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Asymptotic behavior of short trajectories to nonhomogeneous heat-conducting magnetohydrodynamic equations

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Abstract. In this paper, we study the asymptotic behavior of short trajectories of weak solutions to the 2D nonhomogeneous heat-conducting magnetohydrodynamic equations. Several bounds for short trajectories are obtained. An attracting set is constructed, which consists of orbits on $[0, 1]$ of complete bounded solutions. Furthermore, the attracting set is compact in different topologies.

Contents

1. Introduction

Magnetohydrodynamic (MHD) equations describe the motion of a conductive fluid in an electromagnetic field. In the present paper, we study the nonhomogeneous heat-conducting MHD equations in a bounded simply-connected smooth domain $\Omega \subset \mathbb{R}^2$:

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$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu\Delta u + \nabla\pi = (b\cdot\nabla)b + \rho f^1, \\ c_v[(\rho\theta)_t + \operatorname{div}(\rho u\theta)] - \kappa\Delta\theta = 2\mu|d(u)|^2 + \nu|\operatorname{curl} b|^2 + \rho f^2, \\ b_t - \nu\Delta b + (u\cdot\nabla)b - (b\cdot\nabla)u = 0, \\ \operatorname{div} u = \operatorname{div} b = 0, \end{cases} \tag{1.1}$$

where ρ is the mass density; $u = (u^1, u^2)$ and $b = (b^1, b^2)$ are the velocity field and the magnetic field respectively; θ and π represent the temperature and the pressure; $f = (f^1, f^2)$ is the external volume force. Here curl $b = \partial_{x_1} b^2 - \partial_{x_2} b^1$; the term $d(u) = \frac{1}{2}[\nabla u + (\nabla u)^t]$ is the deformation tensor. μ is the viscosity coefficient of the fluid, $\nu > 0$ is the magnetic diffusive coefficient, c_v and κ are the heat capacity and the ratio of the heat conductivity coefficient over the heat capacity.

We consider the initial and boundary conditions as follows:

$$(\rho, u, b, \theta)(x, 0) = (\rho_0, u_0, b_0, \theta_0)(x), \quad \text{in } \Omega,$$

$$u = 0, \quad b\cdot n = 0, \quad \operatorname{curl} b = 0, \quad \theta = 0, \quad \text{on } \partial\Omega,$$

where n is the unit outward normal vector on $\partial\Omega$. For simplicity, we consider $\mu = \nu = \kappa = c_v = 1$ in this paper. The initial density is assumed to be bounded

$$0 < \underline{\rho} \le \rho_0 \le \overline{\rho}, \tag{1.2}$$

where $\underline{\rho}$, $\overline{\rho}$ are positive constants.

If $b = 0$, problem (1.1) reduces to the nonhomogeneous heat-conducting Navier-Stokes equations which have been discussed in numerous studies on the regularity of solutions. W. Wang, H. Yu, and P. Zhang [**22**] proved the global well-posedness of small strong solutions to the 3D nonhomogeneous heat-conducting Navier-Stokes equations in bounded domains. For the nonhomogeneous heat-conducting MHD equations (1.1), H. Wu [**23**] proved the local existence of strong solutions to (1.1) with vacuum under some compatibility conditions. Later, X. Zhong [**27**] obtained the global small strong solutions of 3D initial boundary value problems with vacuum. Recently, X. Zhong [**28**] established the global well-posedness of strong solutions to the 2D Cauchy problem with large initial data and non-vacuum density at infinity. For similar models in fluid mechanics, it is referred to [**5, 21, 24, 25, 26**] and the references therein.

For evolution equations in mathematical physics, it is significant to study the long-time behavior of solutions as $t \longrightarrow +\infty$. If the external force function decays to zero at some rate, the solutions will also decay to zero. In 1985, M. E. Schonbek [**18**] obtained the optimal decay rates of weak solutions to the incompressible Navier-Stokes equations. C. He and D. Zhou [**14**] studied the temporal decay for strong solutions of an incompressible Newtonian flow with intrinsic degree of freedom. L. Brandolese [**3**] showed the space-time decay of left invariant non-stationary Navier-Stokes flows in $\mathbb{R}^d$ $(d = 2, 3)$. For the Navier-Stokes equations in $\mathbb{R}^3$, H. O. Bae and B. J. Jin [**1**] obtained the temporal-spatial decay rates. Moreover, the upper and lower bounds of weighted decay for weak solutions of the Navier-Stokes equations in $\mathbb{R}^n$ were obtained by H. O. Bae and B. J. Jin [**2**].

If the external force does not tend to zero, then the solutions may not decay to zero. In this case, the long-time behavior of solutions can be described in terms of attractors. For the 2D classical homogeneous Navier-Stokes equations ($\rho = 1$ and $\theta = b = 0$ in (1.1)), the theory of global attractors in bounded domains has been widely studied by many scholars (see [**8, 15**]). For systems lack uniqueness, we cannot study global attractors by semiflow methods. An example is the 3D homogeneous Navier-Stokes system ($\rho = 1$ and $\theta = b = 0$ in (1.1)). To overcome this difficulty, G. R. Sell [**19**] developed a new point of view for constructing a global attractor by considering trajectories as *data* and studying trajectories in stead of solution semigroups. V. V. Chepyzhov and M. I. Vishik [**7**] provided the theory of so-called *trajectory attractors* which can be used to study the behavior of solutions without uniqueness. J. Málek, J. Nečas and D. Pražzák [**17**] developed the method of short trajectories using similar ideas. Silimarly, E. Feireisl and H. Petzeltová [**12**] proved the existence of a compact global attractor for the 3D isentropic compressible Navier-Stokes equations. V. V. Chepyzhov and M. I. Vishik [**6**] constructed a uniform trajectory attractor for non-autonomous equations. Up to now, few results for nonhomogeneous problems (ρ is not a constant function) are known. In the present paper, we will use similar techniques to those in [**11**] for compressible problems which is different from the semiflow methods.

