Dynamics of PDE, Vol.19, No.3, 191-206, 2022

Inviscid limit of the inhomogeneous incompressible Navier-Stokes equations under the weak Kolmogorov hypothesis in $\mathbb{R}^3$

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Communicated by Jiahong Wu, received November 10, 2021.

Abstract. In this paper, we consider the inviscid limit of inhomogeneous incompressible Navier-Stokes equations under the weak Kolmogorov hypothesis in $\mathbb{R}^3$. In particular, this limit is a weak solution of the corresponding Euler equations. We first deduce the Kolmogorov-type hypothesis in $\mathbb{R}^3$, which yields the uniform bounds of α^{th}-order fractional derivatives of $\sqrt{\rho^\mu}\mathbf{u}^\mu$ in L^2_x for some $\alpha > 0$, independent of the viscosity. The uniform bounds can provide strong convergence of $\sqrt{\rho^\mu}\mathbf{u}^\mu$ in L^2 space. This shows that the inviscid limit is a weak solution to the corresponding Euler equations.

Contents

1. Introduction

The main purpose of this paper is to study the vanishing viscosity limit of inhomogeneous Navier-Stokes equations in $\mathbb{R}^3$ under the well known hypothesis by Kolmogorov [**9, 10**]. In particular, a weaker version Assumption (KHw) of Assumption (KH) which was derived in [**4, 5**] in a periodic domain, can provide the convergence of weak solutions of the Navier-Stokes equations through a subsequence to a solution of the Euler equations. We are particularly interested in extending these results to the inhomogeneous fluids in the whole space, because

1991 *Mathematics Subject Classification.* 76D05, 35Q31, 35D30.
Key words and phrases. Inviscid limit, Kolmogorov hypothesis, inhomogeneous Navier-Stokes equations, Euler equations.

of the special structure of inhomogeneous equations and different formulation of Kolmogorov hypothesis in $\mathbb{R}^3$.

Particularly we consider the following Navier-Stokes equations for inhomogeneous fluids in $\mathbb{R}^3$:

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P = \operatorname{div}[\mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)] + \rho \mathbf{f}, \\ \operatorname{div}\ \mathbf{u} = 0, \end{cases} \tag{1.1}$$

with the initial condition

$$(\rho, \rho \mathbf{u})|_{t=0} = (\rho_0, \mathbf{m}_0)(\mathbf{x})\ x \in \mathbb{R}^3, \tag{1.2}$$

and the boundary condition

$$\mathbf{u} \to \mathbf{u}_\infty,\ as\ |\mathbf{x}| \to +\infty,\ for\ all\ t \geq 0, \tag{1.3}$$

where $\mathbf{u}_\infty$ is fixed in $\mathbb{R}^3$. Here $\rho, \mathbf{u}, P$ stand for the density, velocity and pressure of the fluid, respectively, $\mu > 0$ is the viscosity coefficient, and $\mathbf{f} = \mathbf{f}(\mathbf{x}, t) = (f_1, f_2, f_3)(\mathbf{x}, t)$ denotes a given external force.

There are many literatures studying the existence/uniqueness of solutions to inhomogeneous incompressible Navier-Stokes equations, see Refs. [**1, 8, 17**] and the references therein. Ladyženskaja and Solonnikov [**11**] first addressed the question of unique solvability of (1.1). The existence of weak solutions to (1.1) was first proved by Lions [**14**]. For clarity of presentation, henceforth we concentrate on the case $\mathbf{u}_\infty = 0$ in this paper. More precisely, for any $T \in (0, \infty)$, we assume $\mathbf{f} \in L^2(0, T; L^2(\mathbb{R}^3))$, and the initial data satisfies the following conditions:

$$\rho_0 \geq 0\ a.e.\ in\ \mathbb{R}^3,\ \rho_0 \in L^\infty(\mathbb{R}^3), \tag{1.4}$$

$$\mathbf{m}_0 \in L^2(\mathbb{R}^3),\ \mathbf{m}_0 = 0\ a.e.\ on\ \{\rho_0 = 0\},\ \frac{\mathbf{m}_0^2}{\rho_0} \in L^1(\mathbb{R}^3), \tag{1.5}$$

$$(1/\sqrt{\rho_0}) 1_{(\rho_0 < \delta_0)} \in L^2(\mathbb{R}^3)\ for\ some\ positive\ constant\ \delta_0, \tag{1.6}$$

Under the above conditions, Lions [**14**] proved the global existence of weak solutions in the following sense:

Definition 1.1 ([**14**]). For any $T > 0$, $(\rho^\mu, \mathbf{u}^\mu)(t, x)$ is a weak solution on $[0, T]$ of (1.1)-(1.2) if

- $\sqrt{\rho^\mu}\mathbf{u}^\mu \in L^\infty(0, T; L^2(\mathbb{R}^3))$, $\nabla \mathbf{u}^\mu \in L^2((0, T) \times \mathbb{R}^3)$, $\rho^\mu \in C([0, T]; L^p(\mathbb{R}^3))$ for $1 \leq p \leq \infty$,
- For any $\boldsymbol{\phi} \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^3; \mathbb{R}^3)$ such that div $\boldsymbol{\phi} = 0$,

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} (\rho^\mu \mathbf{u}^\mu \cdot \boldsymbol{\phi}_t + (\rho^\mu \mathbf{u}^\mu \otimes \mathbf{u}^\mu) : \nabla \boldsymbol{\phi} + \rho^\mu \mathbf{f} \cdot \boldsymbol{\phi})\, d\mathbf{x}\, dt + \int_{\mathbb{R}^3} \mathbf{m}_0(\mathbf{x}) \cdot \boldsymbol{\phi}(0, \mathbf{x})\, d\mathbf{x} \\ =&\frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \mu(\nabla \mathbf{u}^\mu + (\nabla \mathbf{u}^\mu)^T) \cdot (\nabla \boldsymbol{\phi} + (\nabla \boldsymbol{\phi})^T)\, d\mathbf{x}\, dt, \end{aligned} \tag{1.7}$$

- For any $\phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^3; \mathbb{R})$

$$\int_0^T \int_{\mathbb{R}^3} (\rho^\mu \phi_t + \rho^\mu \mathbf{u}^\mu \cdot \nabla \phi)\, d\mathbf{x}\, dt + \int_{\mathbb{R}^3} \rho_0^\mu \phi(0, \mathbf{x})\, d\mathbf{x} = 0. \tag{1.8}$$

- the energy inequality holds for any $t \in [0, T]$

$$\int_{\mathbb{R}^3} \frac{\rho^\mu |\mathbf{u}^\mu|^2}{2} \,\mathrm{d}\mathbf{x} + \int_0^t \int_{\mathbb{R}^3} \mu |\nabla \mathbf{u}^\mu|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}s \le \int_{\mathbb{R}^3} \frac{|\mathbf{m}_0|^2}{2\rho_0} \,\mathrm{d}\mathbf{x} + \int_0^t \int_{\mathbb{R}^3} \rho^\mu \mathbf{u}^\mu \cdot \mathbf{f} \,\mathrm{d}\mathbf{x}\,\mathrm{d}s. \tag{1.9}$$

However the uniqueness of the weak solutions in three-dimensional space is still an interesting open question.

The inviscid limit for Navier-Stokes equations has also been extensively studied (see Refs.[**2, 3, 15**] for instance). In this paper, if the weak solutions of Navier-Stokes equations are under the Kolmogorov hypothesis, the goal is to show that the weak solution for (1.1)-(1.2) converges to a weak solution of the following Euler equations:

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P = \rho \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0, \end{cases} \tag{1.10}$$

with the same initial data (1.2). The global weak solution of (1.10) is given in the following sense:

Definition 1.2. For any $T > 0$, $(\rho, \mathbf{u})(t, x)$ is called a global weak solution on $[0, T]$ of (1.10) with initial data $(\rho_0, \mathbf{m}_0)$ if $(\rho, \mathbf{u})$ satisfies:

- $(\rho, \mathbf{u})$ solves the system (1.10) in the sense of distributions in $[0, T] \times \mathbb{R}^3$, i.e.
  (a). For any $\boldsymbol{\phi} \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^3; \mathbb{R}^3)$ such that div $\boldsymbol{\phi} = 0$,

$$\int_0^T \int_{\mathbb{R}^3} (\sqrt{\rho}\sqrt{\rho}\mathbf{u} \cdot \boldsymbol{\phi}_t + (\sqrt{\rho}\mathbf{u} \otimes \sqrt{\rho}\mathbf{u}) : \nabla \boldsymbol{\phi} + \rho \mathbf{f} \cdot \boldsymbol{\phi}) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\mathbb{R}^3} \mathbf{m}_0(\mathbf{x}) \cdot \boldsymbol{\phi}(0, \mathbf{x}) \,\mathrm{d}\mathbf{x} = 0, \tag{1.11}$$

  (b). For any $\phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^3; \mathbb{R})$

$$\int_0^T \int_{\mathbb{R}^3} (\rho \phi_t + \sqrt{\rho}\sqrt{\rho}\mathbf{u} \cdot \nabla \phi) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\mathbb{R}^3} \rho_0 \phi(0, \mathbf{x}) \,\mathrm{d}\mathbf{x} = 0. \tag{1.12}$$

