl} & \Delta(\frac{u_i}{2}) + \frac{u_i}{2}(g_i)_{u_i}(0, ..., 0, \frac{u_i}{2}, 0, ..., 0) \\ \geq & \frac{1}{2}[\Delta(u_i) + g_i(0, ..., 0, u_i, 0, ..., 0)] \\ \geq & \frac{1}{2}[\Delta(u_i) + g_i(u_1, ..., u_{i-1}, u_i, u_{i+1}, ..., u_N)] \\ = & 0, \end{array}$$

so, $\frac{u_i}{2}$ is a subsolution to

$$\begin{array}{l} \Delta(Z) + Z(g_i)_{u_i}(0, ..., Z, 0, ..., 0) = 0 \text{ in } \Omega, \\ Z|_{\partial\Omega} = 0. \end{array}$$

Any constant $M > c_0$ is a supersolution to

$$\begin{array}{l} \Delta(Z) + Z(g_i)_{u_i}(0, ..., 0, Z, 0, ..., 0) = 0 \text{ in } \Omega, \\ Z|_{\partial\Omega} = 0. \end{array}$$

Hence, by the Lemma 3.2 and uniqueness of positive solution in the Lemma 3.5, $\frac{u_i}{2} \leq \theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}$, in other words,

$$u_i \leq 2\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}. \tag{5.1}$$

For sufficiently small $\epsilon > 0$, by the monotonicity of $(g_i)_{u_i}$,

$$\begin{array}{rl} & \Delta(\epsilon\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}) \\ & +\epsilon\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}(g_i)_{u_i}(0, ..., 0, \epsilon\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}, 0, ..., 0) \\ = & \epsilon[\Delta(\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}) \\ & +\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}(g_i)_{u_i}(0, ..., 0, \epsilon\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}, 0, ..., 0)] \\ > & \epsilon[\Delta(\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}) \\ & +\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}(g_i)_{u_i}(0, ..., 0, \theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}, 0, ..., 0)] \\ = & 0, \end{array}$$

so, $\epsilon\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)}$ is a subsolution to

$$\begin{array}{l} \Delta(Z) + Z(g_i)_{u_i}(0, ..., 0, Z, 0, ..., 0) = 0 \text{ in } \Omega, \\ Z|_{\partial\Omega} = 0. \end{array}$$

Since $(g_i)_{u_i}(0, ..., 0, c_0, 0, ..., 0) \leq 0$, c_0 is a supersolution to

$$\begin{array}{l} \Delta(Z) + Z(g_i)_{u_i}(0, ..., 0, Z, 0, ..., 0) = 0 \text{ in } \Omega, \\ Z|_{\partial\Omega} = 0. \end{array}$$

Therefore, by the Lemma 3.2 and uniqueness of positive solution in the Lemma 3.5,

$$\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)} \leq c_0.$$

Hence, by (5.1), monotonicity of $(g_i)_{u_i}$, and Lemma 4.1,

$$\begin{aligned} & \Delta(u_i) + u_i(g_i)_{u_i}(2c_0, ..., 2c_0, u_i, 2c_0, ..., 2c_0) \\ \leq \ & \Delta(u_i) + u_i(g_i)_{u_i}(2\theta_{(g_1)_{u_1}(\cdot,0,...,0)}, ..., 2\theta_{(g_{i-1})_{u_{i-1}}(0,...,0,\cdot,0,...,0)}, u_i, \\ & 2\theta_{(g_{i+1})_{u_{i+1}}(0,...,0,\cdot,0,...,0)}, ..., 2\theta_{(g_N)_{u_N}(0,...,0,\cdot)}) \\ \leq \ & \Delta(u_i) + u_i(g_i)_{u_i}(u_1, ..., u_N) \\ \leq \ & \Delta(u_i) + (g_i)(u_1, ..., u_N) \\ = \ & 0, \end{aligned}$$

so, u_i is a supersolution to

$$\Delta(Z) + Z(g_i)_{u_i}(2c_0, ..., 2c_0, Z, 2c_0, ..., 2c_0) = 0 \text{ in } \Omega, \quad Z|_{\partial\Omega} = 0.$$

Since $(g_i)_{u_i}(2c_0, ..., 2c_0, 0, 2c_0, ..., 2c_0) > \lambda_1$, by the continuity of $(g_i)_{u_i}$, $(g_i)_{u_i}(2c_0, ..., 2c_0, \epsilon\phi_0, 2c_0, ..., 2c_0) > \lambda_1$ for sufficiently small $\epsilon > 0$, and so,

$$\begin{aligned} & \Delta(\epsilon\phi_0) + \epsilon\phi_0(g_i)_{u_i}(2c_0, ..., 2c_0, \epsilon\phi_0, 2c_0, ..., 2c_0) \\ = \ & \epsilon[\Delta\phi_0 + \phi_0(g_i)_{u_i}(2c_0, ..., 2c_0, \epsilon\phi_0, 2c_0, ..., 2c_0)] \\ > \ & \epsilon[\Delta\phi_0 + \lambda_1\phi_0] \\ = \ & 0, \end{aligned}$$

so, $\epsilon\phi_0$ is a subsolution to

$$\Delta(Z) + Z(g_i)_{u_i}(2c_0, ..., 2c_0, Z, 2c_0, ..., 2c_0) = 0 \text{ in } \Omega, \quad Z|_{\partial\Omega} = 0.$$

Hence, by the Lemma 3.2 and uniqueness of positive solution in the Lemma 3.5, we have

$$\theta_{(g_i)_{u_i}(2c_0,...,2c_0,\cdot,2c_0,...,2c_0)} \leq u_i. \tag{5.2}$$

By (5.1) and (5.2), we conclude the desired inequalities.