In this paper, we set

$$\begin{aligned} C^\infty_{c,\sigma}(\Omega) &= \{\varphi \in C^\infty_c(\Omega) \mid \operatorname{div} \varphi = 0\}, \\ L^2_\sigma(\Omega) &= \text{the closure of } C^\infty_{c,\sigma}(\Omega) \text{ in } L^2(\Omega), \\ H^1_{0,\sigma}(\Omega) &= \text{the closure of } C^\infty_{c,\sigma}(\Omega) \text{ in } H^1_0(\Omega), \\ G(\Omega) &= L^2_\sigma(\Omega) \cap \{b \in H^1(\Omega) \mid \operatorname{curl} b = 0 \text{ on } \partial\Omega\}, \\ H^1(\Omega)/\mathbb{R} &= \left\{\pi \in H^1(\Omega) \,\middle|\, \int_\Omega \pi dx = 0\right\}. \end{aligned}$$

For the external force, we suppose

$$\left\{ \begin{array}{l} \mathcal{F} \subset \{f \in L^2_b(\mathbb{R}; L^2(\Omega)) \mid \|f\|_{L^2_b} \le R_{\mathcal{F}}\}, \\ \mathcal{T}(s)\mathcal{F} \subset \mathcal{F}, \text{ for all } s > 0, \\ \mathcal{F} \text{ is compact in } L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega)), \end{array} \right. \tag{1.3}$$

where $\mathcal{T}(s) : f(\cdot) \mapsto f(\cdot + s)$ is the translation, $R_{\mathcal{F}}$ is a nonnegative constant and

$$\|f\|^2_{L^2_b} = \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(s)\|^2_{L^2} ds.$$

This paper is organized as follows. In Section 2, we shall recall some preliminaries. In Section 3, we will study global weak solutions obtained by approximation. In section 4, we will give the theory of short trajectories. In the fifth section, we shall study trajectory attractors of (1.1).

2. Preliminaries

For the Stokes problem, the following two regularity results hold, see [**20**] and [**27**].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, $f \in L^2(\Omega)$.*

(i) The Stokes problem with the non-slip boundary condition

$$\left\{ \begin{array}{ll} -\Delta u + \nabla \pi = f, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{array} \right.$$

has a unique strong solution $(u,\pi)\in H^2(\Omega)\times[H^1(\Omega)/\mathbb{R}]$ *such that*

$$\|\nabla^2 u\|_{L^2}+\|\nabla\pi\|_{L^2}\le C\|f\|_{L^2},$$

where C *depends on* Ω *only;*

(ii) If Ω *is simply-connected, then the Stokes problem with the Navier-type boundary condition*

$$\begin{cases} -\Delta u+\nabla\pi=f, & \text{in } \Omega,\\ \mathrm{div}\, u=0, & \text{in } \Omega,\\ u\cdot n=\mathrm{curl}\, u=0, & \text{on } \partial\Omega,\end{cases}$$

has a unique strong solution $(u,\pi)\in H^2(\Omega)\times[H^1(\Omega)/\mathbb{R}]$ *such that*

$$\|\nabla^2 u\|_{L^2}+\|\nabla\pi\|_{L^2}\le C\|f\|_{L^2},$$

where C *depends on* Ω *only.*

We also need the following so-called uniform Gronwall inequality, see [**7**].

LEMMA 2.2. *Let* g,h,y *be three positive locally integrable functions on* $(0,+\infty)$ *such that* y' *is locally integrable on* $(0,+\infty)$*, and which satisfy*

$$y'\le gy+h,\quad \text{for all } t\ge 0,$$

and

$$\int_t^{t+\tau} g(s)ds\le a_1,\quad \int_t^{t+\tau} h(s)ds\le a_2,\quad \int_t^{t+\tau} y(s)ds\le a_3,\quad \text{for all } t\ge 0,$$

where τ,a_1,a_2,a_3 *are positive constants. Then*

$$y(t+\tau)\le\left(\frac{a_3}{\tau}+a_2\right)e^{a_1},\quad \text{for all } t\ge 0.$$

For compactness in time with values in weak topologies, we refer to [**16**]. This will be used for convergence of density functions.

LEMMA 2.3. *Let* X *be a separable reflexive Banach space and* Y *be a Banach space such that* $X\hookrightarrow Y$, Y' *is separable and dense in* X'. *Let* $\{f_n\}_{n\in\mathbb{N}}\subset C([0,T];Y)$ *and be bounded in* $L^\infty(0,T;X)$ *for some* $T>0$. *We assume that for all* $\varphi\in Y'$, $\langle\varphi,f_n(t)\rangle_{Y',Y}$ *is uniformly continuous in* $t\in[0,T]$ *uniformly in* $n\ge 1$. *Then,* $\{f_n\}_{n\in\mathbb{N}}$ *is relatively compact in* $C([0,T];X_{weak})$.

3. Global weak solutions

In this section, we shall study global weak solutions obtained by approximation. To this end, we consider the auxiliary problem first.

LEMMA 3.1. *Let* $\Omega\subset\mathbb{R}^2$ *be a smooth simply-connected bounded domain and* $\mathcal{F}$ *satisfy* (1.3). *For any* $(\rho_0,u_0,b_0,\theta_0)\in C^\infty(\overline{\Omega})\times H^1_{0,\sigma}(\Omega)\times G(\Omega)\times L^2(\Omega)$ *satisfying* (1.2), $f\in\mathcal{F}$, *there exists a weak solution* (ρ,u,b,θ), *for all* $T>0$,

$$\begin{cases} 0<\underline{\rho}\le\rho\le\overline{\rho},\\ u\in L^\infty(0,T;H^1_{0,\sigma}(\Omega))\cap L^2(0,T;H^2(\Omega)),\\ b\in L^\infty(0,T;G(\Omega))\cap L^2(0,T;H^2(\Omega)),\\ \theta\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega)),\end{cases}$$

to the system

$$
(3.1)\qquad \begin{cases} \rho_t + div(\rho u_\delta) = 0, \\ (\rho u)_t + div(\rho u_\delta \otimes u) - \Delta u + \nabla \pi = (b_\delta \cdot \nabla) b + \rho f^1, \\ (\rho\theta)_t + div(\rho u_\delta \theta) - \Delta\theta = 2|d(u_\delta)|^2 + |curl\ b_\delta|^2 + \rho f^2, \\ b_t - \Delta b + (u_\delta \cdot \nabla) b - (b_\delta \cdot \nabla) u + \nabla \widetilde{\pi} = 0, \\ div\ u = div\ b = 0, \\ (\rho, u, b, \theta)|_{t=0} = (\rho_0, u_0, b_0, \theta_0), \end{cases}
$$

where u_δ, b_δ is the so-called retarded mollification *of u, b introduced by Caffarelli-Korn-Nirenberg in* [**4**]. *Moreover, there exists a time*

$$t_0 = t_0(\|u_0\|_{H^1}, \|b_0\|_{H^1}, \|\theta_0\|_{L^2}, \|f\|_{L^2_b})$$

such that, for all $t \geq t_0$,

$$\|(u(t), b(t), \theta(t))\|^2_{H^1} \leq C\|f\|^2_{L^2_b} e^{C\|f\|^4_{L^2_b}},$$

and

$$\int_t^{t+1} (\|(u_s, b_s, \theta_s)\|^2_{L^2} + \|(u, b, \theta)\|^2_{H^2}) ds \leq C\|f\|^2_{L^2_b} e^{C\|f\|^4_{L^2_b}},$$

where C is independent of $\rho_0, u_0, b_0, \theta_0, f$.