- (1.2) holds in $\mathcal{D}'(\mathbb{R}^3)$,
- the energy inequality holds for any $t \in (0, \infty)$

$$\int_{\mathbb{R}^3} \frac{\rho |\mathbf{u}|^{\mathbf{2}}}{2} \,\mathrm{d}\mathbf{x} \le \int_{\mathbb{R}^3} \frac{|\mathbf{m}_0|^2}{2\rho_0} \,\mathrm{d}\mathbf{x} + \int_0^t \int_{\mathbb{R}^3} \rho \mathbf{u} \cdot \mathbf{f} \,\mathrm{d}\mathbf{x}\,\mathrm{d}s. \tag{1.13}$$

In this paper, we aim to investigate such limits for global weak solutions from the inhomogeneous incompressible Navier-Stokes equations to the corresponding Euler equations under the weaker Kolmogorov-type hypothesis, which was particularly motivated by [**4**] for incompressible fluids and [**5**] for compressible fluids. Compared to these two models, the special features of inhomogeneous incompressible Navier-Stokes equations bring new difficulties to the mathematical analysis. Specifically, on one hand, the pressure-density relation for the compressible flows does not hold for the inhomogeneous flow anymore. On the other hand, the term $\partial_t(\rho \mathbf{u})$ is nonlinear in $(\rho, \mathbf{u})$ thus requires additional argument to handle the regularity in time, which is not necessary for homogeneous incompressible Euler system. In fact, the same issue arises in the mathematical study of the energy equality for the weak solutions of compressible or inhomogeneous flows, see reference [**6, 7, 12, 18**] and their references. To circumvent this time regularity assumption, one approach is to formulate the equations in terms of the density and energy equality for the momentum $\mathbf{m} = \rho \mathbf{u}$ and obtain the energy equality by multiplying the momentum equation by $(\rho \mathbf{u})^\varepsilon / \rho^\varepsilon$ instead of $\mathbf{u}^\varepsilon$, where ε is a suitable regulation. However, $(\rho \mathbf{u})^\varepsilon / \rho^\varepsilon$ cannot keep the divergence free structure, thus requires a commutator estimate involving the pressure and additional regularity of the pressure is needed. In [**6, 18**], the authors use $(\varphi(t)\mathbf{u}^\varepsilon)^\varepsilon$ where $\varphi(t)$ is sufficient nice function. This function is divergence free so it does not require any additional regularity of the pressure. We adopt this argument to handle the special feature.

In addition to the special feature of inhomogeneous incompressible Navier-Stokes equations, we are mainly interested in the inviscid limit problem in the whole space. We adopt the Fourier integrals to express the Kolmogorov-type hypothesis in $\mathbb{R}^3$, which is different from the case in $\mathbb{T}_P = [-\frac{P}{2}, \frac{P}{2}]^3, P > 0$, in [**4, 5**]. To deal with this problem, we derive the Kolmogorov-type hypothesis for inhomogeneous incompressible fluids in $\mathbb{R}^3$ based on the physical background as described in [**16**]. The details can be founded in section 2. Accordingly, it is interesting to investigate the inviscid limit of the inhomogeneous incompressible Navier-Stokes equations in $\mathbb{R}^3$ under the weaker version of Kolmogorov-type hypothesis. The following theorem is our main result of this paper.

THEOREM 1.3. *Under Assumption (RICKHw)* (2.10)*, the weak solution* $(\rho^\mu, \mathbf{u}^\mu)$ *of* (1.1)-(1.2) *as in Definition 1.1, there exists a subsequence (still denoted)* $(\rho^\mu, \mathbf{u}^\mu)$ *and a function* $(\rho, \mathbf{u})$ *such that as* $\mu \to 0$,

$$\rho^\mu \to \rho \text{ weakly in } L^p((0,T)\times\mathbb{R}^3), \ \sqrt{\rho^\mu}\mathbf{u}^\mu \to \sqrt{\rho}\mathbf{u} \text{ in } L^2((0,T)\times\mathbb{R}^3), \tag{1.14}$$

where $1 < p < \infty$*, and* $(\rho, \mathbf{u})$ *is a weak solution of* (1.10) *with initial data* $(\rho_0, \mathbf{m}_0)$*.*

REMARK 1.4. The inviscid limit for compressible Navier-Stokes equations in $\mathbb{R}^3$ can be obtained by the same method in this paper. However, for homogeneous incompressible Navier-Stokes equations, the total energy $\mathcal{E}(t)$ per unit mass vanishes as defined in Kolmogorov hypothesis in [**16**].

We can establish a similar result in $\mathbb{T}_P = [-\frac{P}{2}, \frac{P}{2}]^3, P > 0$. The proof will be given in Appendix at the end of this paper.

REMARK 1.5. If we consider the fluid in a domain with period $\mathbb{T}_P = [-P/2, P/2]^3 \subset \mathbb{R}^3, P > 0$, one can deduce the same result as Theorem 1.3. However, in this case, the Kolmogorov hypothesis is deduced as follows:

$$\sup_{k\geq k_*}\Big(|\mathbf{k}|^{3+\beta}\int_0^T |\widehat{\sqrt{\rho}\mathbf{u}}(t,\mathbf{k})|^2 \mathrm{d}t\Big) \leq C_T, \ \textit{for some } \beta > 0 \tag{1.15}$$

different from Assumption (RICKHw). In fact, for the domain $\mathbb{T}_P$, the total energy $\mathcal{E}(t)$ per unit mass satisfies

$$\mathcal{E}(t) = \frac{1}{\int_{\mathbb{T}_P}\rho^\mu \mathrm{d}\mathbf{x}}\int_{\mathbb{T}_P}\frac{\rho^\mu|\mathbf{u}^\mu|^2}{2}\,\mathrm{d}\mathbf{x} = \sum_{k\geq 0} E(t,k) = \sum_{k\geq 0} 4\pi q(t,k)k^2, \tag{1.16}$$

and the weighted velocity $\sqrt{\rho}\mathbf{u}(t,\mathbf{x})$ can be expanded to Fourier series, i.e.,

$$\sqrt{\rho}\mathbf{u}(t,\mathbf{x}) = \sum_{\mathbf{k}} \widehat{\sqrt{\rho}\mathbf{u}}(t,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{1.17}$$

The proof highly relies on (1.15) and (1.17), see Appendix 5.

The global existence theory of weak solutions to Euler equations in three dimensional space has not been established yet. This paper can be viewed as an attempt to find a way of obtaining existence result of the inhomogeneous Euler equations under special conditions. In general, the vanishing viscosity limit of Navier-Stokes equations is not a solution to the corresponding Euler equations, since the uniform bounds cannot guarantee the convergence of the nonlinear term $\rho^\mu\mathbf{u}^\mu \otimes \mathbf{u}^\mu$.

In order to pass the limit of the nonlinear term $\rho^\mu\mathbf{u}^\mu \otimes \mathbf{u}^\mu$, if we follow the techniques in [**5**], we will have to require additional regularity on the pressure. However, we do not have a pressure law for inhomogeneous fluids, and the velocity is not good enough to be a test function even after smoothing. Our key idea is to show the strong convergence of $\sqrt{\rho^\mu}\mathbf{u}^\mu$ in $L^2_{t,x}$, which yields that the weak limit of a subsequence is a weak solution to the Euler equations.

The manuscript is organized as follows. In section 2, we present the Kolmogorov hypotheses for the inhomogeneous incompressible fluids in the whole space, and a sufficient weaker version of Assumption (RICKHw) for our work. In section 3, we derive the compactness of weak solutions of the Navier-Stokes equations when the viscosity coefficient vanishes, which is crucial to obtain the weak solution to the Euler equations. In Section 4, we prove our main result based on the compactness results in Section 3. We give an outline of the proof to Remark 1.5 in Section 5 as an appendix.

2. The Kolmogorov hypotheses for the inhomogeneous incompressible fluids

In this section, we introduce the Kolmogorov hypotheses for the inhomogeneous incompressible fluids in $\mathbb{R}^3$ and the corresponding Kolmogorov-type hypothesis (RICKH) in mathematical terms. Note that, Kolmogorov [**9, 10**] first gave the following two assumptions for isotropic incompressible turbulence:

(i) At sufficiently high wavenumbers, the energy spectrum $E(t,k)$ can depend only on the fluid viscosity μ, the dissipation rate ε and the wavenumber k itself.

(ii) $E(t,k)$ should be independent of the viscosity as the Reynolds number tends to infinity:
$$E(t,k) \approx \alpha\varepsilon^{2/3}k^{-5/3}$$
in the limit of infinite Reynolds number, where α may depend on t, but is independent of k, ε.

Based on the above Kolmogorov's two hypotheses, Chen-Glimm [**4, 5**] interpreted in mathematical terms for the incompressible and compressible Kolmogorov-type hypothesis in $\mathbb{T}_P$. We generalize them to $\mathbb{R}^n$ for the fluid equations as follows.