6. Uniqueness

In this section, we prove the uniqueness of positive solution to (4.1) with the following additional growth conditions. For our convenience we denote $A(a, b)$ for the average of real numbers a and b:

(U_1) $(g_i)_{u_iu_j}, i, j = 1, ..., N$ are bounded.
(U_2) $(g_i)_{u_j}(u_i, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) = 0, i, j = 1, ..., N, i \neq j$.
(U_3) If $A(x_i, z_i) \geq A(w_i, q_i)$ and $y_1, ..., y_{i-1}, y_{i+1}, ..., y_N \in R$, then there are $\bar{x}_i$ between x_i and z_i, and $\bar{w}_i$ between w_i and q_i such that $\bar{x}_i - \bar{w}_i \geq A(x_i, z_i) - A(w_i, q_i)$ and

$$\begin{aligned} & g_i(y_1, ..., y_{i-1}, x_i, y_{i+1}, ..., y_N) - g_i(y_1, ..., y_{i-1}, z_i, y_{i+1}, ..., y_N) \\ = \ & (x_i - z_i)(g_i)_{u_i}(y_1, ..., y_{i-1}, \bar{x}_i, y_{i+1}, ..., y_N), \\ & g_i(y_1, ..., y_{i-1}, w_i, y_{i+1}, ..., y_N) - g_i(y_1, ..., y_{i-1}, q_i, y_{i+1}, ..., y_N) \\ = \ & (w_i - q_i)(g_i)_{u_i}(y_1, ..., y_{i-1}, \bar{w}_i, y_{i+1}, ..., y_N) \end{aligned}$$

in virtue of the Mean Value Theorem for $i = 1, ..., N$.

We have the following uniqueness result.

THEOREM 6.1. *If*
(A) $(g_i)_{u_i}(2c_0, ..., 2c_0, 0, 2c_0, ..., 2c_0) > \lambda_1, i = 1, ..., N,$ *and*

(B) $\sup(g_i)_{u_iu_i} < \sum_{j=1,j\neq i}^{N}[\inf(g_i)_{u_iu_j} + 2\inf(g_j)_{u_iu_j} \sup \frac{\theta_{(g_j)_{u_j}(0,\ldots,0,\cdot,0,\ldots,0)}}{\theta_{(g_i)_{u_i}(2c_0,\ldots,2c_0,\cdot,2c_0,\ldots,2c_0)}}]$, $i = 1, ..., N$, *then (4.1) has a unique posotive solution.*

The condition in (B) implies that the species interact strongly among themselves and weakly with others.

Proof. The existence was already proved in the last section. We prove the uniqueness.
Let $(u_1, ..., u_N), (w_1, ..., w_N)$ be positive solutions to (4.1), and let $p_i = u_i - w_i, i = 1, ..., N$. We want to show that $p_i \equiv 0, i = 1, ..., N$.
Since $(u_1, ..., u_N), (w_1, ..., w_N)$ are solutions to (4.1), for $i = 1, ..., N$,

$$\Delta(p_i) + g_i(u_1, ..., u_N) - g_i(w_1, ..., w_N) = 0.$$

Hence, by (U_3) and the Mean Value Theorem, there are $\bar{u}_i, \bar{u}_{ij}, j = 1, ..., N, j \neq i, \tilde{u}_i$ such that $\bar{u}_i$ is between u_i and w_i, $\bar{u}_j$ is between u_{ij} and w_j, $j = 1, ..., N, j \neq i$, $0 \leq \tilde{u}_i \leq u_i$, $\bar{u}_i - \tilde{u}_i \geq \frac{u_i+w_i}{2} - \frac{u_i}{2} = \frac{w_i}{2}$ and

$$\begin{aligned}&\Delta(p_i) + p_i(g_i)_{u_i}(u_1, ..., u_{i-1}, \tilde{u}_i, u_{i+1}, ..., u_N)\\ &+p_i[(g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N) - (g_i)_{u_i}(u_1, ..., u_{i-1}, \tilde{u}_i, u_{i+1}, ..., u_N)\\ &+\sum_{j=1,j\neq i}^{N} p_j(g_i)_{u_j}(w_1, ..., w_{j-1}, \bar{u}_{ij}, u_j, ..., u_{i-1}, w_i, u_{i+1}, ..., u_N) = 0.\end{aligned}$$

Since

$$\Delta(u_i) + (g_i)_{u_i}(u_1, ..., u_{i-1}, \tilde{u}_i, u_{i+1}, ..., u_N)u_i = \Delta(u_i) + g_i(u_1, ..., u_N) = 0,$$

by the Lemma 3.3, we have

$$\int_\Omega -p_i\Delta(p_i) - (g_i)_{u_i}(u_1, ..., u_{i-1}, \tilde{u}_i, u_{i+1}, ..., u_N)p_i^2 dx \geq 0,$$

and so

$$\begin{aligned}&\int_\Omega -[(g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N) - (g_i)_{u_i}(u_1, ..., u_{i-1}, \tilde{u}_i, u_{i+1}, ..., u_N)]p_i^2\\ &-\sum_{j=1,j\neq i}^{N} p_ip_j(g_i)_{u_j}(w_1, ..., w_{j-1}, \bar{u}_{ij}, u_{j+1}, ..., u_{i-1}, w_i, u_{i+1}, ..., u_N)dx \leq 0.\end{aligned}$$

Hence, taking the sum $i = 1, ..., N$, we have

$$\begin{aligned}&\int_\Omega -\sum_{i=1}^{N} p_i^2[(g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N)\\ &-(g_i)_{u_i}(u_1, ..., u_{i-1}, \tilde{u}_i, u_{i+1}, ..., u_N)]\\ &-\sum_{i,j=1,j\neq i}^{N} p_ip_j[(g_i)_{u_j}(w_1, ..., w_{j-1}, \bar{u}_{ij}, u_{j+1}, ..., u_{i-1}, w_i, u_{i+1}, ..., u_N)\\ &-(g_i)_{u_j}(w_1, ..., w_{j-1}, \bar{u}_{ij}, u_{j+1}, ..., u_{i-1}, 0, u_{i+1}, ..., u_N)]dx \leq 0\end{aligned}$$

by (U_2).
By the Mean Value Theorem again, there are $\hat{u}_i, \bar{w}_i, i = 1, ..., N$, such that $\tilde{u}_i \leq \hat{u}_i \leq \bar{u}_i, 0 \leq \bar{w}_i \leq w_i$ and

$$\begin{aligned}&\int_\Omega -\sum_{i=1}^{N} p_i^2(\bar{u}_i - \tilde{u}_i)(g_i)_{u_iu_i}(u_1, ..., u_{i-1}, \hat{u}_i, u_{i+1}, ..., u_N)\\ &-\sum_{i,j=1,j\neq i}^{N} p_ip_jw_i(g_i)_{u_iu_j}(w_1, ..., w_{j-1}, \bar{u}_{ij}, u_{j+1}, ..., u_{i-1}, \bar{w}_i, u_{i+1}, ..., u_N)dx \leq 0.\end{aligned}$$

