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Dual Lyapunov approach to finite time stability for parabolic PDE

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Abstract. We investigate stability and finite time stability properties of the zero solution to a semilinear parabolic equation. To this effect we develop a new, dual approach to Lyapunov concept of stability. The dual Lyapunov function satisfies a dual Hamilton-Jacobi inequality. This is a basis to study finite time stability (finite extinction time) of the original problem.

Contents

1. Introduction

We investigate stability properties of the zero solution $u = 0$ of the parabolic equation

(1.1) $u_t(t,x) - \Delta_x u(t,x) = f(t,x,u(t,x)),$ in $(0,\infty)\times\Omega,$ $u = 0$ in $(0,\infty)\times\partial\Omega,$

under perturbations in $C^0(\Omega)$. We assume that Ω is a bounded domain in $\mathbb{R}^n$ with a sufficiently smooth boundary and $f : [0,+\infty)\times\Omega\times\mathbb{R}\to\mathbb{R}$, $f(t,x,0) = 0$ for all $t \geq 0$ and $x \in \Omega$. We do not impose any assumptions on dimension of Ω and the form of f, as it is usually done, see e.g. [**8**]. However, we will not concentrate on the

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existence of strong or weak solutions to (1.1), we simply assume that strong solutions (not necessary unique) to (1.1) exist on $[0,+\infty)\times\Omega$. It is well known that if f is continuous and locally Lipschitz in u then for any $u_0\in C^{\alpha}(\bar{\Omega})$, $(0<\alpha<1, u_0(x)=u(0,x))$ there exists a unique classical solution of (1.1), for small t (see e.g. [**9**], [**15**]). One of the main tool which is usually applied to study stability or instability of (1.1) is the Lyapunov function, that in the case of (1.1) is the standard energy functional (see e.g. [**3**], [**8**]). In the last two decades appear a different method to study stability by approximation (1.1) with Ordinary Differential Equations (ODEs) using, e.g. Galerkin's method or finite difference, and then applying finite-dimensional optimal control methods [**1**], [**4**], [**7**] or by using Sum-of-Squares optimization to construct Lyapunov function for parabolic equation (see [**18**]). We want to investigate the stability of (1.1) in quite a different and new way. This concept is based on ideas from [**19**] (see also [**21**]). Namely we will extend the space $[0,+\infty)\times\Omega$ to a dual $[0,+\infty)\times\Omega\times\mathcal{P}$, $\mathcal{P}\subset\mathbb{R}$ and then we will consider a dual equation to (1.1) in the dual space $[0,+\infty)\times\Omega\times\mathcal{P}$. Next we form an auxiliary function W satisfying a dual Hamilton-Jacobi inequality. Such a function W is the base for construction a dual Lyapunov function. That approach allows us to come back to original ideas of Lyapunov when the Lyapunov function satisfies some (dual) Hamilton-Jacobi inequality in $[0,+\infty)\times\Omega\times\mathcal{P}$. Moreover, we weaken assumptions on the source f, i.e. f is a Caratheodory function only (measurable in (t,x) and continuous in u). However, our main goal of this paper is to study finite time stability of (1.1) (often named stabilization in finite time or finite extinction time), i.e. we want to formulate sufficient conditions when a solution to (1.1) with initial condition near zero approach zero in finite time. Extinction of solutions is one of the most important features of many evolutionary equations that has been the subject of intensive study during the past few decades. Extinction via fast diffusion was first proved by E. Sabinina [**22**], and since then, there has been increasing interest in this direction (see e.g. [**6**], [**10**], [**11**], [**12**], [**13**], [**16**], [**23**]). At the begining the authors studied the equation

$$u_t-\Delta_x u=u^q,\ \text{in } (0,\infty)\times\Omega,\ u=0 \text{ in } (0,\infty)\times\partial\Omega, \tag{1.2}$$

or

$$u_t-\Delta_x u=f(u),\ \text{in } (0,\infty)\times\Omega,\ u=0 \text{ in } (0,\infty)\times\partial\Omega, \tag{1.3}$$

with the initial condition $u_0(x)$ positive and $f>0$. It was proved that a nontrivial solution of (1.2) vanishes in finite time if and only if $0<q<1$ and in case (1.3) if and only if

$$\int_0^{\epsilon}\frac{ds}{f(s)}<\infty\ (\epsilon>0), \tag{1.4}$$

which means that strong absorption will cause extinction to occur in finite time. When in (1.2) or (1.3) $\Delta_x u$ is replaced by $\Delta_x\phi(u)$ then additional condition of type (1.4) is necessary (see e.g. [**13**]). In [**17**] the authors deal with the case when ϕ and f depend on (t,x) also. However then f, being positive, is estimate below by a function $p(u)$ satisfying condition (1.4) and $\lim_{u\to 0^+}p(u)/u=\infty$. Main used methods, in most of mentioned papers, were comparison principles and energy method (see the expository paper [**5**]). We should stress that still remain unsolved problem for (1.3) with $\Delta_x u$ replaced by $\Delta_x \phi(u)$ when

$$\int_0^\epsilon \frac{ds}{f(s)} = \infty, \ \int_0^\epsilon \frac{ds}{\phi(s)} = \infty, \ \int_0^\epsilon \frac{ds}{f(s)+\phi(s)} < \infty.$$