Assumption (RICKH). For any $T > 0$, there exists $C_T > 0$ and k_*(sufficiently large) depending on $\rho_0, \mathbf{m}_0$ and $\mathbf{f}$ but independent of the viscosity μ such that, for $k = |\mathbf{k}| \geq k_*$,

$$\int_0^T E(t,\mathbf{k})\mathrm{d}t \leq C_T k^{-\frac{5}{3}}. \tag{2.1}$$

On the one hand, McComb [**16**] defined the spectral tensor $Q_{\alpha\beta}(t,\mathbf{k})$, and stated the relationship between the trace of $Q_{\alpha\beta}(t,\mathbf{k})$ and the energy $\mathcal{E}(t)$ per unit mass of fluid at time t, i.e.

$$2\mathcal{E}(t) = trQ_{\alpha\beta}(t,\mathbf{r})|_{r=0} = tr\int_0^\infty q(t,k)k^2\mathrm{d}k\int D_{\alpha\beta}(\mathbf{k})\mathrm{d}\Omega_{\mathbf{k}}, \tag{2.2}$$

where $D_{\alpha\beta}(\mathbf{k})$, $q(t,k)$ and $\mathrm{d}\Omega_{\mathbf{k}}$ denote projection operator, spectral density and the elementary solid angle in wavenumber space respectively. Using (2.2), Leslie [**13**] obtained the following crucial result

$$\begin{aligned}\mathcal{E}(t) =& \frac{4\pi}{3}tr\delta_{\alpha\beta}\int_0^\infty q(t,k)k^2\mathrm{d}k\\ =& \int_0^\infty 4\pi k^2 q(t,k)\mathrm{d}k\end{aligned}$$

$$= \int_0^\infty E(t,k)\mathrm{d}k. \tag{2.3}$$

On the other hand, from the energy inequality (1.9) of the weak solutions $(\rho^\mu, \mathbf{u}^\mu)$, the Gronwall inequality yields to

$$\int_{\mathbb{R}^3} \frac{\rho^\mu |u^\mu|^2}{2} \,\mathrm{d}\mathbf{x} + \int_0^t \int_{\mathbb{R}^3} \mu |\nabla \mathbf{u}^\mu|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}s \le M_T, \tag{2.4}$$

where M_T is a positive constant dependent on intial data, $\mathbf{f}$ and T, but independent of μ. By (2.3) and (2.4), the total energy $\mathcal{E}(t)$ per unit mass at time t for the inhomogeneous turbulence is:

$$\mathcal{E}(t) = \frac{1}{\int_{\mathbb{R}^3} \rho^\mu \mathrm{d}x} \int_{\mathbb{R}^3} \frac{\rho^\mu |\mathbf{u}^\mu|^2}{2} \,\mathrm{d}\mathbf{x} = \int_0^\infty E(t,k)\mathrm{d}k = \int_0^\infty 4\pi q(t,k) k^2 \mathrm{d}k, \tag{2.5}$$

Furthermore, if we consider the domain $\mathbb{T}_P = [-P/2, P/2]^3 \subset \mathbb{R}^3, P > 0$, the wavevector $\mathbf{k} = (k_1, k_2, k_3) = \frac{2\pi}{P}(n_1, n_2, n_3) \in \mathbb{R}^3$, with $n_j = 0, \pm 1, \pm 2, \cdots$, and $j = 1, 2, 3$, is discrete. When the setting is in $\mathbb{R}^3$, intuitively we can view that it is the large box limit as $P \to \infty$. We introduce the Fourier transform for the weighted velocity $\sqrt{\rho}\mathbf{u}(t, \mathbf{x})$ in the $\mathbf{x}$-variable as follows

$$\widehat{\sqrt{\rho}\mathbf{u}}(t, \mathbf{k}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \sqrt{\rho}\mathbf{u}(t, \mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \,\mathrm{d}\mathbf{x}, \tag{2.6}$$

thus

$$\sqrt{\rho}\mathbf{u}(t, \mathbf{x}) = \int_{\mathbb{R}^3} \widehat{\sqrt{\rho}\mathbf{u}}(t, \mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \,\mathrm{d}\mathbf{k}. \tag{2.7}$$

By Parseval identity, we have

$$\| \sqrt{\rho^\mu} u^\mu \|^2_{L^2_{\mathbf{x}}} = \| \widehat{\sqrt{\rho^\mu} u^\mu} \|^2_{L^2_{\mathbf{k}}}. \tag{2.8}$$

Clearly, it is more convenient to use the spherical coordinates (k, θ, φ), where

$$k = |\mathbf{k}|,\ 0 \le \theta \le 2\pi, 0 \le \varphi \le \pi.$$

Given the fact

$$\widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}(t, \mathbf{k}) = \widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}(t, k, \theta, \varphi)$$

and (2.8), we derive that

$$\begin{aligned}
\int_0^\infty E(t,k)\ \mathrm{d}k =& \frac{1}{\int_{\mathbb{R}^3} \rho^\mu \mathrm{d}x} \int_{\mathbb{R}^3} \frac{\rho^\mu |\mathbf{u}^\mu|^2}{2} \,\mathrm{d}\mathbf{x} \\
=& \frac{1}{2\int_{\mathbb{R}^3} \rho_0 \mathrm{d}x} \int_{\mathbb{R}^3} |\widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}(t, \mathbf{k})|^2 \,\mathrm{d}\mathbf{k} \\
=& \frac{1}{2\int_{\mathbb{R}^3} \rho_0 \mathrm{d}x} \int_0^\pi \int_0^{2\pi} \int_0^\infty |\widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}(t, k, \theta, \varphi)|^2\ k^2 \sin\varphi \,\mathrm{d}k\ \mathrm{d}\theta\ \mathrm{d}\varphi,
\end{aligned} \tag{2.9}$$

where we used

$$\int_{\mathbb{R}^3} \rho^\mu(\mathbf{x}, t)\ \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^3} \rho_0(\mathbf{x})\ \mathrm{d}\mathbf{x}.$$

This can be derived from the mass equation in (1.1).

By *Assumption (RICKH)*, $E(t, \mathbf{k}) = E(t, k, \theta, \varphi)$ and the equality (2.9), the following weaker version of *Assumption (RICKHw)* could be obtained for $\rho_0 \in L^1(\mathbb{R}^3)$, which is sufficient for the compactness analysis in this paper.

Assumption (RICKHw). For any $T>0$, there exists $C_T>0$ and k_*(sufficiently large) depending on $\rho_0, \mathbf{m}_0$ and $\mathbf{f}$ but independent of the viscosity μ such that, for $k=|\mathbf{k}|\geq k_*$,
(2.10)

$$\sup_{k\geq k_*}\Big(|\mathbf{k}|^{3+\beta}\int_0^T\int_0^\pi\int_0^{2\pi}|\widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}(t,k,\theta,\varphi)|^2\ k^2\sin\varphi\,\mathrm{d}\theta\ \mathrm{d}\varphi\ \mathrm{d}t\Big)\leq C_T,\ \textit{for some } \beta>0.$$

3. Compactness of weak solutions

In this section, We develop the compactness of weak solutions when the viscosity coefficient vanishes. The following lemma is crucial to show the compactness of weak solutions.

Lemma 3.1. *Under Assumption (RICKHw), for any $T\in(0,+\infty)$, there exists $C>0$, independent of $\mu>0$, such that*

$$\int_0^T\int_{\mathbb{R}^3}|D_x^\alpha(\sqrt{\rho^\mu}\mathbf{u}^\mu)|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\leq C, \tag{3.1}$$

where $\alpha\in(0,1+\frac{\beta}{2})$, C is a generic positive constant depending on $\rho_0,\mathbf{m}_0,\mathbf{f},\alpha,k_*$ and $T>0$, but independent of $\mu>0$.

Remark 3.2. A similar version for periodic domain case were given in [**4, 5**], where the proof relied on the definition of fractional derivatives via Fourier series.

Proof. With the definition of fractional derivatives via Fourier transform, the Parseval identity, the spherical coordinates and *Assumption (RICKHw)*, we have

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^3}|D_x^\alpha(\sqrt{\rho^\mu}\mathbf{u}^\mu)(t,\mathbf{x})|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
=&\int_0^T\int_{\mathbb{R}^3}|D_x^\alpha\widehat{(\sqrt{\rho^\mu}\mathbf{u}^\mu)}(t,\mathbf{k})|^2\,\mathrm{d}\mathbf{k}\mathrm{d}t\\
=&\int_0^T\int_{\mathbb{R}^3}|\mathbf{k}|^{2\alpha}|\widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}(t,\mathbf{k})|^2\,\mathrm{d}\mathbf{k}\mathrm{d}t\\
=&\int_0^T\int_0^\pi\int_0^{2\pi}\int_0^\infty k^{2\alpha}|\widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}(t,k,\theta,\varphi)|^2k^2\sin\varphi\,\mathrm{d}k\ \mathrm{d}\theta\ \mathrm{d}\varphi\ \mathrm{d}t\\
=&\int_0^T\int_0^\pi\int_0^{2\pi}\int_0^{k_*} k^{2\alpha}|\widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}|^2k^2\sin\varphi\,\mathrm{d}k\ \mathrm{d}\theta\ \mathrm{d}\varphi\ \mathrm{d}t\\
&+\int_0^T\int_0^\pi\int_0^{2\pi}\int_{k_*}^\infty k^{2\alpha}|\widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}|^2k^2\sin\varphi\,\mathrm{d}k\ \mathrm{d}\theta\ \mathrm{d}\varphi\ \mathrm{d}t\\
\leq& Ck_*^{2\alpha}\int_0^T\int_{\mathbb{R}^3}|\sqrt{\rho^\mu}\mathbf{u}^\mu|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+C\int_{k_*}^\infty k^{2\alpha-3-\beta}\,\mathrm{d}k\\
\leq& C,
\end{aligned} \tag{3.2}$$

where $\alpha<1+\frac{\beta}{2}$. □

With Lemma 3.1 at hand, we have the following uniform bound of $\sqrt{\rho^\mu}\mathbf{u}^\mu$ in μ.