But, since

$$\bar{u}_i - \tilde{u}_i \geq \frac{w_i}{2}, i = 1, ..., N,$$

we have

$$\begin{aligned}&\int_\Omega -\sum_{i=1}^{N} p_i^2 \tfrac{w_i}{2}(g_i)_{u_iu_i}(u_1, ..., u_{i-1}, \hat{u}_i, u_{i+1}, ..., u_N)\\ &-\sum_{i,j=1,j\neq i}^{N} p_ip_jw_i(g_i)_{u_iu_j}(w_1, ..., w_{j-1}, \bar{u}_{ij}, u_{j+1}, ..., u_{i-1}, \bar{w}_i, u_{i+1}, ..., u_N)dx \leq 0.\end{aligned}$$

If the integrand on the left side is positive definite, then $p_i \equiv 0, i = 1, ..., N$, which means the uniqueness.
But,

$$\begin{aligned} & -(g_i)_{u_i u_j}(w_1, ..., w_{j-1}, \bar{u}_{ij}, u_{j+1}, ..., u_{i-1}, \bar{w}_i, u_{i+1}, ..., u_N) p_i p_j w_i \\ \leq \quad & -(g_i)_{u_i u_j}(w_1, ..., w_{j-1}, \bar{u}_{ij}, u_{j+1}, ..., u_{i-1}, \bar{w}_i, u_{i+1}, ..., u_N)(\frac{p_i^2}{2} + \frac{p_j^2}{2}) w_i \end{aligned}$$

for $i = 1, ..., N$, and so, the integrand is positive definite if

$$-\frac{1}{2}\sup(g_i)_{u_i u_i} > -\sum_{j=1, j\neq i}^{N} [\frac{\inf(g_i)_{u_i u_j}}{2} + \frac{\inf(g_j)_{u_i u_j}}{2}\frac{w_j}{w_i}]$$

for $i = 1, ..., N$, which is true if the condition is satisfied by the solution estimates in the Theorem 5.1.

7. Uniqueness with Perturbation

We consider the model

$$\left\{ \begin{array}{l} \Delta u_i + g_i(u_1, ..., u_N) = 0, \quad \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1, ..., N, \end{array} \right. \tag{7.1}$$

where Ω is a smooth, bounded domain in R^n and the C^2 functions g_i's are relative growth rates satisfying the following conditions:

(P1) $(g_i)_{u_j u_k}, i, j, k = 1, ..., N$ are bounded.
(P2) $(g_i)_{u_i u_i} < 0, i = 1, ..., N$.
(P3) There exist constants $c_0 > 0$ such that $(g_i)_{u_i}(0, ..., 0, u_i, 0, ..., 0) < 0$ for $u_i > c_0$, $i = 1, ..., N$.
(P4) For all u_i, there are $M(j, u_i), j \neq i$ such that $(g_i)_{u_i}(M(1, u_i), ..., M(i-1, u_i), u_i, M(i+1, u_i), ..., M(N, u_i)) \leq 0$, $i = 1, ..., N$.
(P5) $g_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) = 0$ for all $u_1, ..., u_{i-1}, u_{i+1}, ..., u_N$, $i = 1, ..., N$.

Define
$B = \{(\alpha_1, ..., \alpha_N) \in [C^2]^N | (\alpha_i)_{u_j u_k}$ are bounded,$\alpha_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) = 0, i, j, k = 1, ..., N.\}$,
with

$$\| (\alpha_1, ..., \alpha_N) \|_B = \sum_{i=1}^{N} |(\alpha_i)_{u_i}(0, ..., 0)| + \sum_{i,j,k=1}^{N} \sup |(\alpha_i)_{u_j u_k}|$$

for all $(\alpha_1, ..., \alpha_N) \in B$.
Then by the functional analysis theory, $(B, \| \cdot \|_B)$ is a Banach space containing $(g_1, ..., g_N)$.

Let $C \subseteq B$ be the constraint set such that for all $(\alpha_1, ..., \alpha_N) \in C$,
(C1) $(\alpha_i)_{u_j} < 0$, for $i, j = 1, ..., N, i \neq j$, $(\alpha_i)_{u_i u_j} < 0, i, j = 1, ..., N$,
(C2) $\alpha_i(u_1, ..., u_{i-1}, u_i, u_{i+1}, ..., u_N) \leq u_i(\alpha_i)_{u_i}(u_1, ..., u_{i-1}, \frac{u_i}{2}, u_{i+1}, ..., uN)$, for all $u_1, ..., u_N$, $i = 1, ..., N$,
(C3) There exist constants $c_0 > 0$ such that $(\alpha_i)_{u_i}(0, ..., 0, u_i, 0, ..., 0) < 0$ for $u_i > c_0$, $i = 1, ..., N$.

The following theorem is our main result about the perturbation of uniqueness.

THEOREM 7.1. *Suppose*
(A) $(\alpha_i)_{u_i}(2c_0, ..., 2c_0, 0, 2c_0, ..., 2c_0) > \lambda_1, i = 1, ..., N,$
(B) (7.1) has a unique coexistence state $(u_1, ..., u_N)$*,*
(C) the Frechet derivative of (7.1) at $(u_1, ..., u_N)$ *is invertible.*
Then there is a neighborhood V *of* $(g_1, ..., g_N)$ *in* B *such that if* $(\bar{g}_1, ..., \bar{g}_N) \in V \bigcap C$*, then (7.1) with* $(\bar{g}_1, ..., \bar{g}_N)$ *has a unique positive solution.*

Biologically, the first condition in Theorem 7.3 indicates that the rates of reproduction are relatively large. Similarly, the third condition, which requires the invertibility of the Frechet derivative, signifies that the rates of self-limitation are relatively larger than the rates of competition, a relationship that will be established in Lemma 7.2. When these conditions are fulfilled, the conclusion of our theorem asserts that small perturbations of the rates do not affect the existence and uniqueness of the positive steady state. That is, the species implied can continue to coexist even if the factors determining the population densities vary slightly.

Now, at first glance, Theorem 7.3 may appear to be a consequence of the Implicit Function Theorem. However, the Implicit Function Theorem only guarantees local uniqueness. In contrast, our results in Theorem 7.3 guarantee global uniqueness. The techniques we will use in the proof of Theorem 7.3 include the Implicit Function Theorem and a priori estimates on solutions of (7.1).