Proof. The density ρ is solved by the method of characteristics. Let $X = X(\tau)$ be the solution to

$$\begin{cases} X'(\tau) = u_\delta(X, \tau), \\ X|_{\tau=t} = x. \end{cases}$$

Then X describes the trajectories of liquid particles. By ODE theory, such $X = X(\tau; t, x)$ uniquely exists and is smooth. Let $\rho(x,t) = \rho_0(X(0; t, x))$, then ρ is a solution to $(3.1)_1$, and $\underline{\rho} \leq \rho \leq \overline{\rho}$.

Testing $(3.1)_2$ by $2u$, $(3.1)_4$ by $2b$, integrating by parts, and using $\underline{\rho} \leq \rho \leq \overline{\rho}$, we can deduce

$$\frac{d}{dt}(\|\sqrt{\rho} u\|^2_{L^2} + \|b\|^2_{L^2}) + c(\|\nabla u\|^2_{L^2} + \|\text{curl } b\|^2_{L^2}) \leq C\|f\|_{L^2}(\|u\|_{L^2} + \|b\|_{L^2}).$$

Recall the Poincaré inequality (see [**13**])

$$\|u\|_{L^2} \leq C\|\nabla u\|_{L^2}, \quad \text{for } u \in H^1(\Omega) \text{ with } u \cdot n = 0 \text{ on } \partial\Omega,$$

and the following inequality (see [**9**])

$$\|b\|_{H^1} \leq C\|\text{curl } b\|_{L^2}, \quad \text{for } b \in L^2_\sigma(\Omega) \cap H^1(\Omega).$$

We derive

$$
(3.2)\qquad \frac{d}{dt}(\|\sqrt{\rho} u\|^2_{L^2} + \|b\|^2_{L^2}) + c(\|\nabla u\|^2_{L^2} + \|\text{curl } b\|^2_{L^2}) \leq C\|f\|^2_{L^2}.
$$

Furthermore

$$\frac{d}{dt}(\|\sqrt{\rho} u\|^2_{L^2} + \|b\|^2_{L^2}) + c(\|u\|^2_{L^2} + \|b\|^2_{L^2}) \leq C\|f\|^2_{L^2}.$$

Noticing $\underline{\rho} \leq \rho \leq \overline{\rho}$, we have

$$\frac{d}{dt}(\|\sqrt{\rho} u\|^2_{L^2} + \|b\|^2_{L^2}) + c(\|\sqrt{\rho} u\|^2_{L^2} + \|b\|^2_{L^2}) \leq C\|f\|^2_{L^2}.$$

The Gronwall inequality leads to

$$\|\sqrt{\rho} u\|^2_{L^2} + \|b\|^2_{L^2} \leq (\|\sqrt{\rho_0} u_0\|^2_{L^2} + \|b_0\|^2_{L^2}) e^{-ct} + C\int_0^t e^{-c(t-s)} \|f(s)\|^2_{L^2} ds.$$

If we assume $f(t) = 0$ when $t < 0$, then

$$\begin{aligned}\int_0^t e^{-c(t-s)}\|f(s)\|_{L^2}^2 ds &= \int_0^t e^{-cs}\|f(t-s)\|_{L^2}^2 ds\\ &\le \int_0^{+\infty} e^{-cs}\|f(t-s)\|_{L^2}^2 ds\\ &= \sum_{i=0}^{\infty}\int_i^{i+1} e^{-cs}\|f(t-s)\|_{L^2}^2 ds\\ &\le \sum_{i=0}^{\infty} e^{-ci}\int_i^{i+1}\|f(t-s)\|_{L^2}^2 ds\\ &\le \sum_{i=0}^{\infty} e^{-ci}\|f\|_{L_b^2}^2\\ &= (1-e^{-c})^{-1}\|f\|_{L_b^2}^2\\ &\le (1+c^{-1})\|f\|_{L_b^2}^2.\end{aligned}$$

Therefore,

$$\|\sqrt{\rho}u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 \le (\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2)e^{-ct} + C\|f\|_{L_b^2}^2. \tag{3.3}$$

Moreover, we derive from (3.2) that

$$\int_0^T (\|\nabla u\|_{L^2}^2 + \|\text{curl } b\|_{L^2}^2)dt \le C[\|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 + (T+1)\|f\|_{L_b^2}^2]. \tag{3.4}$$

Test $(3.1)_2$ by $2u_t$, then

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + c\|\sqrt{\rho}u_t\|_{L^2}^2 \le C[\|(\rho u_\delta\cdot\nabla)u\|_{L^2} + \|(b_\delta\cdot\nabla)b\|_{L^2} + \|\rho f\|_{L^2}]\|u_t\|_{L^2}.$$

Using $\underline{\rho} \le \rho \le \overline{\rho}$, we obtain

$$\|u_t\|_{L^2} \le C\|\sqrt{\rho}u_t\|_{L^2} \le C\|u_t\|_{L^2}.$$

Thus,

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + c\|u_t\|_{L^2}^2 \le C[\|(\rho u_\delta\cdot\nabla)u\|_{L^2}^2 + \|(b_\delta\cdot\nabla)b\|_{L^2}^2 + \|f\|_{L^2}^2]. \tag{3.5}$$

Multiplying $(3.1)_4$ by $2b_t$ gives

$$\frac{d}{dt}\|\text{curl } b\|_{L^2}^2 + c\|b_t\|_{L^2}^2 \le C[\|(u_\delta\cdot\nabla)b\|_{L^2}^2 + \|(b_\delta\cdot\nabla)u\|_{L^2}^2]. \tag{3.6}$$

Recalling

$$\|\varphi\|_{L^4} \le C\|\varphi\|_{L^2}^{\frac12}(\|\nabla\varphi\|_{L^2} + \|\varphi\|_{L^2})^{\frac12}, \quad \text{for all } \varphi \in H^1(\Omega), \tag{3.7}$$

and $\underline{\rho} \le \rho \le \overline{\rho}$, we deduce

$$\begin{aligned}\|(\rho u_\delta\cdot\nabla)u\|_{L^2}^2 &\le C\|u_\delta\|_{L^4}^2\|\nabla u\|_{L^4}^2\\ &\le C\|u_\delta\|_{L^2}\|\nabla u_\delta\|_{L^2}\|\nabla u\|_{L^2}(\|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^2}).\end{aligned} \tag{3.8}$$