Corollary 3.3. Under *Assumption (RICKHw)*, for any $T>0$, there exists $C>0$, independent of $\mu>0$, such that for $q=q(\beta)>2$,

$$\|\sqrt{\rho^\mu}\mathbf{u}^\mu\|_{L^q((0,T)\times\mathbb{R}^3)}\leq C. \tag{3.3}$$

Proof. In view of Gagliardo-Nirenberg interpolation inequality and Lemma 3.1, we have

$$\|\sqrt{\rho^\mu}\mathbf{u}^\mu\|_{L^q} \leq \|\sqrt{\rho^\mu}\mathbf{u}^\mu\|_{L^2}^{\alpha_1}\|\sqrt{\rho^\mu}\mathbf{u}^\mu\|_{H^\alpha}^{1-\alpha_1}, \tag{3.4}$$

where $\frac{1}{q} = \frac{\alpha_1}{2} + \left(\frac{1}{2} - \frac{\alpha}{3}\right)(1-\alpha_1)$ for $0 < \alpha_1 < 1$. Note that $\sqrt{\rho^\mu}\mathbf{u}^\mu$is uniformly bounded in $L^2(0,T;H^\alpha(\mathbb{R}^3))$, we get $q = 2 + \frac{4\alpha}{3} > 2$. □

Lemma 3.4. *Under Assumption (RICKHw), for any $T \in (0,+\infty)$, $\sqrt{\rho^\mu}\mathbf{u}^\mu$ is equicontinuous with respect to the space variable $\mathbf{x}$ in $L^2((0,T)\times\mathbb{R}^3)$, independent of μ, i.e.,*

$$\int_0^T \int_{\mathbb{R}^3} |\sqrt{\rho^\mu}\mathbf{u}^\mu(t,\mathbf{x}+\Delta\mathbf{x}) - \sqrt{\rho^\mu}\mathbf{u}^\mu(t,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \to 0, \text{ as } \Delta\mathbf{x} \to 0. \tag{3.5}$$

Proof. Using Lemma 3.1, Hölder's inequality, Parseval identity and the properties of the Fourier transform, we obtain for any $\alpha \leq 1$

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} |\sqrt{\rho^\mu}\mathbf{u}^\mu(t,\mathbf{x}+\Delta\mathbf{x}) - \sqrt{\rho^\mu}\mathbf{u}^\mu(t,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
=&\int_0^T \int_{\mathbb{R}^3} \Big| \int_{\mathbb{R}^3} \widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}(t,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}(e^{i\mathbf{k}\cdot\Delta\mathbf{x}}-1)\,\mathrm{d}\mathbf{k}\Big|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
\leq& C\int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{2i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{k} \int_{\mathbb{R}^3} |\widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}(t,\mathbf{k})|^2 (e^{i\mathbf{k}\cdot\Delta\mathbf{x}}-1)^2 \,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
\leq& C|\Delta\mathbf{x}|^{2\alpha} \int_{\mathbb{R}^3} \delta(2\mathbf{x})\,\mathrm{d}\mathbf{x} \int_0^T \int_{\mathbb{R}^3} |\mathbf{k}|^{2\alpha} |\widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}(t,\mathbf{k})|^2 \,\mathrm{d}\mathbf{k}\,\mathrm{d}t\\
\leq& C|\Delta\mathbf{x}|^{2\alpha} \int_0^T \int_{\mathbb{R}^3} |D^\alpha_{\mathbf{x}}(\sqrt{\rho^\mu}\mathbf{u}^\mu)|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
\leq& C|\Delta\mathbf{x}|^{2\alpha},
\end{aligned}$$

where $\delta(\mathbf{x})$ is the Dirac delta function, and satisfies $\int_{\mathbb{R}^3} e^{i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{k} = \delta(\mathbf{x})$. □

To show the strong convergence of $\sqrt{\rho^\mu}\mathbf{u}^\mu$ in L^2 space, it is crucial to have the following lemma.

Lemma 3.5. *Under Assumption (RICKHw), for any $T > 0$, $\sqrt{\rho^\mu}\mathbf{u}^\mu$ is equicontinuous with respect to the time variable t in $L^2((0,T-\Delta t)\times\mathbb{R}^3)$, independent of μ, i.e.*

$$\int_0^{T-\Delta t} \int_{\mathbb{R}^3} |\sqrt{\rho^\mu}\mathbf{u}^\mu(t+\Delta t,\mathbf{x}) - \sqrt{\rho^\mu}\mathbf{u}^\mu(t,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \to 0, \text{ as } \Delta t \to 0. \tag{3.6}$$

Proof. For simplicity, we drop the superscript μ of $\sqrt{\rho^\mu}\mathbf{u}^\mu$ in the following proof. For any $\varphi(t) \in \mathcal{D}(0,+\infty)$, we have

$$\begin{aligned}
&\int_0^{T-\Delta t} \int_{\mathbb{R}^3} |\sqrt{\rho}\mathbf{u}(t+\Delta t,\mathbf{x}) - \sqrt{\rho}\mathbf{u}(t,\mathbf{x})|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
=&\int_0^{T-\Delta t} \int_{\mathbb{R}^3} [\rho\mathbf{u}(t+\Delta t,\mathbf{x}) - \rho\mathbf{u}(t,\mathbf{x})][\mathbf{u}(t+\Delta t,\mathbf{x}) - \mathbf{u}(t,\mathbf{x})] \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&+\int_0^{T-\Delta t} \int_{\mathbb{R}^3} \sqrt{\rho}\mathbf{u}(t+\Delta t,\mathbf{x})\mathbf{u}(t,\mathbf{x})[\sqrt{\rho}(t+\Delta t,\mathbf{x}) - \sqrt{\rho}(t,\mathbf{x})] \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
&+\int_0^{T-\Delta t} \int_{\mathbb{R}^3} \sqrt{\rho}\mathbf{u}(t,\mathbf{x})\mathbf{u}(t+\Delta t,\mathbf{x})[\sqrt{\rho}(t,\mathbf{x}) - \sqrt{\rho}(t+\Delta t,\mathbf{x})] \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
=&\int_0^{T-\Delta t} \int_{\mathbb{R}^3} [\rho\mathbf{u}(t+\Delta t,\mathbf{x}) - \rho\mathbf{u}(t,\mathbf{x})]\Big\{[\mathbf{u}(t+\Delta t,\mathbf{x}) - \varphi(t)\mathbf{u}^\epsilon(t+\Delta t,\mathbf{x})]
\end{aligned}$$

$$
\begin{aligned}
&- [\mathbf{u}(t,\mathbf{x}) - \varphi(t)\mathbf{u}^\epsilon(t,\mathbf{x})]\Big\} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \int_0^{T-\Delta t}\int_{\mathbb{R}^3} [\rho\mathbf{u}(t+\Delta t,\mathbf{x}) - \rho\mathbf{u}(t,\mathbf{x})][\varphi(t)\mathbf{u}^\epsilon(t+\Delta t,\mathbf{x}) - \varphi(t)\mathbf{u}^\epsilon(t,\mathbf{x})]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \int_0^{T-\Delta t}\int_{\mathbb{R}^3} \sqrt{\rho}\mathbf{u}(t+\Delta t,\mathbf{x})\mathbf{u}(t,\mathbf{x})[\sqrt{\rho}(t+\Delta t,\mathbf{x}) - \sqrt{\rho}(t,\mathbf{x})]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \int_0^{T-\Delta t}\int_{\mathbb{R}^3} \sqrt{\rho}\mathbf{u}(t,\mathbf{x})\mathbf{u}(t+\Delta t,\mathbf{x})[\sqrt{\rho}(t,\mathbf{x}) - \sqrt{\rho}(t+\Delta t,\mathbf{x})]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&= I_1 + I_2 + I_3 + I_4, \qquad (3.7)
\end{aligned}
$$

where

$$
\mathbf{u}^\epsilon(t,\mathbf{x}) = (j_\epsilon * \mathbf{u})(t,\mathbf{x}) = \int_{|t-s|\le\epsilon}\int_{|\mathbf{x}-\mathbf{y}|\le\epsilon} j_\epsilon(t-s,\mathbf{x}-\mathbf{y})\mathbf{u}(s,\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}s,
$$

and j_ϵ is a standard mollifier. Note that $\varphi(t)\mathbf{u}^\epsilon(t+\Delta t,\mathbf{x})$, $\varphi(t)\mathbf{u}^\epsilon(t,\mathbf{x})$ are well defined in $\mathcal{D}(0,+\infty)$.