PROOF. Since the Frechet derivative of (7.1) at $(u_1, ..., u_N)$ is invertible, by the Implicit Function Theorem, there is a neighborhood V of $(g_1, ..., g_N)$ in B and a neighborhood W of $(u_1, ..., u_N)$ in $[C_0^{2,\alpha}(\bar{\Omega})]^N$ such that for all $(\bar{g}_1, ..., \bar{g}_N) \in V$ there is a unique positive solution $(\bar{u}_1, ..., \bar{u}_N) \in W$ of (7.1) with $(\bar{g}_1, ..., \bar{g}_N)$. Thus, the local uniqueness of the solution is guaranteed.
To prove global uniqueness, suppose that the conclusion of Theorem 7.3 is false. Then there are sequences $(g_{1n}, ..., g_{Nn}, u_{1n}, ..., u_{Nn}), (g_{1n}, ..., g_{Nn}, \bar{u}_{1n}, ..., \bar{u}_{Nn})$ in $(V \bigcap C) \times [C_0^{2,\alpha}(\bar{\Omega})]^N$ such that $(u_{1n}, ..., u_{Nn})$ and $(\bar{u}_{1n}, ..., \bar{u}_{Nn})$ are positive solutions of (7.1) with $(g_{1n}, ..., g_{Nn})$ and $(u_{1n}, ..., u_{Nn}) \neq (\bar{u}_{1n}, ..., \bar{u}_{Nn})$ and $(g_{1n}, ..., g_{Nn}) \to (g_1, ..., g_N)$. By Schauder's estimate in elliptic theory, the convergence of $(g_{1n}, ..., g_{Nn})$, and the solution estimate in the Theorem 5.1, there are constants $k_1 > 0, k_2 > 0, k_3 > 0$ such that

$$|u_{in}|_{2,\alpha} \leq k_1[|g_{in}(u_{1n}, ..., u_{Nn})|_\alpha + \sup_{x\in\bar{\Omega}}(u_{in}(x))] \leq k_1(k_2 + k_3), i = 1, ..., N$$

for all $n = 1, 2, ...,$ and so, we conclude that $|u_{in}|_{2,\alpha}, i = 1, ..., N$, are uniformly bounded. Therefore, there are uniformly convergent subsequences of $u_{in}, i = 1, ..., N$, which again will be denoted by u_{in}, $i = 1, ..., N$.
Thus, let

$$(u_{1n}, ..., u_{Nn}) \to (\hat{u}_1, ..., \hat{u}_N),$$
$$(\bar{u}_{1n}, ..., \bar{u}_{Nn}) \to (\bar{u}_1, ..., \bar{u}_N).$$

Then $(\hat{u}_1, ..., \hat{u}_N), (\bar{u}_1, ..., \bar{u}_N) \in (C^{2,\alpha})^N$ are also solutions of (7.1) with $(g_1, ..., g_N)$. We claim that $\hat{u}_1 > 0, ..., \hat{u}_N > 0, \bar{u}_1 > 0, ..., \bar{u}_N > 0$. By (P4) and the Maximum Principles, we claim $\hat{u}_1, ..., \hat{u}_N, \bar{u}_1, ..., \bar{u}_N$ are not identically zero
Suppose that it is not true. Without loss of generality assume $\hat{u}_1 \equiv 0$.
Then let $\tilde{u}_{1n} = \frac{u_{1n}}{\|u_{1n}\|_\infty}$ and $\tilde{u}_{in} = u_{in}$, $i = 2, ..., N$. By the elliptic theory again,

there is $\tilde{u}_1$ such that $\tilde{u}_{1n} \to \tilde{u}_1$. Hence,

$$\begin{array}{rl} & \Delta\tilde{u}_{1n} + \tilde{u}_{1n}(g_{1n})_{u_1}(u_{1n}, \tilde{u}_{2n}, ..., \tilde{u}_{Nn}) \\ = & \frac{1}{\|u_{1n}\|_\infty}[\Delta u_{1n} + u_{1n}(g_{1n})_{u_1}(u_{1n}, u_{2n}, ..., u_{Nn})] \\ \leq & \frac{1}{\|u_{1n}\|_\infty}[\Delta u_{1n} + g_{1n}(u_{1n}, ..., .u_{Nn})] \\ = & 0, \end{array}$$

and

$$\begin{array}{rl} & \Delta\tilde{u}_{1n} + \tilde{u}_{1n}(g_{1n})_{u_1}(\frac{u_{1n}}{2}, \tilde{u}_{2n}, ..., \tilde{u}_{Nn}) \\ = & \frac{1}{\|u_{1n}\|_\infty}[\Delta u_{1n} + u_{1n}(g_{1n})_{u_1}(\frac{u_{1n}}{2}, u_{2n}, ..., u_{Nn})] \\ \geq & \frac{1}{\|u_{1n}\|_\infty}[\Delta u_{1n} + g_{1n}(u_{1n}, ..., u_{Nn})] \\ = 0. & \end{array}$$

But, since

$$\begin{array}{c} \Delta\tilde{u}_{1n} + \tilde{u}_{1n}(g_{1n})_{u_1}(u_{1n}, \tilde{u}_{2n}, ..., \tilde{u}_{Nn}) \to \Delta\tilde{u}_1 + \tilde{u}_1(g_1)_{u_1}(0, \hat{u}_2, ..., \hat{u}_N), \\ \Delta\tilde{u}_{1n} + \tilde{u}_{1n}(g_{1n})_{u_1}(\frac{u_{1n}}{2}, \tilde{u}_{2n}, ..., \tilde{u}_{Nn}) \to \Delta\tilde{u}_1 + \tilde{u}_1(g_1)_{u_1}(0, \hat{u}_2, ..., \hat{u}_N), \end{array}$$

we conclude that

$$\Delta\tilde{u}_1 + \tilde{u}_1(g_1)_{u_1}(0, \hat{u}_2, ..., \hat{u}_N) = 0.$$