In this paper we use a dual method with an auxiliary function W satisfying a stronger dual Hamilton-Jacobi inequality which is the base for constructing a dual Lyapunov function. Such a new approach to study the finite extinction time allows us to look in a different way on the assumptions on the data in particular we do not assume for the source term f any form of (1.4) in an explicit way. It is replaced by different type of assumptions as we do not require that f and initial condition $u_0(x)$ are positive and we use a completely new approach to study the finite stability of (1.1). A consequence of that new approach is that we get estimation of the settling-time function. However, we should underline that our dual approach has nothing common with the dual method of [**6**]. In [**6**] duality relates to the "dual" equation for

$$u_t - \Delta_x \phi(u) = 0, \text{ in } (0,\infty)\times\Omega, \ u = 0 \text{ in } (0,\infty)\times\partial\Omega, \tag{1.5}$$

i.e.

$$v_t + \phi(-\Delta_x v) = 0, \text{ in } (0,\infty)\times\Omega, \ u = 0 \text{ in } (0,\infty)\times\partial\Omega \tag{1.6}$$

and for (1.6) the finite time extinction is established using comparison method and then duality arguments considered between spaces were used to get stability for original problem (1.5). Our dual Hamilton-Jacobi inequality is completely different (see (2.1) below) and its solution builds a dual Lyapunov function enabling to get stability in dual space.

2. Stability in dual space

Let Ω be a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary and $f : [0,+\infty)\times\Omega\times\mathbb{R}\to\mathbb{R}$, $f(t,x,0) = 0$ for all $t \geq 0$ and $x \in \Omega$, $f \not\equiv 0$, measurable in (t,x) and continuous with respect to the third variable u. We consider an open set $\mathcal{P} \subset\mathbb{R}$ containing zero and an auxiliary function $W : [0,+\infty)\times\bar{\Omega}\times\mathcal{P} \to \mathbb{R}$. We call $\mathcal{P}$ a dual set. We assume that $W \in C^2([0,+\infty)\times\Omega\times\mathcal{P})\cap C([0,+\infty)\times\bar{\Omega}\times\mathcal{P})$ and that $\Delta_x(W_y(t,x,y))$ exists for almost all $x \in \Omega$ and all $(t,y) \in [0,+\infty)\times\mathcal{P}$. Furthermore, we assume that for all $t \geq 0$ and $x \in \Omega$, $W(t,x,0) = 0$, $W_y(t,x,0) = 0$ and that for almost all $x \in \Omega$ and all $(t,y) \in [0,+\infty)\times\mathcal{P}$ the function W satisfies the following dual Hamilton-Jacobi inequality:

$$W_t(t,x,y) - y\cdot\Delta_x(W_y(t,x,y)) + yf(t,x,-W_y(t,x,y)) \leq 0. \tag{2.1}$$

In order to formulate the problem of stability in the dual space we introduce a dual to u solution of (1.1). We call $y(t,x)$, $(t,x) \in [0,+\infty)\times\Omega$, $y \in C^1([0,+\infty)\times\Omega)$, the dual solution to $u(t,x)$ of (1.1) if it satisfies, for the above W, the following dual equation:

$$\frac{d}{dt}(W_y(t,x,y(t,x))) - \Delta_x(W_y(t,x,y(t,x))) = -f(t,x,-W_y(t,x,y(t,x))),$$

$$\text{for a.a. } (t,x) \in [0,+\infty)\times\Omega, \ y = 0 \text{ in } [0,+\infty)\times\partial\Omega. \tag{2.2}$$

Notice that by our assumptions on f and W, $y = 0$ is a solution to (2.2). Denote by $\mathbb{P}$ the set of all dual solutions for all solutions of (1.1). Thus we have the relation:

for each $y \in \mathbb{P}$ there exists a solution $u(t,x)$ to (1.1) such that

$$u(t,x) = -W_y(t,x,y(t,x)),\ (t,x) \in [0,+\infty) \times \Omega. \tag{2.3}$$

The set of all solutions $u(t,x)$ of (1.1) satisfying (2.3) we denote by $\mathbb{U}$.

Now we introduce a dual Lyapunov function $T : [0,+\infty) \times \Omega \times \mathcal{P} \to \mathbb{R}$ as follows:

$$T(t,x,y) = W(t,x,y) - yW_y(t,x,y). \tag{2.4}$$

Definition 2.1. We say that for an arbitrary $x \in \Omega$ the function T is nonincreasing along $y \in \mathbb{P}$, if the function $[0,+\infty) \ni t \mapsto T(t,x,y(t,x)) = T_y(t,x)$ is nonincreasing, for each fixed $x \in \Omega$.

Proposition 2.2. Under our assumptions on the function $W : [0,+\infty) \times \bar{\Omega} \times \mathcal{P} \to \mathbb{R}$, for each fixed $x \in \Omega$ and every function $y \in \mathbb{P}$, the function $t \to T_y(t,x)$ is noincreasing in $[0,+\infty)$.