We divide the equality (3.7) into four terms, and each term will be considered separately. First, we can rewrite I_3 as following

$$
\begin{aligned}
I_3 = &\int_0^{T-\Delta t}\int_{\mathbb{R}^3} \sqrt{\rho}\mathbf{u}(t+\Delta t,\mathbf{x})\mathbf{u}(t,\mathbf{x})\Big\{[\sqrt{\rho}(t+\Delta t,\mathbf{x}) - (\sqrt{\rho})^\epsilon(t+\Delta t,\mathbf{x})] \\
&- [\sqrt{\rho}(t,\mathbf{x}) - (\sqrt{\rho})^\epsilon(t,\mathbf{x})]\Big\}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \int_0^{T-\Delta t}\int_{\mathbb{R}^3} \sqrt{\rho}\mathbf{u}(t+\Delta t,\mathbf{x})\mathbf{u}(t,\mathbf{x})[(\sqrt{\rho})^\epsilon(t+\Delta t,\mathbf{x}) - (\sqrt{\rho})^\epsilon(t,\mathbf{x})]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
= &\int_0^{T-\Delta t}\int_{\mathbb{R}^3} \sqrt{\rho}\mathbf{u}(t+\Delta t,\mathbf{x})\mathbf{u}(t,\mathbf{x})[\sqrt{\rho}(t+\Delta t,\mathbf{x}) - (\sqrt{\rho})^\epsilon(t+\Delta t,\mathbf{x})]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \int_0^{T-\Delta t}\int_{\mathbb{R}^3} \sqrt{\rho}\mathbf{u}(t+\Delta t,\mathbf{x})\mathbf{u}(t,\mathbf{x})[(\sqrt{\rho})^\epsilon(t,\mathbf{x}) - \sqrt{\rho}(t,\mathbf{x})]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \int_0^{T-\Delta t}\int_{\mathbb{R}^3} \sqrt{\rho}\mathbf{u}(t+\Delta t,\mathbf{x})\mathbf{u}(t,\mathbf{x})[(\sqrt{\rho})^\epsilon(t+\Delta t,\mathbf{x}) - (\sqrt{\rho})^\epsilon(t,\mathbf{x})]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
= &I_3^1 + I_3^2 + I_3^3. \qquad (3.8)
\end{aligned}
$$

Here we decompose $\mathbf{u}$ into $\mathbf{u} = \mathbf{u}1_{\{\rho<\delta_0\}} + \mathbf{u}1_{\{\rho\ge\delta_0\}} = \mathbf{u}_1 + \mathbf{u}_2$, then in view of $\frac{1}{\sqrt{\rho_0}}1_{\{\rho_0<\delta_0\}} \in L^2(\mathbb{R}^3)$, we can get

$$
\|\mathbf{u}1_{\{\rho<\delta_0\}}\|_{L^l} \le \Big\|\frac{1}{\sqrt{\rho}}1_{\{\rho<\delta_0\}}\Big\|_{L^2}\|\sqrt{\rho}\mathbf{u}\|_{L^q},\ l = \frac{2q}{q+2}. \tag{3.9}
$$

Thus $\mathbf{u}_1 \in L^l((0,T)\times\mathbb{R}^3)$. As for $\mathbf{u}_2$, since $\sqrt{\rho}\mathbf{u} \in L^\infty(0,T;L^2(\mathbb{R}^3))$, we have $\mathbf{u}_2 \in L^\infty(0,T;L^2(\mathbb{R}^3))$. Moreover, it holds that as $\epsilon \to 0$,

$$
\|\mathbf{u}_1 - \mathbf{u}_1^\epsilon\|_{L^l(0,T;L^l(\mathbb{R}^3))} \to 0,\ \|\mathbf{u}_2 - \mathbf{u}_2^\epsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \to 0. \tag{3.10}
$$

For $I_3^1 + I_3^2$, using $(\sqrt{\rho})^\epsilon \to \sqrt{\rho}$ in $C([0,T],L^p(\mathbb{R}^3))$, $2 \le p < \infty$, as $\epsilon = \epsilon(\Delta t) \to 0$, we have

$$
\begin{aligned}
I_3^1 + I_3^2 \le &\int_0^{T-\Delta t}\int_{\mathbb{R}^3} \sqrt{\rho}\mathbf{u}\ \mathbf{u}_1[(\sqrt{\rho})^\epsilon - \sqrt{\rho}]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \int_0^{T-\Delta t}\int_{\mathbb{R}^3} \sqrt{\rho}\mathbf{u}\ \mathbf{u}_2[(\sqrt{\rho})^\epsilon - \sqrt{\rho}]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
\le &C\|\sqrt{\rho}\mathbf{u}\|_{L^q(0,T;L^q(\mathbb{R}^3))}\|\mathbf{u}_1\|_{L^l(0,T;L^l(\mathbb{R}^3))}\|(\sqrt{\rho})^\epsilon - \sqrt{\rho}\|_{L^\infty(0,T;L^{p_1}(\mathbb{R}^3))}
\end{aligned}
$$

$$+ C\|\sqrt{\rho}\mathbf{u}\|_{L^q(0,T;L^q(\mathbb{R}^3))}\|\mathbf{u}_2\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\|(\sqrt{\rho})^\epsilon - \sqrt{\rho}\|_{L^\infty(0,T;L^{p_2}(\mathbb{R}^3))}$$
$$\to 0, \ as \ \Delta t \to 0,$$

where $q > 4$, p_1 and p_2 satisfy $\frac{1}{q} + \frac{1}{l} + \frac{1}{p_1} = 1$ and $\frac{1}{q} + \frac{1}{2} + \frac{1}{p_2} = 1$ respectively.
For I_3^3, using Hölder inequality, we obtain

$$I_3^3 \le C\|\sqrt{\rho}\mathbf{u}\|_{L^q(0,T;L^q(\mathbb{R}^3))}\|\mathbf{u}_1\|_{L^l(0,T;L^l(\mathbb{R}^3))}\|(\sqrt{\rho})^\epsilon(t+\Delta t,\mathbf{x}) - (\sqrt{\rho})^\epsilon(t,\mathbf{x})\|_{L^{\frac{2q}{q-4}}([0,T]\times\mathbb{R}^3)}$$
$$+ C\|\sqrt{\rho}\mathbf{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\|\mathbf{u}_2\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\|(\sqrt{\rho})^\epsilon(t+\Delta t,\mathbf{x}) - (\sqrt{\rho})^\epsilon(t,\mathbf{x})\|_{L^\infty([0,T]\times\mathbb{R}^3)}$$
$$\to 0, \ as \ \Delta t \to 0,$$

where $q > 4$.

Similarly, we have as $\Delta t \to 0, I_4 \to 0$, independent of μ.

For I_2, from div $\mathbf{u}^\epsilon = 0$ and Definition 1.1 , we get

$$\begin{aligned} I_2 =& \int_0^{T-\Delta t}\int_{\mathbb{R}^3}\int_t^{t+\Delta t}(\rho\mathbf{u})_s(s,\mathbf{x})\cdot[\varphi(t)\mathbf{u}^\epsilon(t+\Delta t,\mathbf{x}) - \varphi(t)\mathbf{u}^\epsilon(t,\mathbf{x})]\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ =& \int_0^{T-\Delta t}\int_{\mathbb{R}^3}\int_t^{t+\Delta t}(\rho\mathbf{u}\otimes\mathbf{u})(s,\mathbf{x}) : \nabla[\varphi(t)\mathbf{u}^\epsilon(t+\Delta t,\mathbf{x}) - \varphi(t)\mathbf{u}^\epsilon(t,\mathbf{x})]\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ & - \int_0^{T-\Delta t}\int_{\mathbb{R}^3}\int_t^{t+\Delta t}\mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)(s,\mathbf{x})\cdot\nabla[\varphi(t)\mathbf{u}^\epsilon(t+\Delta t,\mathbf{x}) - \varphi(t)\mathbf{u}^\epsilon(t,\mathbf{x})]\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ & + \int_0^{T-\Delta t}\int_{\mathbb{R}^3}\int_t^{t+\Delta t}(\rho\mathbf{f})(s,\mathbf{x})\cdot[\varphi(t)\mathbf{u}^\epsilon(t+\Delta t,\mathbf{x}) - \varphi(t)\mathbf{u}^\epsilon(t,\mathbf{x})]\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ =& I_2^1 + I_2^2 + I_2^3. \end{aligned} \tag{3.11}$$