Similarly,

$$\|(b_\delta\cdot\nabla)b\|_{L^2}^2 \le C\|b_\delta\|_{L^2}\|\text{curl } b_\delta\|_{L^2}\|\text{curl } b\|_{L^2}(\|\nabla^2 b\|_{L^2} + \|\text{curl } b\|_{L^2}). \tag{3.9}$$

We deduce from $(3.1)_2$ and Lemma 2.1 that

$$\|\nabla^2 u\|_{L^2} \le C(\|\rho u_t\|_{L^2} + \|(\rho u_\delta \cdot \nabla)u\|_{L^2} + \|(b_\delta \cdot \nabla)b\|_{L^2} + \|\rho f^1\|_{L^2}). \tag{3.10}$$

Analogously,

$$\|\nabla^2 b\|_{L^2} \le C(\|b_t\|_{L^2} + \|(b_\delta \cdot \nabla)u\|_{L^2} + \|(u_\delta \cdot \nabla)b\|_{L^2}). \tag{3.11}$$

Taking (3.8)–(3.11) into (3.5), (3.6), we derive

$$\begin{aligned}&\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\text{curl }b\|_{L^2}^2) + c(\|u_t\|_{L^2}^2 + \|b_t\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2)\\ &\quad \le C(1 + \|u_\delta\|_{L^2}^2\|\nabla u_\delta\|_{L^2}^2 + \|b_\delta\|_{L^2}^2\|\text{curl }b_\delta\|_{L^2}^2)(\|\nabla u\|_{L^2}^2 + \|\text{curl }b\|_{L^2}^2).\end{aligned} \tag{3.12}$$

Because u_δ, b_δ is the retarded mollification of u, b, estimates (3.3) and (3.4) are also valid for them. Thus,

$$\int_0^T (\|u_\delta\|_{L^2}^2\|\nabla u_\delta\|_{L^2}^2 + \|b_\delta\|_{L^2}^2\|\nabla b_\delta\|_{L^2}^2)dt \le C(T, \|u_0\|_{L^2}, \|b_0\|_{L^2}, \|f\|_{L_b^2}).$$

Using the Gronwall inequality, we deduce

$$\sup_{0\le t\le T}(\|\nabla u(t)\|_{L^2}^2 + \|\text{curl }b(t)\|_{L^2}^2) \le C(T, \|u_0\|_{H^1}, \|b_0\|_{H^1}, \|f\|_{L_b^2}), \tag{3.13}$$

$$\int_0^T (\|\partial_t u(t)\|_{L^2}^2 + \|\partial_t b(t)\|_{L^2}^2)dt \le C(T, \|u_0\|_{H^1}, \|b_0\|_{H^1}, \|f\|_{L_b^2}), \tag{3.14}$$

$$\int_0^T (\|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^2 b(t)\|_{L^2}^2)dt \le C(T, \|u_0\|_{H^1}, \|b_0\|_{H^1}, \|f\|_{L_b^2}). \tag{3.15}$$

Testing $(3.1)_3$ by 2θ and integrating by parts, we can deduce

$$\begin{aligned}\frac{d}{dt}\|\sqrt{\rho}\theta\|_{L^2}^2 + c\|\nabla\theta\|_{L^2}^2 &\le C(\|\nabla u_\delta\|_{L^4}^2 + \|\text{curl }b_\delta\|_{L^4}^2 + \|f\|_{L^2})\|\theta\|_{L^2}\\ &\le C(\|\nabla u_\delta\|_{L^4}^2 + \|\text{curl }b_\delta\|_{L^4}^2 + \|f\|_{L^2})\|\nabla\theta\|_{L^2}.\end{aligned}$$

Hence,

$$\frac{d}{dt}\|\sqrt{\rho}\theta\|_{L^2}^2 + c\|\nabla\theta\|_{L^2}^2 \le C(\|\nabla u_\delta\|_{L^4}^4 + \|\text{curl }b_\delta\|_{L^4}^4 + \|f\|_{L^2}^2). \tag{3.16}$$

Combining (3.13) and (3.15), we deduce from (3.7) that

$$\begin{aligned}\int_0^T (\|\nabla u_\delta\|_{L^4}^4 + \|\text{curl }b_\delta\|_{L^4}^4)dt &\le C\int_0^T \|\nabla u_\delta\|_{L^2}^2(\|\nabla^2 u_\delta\|_{L^2}^2 + \|\nabla u_\delta\|_{L^2}^2)dt\\ &\quad + C\int_0^T \|\text{curl }b_\delta\|_{L^2}^2(\|\nabla^2 b_\delta\|_{L^2}^2 + \|\text{curl }b_\delta\|_{L^2}^2)dt\\ &\le C(T, \|u_0\|_{H^1}, \|b_0\|_{H^1}, \|f\|_{L_b^2}).\end{aligned}$$

Thus we deduce from (3.16) by the Gronwall inequality that

$$\sup_{0\le t\le T}\|\sqrt{\rho}\theta\|_{L^2}^2 + \int_0^T \|\nabla\theta\|_{L^2}^2 dt \le C(T, \|u_0\|_{H^1}, \|b_0\|_{H^1}, \|\theta_0\|_{L^2}, \|f\|_{L_b^2}). \tag{3.17}$$

Moreover, from (3.2) and (3.3), there exists $C_1 > 0$ and

$$t_1 = t_1(C_1, \|u_0\|_{L^2}, \|b_0\|_{L^2}, \|f\|_{L_b^2}),$$

such that for all $t \ge t_1$,

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 \le C_1\|f\|_{L_b^2}^2,$$

and

$$\int_t^{t+1}(\|\nabla u\|_{L^2}^2+\|\text{curl } b\|_{L^2}^2)ds\le C_1\|f\|_{L_b^2}^2.$$

Applying Lemma 2.2 to (3.12), we deduce for all $t\ge t_1+1$ that

$$\|\nabla u(t)\|_{L^2}^2+\|\text{curl } b(t)\|_{L^2}^2\le C\|f\|_{L_b^2}^2 e^{C\|f\|_{L_b^2}^4}.$$

Taking it into (3.12) then

$$\int_t^{t+1}(\|u_s\|_{L^2}^2+\|b_s\|_{L^2}^2+\|\nabla^2 u\|_{L^2}^2+\|\nabla^2 b\|_{L^2}^2)ds\le C\|f\|_{L_b^2}^2 e^{C\|f\|_{L_b^2}^4}.$$

Thus the right-hand-side of (3.16) possesses the estimate

$$\int_t^{t+1}(\|\nabla u_\delta\|_{L^4}^4+\|\text{curl } b_\delta\|_{L^4}^4)ds\le C\|f\|_{L_b^2}^4 e^{C\|f\|_{L_b^2}^4}.$$