Proof. Let us fix $x \in \Omega$ and $y \in \mathbb{P}$. Due to (2.2), (2.1) and (5.6) we have

$$\begin{aligned}
0 \geq\ & W_t(t,x,y(t,x)) - y(t,x) \cdot \Delta_x W_y(t,x,y(t,x)) \\
& + y(t,x) f(t,x,-W_y(t,x,y(t,x))) \\
=\ & W_t(t,x,y(t,x)) - y(t,x) \cdot \tfrac{d}{dt}(W_y(t,x,y(t,x))) \\
=\ & W_t(t,x,y(t,x)) + W_y(t,x,y(t,x)) \cdot \tfrac{d}{dt}(y(t,x)) \\
-\ & \tfrac{d}{dt}(y(t,x)) \cdot W_y(t,x,y(t,x)) - y(t,x) \cdot \tfrac{d}{dt}(W_y(t,x,y(t,x))) \\
=\ & \tfrac{d}{dt}(W(t,x,y(t,x))) - \tfrac{d}{dt}(y(t,x)W_y(t,x,y(t,x))) \\
=\ & \tfrac{d}{dt}(W(t,x,y(t,x)) - y(t,x)W_y(t,x,y(t,x))) \\
=\ & \tfrac{d}{dt}T(t,x,y(t,x)) \text{ for almost all } t \geq 0.
\end{aligned}$$

Integrating $\frac{d}{dt}T(t,x,y(t,x)) \leq 0$ on an arbitrary interval $[a,b] \in [0,+\infty)$, we obtain

$$\int_a^b \frac{d}{dt} T(t,x,y(t,x)) \leq 0$$

and finally we get

$$T(b,x,y(b,x)) - T(a,x,y(a,x)) \leq 0.$$

The arbitrariness of $a < b$ completes the proof. □

We say that a function $\gamma : [0,+\infty) \to [0,+\infty)$ belongs to a class $\mathcal{K}_1$, if it is increasing and $\gamma(0) = 0$.
We assume that for any $d_1 \in \mathcal{K}_1$ the function T fulfills the following inequality:

$$T(t,x,y) \geq d_1(|y|) \tag{2.5}$$

for all $t \geq 0$, $x \in \Omega$, $y \in \mathcal{P}$.

Definition 2.3. We say that equation (2.2) is stable at the origin, if for any $t_0 \geq 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for each $y \in \mathbb{P}$ satisfying $|y(t_0,x)| < \delta$ for all $x \in \Omega$ we have $|y(t,x)| < \varepsilon$ for $t \geq t_0$, $x \in \Omega$.

Theorem 2.4. *Let the function $W : [0,+\infty) \times \Omega \times \mathcal{P} \to \mathbb{R}$ be given as above and the function T defined in (5.6) satisfies (2.5), then a dual solution of (2.2) is stable at the origin.*

PROOF. Take any fixed $x \in \Omega$. Let $t_0 \geq 0$, $\varepsilon > 0$ and let S_ε be the sphere of the ball $\overline{B}_\varepsilon(0) \subset \mathcal{P}$ with center at zero. Then, due to (2.5) we can see that there is a function $d_1 \in \mathcal{K}_1$ such that we have

$$\eta = \max_{s \in \bar{\Omega}} \inf_{t \geq t_0} \inf_{y \in S_\varepsilon} T(t, s, y) \geq d_1(\varepsilon) > 0.$$

We know that $W(t_0, x, 0) = 0$, so at the same time also $T(t_0, x, 0) = 0$. Moreover, because T is continuous at $(t_0, x, 0)$, so there exists $\delta > 0$ (we can choose $\delta < \varepsilon$) such that for $|y| < \delta$ we have $0 \leq T(t_0, x, y) < \eta$. Now, let us take any $y \in \mathbb{P}$, satisfying $|y(t_0, s)| \leq \delta$ for all $s \in \Omega$. Then $T(t_0, x, y(t_0, x)) < \eta$. The facts that the function $t \mapsto T(t, x, y(t, x))$ is nonincreasing along $y(t, x)$ (Proposition 2.2) and y is continuous guarantee that y cannot intersect S_ε. Therefore $|y(t, x)| < \varepsilon$ for $t \geq t_0$. So, (2.2) is stable at the origin. Let us notice that this reasoning is independent of the choice of x, and thus the stability is uniformly with respect to x. □

DEFINITION 2.5. We say that the equation (2.2) is uniformly stable at the origin, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for each $t_0 \geq 0$, $x \in \Omega$ and each $y \in \mathbb{P}$ satisfying $|y(t_0, x)| < \delta$ we have $|y(t, x)| < \varepsilon$ for $t \geq t_0, x \in \Omega$.

To get uniform stability of (2.2) we have to assume just that T is bounded from above by time-independent function $d_2(|y|)$ belonging to class $\mathcal{K}_1$ and continuous in 0. Since this is a classical result, so we only formulate a theorem and skip the proof.

THEOREM 2.6. *Let the function* $W : [0, +\infty) \times \Omega \times \mathcal{P} \to \mathbb{R}$ *be given as above and the functon* T *defined in (5.6) satisfies (2.5) and*

$$T(t, x, y) \leq d_2(|y|),$$

for $t \geq 0$, $x \in \Omega$, $y \in \mathcal{P}$ *and some* $d_2 \in \mathcal{K}_1$ *continuous in* 0, *then (2.2) is uniformly stable at the origin.*

3. Stability in primary space

Now, we want to prove theorems about stability and uniform stability in primary space, i.e. stability, uniform stability of the equation (1.1) at the origin.
Denote by $\mathcal{U}$ the set of all continuous functions $u : [0, +\infty) \times \bar{\Omega} \to \mathbb{R}$ having for almost all $x \in \Omega$ derivative in t and for which for almost all t exists $\Delta_x u(t, x)$. We will consider only strong solutions to (1.1) i.e. functions $u \in \mathcal{U}$ satisfying (1.1) almost everewhere in $[0, b) \times \Omega$, $b > 0$ and this set we denote by $\mathcal{U}_s$. By $\mathcal{U}_0$ we denote the set of all continuous functions $u : \bar{\Omega} \to \mathbb{R}$ with the norm $\|u\| = \max_{x \in \bar{\Omega}} |u(x)|$.