If $\hat{u}_i \equiv 0$ for all $i = 2, ..., N$, then

$$\begin{array}{rl} & \lambda_1 - (g_1)_{u_1}(0, 2c_0, ..., 2c_0) \\ = & \lambda_1[-(g_1)_{u_1}(0, 2c_0, ..., 2c_0)] \\ \geq & \lambda_1[-(g_1)_{u_1}(0, 2\theta_{(g_2)_{u_2}(0,\cdot,0,...,0)}), ..., 2\theta_{(g_N)_{u_N}(0,...,0,\cdot)}] \\ > & \lambda_1[-(g_1)_{u_1}(0, ..., 0)] \\ = & 0, \end{array}$$

which contradicts to our assumption.
If $\hat{u}_i > 0$ for some $i = 2, ..., N$, then since

$$\begin{array}{rl} & \Delta\tilde{u}_{in} + \tilde{u}_{in}(g_{in})_{u_i}(u_{1n}, u_{2n}, ..., u_{(i-1)n}, \frac{\tilde{u}_{in}}{2}, u_{(i+1)n}, ..., u_{Nn}) \\ = & \Delta(u_{in}) + u_{in}(g_{in})_{u_i}(u_{1n}, u_{2n}, ..., u_{(i-1)n}, \frac{u_{in}}{2}, u_{(i+1)n}, ..., u_{Nn}) \\ \geq & \Delta(u_{in}) + g_{in}(u_{1n}, u_{2n}, ..., u_{Nn}) \\ = 0, & \end{array}$$

and

$$\begin{array}{rl} & \Delta\tilde{u}_{in} + \tilde{u}_{in}(g_{in})_{u_i}(u_{1n}, u_{2n}, ..., u_{(i-1)n}, \frac{\tilde{u}_{in}}{2}, u_{(i+1)n}, ..., u_{Nn}) \\ \to & \Delta\hat{u}_i + \hat{u}_i(g_i)_{u_i}(0, \hat{u}_2, ..., \hat{u}_{i-1}, \frac{\hat{u}_i}{2}, \hat{u}_{i+1}, ..., \hat{u}_N), \end{array}$$

$$\begin{array}{rl} & \Delta\hat{u}_i + \hat{u}_i(g_i)_{u_i}(0, ..., 0, \frac{\hat{u}_i}{2}, 0, ..., 0) \\ \geq & \Delta\hat{u}_i + \hat{u}_i(g_i)_{u_i}(0, \hat{u}_2, ..., \hat{u}_{i-1}, \frac{\hat{u}_i}{2}, \hat{u}_{i+1}, ..., \hat{u}_N) \\ \geq & 0, \end{array}$$

and so $\frac{\hat{u}_i}{2}$ is a subsolution to

$$\left\{ \begin{array}{rcl} \Delta(\omega) + \omega(g_i)_{u_i}(0, ..., 0, \omega, 0, ..., 0) & = & 0, \\ \omega_{\partial\Omega} & = & 0. \end{array} \right.$$

But, since any large enough constant is a supersolution to

$$\left\{ \begin{array}{rcl} \Delta(\omega) + \omega(g_i)_{u_i}(0, ..., 0, \omega, 0, ..., 0) & = & 0, \\ \omega_{\partial\Omega} & = & 0, \end{array} \right.$$

by the Lemma 3.2, we have

$$\frac{\hat{u}_i}{2} \leq \theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)},$$

and so, by the monotonicity

$$\begin{aligned} & \lambda_1 - (g_1)_{u_1}(0, 2c_0, ..., 2c_0) \\ = & \lambda_1[-(g_1)_{u_1}(0, 2c_0, ..., 2c_0)] \\ \geq & \lambda_1[-(g_1)_{u_1}(0, 2\theta_{(g_2)_{u_2}(0,\cdot,0,...,0)}), ..., 2\theta_{(g_N)_{u_N}(0,...,0,\cdot)}] \\ \geq & \lambda_1[-(g_1)_{u_1}(0, \hat{u}_2, ..., \hat{u}_N)] \\ = & 0, \end{aligned}$$

which is a contradiction.
By the same procedure with the sequence $(\bar{u}_{1n}, ..., \bar{u}_{Nn})$, we also have $\bar{u}_1 > 0, ..., \bar{u}_N > 0$.
In conclusion, both $(\hat{u}_1, ..., \hat{u}_N)$ and $(\bar{u}_1, ..., \bar{u}_N)$ are positive solutions to (7.1) with $(g_1, ..., g_N)$. But, by condition (B), $\hat{u}_1 = \bar{u}_1 = u_1, ..., \hat{u}_N = \bar{u}_N = u_N$. This is a contradiction to the Implicit Function Theorem, since $(u_{1n}, ..., u_{Nn}) \neq (\bar{u}_{1n}, ..., \bar{u}_{Nn})$.

In biological terms, the proof of our theorem indicates that if one of the species living in the same domain becomes extinct, that is, if one species is excluded by the others, then the reproduction rates of both must be small. In other words, the region condition of reproduction rates (A) is reasonable.

Now, the condition (C) in Theorem 7.3 requiring the invertibility of the Frechet derivative is too artificial to have any direct biological implications. We therefore turn our attention to more applicable conditions that will guarantee the invertibility of the Frechet derivative. We then obtain the following relationship:
We consider the model

$$\begin{cases} \Delta u_i + g_i(u_1, ..., u_N) = 0, & \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1, ..., N, \end{cases} \tag{7.2}$$

where Ω is a smooth, bounded domain in R^n and the C^2 functions g_i's are such that
(I1) $(g_i)_{u_i u_j}, i, j = 1, ..., N$ are bounded,
(I2) $(g_i)_{u_j} < 0, i, j = 1, ..., N, i \neq j$, $(g_i)_{u_i u_j} < 0, i, j = 1, ..., N$,
(I3) $g_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) = 0$ for all $u_1, ..., u_{i-1}, u_{i+1}, ..., u_N \in R$, $i = 1, ..., N$,
(I4) If $A(x_i, z_i) \geq A(w_i, q_i)$ and $y_1, ..., y_{i-1}, y_{i+1}, ..., y_N \in R$, then there are $\bar{x}_i$ between x_i and z_i, and $\bar{w}_i$ between w_i and q_i such that $\bar{x}_i - \bar{w}_i \geq A(x_i, z_i) - A(w_i, q_i)$ and

$$\begin{aligned} & g_i(y_1, ..., y_{i-1}, x_i, y_{i+1}, ..., y_N) - g_i(y_1, ..., y_{i-1}, z_i, y_{i+1}, ..., y_N) \\ = & (x_i - z_i)(g_i)_{u_i}(y_1, ..., y_{i-1}, \bar{x}_i, y_{i+1}, ..., y_N), \\ & g_i(y_1, ..., y_{i-1}, w_i, y_{i+1}, ..., y_N) - g_i(y_1, ..., y_{i-1}, q_i, y_{i+1}, ..., y_N) \\ = & (w_i - q_i)(g_i)_{u_i}(y_1, ..., y_{i-1}, \bar{w}_i, y_{i+1}, ..., y_N) \end{aligned}$$

in virtue of the Mean Value Theorem for $i = 1, ..., N$,
(I5) $g_i(u_1, ..., u_i, ..., u_N) \leq u_i(g_i)_{u_i}(u_1, ..., u_{i-1}, \frac{u_i}{2}, u_{i+1}, ..., u_N)$ for all $u_1, ..., u_N$, $i = 1, ..., N$,
(I6) there is $c_0 > 0$ such that $(g_i)_{u_i}(0, ..., 0, u_i, 0, ..., 0) < 0$ for all $u_i > c_0$, $i = 1, ..., N$.