By similar procedures to (3.3), we derive from (3.16) that

$$\|\sqrt{\rho}\theta(t)\|_{L^2}^2\le\|\sqrt{\rho(t_1+1)}\theta(t_1+1)\|_{L^2}^2 e^{-c(t-t_1-1)}+C\|f\|_{L_b^2}^2 e^{C\|f\|_{L_b^2}^4}.$$

Inequality (3.17) says that

$$\|\sqrt{\rho(t_1+1)}\theta(t_1+1)\|_{L^2}^2\le C(t_1,\|u_0\|_{H^1},\|b_0\|_{H^1},\|\theta_0\|_{L^2},\|f\|_{L_b^2}).$$

Therefore, there exists $C_2>0$ and $t_2=t_2(C_1,C_2,\|u_0\|_{H^1},\|b_0\|_{H^1},\|\theta_0\|_{L^2},\|f\|_{L_b^2})$, such that for all $t\ge t_2$,

$$\|\theta(t)\|_{L^2}^2\le C_2\|f\|_{L_b^2}^2 e^{C_2\|f\|_{L_b^2}^4},$$

and by (3.16),

$$\int_t^{t+1}\|\nabla\theta\|_{L^2}^2 ds\le C_2\|f\|_{L_b^2}^2 e^{C_2\|f\|_{L_b^2}^4}.$$

Multiplying $(3.1)_3$ by $2\theta_t$, then

$$\frac{d}{dt}\|\nabla\theta\|_{L^2}^2+c\|\sqrt{\rho}\theta_t\|_{L^2}^2\le C[\|(\rho u_\delta\cdot\nabla)\theta\|_{L^2}^2+\|\nabla u_\delta\|_{L^4}^4+\|\text{curl } b_\delta\|_{L^4}^4+\|\rho f\|_{L^2}^2].$$

Using $\underline{\rho}\le\rho\le\overline{\rho}$ and the standard elliptic regularity theory instead of Lemma 2.1, we obtain from Lemma 2.2 that, for all $t\ge t_2+1$,

$$\|\nabla\theta(t)\|_{L^2}^2\le C\|f\|_{L_b^2}^2 e^{C\|f\|_{L_b^2}^4},$$

and

$$\int_t^{t+1}(\|\theta_s\|_{L^2}^2+\|\nabla^2\theta\|_{L^2}^2)ds\le C\|f\|_{L_b^2}^2 e^{C\|f\|_{L_b^2}^4}.$$

□

Now we construct weak solutions to (3.1) with initial data $\rho_0\in L^\infty(\Omega)$.

THEOREM 3.2. *Let $\Omega\subset\mathbb{R}^2$ be a smooth simply-connected bounded domain and $\mathcal{F}$ satisfy* (1.3). *For any $(\rho_0,u_0,b_0,\theta_0)\in L^\infty(\Omega)\times H_{0,\sigma}^1(\Omega)\times G(\Omega)\times L^2(\Omega)$ satisfying* (1.2), *$f\in\mathcal{F}$, there exists a weak solution (ρ,u,b,θ) to* (1.1), *for all $T>0$,*

$$\begin{cases} 0<\underline{\rho}\le\rho\le\overline{\rho},\\ u\in L^\infty(0,T;H_{0,\sigma}^1(\Omega))\cap L^2(0,T;H^2(\Omega)),\\ b\in L^\infty(0,T;G(\Omega))\cap L^2(0,T;H^2(\Omega)),\\ \theta\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega)).\end{cases}$$

Moreover, there exists $t_0 = t_0(\|u_0\|_{H^1}, \|b_0\|_{H^1}, \|\theta_0\|_{L^2}, \|f\|_{L^2_b})$ *such that for all* $t \geq t_0$,

$$\|(u(t), b(t), \theta(t))\|_{H^1}^2 \leq C\|f\|_{L^2_b}^2 e^{C\|f\|_{L^2_b}^4},$$

and

$$\int_t^{t+1} (\|(u_s, b_s, \theta_s)\|_{L^2}^2 + \|(u, b, \theta)\|_{H^2}^2) ds \leq C\|f\|_{L^2_b}^2 e^{C\|f\|_{L^2_b}^4},$$

where C *is independent of* $\rho_0, u_0, b_0, \theta_0, f$.

Proof. Let $(\rho^\delta, u^\delta, b^\delta, \theta^\delta)$ $(0 < \delta < 1)$ be the weak solution obtained in Lemma 3.1 with initial data $(\rho_{0,\delta}, u_0, b_0, \theta_0)$, where $\rho_{0,\delta}$ is the standard mollification of ρ_0. It is not difficult to verify that all estimates in the proof of Lemma 3.1 are independent of δ, which means that $(\rho^\delta, u^\delta, b^\delta, \theta^\delta)$ converges to (ρ, u, b, θ) as $\delta \longrightarrow 0^+$ in some sense by the weak convergence theorem. Meanwhile, (3.13)–(3.15) give

$$\int_0^T [\|\partial_t(\nabla u^\delta)\|_{H^{-1}}^2 + \|\partial_t(\nabla b^\delta)\|_{H^{-1}}^2] dt \leq C(T, \|u_0\|_{H^1}, \|b_0\|_{H^1}, \|f\|_{L^2_b}),$$

and

$$\int_0^T (\|\nabla u^\delta\|_{H^1}^2 + \|\nabla b^\delta\|_{H^1}^2) dt \leq C(T, \|u_0\|_{H^1}, \|b_0\|_{H^1}, \|f\|_{L^2_b}).$$

Noticing that the imbedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, we can derive from the well-known Aubin-Lions lemma that

$$\nabla u^\delta \longrightarrow \nabla u \text{ strongly in } L^2(0, T; L^2(\Omega)),$$
$$\nabla b^\delta \longrightarrow \nabla b \text{ strongly in } L^2(0, T; L^2(\Omega)),$$

which gives the strong convergence of $|d((u^\delta)_\delta)|^2$ and $|\text{curl } (b^\delta)_\delta|^2$. We can also pass the other terms in (3.1) to the limit by similar procedures to the inhomogeneous Navier-Stokes equations (see [**16**, Chapter II]). Moreover, $\nabla\widetilde{\pi} = 0$ (in the limit equations), see [**10**]. We omit the details here. □

Lemma 3.3. *For any weak solution* (ρ, u, b, θ) *obtained in Theorem 3.2, we have* $\rho \in C([0, T]; L^p(\Omega))$, *for any* $1 \leq p < \infty$ *and* $T > 0$. *The norm* $\|\rho(t)\|_{L^p}$ *is independent of* t.