DEFINITION 3.1. We say that the equation (1.1) is stable at the origin, if for any $t_0 \geq 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that any solution $u \in \mathcal{U}_s$ of (1.1) satisfying $|u(t_0, x)| < \delta$ for each $x \in \Omega$, is extendable to $[t_0, +\infty)$ and $|u(t, x)| < \varepsilon$ for $t \geq t_0$, $x \in \Omega$.

DEFINITION 3.2. We say that the equation (1.1) is uniformly stable at the origin, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for each $t_0 \geq 0$, $x \in \Omega$ and any solution $u \in \mathcal{U}_s$ of (1.1) satisfying $|u(t_0, x)| < \delta$ and $|u(t, x)| < \varepsilon$ for $t \geq t_0$, $x \in \Omega$.

Moreover, we assume on W defined in the former section that it satisfies in addition the following assumptions:

Assumption 3.3. For each $\varepsilon > 0$ and each $t_0 \geq 0$ there exists $\varepsilon_1 > 0$ such that for all $x \in \Omega$ and $t \geq t_0$ and $y \in \mathcal{P}$, if only $|y| < \varepsilon_1$, then $|W_y(t,x,y)| < \varepsilon$.

Assumption 3.4. For all $t_0 \geq 0$ and $\delta_1 > 0$ there exists $\delta > 0$ such that for each $u(t_0,\cdot) \in \mathcal{U}_0$, if only $|u(t_0,x)| < \delta$, for each $x \in \Omega$, then there exists $y \in \mathbb{P}$, $u(t_0,x) = -W_y(t_0,x,y(t_0,x))$ such that $|y(t_0,x)| < \delta_1$ for each $x \in \Omega$.

Theorem 3.5. *Let the function $W : [0,+\infty) \times \Omega \times \mathcal{P} \to \mathbb{R}$ be given as in the former section. Let the Assumptions 3.3, 3.4 hold and assume that equation (2.2) in the dual space is stable at the origin, then the corresponding equation (1.1) in the primary space is also stable at the origin.*

Proof. Let us assume that equation (2.2) is stable at the origin. This means that for any $t_0 \geq 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x \in \Omega$ and each solution of (2.2) satisfying $|y(t_0,x)| < \delta$ we have $|y(t,x)| < \varepsilon$ for $t \geq t_0$. Let $t_0 \geq 0$ and $\varepsilon > 0$. By Assumption 3.3 there exists $\varepsilon_1 > 0$ such that for all $x \in \Omega$ and $t \geq t_0$ and $y \in \mathcal{P}$, if only $|y| < \varepsilon_1$, then $|W_y(t,x,y)| < \varepsilon$. Because $0 \in \mathbb{P}$ is stable, so for ε_1 there exists $\delta_1 > 0$ such that for each $x \in \Omega$ if $|y(t_0,x)| < \delta_1$, then $|y(t,x)| < \varepsilon_1$ for $t \geq t_0, x \in \Omega$. According to Assumption 3.4, there exists $\delta > 0$ such that if u is such a solution of (1.1) that $|u(t_0,x)| < \delta$, then there exists $y \in \mathbb{P}$ such that $|y(t_0,x)| < \delta_1$ and $u(t_0,x) = -W_y(t_0,x,y(t_0,x))$, $x \in \Omega$. Again, according to Assumption 3.4, we may conclude that $|u(t,x)| = |W_y(t,x,y(t,x))| < \varepsilon$ for $t \geq t_0$, which means that equation (1.1) is stable at the origin. □

Remark 3.6. Let us observe, that from Theorems 2.4 and 3.5 we have the following conclusion:
Let the function $W : [0,+\infty) \times \Omega \times \mathcal{P} \to \mathbb{R}$ be given as in the former section and satisfying (2.1). Assume the function T, defined in (5.6), satisfies (2.5), Assumptions 3.3, 3.4. Then equation (1.1) in the primary space is stable at the origin.

4. Using Theorems 2.4, 3.5 to get stability of (1.1)

Applying Theorems 4 and 11 we derive conditions for the function f and give explicit formula for the function W to prove stability of (1.1).

Theorem 4.1. *Assume that there exist number $\beta > 1$, a function $d(\cdot) : (0,\infty) \times \Omega \to \boldsymbol{R}^+$, $b(\cdot), g(\cdot) : (0,\infty) \times \Omega \to \boldsymbol{R}^-$ measurable, $a(\cdot) : (0,\infty) \times \Omega \to \boldsymbol{R}^+$ integrable, twice differentiable in x such that*

$$\int_0^t a_x(s,x)ds < b(t,x) < 0,$$

$$\int_0^t \Delta_x a(s,x)ds < g(t,x) < 0,$$

$$yf(t,x,-W_y(t,x,y)) \leq d(t,x)\,|y|^\beta \tag{4.1}$$

for $t \in (0,\infty)$, $y \in B(0,1)$ and $W(t,x,y) = C(t,x)|y|^\beta$, $t \in (0,\infty)$, $x \in \Omega$, $y \in \mathcal{P} = B(0,1)$, where $C(t,x) = -\exp(D\int_0^t a(s,x)ds)$, $D > 1$ sufficiently large. Then equation (1.1) is stable at the origin.