Notice that, for any $x \in \mathbb{R}^3$,

$$\begin{aligned} |\varphi(t)\mathbf{u}^\epsilon(t,\mathbf{x})| \le& \frac{C}{\epsilon^4}\Big|\int_{|t-s|\le\epsilon}\int_{|\mathbf{x}-\mathbf{y}|\le\epsilon} j\big(\frac{t-s}{\epsilon}, \frac{\mathbf{x}-\mathbf{y}}{\epsilon}\big)\mathbf{u}(s,\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}s\Big| \\ \le& \frac{C}{\epsilon^4}\|\mathbf{u}_1\|_{L^l(0,T;L^l(\mathbb{R}^3))}\Big(\int_{|\tau|\le 1}\int_{|\mathbf{z}|\le 1}|j(\tau,\mathbf{z})|^{l'}\epsilon^4\,\mathrm{d}\mathbf{z}\,\mathrm{d}\tau\Big)^{1/l'} \\ & + \frac{C}{\epsilon^4}\|\mathbf{u}_2\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\Big(\int_{|\tau|\le 1}\int_{|\mathbf{z}|\le 1}|j(\tau,\mathbf{z})|^2\epsilon^4\,\mathrm{d}\mathbf{z}\,\mathrm{d}\tau\Big)^{1/2} \\ \le& \frac{C}{\epsilon^{2(q+2)/q}}, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} |\varphi(t)\nabla\mathbf{u}^\epsilon(t,\mathbf{x})| \le& \frac{C}{\epsilon^5}\Big|\int_{|t-s|\le\epsilon}\int_{|\mathbf{x}-\mathbf{y}|\le\epsilon}\partial_2 j\big(\frac{t-s}{\epsilon}, \frac{\mathbf{x}-\mathbf{y}}{\epsilon}\big)\mathbf{u}(s,\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}s\Big| \\ \le& \frac{C}{\epsilon^5}\|\mathbf{u}_1\|_{L^l(0,T;L^l(\mathbb{R}^3))}\Big(\int_{|\tau|\le 1}\int_{|\mathbf{z}|\le 1}|\partial_z j(\tau,\mathbf{z})|^{l'}\epsilon^4\,\mathrm{d}\mathbf{z}\,\mathrm{d}\tau\Big)^{1/l'} \\ & + \frac{C}{\epsilon^5}\|\mathbf{u}_2\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\Big(\int_{|\tau|\le 1}\int_{|\mathbf{z}|\le 1}|\partial_z j(\tau,\mathbf{z})|^2\epsilon^4\,\mathrm{d}\mathbf{z}\,\mathrm{d}\tau\Big)^{1/2} \\ \le& \frac{C}{\epsilon^{(3q+4)/q}}. \end{aligned} \tag{3.13}$$

For $I_2^1 + I_2^3$, by (3.12),(3.13) and Hölder inequality, we have

$$\begin{aligned} I_2^1 + I_2^3 \le& C\Delta t\ \|\sqrt{\rho}\mathbf{u}\|_{L^2}^2|\varphi\nabla\mathbf{u}^\epsilon| + C\Delta t\ \|\rho\|_{L^2}\|\mathbf{f}\|_{L^2}|\varphi\mathbf{u}^\epsilon| \\ \le& \frac{C\Delta t}{\epsilon^{(3q+4)/q}}. \end{aligned} \tag{3.14}$$

For I_2^2, in view of Hölder inequality and the energy inequality in Definition 1.1, we obtain

$$\begin{aligned}
I_2^2 \leq & C\Big|\int_0^{T-\Delta t}\int_{\mathbb{R}^3}\int_t^{t+\Delta t}\sqrt{\mu}(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)(s,\mathbf{x})\,\mathrm{d}s\cdot[\sqrt{\mu}\nabla\mathbf{u}^\epsilon(t+\Delta t,\mathbf{x})-\sqrt{\mu}\nabla\mathbf{u}^\epsilon(t,\mathbf{x})]\mathrm{d}\mathbf{x}\mathrm{d}t\Big| \\
\leq & C\Big[\int_0^{T-\Delta t}\int_{\mathbb{R}^3}\Big(\int_t^{t+\Delta t}\sqrt{\mu}(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)(s,\mathbf{x})\,\mathrm{d}s\Big)^2\mathrm{d}\mathbf{x}\,\mathrm{d}t\Big]^{\frac{1}{2}}\Big[\int_0^{T-\Delta t}\int_{\mathbb{R}^3}\mu|\nabla\mathbf{u}^\epsilon|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\Big]^{\frac{1}{2}} \\
\leq & C(\Delta t)^{\frac{1}{2}}\Big[\int_0^{T-\Delta t}\int_{\mathbb{R}^3}\int_t^{t+\Delta t}\mu(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)^2(s,\mathbf{x})\,\mathrm{d}s\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\Big]^{\frac{1}{2}} \\
\leq & C\Delta t. \qquad (3.15)
\end{aligned}$$

Plugging (3.14) and (3.15) into (3.11) yields

$$I_2 \leq \frac{C\Delta t}{\epsilon^{(3q+4)/q}} + C\Delta t \leq \frac{C\Delta t}{\epsilon^{(3q+4)/q}}. \tag{3.16}$$

By choosing $\epsilon = (C\Delta t)^{\frac{q}{6q+8}}$, we deduce that

$$I_2 \leq \Delta t^{\frac{1}{2}} \to 0, \ \ as\ \Delta t \to 0. \tag{3.17}$$

To handle the term I_1, we choose a C^∞ nonnegative cut-off function as the following

$$\varphi(t) = \begin{cases} 0, & t \leq \Delta t \\ 1, & 2\Delta t \leq t \leq T - 2\Delta t \\ 0, & t \geq T - \Delta t, \end{cases}$$

where $\Delta t > 0$ is small and $|\varphi'(t)| \leq \frac{1}{\Delta t}$. Notice that $\varphi(t) \in \mathcal{D}(0, +\infty)$, and $\varphi(t)$ converges to 1 almost everywhere as $\Delta t \to 0$.

By using Hölder inequality, Corollary 3.3 and (3.10), as $\Delta t \to 0$, we have

$$\begin{aligned}
I_1 = & \int_0^{T-\Delta t}\int_{\mathbb{R}^3}[\rho\mathbf{u}(t+\Delta t,\mathbf{x})-\rho\mathbf{u}(t,\mathbf{x})]\Big\{\big[\ \mathbf{u}(t+\Delta t,\mathbf{x})-\varphi(t)\mathbf{u}^\epsilon(t+\Delta t,\mathbf{x})\big] \\
& -\big[\ \mathbf{u}(t,\mathbf{x})-\varphi(t)\mathbf{u}^\epsilon(t,\mathbf{x})\big]\Big\}\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
\leq & C\int_0^{T-\Delta t}\|\sqrt{\rho}\mathbf{u}\|_{L^q}\big[\ \|\mathbf{u}_1-\mathbf{u}_1^\epsilon\|_{L^l}\|\sqrt{\rho}\|_{L^{p_1}}+\|(1-\varphi(t))\mathbf{u}_1^\epsilon\|_{L^l}\|\sqrt{\rho}\|_{L^{p_1}}\big]\,\mathrm{d}t \\
& + C\int_0^{T-\Delta t}\|\sqrt{\rho}\mathbf{u}\|_{L^2}\big[\ \|\mathbf{u}_2-\mathbf{u}_2^\epsilon\|_{L^2}+\|(1-\varphi(t))\mathbf{u}_2^\epsilon\|_{L^2}\big]\,\mathrm{d}t \\
\leq & C\|\sqrt{\rho}\mathbf{u}\|_{L^q(0,T;L^q(\mathbb{R}^3))}\big[\ \|\mathbf{u}_1-\mathbf{u}_1^\epsilon\|_{L^l(0,T;L^l(\mathbb{R}^3))}+\|\mathbf{u}_1^\epsilon\|_{L^l(0,T;L^l(\mathbb{R}^3))}\|1-\varphi\|_{L^{p_1}(0,T)}\big] \\
& +C\|\sqrt{\rho}\mathbf{u}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\big[\ \|\mathbf{u}_2-\mathbf{u}_2^\epsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}+\|\mathbf{u}_2^\epsilon\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\|1-\varphi\|_{L^\infty(0,T)}\big] \\
\to & 0, \qquad (3.18)
\end{aligned}$$

where $\frac{1}{q}+\frac{1}{l}+\frac{1}{p_1}=1, q>4$.

Combining (3.8),(3.17) and (3.18), we complete the proof. □

PROPOSITION 3.6. Under *Assumption (RICKHw)*, for any $T > 0$, there exists a subsequence (still denoted) $\sqrt{\rho}^\mu\mathbf{u}^\mu$ and a function $\sqrt{\rho}\mathbf{u} \in L^2((0,T)\times\mathbb{R}^3)$ such that

$$\sqrt{\rho}^\mu\mathbf{u}^\mu \to \sqrt{\rho}\mathbf{u}\ \ in\ L^2((0,T)\times\mathbb{R}^3)\ \ as\ \mu\to 0. \tag{3.19}$$

PROOF. Thanks to Lemmas 3.4-3.5, we have L^2-equicontinuity of $\sqrt{\rho}^\mu\mathbf{u}^\mu$. Thus (3.19) follows. □

4. Vanishing viscosity limit

In this section, we prove our main result by the compactness argument as μ tends to zero.

From Definition 1.1, we know that ρ^μ is bounded in $C([0,T];L^p(\mathbb{R}^3))$. Here we apply the renormalized techniques in [**14**] to show that this bound is uniform with respect to μ. Using the standard mollifier $\eta_\epsilon(\mathbf{x})$ to test $(1.1)_1$, we have

$$\partial_t\langle\rho^\mu\rangle_\epsilon+\nabla\langle\rho^\mu\rangle_\epsilon\cdot\mathbf{u}^\mu=\operatorname{div}(\langle\rho^\mu\rangle_\epsilon\mathbf{u}^\mu)-\langle\operatorname{div}(\rho^\mu\mathbf{u}^\mu)\rangle_\epsilon,\ a.e.\ on\ U\subset\subset(0,T)\times\mathbb{R}^3, \tag{4.1}$$

where $\langle f\rangle_\epsilon:=\int_{\mathbb{R}^3}f(\mathbf{x}-\mathbf{y})\eta_\epsilon(\mathbf{y})\,\mathrm{d}\mathbf{y}$.