Proof. We consider $p > 1$ first. Because $u \in L^2(0, T; H^1_0(\Omega)) \subset L^2(0, T; L^p(\Omega))$, we deduce $\rho_t = -\text{div } (\rho u) \in L^2(0, T; W^{-1,p}(\Omega))$ and $\rho \in C([0, T]; W^{-1,p}(\Omega))$. According to $\underline{\rho} \leq \rho \leq \overline{\rho}$, we derive $\rho \in L^\infty(0, T; L^p(\Omega))$ and then $\rho \in C([0, T]; L^p_{\text{weak}}(\Omega))$. Next, by extension and mollification, we can obtain $\rho \in C([0, T]; L^p(\Omega))$ from the continuity equation $(1.1)_1$ (see [**16**, Chapter II, Lemma 2.3] and what follows).

For $p = 1$, we take $p_0 > 1$ and then obtain from the Hölder inequality that

$$\|\rho(t + \Delta t) - \rho(t)\|_{L^1} \leq \|\rho(t + \Delta t) - \rho(t)\|_{L^{p_0}} |\Omega|^{1 - \frac{1}{p_0}} \longrightarrow 0, \quad \text{as } \Delta t \longrightarrow 0,$$

for any $t \in [0, T]$. So $\rho \in C([0, T]; L^1(\Omega))$.

Equation $(1.1)_1$ together with the incompressibility condition $(1.1)_5$ implies that the distribution function of $\rho(t)$ considered as a function of x is independent of t, because of the celebrated Liouville theorem, see [**16**, Section 2.3]. Thus, $\|\rho(t)\|_{L^p}$ is independent of t. □

4. Short trajectories

We first study short trajectories to (1.1). Define

$$U^s[E_0,\mathcal{F}](t)=\{(\rho,u,b,\theta)(\tau),\tau\in[0,1]\mid(\rho,u,b,\theta)(\tau)=(\widetilde{\rho},\widetilde{u},\widetilde{b},\widetilde{\theta})(\tau+t),$$
$$\text{for some weak solution } (\widetilde{\rho},\widetilde{u},\widetilde{b},\widetilde{\theta}) \text{ obtained in Theorem 3.2}$$
$$\text{with initial energy } \|(\widetilde{u},\widetilde{b})(0)\|_{H^1}^2+\|\widetilde{\theta}(0)\|_{L^2}^2\le E_0\},$$

and

$$\mathcal{F}^+=\bigcap_{s\ge 0}\overline{\mathcal{T}(s)\mathcal{F}},$$

where the closures are taken in $L^2_{\text{loc}}(\mathbb{R};L^2(\Omega))$. Under the assumption (1.3), $\mathcal{F}^+\subset\mathcal{F}$ is nonempty and compact in $L^2_{\text{loc}}(\mathbb{R};L^2(\Omega))$.

The set $U^s[E_0,\mathcal{F}](t)$ describes the behavior of weak solutions on $[t,t+1]$. We can deduce the following proposition which can be regarded as the behavior of $U^s[E_0,\mathcal{F}](t)$ as $t\longrightarrow+\infty$ in some sense.

Proposition 4.1. *Let $\Omega\subset\mathbb{R}^2$ be a smooth simply-connected bounded domain and $\mathcal{F}$ satisfy* (1.3).

$$(\rho_n,u_n,b_n,\theta_n)\in U^s[E_0,\mathcal{F}](t_n),$$

for a sequence $t_n\longrightarrow+\infty$. Then, there is a subsequence $\{(\rho_{n_k},u_{n_k},b_{n_k},\theta_{n_k})\}_{k\in\mathbb{N}}$, such that

$$\rho_{n_k}\longrightarrow\rho_* \ \textit{in } C([0,1];L^p_{weak}(\Omega)), \tag{4.1}$$

$$(u_{n_k},b_{n_k},\theta_{n_k})\longrightarrow(u_*,b_*,\theta_*) \ \textit{in } C([0,1];H^1_{weak}(\Omega)), \tag{4.2}$$
$$(u_{n_k},b_{n_k},\theta_{n_k})\longrightarrow(u_*,b_*,\theta_*) \ \textit{weakly}-* \ \textit{in } L^\infty(0,1;H^1(\Omega)), \tag{4.3}$$
$$(u_{n_k},b_{n_k},\theta_{n_k})\longrightarrow(u_*,b_*,\theta_*) \ \textit{weakly in } L^2(0,1;H^2(\Omega)), \tag{4.4}$$
$$(u_{n_k},b_{n_k},\theta_{n_k})\longrightarrow(u_*,b_*,\theta_*) \ \textit{strongly in } L^2(0,1;H^1(\Omega)), \tag{4.5}$$

as $k\longrightarrow\infty$, for any $1<p<\infty$, where $(\rho_*,u_*,b_*,\theta_*)$ is a weak solution of (1.1) *in $\Omega\times\mathbb{R}$ with $f_*\in\mathcal{F}^+$, such that for all $t\ge s$,*

$$\|(u_*,b_*,\theta_*)(t)\|_{H^1}^2\le CR_{\mathcal{F}}^2e^{CR_{\mathcal{F}}^4}, \tag{4.6}$$

and

$$\int_s^t(\|(\partial_\tau u_*,\partial_\tau b_*,\partial_\tau\theta_*)\|_{L^2}^2+\|(u_*,b_*,\theta_*)\|_{H^2}^2)d\tau\le C(t-s,R_{\mathcal{F}}). \tag{4.7}$$

Proof. From Theorem 3.2, we know that there exists $t_0=t_0(E_0)$ such that, for all $t\ge t_0$,

$$\|(u_n(t),b_n(t),\theta_n(t))\|_{H^1}^2\le CR_{\mathcal{F}}^2e^{CR_{\mathcal{F}}^4},$$

and

$$\int_0^1(\|(\partial_t u_n,\partial_t b_n,\partial_t\theta_n)\|_{L^2}^2+\|(u_n,b_n,\theta_n)\|_{H^2}^2)ds\le CR_{\mathcal{F}}^2e^{CR_{\mathcal{F}}^4},$$

where C is independent of $\rho_0,u_0,b_0,\theta_0,\mathcal{F}$. Then (4.3)–(4.5) can be obtained for some subsequence $\{(u_{n_k},b_{n_k},\theta_{n_k})\}_{k\in\mathbb{N}}$.