Proof. Let $W(t,x,y) = C(t,x)|y|^\beta$, $t \in (0,\infty)$, $x \in \Omega$, $y \in \mathcal{P} = B(0,1)$, $D > 1$ be sufficiently large, $C(t,x) = -\exp(D\int_0^t a(s,x)ds)$. Then for $\mathcal{P} = B(0,1)$, $x \in \Omega$

all assumptions on W are satisfied. We have to check that (2.1) is satisfied too. Observe that we have the following estimation

$$\begin{aligned} &W_t(t,x,y) - y\cdot\Delta_x W_y(t,x,y) + yf(t,x,-W_y(t,x,y)) \\ &= C(t,x)Da(t,x)|y|^\beta - y\Delta_x C(t,x)\beta y|y|^{\beta-2} + yf(t,x,-C(t,x)\beta y|y|^{\beta-2}) \\ &\le C(t,x)Da(t,x)|y|^\beta - yC(t,x)D(Db(t,x)+g(t,x)))\beta y|y|^{\beta-2} + d(t,x)\,|y|^\beta \\ &= |y|^\beta(C(t,x)Da(t,x) - \beta C(t,x)D(Db(t,x)+g(t,x)) + d(t,x)) \end{aligned}$$

Notice that the above inequality is negative for $D > 1$ sufficiently large. Thus, by Remark 3.6 we get the stability of equation (1.1) in the primary space. □

5. Finite-time stability of the equation (1.1) at the origin

Definition 5.1. For any $t_0 \ge 0$ and $u_0(x) \in \mathbb{R}\setminus\{0\}$, $x \in \Omega$, $u_0(\cdot) \in \mathcal{U}_0$ and $\varphi \in \mathcal{U}$ such that $\varphi(t_0,x) = u_0(x)$ for $x \in \Omega$, let us denote $c_\varphi(t_0,u_0(\cdot))$ a finite number (if exists) such that $\varphi(t,x) \in \mathbb{R}$ for $t \in (t_0, c_\varphi(t_0,u_0(\cdot)))$ and $x \in \Omega$, and

$$\lim_{t\to c_\varphi(t_0,u_0(\cdot))^-} (\sup_{x\in\Omega} \mid \varphi(t,x) \mid) = 0.$$

Let

$$\tau_\varphi(t_0,u_0(\cdot)) = \begin{cases} c_\varphi(t_0,u_0(\cdot)), & \text{if exists,} \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.1}$$

Definition 5.2. A function $S : [0,+\infty)\times\mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ is called settling-time function if

- $S(t_0,0) = t_0$ for any $t_0 \ge 0$,
- for any $u_0 \in \mathcal{U}_0$, $t_0 \ge 0$ and $u_0(x) \in \mathbb{R}\setminus\{0\}$, $x \in \Omega$,
  $S(t_0,u_0(\cdot)) = \sup_{\substack{\varphi\in\mathcal{U}_s,\\ \varphi(t_0,\cdot)=u_0(\cdot)}} \{\tau_\varphi(t_0,u_0(\cdot))\}.$

Definition 5.3. We say that equation (1.1) is finite-time stable at the origin, if it is stable at the origin and for each $t_0 \ge 0$ there exists $\delta > 0$ such that for any solution of (1.1) satisfying $|u(t_0,x)| < \delta$, $x \in \Omega$, $S(t_0,u(t_0,\cdot))$ is finite.

We say that a function $c : [0,+\infty)\times\Omega \to [0,+\infty)$ belongs to a class $\mathcal{M}$, if it is measurable, bounded and there exists $t_c \ge 0$ such that $\int_{t_c}^{+\infty} c(s,x)ds = +\infty$ for each $x \in \Omega$.

Proposition 5.4. Assume that a function $W : [0,+\infty)\times\Omega\times\mathcal{P} \to \mathbb{R}$ satisfies all assumptions from the section 2 and the function T defined in (5.6) satisfies (2.5), then for each $t_0 \in [0,+\infty)$, $x \in \Omega$ and arbitrary solution $u \in \mathcal{U}_s$, such that $u(s,\cdot) = 0$ for some $s \in [t_0,b)$ (b can be infinity) we have $u(t,x) = 0$ for all $t \in [s,\infty)$, $x \in \Omega$.

Proof. Let us fix any $t_0 \in [0,+\infty)$, $x \in \Omega$, $u \in \mathcal{U}_s$ such that $u(s,x) = 0$ for some $s \in [t_0,b)$. Let $y \in \mathbb{P}$ be such that $u(t,x) = -W_y(t,x,y(t,x))$ and $y(s,x) = 0$. The function $T_{x,y} : t \to T(t,x,y(t,x))$ is continuous. If there were $t_1 > s$ such that $u(t_1,x) \ne 0$ then also $y(t_1,x) \ne 0$. Hence and from (2.5) we would have that $T_{x,y}(t_1) > 0$. Put $a = \frac{1}{2}T_{x,y}(t_1)$ and $D = \{t \in (s,t_1) : T_{x,y}(t) = a\}$. It is obvious that $D \ne \emptyset$. Let $\bar{t} = \inf D$. We know that $\bar{t} \ge s$. But if $\bar{t} = s$, then there exists a sequence (s_n), $s_n \in D$, such that $s_n \to s$. But from continuity of $T_{x,y}$ we obtain $0 = T(s,x,0) = T(s,x,y(s,x)) = T_{x,y}(s) = \lim_{n\to\infty} T_{x,y}(s_n) = T_{x,y}(s) = a > 0$, which is impossible. So we can see that $\bar{t} > s$, i.e. $\bar{t} \in (s,t_1)$ and $T_{x,y}(\bar{t}) = a$. Then, again by the continuity of $T_{x,y}$ we have $T_{x,y}(\bar{t}) = a < T_{x,y}(t_1)$. However this is in contradiction with Proposition 2.2 which asserts that $T_{x,y}(t_1) \leq T_{x,y}(\bar{t})$. So, we can deduce that $y(t,x) = 0$ for all $t \in [s,b)$ and by the properties of W_y we get that $u(t,x) = 0$ for all $t \in [s,b)$. □