For $\beta\in C^1([0,\infty),\mathbb{R})$, multiplying (4.1) by $\beta'(\langle\rho^\mu\rangle_\epsilon)$, we get

$$\begin{aligned}&\int_{\mathbb{R}^3}\beta(\langle\rho^\mu\rangle_\epsilon)\,dx-\int_{\mathbb{R}^3}\beta(\rho_0)\mathrm{d}\mathbf{x}+\int_0^T\int_{\mathbb{R}^3}\operatorname{div}(\beta(\langle\rho^\mu\rangle_\epsilon)\cdot\mathbf{u}^\mu)\,dx\,dt\\&=\int_0^T\int_{\mathbb{R}^3}\beta'(\langle\rho^\mu\rangle_\epsilon)[\operatorname{div}(\langle\rho^\mu\rangle_\epsilon\mathbf{u}^\mu)-\langle\operatorname{div}(\rho^\mu\mathbf{u}^\mu)\rangle_\epsilon]\,dx\,dt.\end{aligned} \tag{4.2}$$

Note that $\nabla\mathbf{u}^\mu\in L^2((0,T)\times\mathbb{R}^3),\ \rho^\mu\in C([0,T];L^p(\mathbb{R}^3))\ for\ 1\le p<\infty$, in Definition 1.1, we can show that the right-hand side of (4.2) tends to zero as $\epsilon\to0$. Thus, we obtain as $\epsilon\to0$

$$\int_{\mathbb{R}^3}\beta(\rho^\mu)\mathrm{d}\mathbf{x}=\int_{\mathbb{R}^3}\beta(\rho_0)\mathrm{d}\mathbf{x}. \tag{4.3}$$

we further choose $\beta(s)=0$ for any $0\le s\le\rho_0$ and $\beta(s)>0$ for any $s>\rho_0$. Thus, by (4.3), we have

$$\|\rho\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\le\|\rho_0\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))},$$

this implies that ρ^μ is uniformly bounded in $L^\infty([0,T]\times\mathbb{R}^3)$.

Note that ρ^μ and $\sqrt{\rho^\mu}$ are uniformly bounded in $L^\infty([0,T]\times\mathbb{R}^3)\cap C(0,T;L^p(\mathbb{R}^3))$, where $1\le p<\infty$, thus we have

$$\begin{aligned}&\rho^\mu\to\rho\quad\text{weakly in } L^p([0,T]\times\mathbb{R}^3),\ 1<p<\infty,\\&\sqrt{\rho^\mu}\to\sqrt{\rho}\quad\text{weakly in } L^p([0,T]\times\mathbb{R}^3),\ 1<p<\infty,\end{aligned} \tag{4.4}$$

as $\mu\to0$. With aid of Proposition 3.6, for any test function $\boldsymbol{\phi}\in C_0^\infty(\mathbb{R}_+\times\mathbb{R}^3;\mathbb{R}^3)$, we have

$$\int_0^T\int_{\mathbb{R}^3}\sqrt{\rho^\mu}\sqrt{\rho^\mu}\mathbf{u}^\mu\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\to\int_0^T\int_{\mathbb{R}^3}\sqrt{\rho}\sqrt{\rho}\mathbf{u}\cdot\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{4.5}$$

We recall Proposition 3.6 again to see that

$$\int_0^T\int_{\mathbb{R}^3}\left(\sqrt{\rho^\mu}\mathbf{u}^\mu\otimes\sqrt{\rho^\mu}\mathbf{u}^\mu\right):\nabla\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\to\int_0^T\int_{\mathbb{R}^3}(\sqrt{\rho}\mathbf{u}\otimes\sqrt{\rho}\mathbf{u}):\nabla\boldsymbol{\phi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{4.6}$$

The viscous term in the Navier-Stokes vanishes by letting μ tend to zero, in the following way

$$\left|\int_0^T\int_{\mathbb{R}^3}\mu(\nabla\mathbf{u}^\mu+(\nabla\mathbf{u}^\mu)^T)\cdot(\nabla\boldsymbol{\phi}+(\nabla\boldsymbol{\phi})^T)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right|\le C\sqrt{\mu}\,\|\sqrt{\mu}\nabla\mathbf{u}^\mu\|_{L^2(0,T;L^2(\mathbb{R}^3))}\to0. \tag{4.7}$$

Note that in Definition 1.1, the weak solutions $(\rho^\mu,\mathbf{u}^\mu)$ of the Navier-Stokes equations satisfy the following weak formulation

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^3}(\rho^\mu\mathbf{u}^\mu\cdot\boldsymbol{\phi}_t+(\rho^\mu\mathbf{u}^\mu\otimes\mathbf{u}^\mu):\nabla\boldsymbol{\phi}+\rho^\mu\mathbf{f}\cdot\boldsymbol{\phi})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+\int_{\mathbb{R}^3}\mathbf{m}_0(\mathbf{x})\cdot\boldsymbol{\phi}(0,\mathbf{x})\,\mathrm{d}\mathbf{x}\\&=\frac12\int_0^T\int_{\mathbb{R}^3}\mu(\nabla\mathbf{u}^\mu+(\nabla\mathbf{u}^\mu)^T)\cdot(\nabla\boldsymbol{\phi}+(\nabla\boldsymbol{\phi})^T)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,\end{aligned} \tag{4.8}$$

and

$$\int_0^T \int_{\mathbb{R}^3} \rho^\mu \phi_t + \rho^\mu \mathbf{u}^\mu \cdot \nabla \phi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\mathbb{R}^3} \rho_0^\mu \phi(0,\mathbf{x})\,\mathrm{d}\mathbf{x} = 0. \tag{4.9}$$

With (4.4)-(4.7) at hand, letting $\mu \to 0$ in (4.8) and (4.9), we have

$$\int_0^T \int_{\mathbb{R}^3} (\sqrt{\rho}\sqrt{\rho}\mathbf{u}\cdot\boldsymbol{\phi}_t + (\sqrt{\rho}\mathbf{u}\otimes\sqrt{\rho}\mathbf{u}) : \nabla\boldsymbol{\phi} + \rho\mathbf{f}\cdot\boldsymbol{\phi})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\mathbb{R}^3} \mathbf{m}_0(\mathbf{x})\cdot\boldsymbol{\phi}(0,\mathbf{x})\,\mathrm{d}\mathbf{x} = 0, \tag{4.10}$$

and

$$\int_0^T \int_{\mathbb{R}^3} \rho\phi_t + \sqrt{\rho}\sqrt{\rho}\mathbf{u}\cdot\nabla\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\mathbb{R}^3} \rho_0\phi(0,\mathbf{x})\,\mathrm{d}\mathbf{x} = 0. \tag{4.11}$$

Since $\int_0^t \int_{\mathbb{R}^3} \mu|\nabla\mathbf{u}^\mu|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \geq 0$, from (1.9) we obtain

$$\int_{\mathbb{R}^3} \frac{\rho^\mu|\mathbf{u}^\mu|^2}{2}\,\mathrm{d}\mathbf{x} \leq \int_{\mathbb{R}^3} \frac{|\mathbf{m}_0|^2}{2\rho_0}\,\mathrm{d}\mathbf{x} + \int_0^t \int_{\mathbb{R}^3} \rho^\mu\mathbf{u}^\mu\cdot\mathbf{f}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s. \tag{4.12}$$

By Proposition 3.6 and (4.4), we have

$$\int_0^t \int_{\mathbb{R}^3} \rho^\mu\mathbf{u}^\mu\cdot\mathbf{f}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \to \int_0^t \int_{\mathbb{R}^3} \rho\mathbf{u}\cdot\mathbf{f}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s.$$

Thus, by letting $\mu \to 0$ in (4.12), we deduce the following energy inequality

$$\int_{\mathbb{R}^3} \frac{\rho|\mathbf{u}|^{\mathbf{2}}}{2}\,\mathrm{d}\mathbf{x} \leq \int_{\mathbb{R}^3} \frac{|\mathbf{m}_0|^2}{2\rho_0}\,\mathrm{d}\mathbf{x} + \int_0^t \int_{\mathbb{R}^3} \rho\mathbf{u}\cdot\mathbf{f}\,\mathrm{d}\mathbf{x}\,\mathrm{d}s. \tag{4.13}$$

It is clear that (4.11), (4.10) and (4.13) meet with Definition 1.2. Therefore we conclude that $(\rho, \mathbf{u})$ is a weak solution of (1.10) with initial data $(\rho_0, \mathbf{m}_0)$.