Because $\underline{\rho}\le\rho_n\le\overline{\rho}$, for all $1<p\le\infty$,

$$\rho_{n_k}\longrightarrow\rho_* \ \text{weakly}-* \ \text{in } L^\infty(0,1;L^p(\Omega)), \tag{4.8}$$

as $k \longrightarrow \infty$ for some $\rho_* \in L^\infty(\Omega \times (0,1))$. By $(1.1)_1$, we have for all $1 < p < \infty$,

$$\|\partial_t \rho_n\|_{L^\infty(0,1;W^{-1,p}(\Omega))} \le \|\rho_n u_n\|_{L^\infty(0,1;L^p(\Omega))} \le C\|u_n\|_{L^\infty(0,1;H^1(\Omega))} \le C. \tag{4.9}$$

Then,

$$\begin{aligned} |\langle \rho_n(t+\Delta t), \varphi\rangle - \langle \rho_n(t), \varphi\rangle| &= \left| \int_t^{t+\Delta t} \langle \partial_\tau \rho_n(x,\tau), \varphi(x)\rangle d\tau \right| \\ &\le \|\partial_t \rho_n\|_{L^\infty(0,1;W^{-1,p})} \|\varphi\|_{W_0^{1,p'}} |\Delta t| \\ &\le C\|\varphi\|_{W_0^{1,p'}} |\Delta t|, \end{aligned} \tag{4.10}$$

for all $\varphi \in W_0^{1,p'}(\Omega)$ and $t, t+\Delta t \in [0,1]$. So

$$\|\rho_n(t+\Delta t) - \rho_n(t)\|_{W^{-1,p}} \le C|\Delta t|.$$

Taking $X = L^p(\Omega)$, $Y = W^{-1,p}(\Omega)$, then (4.1) follows from Lemma 2.3. For all $\varphi \in L^2(\Omega)$ and $t, t+\Delta t \in [0,1]$, we obtain

$$\begin{aligned} |\langle u_n(t+\Delta t), \varphi\rangle - \langle u_n(t), \varphi\rangle| &= \left| \int_t^{t+\Delta t} \langle \partial_\tau u_n(x,\tau), \varphi(x)\rangle d\tau \right| \\ &\le \|\partial_t u_n\|_{L^2(0,1;L^2)} \|\varphi\|_{L^2} \sqrt{|\Delta t|} \\ &\le C\|\varphi\|_{L^2} \sqrt{|\Delta t|}. \end{aligned}$$

So

$$\|u_n(t+\Delta t) - u_n(t)\|_{L^2} \le C\sqrt{|\Delta t|}.$$

Taking $X = H^1(\Omega)$, $Y = L^2(\Omega)$, then we deduce from Lemma 2.3 that

$$u_{n_k} \longrightarrow u_* \text{ in } C([0,1]; H^1_{\text{weak}}(\Omega)).$$

Similarly,

$$(b_{n_k}, \theta_{n_k}) \longrightarrow (b_*, \theta_*) \text{ in } C([0,1]; H^1_{\text{weak}}(\Omega)).$$

Because the imbedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ is continuous, then from (4.5),

$$(u_{n_k}, b_{n_k}, \theta_{n_k}) \longrightarrow (u_*, b_*, \theta_*) \text{ strongly in } L^2(0,1; L^4(\Omega)). \tag{4.11}$$

This together with (4.8) shows that

$$(\rho_{n_k} u_{n_k}, \rho_{n_k}\theta_{n_k}) \longrightarrow (\rho_* u_*, \rho_*\theta_*) \text{ weakly in } L^2(0,1; L^4(\Omega)). \tag{4.12}$$

Combining (4.11) and (4.12) gives

$$(\rho_{n_k} u_{n_k} \otimes u_{n_k}, \rho_{n_k} u_{n_k}\theta_{n_k}) \longrightarrow (\rho_* u_* \otimes u_*, \rho_* u_*\theta_*) \text{ weakly in } L^1(0,1; L^2(\Omega)).$$

Combining (4.5) and (4.11) yields

$$(u_{n_k} \cdot \nabla) b_{n_k} \longrightarrow (u_* \cdot \nabla) b_* \text{ strongly in } L^1(0,1; L^{\frac{4}{3}}(\Omega)).$$

Similarly,

$$\begin{aligned} (b_{n_k} \cdot \nabla) u_{n_k} &\longrightarrow (b_* \cdot \nabla) u_* \text{ strongly in } L^1(0,1; L^{\frac{4}{3}}(\Omega)), \\ (b_{n_k} \cdot \nabla) b_{n_k} &\longrightarrow (b_* \cdot \nabla) b_* \text{ strongly in } L^1(0,1; L^{\frac{4}{3}}(\Omega)). \end{aligned}$$

We deduce from (4.5) that

$$(|d(u_{n_k})|^2, |\text{curl } b_{n_k}|^2) \longrightarrow (|d(u_*)|^2, |\text{curl } b_*|^2) \text{ strongly in } L^1(0,1; L^1(\Omega)).$$

Finally, because of (1.3), we can take some $f_* \in \mathcal{F}^+$ such that

$$f_{n_k} \longrightarrow f_* \text{ strongly in } L^2(0,1; L^2(\Omega)).$$

This together with (4.8) yields

$$\rho_{n_k} f_{n_k} \longrightarrow \rho_* f_* \text{ weakly in } L^2(0,1;L^2(\Omega)).$$

Thanks to the above convergences, we can pass (1.1) of $(\rho_{n_k}, u_{n_k}, b_{n_k}, \theta_{n_k})$ to the limit. Thus $(\rho_*, u_*, b_*, \theta_*)$ is a weak solution to (1.1) in $\Omega \times (0,1)$.

It is easy to verify that, for any $N \in \mathbb{N}_+$, we can obtain similar bounds for $(\rho_{n_k}, u_{n_k}, b_{n_k}, \theta_{n_k})$ in $\Omega \times [-N, -N+1]$ and in $\Omega \times [N, N+1]$, when k is sufficiently large such that $t_{n_k} > t_0 + 2N$. Thus, we can take a subsequence which satisfies analogous convergent properties in $\Omega \times [-N, N+1]$. By a diagonal process, $(\rho_*, u_*, b_*, \theta_*)$ satisfies (1.1) in the sense of distributions in $\Omega \times \mathbb{R}$. Inequalities (4.6) and (4.7) are natural according to Theorem 3.2. □

Proposition 4.1 gives a strong convergence of $\{(u_{n_k}(t), b_{n_k}(t), \theta_{n_k}(t))\}_{k\in\mathbb{N}}$ in $H^1(\Omega)$ for a.e. $t \in [0,1]$. But we do not know whether it converges strongly at some desired fixed time, for example, $t = 0$. We can obtain a strong convergence at endpoints as follows.