Let $c \in \mathcal{M}$, $u_0 \in \mathcal{U}_0$, $t \in [0,+\infty)$, $\sigma \in (0,1)$ and let us consider the following Cauchy problem:

$$\begin{cases} \dot{u} = -c(s,x) \operatorname{sign}(u) \left|u\right|^{\sigma}, & s \in [t,+\infty), \\ u(t,x) = \operatorname{sign}(u_0(x)) \left|u_0(x)\right|, & x \in \Omega. \end{cases} \tag{5.2}$$

Note that the right hand side is measurable function with respect to $s \in [0,+\infty)$ and continuous with respect to $u \in \boldsymbol{R}$.

REMARK 5.5. Let us notice, that for any $c \in \mathcal{M}$ and $t \geq 0$ the function $C_t : s \mapsto \int_t^s c(\tau,x)d\tau,\ s \in [0,+\infty)$, is nonincreasing and absolutely continuous and that $\int_t^\infty c(\tau,x)d\tau = \infty$. So, for any $u_0 \in \mathcal{U}_0$ and $\sigma \in (0,1)$ there exists $\bar{t} \geq t$, such that

$$C_t(\bar{t}) = \int_t^{\bar{t}} c(\tau,x)d\tau = \frac{\left\|u_0\right\|^{1-\sigma}}{1-\sigma}. \tag{5.3}$$

Let

$$t_{c,u_0} = \inf\left\{\bar{t} \geq t : \int_t^{\bar{t}} c(\tau,x)d\tau = \frac{\left\|u_0\right\|^{1-\sigma}}{1-\sigma}\right\}. \tag{5.4}$$

It is easy to verify, that the following function is a solution of Cauchy problem (5.2):
(5.5)

$$\mu_{t,u_0}(s,x) = \begin{cases} \operatorname{sign}(u_0(x)) \left(\left|u_0(x)\right|^{1-\sigma} - (1-\sigma)\int_t^s c(\tau,x)d\tau\right)^{\frac{1}{1-\sigma}} & s \in [t,t_{c,u_0}),\ u_0(x) \neq 0,\ x \in \Omega \\ 0, & s \geq t_{c,u_0},\ u_0(x) \neq 0,\ x \in \Omega \\ 0, & s \geq t,\ u_0(x) = 0,\ x \in \Omega. \end{cases}$$

Moreover, note that $\mu_{t,u_0}(s,x) \neq 0$ for $t \geq 0$, $s \in [t,t_{c,u_0})$, $x \in \Omega$ and $u_0(x) \in \mathbb{R} \setminus \{0\}$, $x \in \Omega$.

Let us observe that the settling-time function for (5.2) is given by:

$$S(t,u_0(\cdot)) = \inf\left\{C_t^{-1}\left(\frac{\left\|u_0\right\|^{1-\sigma}}{1-\sigma}\right)\right\}$$

for $u_0 \in \mathcal{U}_0$, $u_0(x) \in \mathbb{R} \setminus \{0\}$, $x \in \Omega$, $\sigma \in (0,1)$ and C_t given by (5.3). For given t, $y \in \mathbb{P}$ and $T(t,x,y)$ let us put

$$T_{t,y}(x) = T(t,x,y(t,x)). \tag{5.6}$$

THEOREM 5.6. *Assume that a function $W : [0,+\infty) \times \Omega \times \mathcal{P} \to \mathbb{R}$ satisfies all assumptions from the section 2, Assumptions 3.3, 3.4, the function T defined in (5.6) satisfies (2.5) and moreover there exist $c \in \mathcal{M}$ and $\alpha \in (0,1)$ such that for $(t,x,y) \in [0,+\infty) \times \Omega \times \mathcal{P}$*

$$W_t(t,x,y) - y \cdot \Delta_x W_y(t,x,y) + yf(t,x,-W_y(t,x,y)) + c(t,x)(T(t,x,y))^\alpha \le 0, \tag{5.7}$$

then (1.1) is finite-time stable at the origin.
The settling-time function for a trajectory starting at $(t_0, u_0(\cdot))$ near $(t_0, 0)$ $(u_0(\cdot) \in \mathcal{U}_0)$ is estimated by

$$S(t_0, u_0(\cdot)) \le t_{c,u_0},$$

where $y_0(x)$ is a point in the dual space such that $u_0(x) = -W_y(t_0, x, y_0(x))$, $x \in \Omega$.