LEMMA 7.2. *Suppose* $(u_1, ..., u_N)$ *is a positive solution to (7.2). If*
(A) $(g_i)_{u_i}(2c_0, ..., 2c_0, 0, 2c_0, ..., 2c_0) > \lambda_1, i = 1, ..., N$,

(B) $\sup(g_i)_{u_iu_i} < \sum_{j=1,j\neq i}^{N}[\inf(g_i)_{u_iu_j} + 2\inf(g_j)_{u_iu_j}\sup\frac{\theta_{(g_j)_{u_j}(0,\ldots,0,\cdot,0,\ldots,0)}}{\theta_{(g_i)_{u_i}(2c_0,\ldots,2c_0,\cdot,2c_0,\ldots,2c_0)}}]$, $i = 1, ..., N$, *then the Frechet derivative of (7.2) at* $(u_1, ..., u_N)$ *is invertible.*

Proof. The Frechet derivative of (7.2) at $(u_1, ..., u_N)$ is $A = (A_{ij})$, where

$$A_{ij} \quad = \quad \begin{cases} \Delta + (g_i)_{u_i}(u_1, ..., u_N), & i = j \\ (g_i)_{u_j}(u_1, ..., u_N), & i \neq j \end{cases}$$

for $i, j = 1, ..., N$. We need to show that $N(A) = \{0\}$ by the Lemma 3.8, where $N(A)$ is the null space of A.

By (I4), there are $\bar{u}_i$ such that $u_i - \bar{u}_i \geq \frac{u_i}{2}$ and

$$\begin{aligned} g_i(u_1, ..., u_N) &= g_i(u_1, ..., u_{i-1}, u_i, u_{i+1}, ..., u_N) - g_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) \\ &= u_i(g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N), i = 1, ..., N. \end{aligned}$$

If

$$\Delta\phi_i + (g_i)_{u_i}(u_1, ..., u_N)\phi_i + \textstyle\sum_{j=1,j\neq i}^{N}(g_i)_{u_j}(u_1, ..., u_N)\phi_j = 0, i = 1, ..., N,$$

then

$$\begin{aligned} &\Delta\phi_i + (g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N)\phi_i + [(g_i)_{u_i}(u_1, ..., u_N) \\ &-(g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N)]\phi_i + \textstyle\sum_{j=1,j\neq i}^{N}(g_i)_{u_j}(u_1, ..., u_N)\phi_j = 0, \\ &i = 1, ..., N. \end{aligned}$$

Hence,

$$\begin{aligned} &\textstyle\int_\Omega |\nabla\phi_i|^2 - (g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N)\phi_i^2 - [(g_i)_{u_i}(u_1, ..., u_N) \\ &-(g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N)]\phi_i^2 - \textstyle\sum_{j=1,j\neq i}^{N}(g_i)_{u_j}(u_1, ..., u_N)\phi_i\phi_j dx = 0, \\ &i = 1, ..., N. \end{aligned}$$

But, since

$$\begin{aligned} &\Delta(u_i) + u_i(g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., , u_N) = \Delta(u_i) + g_i(u_1, ..., u_N) = 0, \\ &i = 1, ..., N, \end{aligned}$$

$$\lambda_1[-(g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N)] = 0, i = 1, ..., N.$$

Hence, by the Lemma 3.3, we have

$$\textstyle\int_\Omega |\nabla\phi_i|^2 - (g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N)\phi_i^2 dx \geq 0, i = 1, ..., N.$$

Therefore,

$$\begin{aligned} &\textstyle\int_\Omega -\sum_{i=1}^{N}[(g_i)_{u_i}(u_1, ..., u_N) - (g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N)]\phi_i^2 \\ &-\textstyle\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}[(g_i)_{u_j}(u_1, ..., u_N) - (g_i)_{u_j}(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N)]\phi_i\phi_j dx \leq 0 \end{aligned}$$

by (I3). But, by the Mean Value Theorem, there are $\tilde{u}_i, \hat{u}_{ij}$ such that

$$\bar{u}_i \leq \tilde{u}_i \leq u_i, 0 \leq \hat{u}_{ij} \leq u_i$$

and

$$\begin{aligned} &(g_i)_{u_i}(u_1, ..., u_{i-1}, u_i, u_{i+1}, ..., u_N) - (g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N) \\ = \;&(u_i - \bar{u}_i)(g_i)_{u_iu_i}(u_1, ..., u_{i-1}, \tilde{u}_i, u_{i+1}, ..., u_N), \\ &(g_i)_{u_j}(u_1, ..., u_{i-1}, u_i, u_{i+1}, ..., u_N) - (g_i)_{u_j}(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) \\ = \;&u_i(g_i)_{u_iu_j}(u_1, ..., u_{i-1}, \hat{u}_{ij}, u_{i+1}, ..., u_N), i, j = 1, ..., N, j \neq i. \end{aligned}$$

Therefore,

$$\textstyle\int_\Omega -\sum_{i=1}^{N}(u_i - \bar{u}_i)(g_i)_{u_iu_i}(u_1, ..., u_{i-1}, \tilde{u}_i, u_{i+1}, ..., u_N)\phi_i^2$$

$-\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} u_i(g_i)_{u_iu_j}(u_1,...,u_{i-1},\hat{u}_{ij},u_{i+1},...,u_N)\phi_i\phi_j dx \leq 0.$
But, since $u_i - \bar{u}_i \geq \frac{u_i}{2}, i = 1,...,N$, we have
$\int_\Omega -\sum_{i=1}^{N} \frac{u_i}{2}(g_i)_{u_iu_i}(u_1,...,u_{i-1},\tilde{u}_i,u_{i+1},...,u_N)\phi_i^2$
$-\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} u_i(g_i)_{u_iu_j}(u_1,...,u_{i-1},\hat{u}_i,u_{i+1},...,u_N)\phi_i\phi_j dx \leq 0.$
Hence, $\phi_i \equiv 0, i = 1,...,N$ if the integrand is positive definite, which is true by the condition and the same proof as in the Theorem 6.1.

Combining Theorems 6.1 and 7.3, and Lemma 7.2, we have the following theorem which is actually the main result in this section.