5. Appendix

In this section, we sketch the proof of Remark 1.5. In the periodic domain $\mathbb{T}_P$, $\mathbf{k} = (k_1, k_2, k_3) = \frac{2\pi}{P}(n_1, n_2, n_3) \in \mathbb{R}^3$, with $n_j = 0, \pm1, \pm2, \cdots$, and $j = 1, 2, 3$, is the discrete wavevector. Thus the total energy $\mathcal{E}(t)$ per unit mass at time t for the inhomogeneous turbulence in $\mathbb{T}_P$ is:

$$\mathcal{E}(t) = \frac{1}{\int_{\mathbb{T}_P}\rho_0\mathrm{d}\mathbf{x}} \int_{\mathbb{T}_P} \frac{\rho^\mu|\mathbf{u}^\mu|^2}{2}\,\mathrm{d}\mathbf{x} = \sum_{k\geq0} E(t,k) = \sum_{k\geq0} 4\pi q(t,k)k^2. \tag{5.1}$$

And the Fourier transform of the weighted velocity $\sqrt{\rho}\mathbf{u}(t,\mathbf{x})$ in the $\mathbf{x}$-variable is

$$\widehat{\sqrt{\rho}\mathbf{u}}(t,\mathbf{k}) = \frac{1}{|\mathbb{T}_P|}\int_{\mathbb{T}_P} \sqrt{\rho}\mathbf{u}(t,\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x},$$

then we have

$$\sqrt{\rho}\mathbf{u}(t,\mathbf{x}) = \sum_{\mathbf{k}} \widehat{\sqrt{\rho}\mathbf{u}}(t,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}.$$

We have the following weaker version of *Assumption (ICKHw)* in $\mathbb{T}_P$:

Assumption (ICKHw). For any $T > 0$, there exist $C_T > 0$ and k_*(sufficiently large) depending on $\rho_0, \mathbf{m}_0, P$ and $\mathbf{f}$ but independent of the viscosity μ such that, for $k = |\mathbf{k}| \geq k_*$,

$$\sup_{k\geq k_*}\left(|\mathbf{k}|^{3+\beta}\int_0^T |\widehat{\sqrt{\rho}\mathbf{u}}(t,\mathbf{k})|^2\mathrm{d}t\right) \leq C_T,\ \textit{for some}\ \beta > 0. \tag{5.2}$$

By the *Assumption (ICKHw)* and the Fourier transform, we can get the following uniform bound of $\sqrt{\rho^\mu}\mathbf{u}^\mu$ in $L^2(0,T;H^\alpha(\mathbb{T}_P))$ for $\alpha\in(0,1+\frac{\beta}{2})$. For completeness, we present the proof which is similar to [**4, 5**].

LEMMA 5.1. *Under Assumption (ICKHw), for any $T\in(0,+\infty)$, there exists $C>0$, independent of $\mu>0$, such that*

$$\int_0^T\int_{\mathbb{T}_P}|D_x^\alpha(\sqrt{\rho^\mu}\mathbf{u}^\mu)|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\le C, \tag{5.3}$$

where $\alpha\in(0,1+\frac{\beta}{2})$.

PROOF. Note that the definition of fractional derivatives via the Fourier transform, the Parseval identity and *Assumption (ICKHw)* imply that

$$\begin{aligned}\int_0^T\int_{\mathbb{T}_P}|D_x^\alpha(\sqrt{\rho^\mu}\mathbf{u}^\mu)|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \le& C\int_0^T\sum_{\mathbf{k}}|\widehat{D_x^\alpha(\sqrt{\rho^\mu}\mathbf{u}^\mu)}|^2\,\mathrm{d}t\\
\le& C\int_0^T\sum_{\mathbf{k}}|\mathbf{k}|^{2\alpha}|\widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}|^2\,\mathrm{d}t\\
\le& C\int_0^T\sum_{0\le|\mathbf{k}|\le k_*}|\mathbf{k}|^{2\alpha}|\widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}|^2\,\mathrm{d}t+C\int_0^T\sum_{|\mathbf{k}|\ge k_*}|\mathbf{k}|^{2\alpha}|\widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}|^2\,\mathrm{d}t\\
\le& Ck_*^{2\alpha}\int_0^T\int_{\mathbb{T}_P}|\sqrt{\rho^\mu}\mathbf{u}^\mu|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t+C\sum_{|\mathbf{k}|\ge k_*}|\mathbf{k}|^{2\alpha-3-\beta}\\
\le& C,\end{aligned}\tag{5.4}$$

where $\alpha<1+\frac{\beta}{2}$. □

With Lemma 5.1 at hand, we deduce the following high integrability of $\sqrt{\rho^\mu}\mathbf{u}^\mu$, i.e.,

$$\|\sqrt{\rho^\mu}\mathbf{u}^\mu\|_{L^q((0,T)\times\mathbb{T}_P)}\le C,\ \ for\ q=q(\beta,r)>2.$$

It also gives us the L^2-equicontinuity of $\sqrt{\rho^\mu}\mathbf{u}^\mu$ in space variable $\mathbf{x}$ in $L^2((0,T)\times\mathbb{T}_P)$, independent of μ.

LEMMA 5.2. *Under Assumption (ICKHw), for any $T\in(0,+\infty)$, we get the equicontinuity of $\sqrt{\rho^\mu}\mathbf{u}^\mu$ with respect to the space variable $\mathbf{x}$ in $L^2((0,T)\times\mathbb{T}_P)$, independent of μ, i.e.,*

$$\int_0^T\int_{\mathbb{T}_P}|\sqrt{\rho^\mu}\mathbf{u}^\mu(t,\mathbf{x}+\Delta\mathbf{x})-\sqrt{\rho^\mu}\mathbf{u}^\mu(t,\mathbf{x})|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\to0,\ \ as\ \Delta\mathbf{x}\to0. \tag{5.5}$$

PROOF. Using Lemma 5.1 and Parseval identity, we obtain

$$\begin{aligned}\int_0^T\int_{\mathbb{T}_P}|\sqrt{\rho^\mu}\mathbf{u}^\mu(t,\mathbf{x}+\Delta\mathbf{x})-\sqrt{\rho^\mu}\mathbf{u}^\mu(t,\mathbf{x})|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t=&\int_0^T\sum_{\mathbf{k}}|\widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}|^2(e^{i\mathbf{k}\cdot\Delta\mathbf{x}}-1)^2\,dt\\
\le& C|\Delta\mathbf{x}|^{2\alpha}\int_0^T\sum_{\mathbf{k}}|\mathbf{k}|^{2\alpha}|\widehat{\sqrt{\rho^\mu}\mathbf{u}^\mu}|^2\,dt\\
\le& C|\Delta\mathbf{x}|^{2\alpha}\int_0^T\int_{\mathbb{T}_P}|D_{\mathbf{x}}^\alpha(\sqrt{\rho^\mu}\mathbf{u}^\mu)|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\
\le& C|\Delta\mathbf{x}|^{2\alpha}.\end{aligned}$$

□

For L^2- equicontinuity of $\sqrt{\rho^\mu}\mathbf{u}^\mu$ with respect to t, the proof is similar to Lemma 3.5 with slight modification. Since the domain considered is a domain with period $\mathbb{T}_P = [-P/2, P/2]^3 \subset \mathbb{R}^3, P > 0$. Hence, one does not need to estimate I_i $(i = 1, 2, 3, 4)$ by dividing $\mathbf{u}$ into $\mathbf{u}_1$ and $\mathbf{u}_2$. We here only state the following lemma.

LEMMA 5.3. *Under Assumption (ICKHw), for any $T > 0$, we have the equicontinuity of $\sqrt{\rho^\mu}\mathbf{u}^\mu$ with respect to the time variable t in $L^2((0, T - \Delta t) \times \mathbb{T}_P)$, independent of μ, i.e.*

$$\int_0^{T-\Delta t} \int_{\mathbb{T}_P} |\sqrt{\rho^\mu}\mathbf{u}^\mu(t + \Delta t, \mathbf{x}) - \sqrt{\rho^\mu}\mathbf{u}^\mu(t, \mathbf{x})|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \to 0, \text{ as } \Delta t \to 0. \tag{5.6}$$

With the L^2- equicontinuity of $\sqrt{\rho^\mu}\mathbf{u}^\mu$ with respect to x and t, we directly deduce that there exists a subsequence (still denoted) $\sqrt{\rho}^\mu\mathbf{u}^\mu$ and a function $\sqrt{\rho}\mathbf{u} \in L^2((0, T) \times \mathbb{T}_P)$ such that

$$\sqrt{\rho}^\mu\mathbf{u}^\mu \to \sqrt{\rho}\mathbf{u} \text{ in } L^2((0, T) \times \mathbb{T}_P) \text{ as } \mu \to 0.$$

Finally we can get a same theorem as Theorem 1.3 as follows:

THEOREM 5.4. *Under Assumption (ICKHw)* (5.2)*, for the weak solution* $(\rho^\mu, \mathbf{u}^\mu)$ *of* (1.1)-(1.2) *as in Definition 1.1, there exists a subsequence (still denote)* $(\rho^\mu, \mathbf{u}^\mu)$ *and a function* $(\rho, \mathbf{u})$ *such that as* $\mu \to 0$,

$$\rho^\mu \to \rho \text{ weakly in } L^p((0, T) \times \mathbb{T}_P),\ \sqrt{\rho^\mu}\mathbf{u}^\mu \to \sqrt{\rho}\mathbf{u}, \text{ in } L^2((0, T) \times \mathbb{T}_P), \tag{5.7}$$

where $1 < p < \infty$*, and* $(\rho, \mathbf{u})$ *is a weak solution of* (1.10) *with the initial data* $(\rho_0, \mathbf{m}_0)$.

Acknowledgments

Cheng Yu is partially supported by Collaboration Grants for Mathematicians from Simons Foundation with award Number: 637792. Xinhua Zhao is supported by the National Natural Science Foundation of China #12101140 and the Talent Special Project of Guangdong Polytechnic Normal University #99166030406.

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