PROOF. First, let us note that (5.7) implies (2.1), thus equation (1.1) is stable at the origin (see Theorem 2.4 and 3.5). Let $t_0 \ge 0$, $\delta_1 > 0$ and take any $u \in \mathcal{U}_s$, $\delta > 0$ such that $|u(t_0,x)| < \delta$, $x \in \Omega$. From Assumption 3.4 we have there exists $y \in \mathbb{P}$ such that $|y(t_0,x)| < \delta_1$, $x \in \Omega$ and from (2.3) that $u(t,x) = -W_y(t,x,y(t,x))$ for $t \in [0,+\infty)$, $x \in \Omega$. Let $\beta > 0$, $B(0,\beta) \subset \mathcal{P}$ and let S_β be a sphere of $B(0,\beta)$. Let

$$0 < \gamma = \inf_{t \ge t_0} \inf_{y \in S_\beta} \max_{x \in \bar{\Omega}} T(t,x,y).$$

Let us fix any $x \in \Omega$. Then, similarly as in the proof of Theorem 2.4 we have that $T(t_0, x, y(t_0,x)) < \gamma$. Proposition 2.2 guarantees that the function $t \to T(t,x,y(t,x))$ is nonincreasing along $t \to y(t,x)$, which gives that

$$T(t,x,y(t,x)) < T(t_0,x,y(t_0,x)) < \gamma$$

for $t \in (t_0, b)$, $b \in (t_0,+\infty) \cup \{+\infty\}$ and this means that $|y(t,x)| < \beta$ for $t \ge t_0$ (again see proof of Theorem 2.4).
From (5.7) we have that

$$\frac{d}{dt} T(t,x,y(t,x)) \le -c(t,x)(T(t,x,y(t,x)))^\alpha$$

for $t \in [t_0,+\infty)$. Fix t. Let us assume in Remark 5.5 $u_0(x) = T_{t,y}(x)$. Then $\mu_{t,T_{t,y}}$ given by formula (5.5), be a solution of (5.2) with the initial condition

$$\mu_{t,T_{t,y}}(t,x) = T_{t,y}(x).$$

Since

$$T_{t,y}(x) = T(t,x,y(t,x)) = \mu_{t,T_{t,y}}(t,x),$$

so from well known comparison lemma applied to (5.2) and $T_{t,y}(x)$ we obtain

$$T_{s,y}(x) \le \mu_{t,T_{t,y}}(s,x) \quad \text{for } s \in [t,+\infty). \tag{5.8}$$

Since T is nonnegative and by properties of W, W_y, it follows that $T_{s,y}(x) = 0$ for $s \in [t_{c,T_{t,y}}, +\infty)$, where $t_{c,T_{t,y}}$ is given by formula (5.4). Because of (2.5), for $s \in [t_{c,T_{t,y}}, +\infty)$ we have that $y(s,x) = 0$. As a consequence of the assumptions on W, W_y we get that $u(s,x) = -W_y(s,x,0) = 0$ and by Proposition 5.4 we obtain

$u(s,x) = 0$ for $s \in [t_{c,T_{t,y}}, +\infty)$. Hence, we can conclude that $S(t, u(t,\cdot)) \le t_{c,T_{t,y}} < \infty$ and for a trajectory starting at $(t_0, u_0(\cdot))$ near $(t_0, 0)$ that

$$S(t_0, u_0) \le t_{c,u_0}.$$

□

6. Application of Theorem 5.6

In this section we show that the dual approach to stability can give different sufficient conditions for finite stability of (1.1).

THEOREM 6.1. *Assume that the following inequality*

$$yf(t,x,-W_y(t,x,y)) \le \frac{1}{4}(e^{-\frac{t}{50}} + 1)\,|y|^2\, e^{-|y|} \tag{6.1}$$

holds for $t \in (0,\infty)$, $y \in B(0,\frac{3}{10})$, $x \in \Omega = B(0,1)\backslash B(0,\frac{1}{2})$,

$$W(t,x,y) = \frac{1}{2}(e^{-\frac{t}{50}} + 1)\,|x|^2\,(-\frac{1}{2} + \frac{1}{2}\sin|y| + \frac{1}{2}e^{|y|} - y^2 e^{-|y|}),$$

$t \in (0,\infty)$, $x \in \Omega$, $y \in \mathcal{P} = B(0,\frac{3}{10})$. *Then the equation (1.1) is finite time stable at the origin. Moreover, for a trajectory starting at* (t_0, u_0) *near* $(t_0, 0)$ *we have for settling time the following estimation*

$$S(t_0, u_0) \le t_{c,T_{t_0,y_0}}$$

where $t_{c,T_{t_0,y_0}}$ *is defined by (5.4) with suitable changing for* c *and* $u_0 = T_{t_0,y_0}$, *and* $y_0(\cdot)$ *is a point in the dual space such that* $u_0(x) = -W_y(t_0,x,y_0(x))$, $x \in \Omega$.

PROOF. Let $W(t,x,y) = \frac{1}{2}(e^{-\frac{t}{50}} + 1)\,|x|^2\,(-\frac{1}{2} + \frac{1}{2}\sin|y| + \frac{1}{2}e^{|y|} - y^2 e^{-|y|})$, $t \in (0,\infty)$, $x \in \Omega = B(0,1)\backslash B(0,\frac{1}{2})$, $y \in \mathcal{P} = B(0,\frac{3}{10})$ and take $\alpha = \frac{9}{10}$. Then for $\mathcal{P} = B(0,\frac{1}{2})$, $x \in \Omega$ all assumptions on W are satisfied. We have to check that (5.7) is satisfied too. To this effect let us use notations and the assumptions of the theorem, then we receive the following estimations for $c(t,x) = \frac{1}{100}(\frac{1}{2}(e^{-\frac{t}{50}} + 1)\,|x|^2)^{-\alpha}$,

$$\begin{aligned} W_t(t,x,y) - y\cdot\Delta_x W_y(t,x,y) + yf(t,x,-W_y(t,x,y)) + c(t,x)(T(t,x,y))^\alpha \\ \le -\frac{1}{100}e^{-\frac{t}{50}}|x|^2(-\frac{1}{2} + \frac{1}{2}\sin|y| + \frac{1}{2}e^{|y|} - y^2e^{-|y|}) \\ -(e^{-\frac{t}{50}} + 1)(\frac{1}{2}|y|\cos|y| + \frac{1}{2}|y|e^{|y|} - y^2e^{-|y|}(2-|y|)) \\ +\frac{1}{4}(e^{-\frac{t}{50}} + 1)\,|y|^2\,e^{-|y|} \qquad (6.2) \\ +\frac{1}{100}(-\frac{1}{2} + \frac{1}{2}\sin|y| - \frac{1}{2}\,|y|\cos|y| + y^2e^{-|y|}(1-|y|) + \frac{1}{2}e^{|y|}(1-|y|))^\alpha \le 0. \end{aligned}$$