THEOREM 7.3. *If (P1)-(P5) and (I1) - (I6) are satisfied and*
(A) $(g_i)_{u_i}(2c_0,...,2c_0,0,2c_0,...,2c_0) > \lambda_1, i = 1,...,N$, *and*
(B) $\sup(g_i)_{u_iu_i} < \sum_{j=1,j\neq i}^{N}[\inf(g_i)_{u_iu_j} + 2\inf(g_j)_{u_iu_j} \sup \frac{\theta_{(g_j)_{u_j}(0,...,0,\cdot,0,...,0)}}{\theta_{(g_i)_{u_i}(2c_0,...,2c_0,\cdot,2c_0,...,2c_0)}}]$,
$i = 1,...,N$, *then there is a neighborhood* V *of* $(g_1,...,g_N)$ *in* B *such that if* $(\bar{g}_1,...,\bar{g}_N) \in V \bigcap C$, *then (7.1) with* $(\bar{g}_1,...,\bar{g}_N)$ *has a unique positive solution.*

8. Uniqueness with Perturbation of Region

We consider the model

$$(8.1) \qquad \begin{cases} \Delta u_i + g_i(u_1,...,u_N) = 0, & \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1,...,N, \end{cases}$$

where Ω is a smooth, bounded domain in R^n and $g_1,...,g_N \in C^2$.

Let Γ be a closed, bounded, convex region in B such that for all $(g_1,...,g_N) \in \Gamma$,
(PR1) $(g_i)_{u_ju_k}, i,j,k = 1,...,N$ are bounded,
(PR2) $(g_i)_{u_iu_i} < 0, i = 1,...,N$,
(PR3) there exists a constant $c_0 > 0$ such that $(g_i)_{u_i}(u_i,0) < 0, i = 1,...,N$ for $u_i > c_0$,
(PR4) for all u_i, there are $M(j,u_i), j \neq i$ such that $(g_i)_{u_i}(M(1,u_i),...,M(i-1,u_i),u_i,M(i+1,u_i),...,M(N,u_i)) \leq 0$, $i = 1,...,N$,
(PR5) $g_i(u_1,...,u_{i-1},0,u_{i+1},...,u_N) = 0$ for all $u_1,...,u_{i-1},u_{i+1},...,u_N, i = 1,...,N$.

The following Theorem is the main result.

THEOREM 8.1. *Suppose*
(A) For all $(g_1,...,g_N) \in \Gamma$, $(g_i)_{u_i}(2c_0,...,2c_0,0,2c_0,...,2c_0) > \lambda_1, i = 1,...,N$,
(B) (8.1) has a unique positive solution for every $(g_1,...,g_N) \in \partial_L\Gamma$, *where*
$\partial_L\Gamma = \{(\lambda_{g_2,...,g_N}, g_2,...,g_N) \in \Gamma|$ *for any fixed* $g_2,...,g_N, \lambda_{g_2,...,g_N} = \inf\{\| g_1 \| \,|(g_1,g_2,...,g_N) \in \Gamma\}.\}$,
(C) for all $(g_1,...,g_N) \in \Gamma$, *the Frèchet derivative of (8.1) at every positive solution to (8.1) is invertible,*
(D) $\Gamma - \partial\Gamma \subseteq C$.
Then for all $(g_1,...,g_N) \in \Gamma$, *(8.1) has a unique positive solution. Furthermore, there is an open set* W *in* B *such that* $\Gamma \subseteq W$ *and for every* $(g_1,...,g_N) \in W \bigcap C$, *(8.1) has a unique positive solution.*

Theorem 8.1 goes even further than Theorem 7.3 which states uniqueness in the whole region of $(g_1,...,g_N)$ whenever we have uniqueness on the left boundary and invertibility of the linearized operator at any particular solution inside the domain.

Proof. For each fixed $g_2, ..., g_N$, consider $(g_1, g_2, ..., g_N) \in \partial_L \Gamma$ and $(\bar{g}_1, g_2, ..., g_N) \in \Gamma$. We need to show that for all $0 \le t \le 1$, (8.1) with $(1 - t)(g_1, g_2, ..., g_N) + t(\bar{g}_1, g_2, ..., g_N)$ has a unique positive solution. Since (8.1) with $(g_1, ..., g_N)$ has a unique positive solution $(u_1, ..., u_N)$ and the Frechet derivative of (8.1) at $(u_1, ..., u_N)$ is invertible, Theorem 7.3 implies that there is an open neighborhood V of $(g_1, ..., g_N)$ in B such that if $(g_{10}, g_{20}, ..., g_{N0}) \in V \bigcap \Gamma$, then (8.1) with $(g_{10}, g_{20}, ..., g_{N0})$ has a unique positive solution. Let
$\lambda_s = \sup\{0 \le \lambda \le 1 |\ \text{(8.1) with } (1-t)(g_1, ..., g_N) + t(\bar{g}_1, g_2, ..., g_N)$
has a unique coexistence state for $0 \le t \le \lambda.\}$.
We need to show that $\lambda_s = 1$.
Suppose $\lambda_s < 1$. From the definition of λ_s, there is a sequence $\{\lambda_n\}$ such that $\lambda_n \to \lambda_s^-$ and there is a sequence $(u_{1n}, u_{2n}, ..., u_{Nn})$ of the unique positive solutions of (8.1) with $(1 - \lambda_n)(g_1, ..., g_N) + \lambda_n(\bar{g}_1, g_2, ..., g_N)$. Then by elliptic theory, there is $(u_{10}, u_{20}, ..., u_{N0})$ such that $(u_{1n}, ..., u_{Nn})$ converges to $(u_{10}, ..., u_{N0})$ uniformly and $(u_{10}, ..., u_{N0})$ is a solution of (8.1) with $(1 - \lambda_s)(g_1, ..., g_N) + \lambda_s(\bar{g}_1, g_2, ..., g_N)$. But, by the same proof as in the Section 7, $u_{10} > 0, ..., u_{N0} > 0$.
We claim that (8.1) has a unique coexistence state with $(1 - \lambda_s)(g_1, ..., g_N) + \lambda_s(\bar{g}_1, g_2, ..., g_N)$. In fact, if not, assume that $(\bar{u}_{10}, ..., \bar{u}_{N0}) \neq (u_{10}, ..., u_{N0})$ is another coexistence state. By the Implicit Function Theorem, there exists $0 < \tilde{a} < \lambda_s$ and very close to λ_s such that (8.1) with $(1 - \tilde{a})(g_1, ..., g_N) + \tilde{a}(\bar{g}_1, g_2, ..., g_N)$ has a coexistence state very close to $(\bar{u}_{10}, ..., \bar{u}_{N0})$, which means that (8.1) with $(1 - \tilde{a})(g_1, ..., g_N) + \tilde{a}(\bar{g}_1, g_2, ..., g_N)$ has more than one coexistence state. This is a contradiction to the definition of λ_s. But, since (8.1) with $(1 - \lambda_s)(g_1, ..., g_N) + \lambda_s(\bar{g}_1, g_2, ..., g_N)$ has a unique coexistence state and the Frechet derivative is invertible, Theorem 7.3 implies that λ_s can not be as defined. Therefore, for each $(g_1, ..., g_N) \in \Gamma$, (8.1) with $(g_1, ..., g_N)$ has a unique coexistence state $(u_1, ..., u_N)$. Furthermore, by the assumption, for each $(g_1, ..., g_N) \in \Gamma$, the Frechet derivative of (8.1) with $(g_1, ..., g_N)$ at the unique solution $(u_1, ..., u_N)$ is invertible. Hence, Theorem 7.3 concluded that there is an open neighborhood $V_{(g_1,..,g_N)}$ of $(g_1, ..., g_N)$ in B such that if $(\bar{g}_1, ..., \bar{g}_N) \in V_{(g_1,...,g_N)} \bigcap C$, then (8.1) with $(\bar{g}_1, ..., \bar{g}_N)$ has a unique coexistence state. Let $W = \bigcup_{(g_1,...,g_N)\in\Gamma} V_{(g_1,...,g_N)}$. Then W is an open set in B such that $\Gamma \subseteq W$ and for each $(\bar{g}_1, ..., \bar{g}_N) \in W \bigcap C$, (8.1) with $(\bar{g}_1, ..., \bar{g}_N)$ has a unique coexistence state.