Thus, by Theorem 5.6 from the former section we get the assertion of the theorem.

□

Let us note, that condition (6.1) is imposed for f, but in fact, in the dual space i.e. for $(t,x,y) \in [0,+\infty)\times\Omega\times\mathcal{P}$. However, we are also interested to have sufficient conditions for stability imposed for f in our original space $[0,+\infty)\times\Omega\times\mathbb{R}$. We do it in the next theorem however with some restrictions for structure of f.

THEOREM 6.2. *Assume that f is of the form $f(t,x,u) = a(t,x)u + f_1(t,x,u)$, $a(\cdot) : (0,\infty) \times \Omega \to \mathbb{R}^+$-measurable, $\Omega = B(0,1)$, there exists the function $d(\cdot) : (0,\infty) \times \Omega \to \mathbb{R}^+$-measurable, such that*

$$a(t,x) + d(t,x) > \frac{1}{2}, \;\; a(t,x) + d(t,x) < 2e^{-\frac{1}{2}},$$

$$|f_1(t,x,u)| \leq d(t,x)\,|u|\,, \tag{6.3}$$

for $t \in (0,\infty)$, $x \in \Omega = B(0,1)\backslash B(0,\frac{1}{2})$, $u \in \mathbb{R}$ and let $W(t,x,y) = (e^{-t}+1)(1 + \frac{1}{2}|x|^2|y|^2 - e^{\frac{1}{2}|y|^2})$, $t \in (0,\infty)$, $x \in \Omega$, $y \in \mathcal{P} = B(0,\frac{1}{2})$. Then equation (1.1) is finite time stable at the origin. Moreover for a trajectory starting at (t_0,u_0) near $(t_0,0)$ we have for settling time the following estimation

$$S(t_0,u_0) \leq t_{c,T_{t_0,y_0}},$$

where $t_{c,T_{t_0,y_0}}$ is defined by (5.4) taking for $c(t,x) = \beta e^{-t}(e^{-t}+1)^{-\alpha}$, $0 < \beta < 1$ small, while $u_0(x) = T(t_0,x,y_0(x))$ and $y_0(\cdot)$ is a point in the dual space such that

$$u_0(x) = -W_y(t_0,x,y_0(x)), \;\; x \in \Omega.$$

PROOF. Take $0 < \alpha < 1$ and

$$W(t,x,y) = (e^{-t}+1)(1 + \frac{1}{2}|x|^2|y|^2 - e^{\frac{1}{2}|y|^2}),$$

$t \in (0,\infty)$, $x \in \Omega$ $\;y \in \mathcal{P} = B(0,1)$. Then for $\mathcal{P} = B(0,\frac{1}{2})$, $x \in \Omega$ all assumptions on W are satisfied. We have to check that (5.7) is satisfied too. To this effect let us use notations and the assumptions of the theorem. Then we receive the following estimations

$$\begin{aligned}
&W_t(t,x,y) - y \cdot \Delta_x W_y(t,x,y) + yf(t,x,-W_y(t,x,y)) + c(t,x)(T(t,x,y))^{\alpha} \\
&\leq -e^{-t}(1 + \frac{1}{2}|x|^2|y|^2 - e^{\frac{1}{2}|y|^2}) - 2(e^{-t}+1)\,|y|^2 \\
&\quad -a(t,x)(e^{-t}+1)\,|y|^2\,(|x|^2 - e^{\frac{1}{2}|y|^2}) \\
&\quad +(e^{-t}+1)d(t,x)\,|y|^2\left||x|^2 - e^{\frac{1}{2}|y|^2}\right| \\
&\quad +\beta e^{-t}(e^{-t}+1)^{-\alpha}((e^{-t}+1)(1 - \frac{1}{2}|x|^2|y|^2 \\
&\quad -e^{\frac{1}{2}|y|^2}(1-|y|^2))^{\alpha} \qquad (6.4) \\
&\leq -e^{-t}(1-\beta) + (e^{-t}+1)(\frac{1}{2} - a(t,x) - d(t,x))\,|x|^2|y|^2 \\
&\quad +(e^{-t}+1)(|y|^2(-2 + a(t,x)e^{\frac{1}{2}} + d(t,x)e^{\frac{1}{2}}).
\end{aligned}$$

Notice that the above inequality is negative for $0 < \beta < 1$. Thus, by Theorem 5.6 from the former section we get the assertion of the theorem. □

REMARK 6.3. Notice that assumption on structure of f in the above theorem is not necessary, we can simply add and subtract $a(t,x)u$. However, then we need different assumption in (6.3) which is less readable.

REMARK 6.4. The approach presented here uses in essential way the diffusion term in an explicit form - see assumption (6.3). Just this term allows us to get better estimation for nonlinear source f_1.

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