Apparently, Theorem 8.1 generalizes Theorem 7.3.

9. Conclusions

In this paper, our investigation of the effects of perturbations on the general competition model resulted in the development and proof of Theorem 7.3, Lemma 7.2, and Theorem 7.3, as detailed above. The three together assert that given the existence of a unique solution $(u_1, ..., u_N)$ to the system (7.1), perturbations of the functions $(g_1, ..., g_N)$ within a specified neighborhood will maintain the existence and uniqueness of the positive steady state. Indeed, our results specifically outline conditions sufficient to maintain the positive, steady state solution when the general competition model is perturbed within some region.

Applying this mathematical result to real world situations, our results establish that the species residing in the same environment can vary their interactions within certain bounds, and continue to survive together indefinitely at unique densities. The conditions necessary for coexistence, as described in the theorem, simply require that members of each species interact strongly with themselves and weakly with members of the other species.

The research presented in this paper has a number of strengths, which confirm both the validity and the applicability of the project. First, the mathematical conditions required in Theorem 7.3 are identical to those required in Theorem 6.1. However, in the Theorem 6.1, we used these conditions to prove the existence and uniqueness of the positive steady state solution for the general competition model. In contrast, the Theorem 7.3 employs the same conditions to establish that the existence and uniqueness of this solution is maintained when the model is perturbed within some neighborhood. Thus, our findings extend and improve established mathematical theory.

Secondly, perturbations of the general model render its implications more applicable both mathematically and biologically. Because our theorem extends the steady state to any functions within some neighborhood of $(g_1, ..., g_N)$, results for the general model pertain to a far wider variety of functions. Biologically, perturbations extend the model's description to species affected by factors that vary slightly yet erratically. Thus, the description of competitive interactions given by the model becomes a closer approximation of real-world population dynamics.

While our research therefore represents a progression in the field, the results obtained have an important limitation. Theorem 7.3, Lemma 7.2, and Theorem 7.3 establish that a region of perturbation exists within which the coexistence state is maintained for the general competition model. However, the exact extent of that region remains unknown.

Therefore, the results presented in this paper may serve as a platform for research of the question given above. Mathematicians should now attempt to establish the exact extent of the perturbation region in which coexistence is maintained for the general model. Such information would prove very useful not only mathematically but also biologically. Specifically, knowledge of the extent of the region would imply exactly how far the species can relax and yet continue to coexist. Thus, the results achieved through our research will enable the field to continue the development of theory on competitive interaction of populations.

References

[1] S. W. Ali and C. Cosner, On the uniqueness of the positive steady state for Lotka - Volterra Models with diffusion, *Journal of Mathematical Analysis and Application*, 168(1992), 329-341.

[2] R. S. Cantrell and C. Cosner, On the steady - state problem for the Volterra - Lotka competition model with diffusion, *Houston Journal of mathematics*, 13(1987), 337-352.

[3] R. S. Cantrell and C. Cosner, On the uniqueness and stability of positive solutions in the Volterra-Lotka competition model with diffusion, *Houston J. Math.*, 15(1989) 341-361.

[4] C. Cosner and A. C. Lazer, Stable coexistence states in the Volterra-Lotka competition model with diffusion, *Siam J. Appl. Math.*, 44(1984), 1112-1132.

[5] C. Gui and Y. Lou, Uniqueness and nonuniqueness of coexistence states in the Lotka-Volterra competition model, *Comm Pure and Appl. Math.*, XVL2(1994), No.12, 1571-1594.
[6] J. L. Gomez and J. P. Pardo, Existence and uniqueness for some competition models with diffusion, *C.R. Acad. Sci. Paris*, 313(1991) Série 1, 933-938.
[7] P. Hess, On uniqueness of positive solutions of nonlinear elliptic boundary value problems, *Math. Z.*, 165(1977), 17-18.
[8] A. Leung, Equilibria and stabilities for competing-species, reaction-diffusion equations with Dirichlet boundary data, *J. Math. Anal. Appl.*, 73(1980), 204-218.
[9] K. Lizarraga, J. Kang, J. Lee, Perturbation of a Nonlinear Elliptic Biological Interacting Model, *Dynamics of PDE*, 3(2006), No.4, 281-293.
[10] J. Lee, J. Kang, Y. Oh, The existence, nonexistence and uniqueness of global positive coexistence of a Nonlinear elliptic biological interacting model, *Kangweon-Kyungki Math. Jour.*, 12(2004), No.1, 77-90.
[11] L. Li and R. Logan, Positive solutions to general elliptic competition models, *Differential and Integral Equations*, 4(1991), 817-834.

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