Corollary 4.2. *For $(u_{n_k}, b_{n_k}, \theta_{n_k})$ and (u_*, b_*, θ_*) in Proposition 4.1, $1 \leq r < \infty$,*

$$(u_{n_k}(0), b_{n_k}(0), \theta_{n_k}(0)) \longrightarrow (u_*(0), b_*(0), \theta_*(0)) \text{ strongly in } L^r(\Omega).$$

Proof. From (4.2),

$$(u_{n_k}(0), b_{n_k}(0), \theta_{n_k}(0)) \longrightarrow (u_*(0), b_*(0), \theta_*(0)) \text{ weakly in } H^1(\Omega).$$

Because $H^1(\Omega) \hookrightarrow L^r(\Omega)$ is compact, the strong convergence in Corollary 4.2 is valid. □

Proposition 4.1 does not give a good strong convergence of ρ_{n_k} (e.g. in some $C([0,1];L^p(\Omega))$) while $(u_{n_k}, b_{n_k}, \theta_{n_k})$ possesses a strong convergence, which is different from the compressible Navier-Stokes equations (see [**12**]). The following example shows that we could not expect for a strong convergence in some $C([0,1];L^p(\Omega))$.

Example 4.3. Let Ω be the unit ball in $\mathbb{R}^2$, $f^1 = (1,0)$, $f^2 = 0$. Then, $(\rho_n, 0, 0, 0) \in U^s[E_0, \mathcal{F}](t)$, for all $t \geq 0$, where

$$\rho_n(x_1, x_2, \tau) = \begin{cases} 3, & x_1 \in (-1 + \frac{2k-2}{2^n}, -1 + \frac{2k-1}{2^n}), \quad k = 1, \cdots, 2^n, \\ 1, & x_1 \in (-1 + \frac{2k-1}{2^n}, -1 + \frac{2k}{2^n}), \quad k = 1, \cdots, 2^n. \end{cases}$$

Obviously, $|\rho_n| \leq 3$. For any $\varphi \in C_c^\infty(\Omega)$, set

$$\Phi(x_1) = \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \varphi(x_1, x_2) dx_2 \in C([-1,1]).$$

Then

$$\int_\Omega (\rho_n - 1)\varphi dx = \sum_{k=1}^{2^n} 2 \int_{-1+\frac{2k-2}{2^n}}^{-1+\frac{2k-1}{2^n}} \Phi(x_1) dx_1.$$

Let

$$m_k = \inf_{(-1+\frac{2k-2}{2^n}, -1+\frac{2k}{2^n})} \Phi, \quad \text{and} \quad M_k = \sup_{(-1+\frac{2k-2}{2^n}, -1+\frac{2k}{2^n})} \Phi.$$

Then,

$$\sum_{k=1}^{2^n} 2^{1-n} m_k \leq \int_\Omega (\rho_n - 1)\varphi dx \leq \sum_{k=1}^{2^n} 2^{1-n} M_k.$$

By the definition of Riemannian integral, the both sides of the above inequalities converge to $\int_{-1}^{1} \Phi(x_1)dx_1 = \int_\Omega \varphi(x)dx$, which implies

$$\lim_{n\to\infty} \int_\Omega (\rho_n - 1)\varphi dx = \int_\Omega \varphi dx.$$

Therefore, for all $1 < p < \infty$, $\rho_n(\tau) - 1$ converges to 1 weakly in $L^p(\Omega)$ as $n \longrightarrow \infty$, by density of $C_c^\infty(\Omega)$ in $L^{p'}(\Omega)$. Then $\rho_n(\tau)$ converges to 2 in $L^p(\Omega)$ weakly. If ρ_n converges in some $C([0,1]; L^q(\Omega))$, then $\rho_n(\tau)$ converges to 2 in $L^q(\Omega)$ and a.e. in Ω which contradicts to $\rho_n(x,\tau) \in \{1,3\}$. So, ρ_n does not converge in any $C([0,1]; L^q(\Omega))$, $1 \le q \le \infty$.

If ρ_n satisfies some additional compactness conditions, then we can derive strong convergence for short trajectories. Let K be a compact set in $L^1(\Omega)$ and define

$$U^s_K[E_0, \mathcal{F}](t) = \{(\rho, u, b, \theta) \in U^s[E_0, \mathcal{F}](t) \mid \rho(\tau) \in K \text{ for some } \tau \in [0,1]\}.$$

Proposition 4.4. *Under the assumptions of Proposition 4.1.*

$$(\rho_n, u_n, b_n, \theta_n) \in U^s_K[E_0, \mathcal{F}](t_n),$$

for a sequence $t_n \longrightarrow +\infty$. *Then, for* $(\rho_*, u_*, b_*, \theta_*)$ *obtained in Proposition 4.1,*

$$\rho_{n_k} \longrightarrow \rho_* \text{ strongly in } C([0,1]; L^p(\Omega)),$$

as $k \longrightarrow \infty$, *for any* $1 < p < \infty$.

Proof. We first assume that

$$\rho_{n_k}(1) \longrightarrow \rho_{**} \text{ strongly in } L^1(\Omega), \tag{4.13}$$

for some $\rho_{**} \in K \cap L^\infty(\Omega)$. By $\underline{\rho} \le \rho_n \le \overline{\rho}$, this convergence can be revised to

$$\rho_{n_k}(1) \longrightarrow \rho_{**} \text{ strongly in } L^p(\Omega),$$

for all $1 < p < \infty$. From (4.1),

$$\rho_{n_k}(1) \longrightarrow \rho_*(1) \text{ weakly in } L^p(\Omega).$$

Then $\rho_*(1) = \rho_{**}$. Lemma 3.3 shows that $\|\rho_n(t)\|_{L^p} = \|\rho_n(1)\|_{L^p}$, $t \in [0,1]$. The proof of Lemma 3.3 can also be applied to ρ_*. Thus,

$$\lim_{k\to\infty} \|\rho_{n_k}(t)\|_{L^p} = \lim_{k\to\infty} \|\rho_{n_k}(1)\|_{L^p} = \|\rho_*(1)\|_{L^p} = \|\rho_*(t)\|_{L^p},$$

and

$$\begin{aligned}\|\rho_{n_k}(t) - \rho_*(t)\|^2_{L^2} &= \|\rho_{n_k}(t)\|^2_{L^2} + \|\rho_*(t)\|^2_{L^2} - 2(\rho_{n_k}(t), \rho_*(t))_{L^2} \\ &= \|\rho_{n_k}(1)\|^2_{L^2} + \|\rho_*(1)\|^2_{L^2} - 2(\rho_{n_k}(t), \rho_*(t))_{L^2}.\end{aligned}$$

For any $\varepsilon > 0$, from $\rho_* \in C([0,1]; L^2(\Omega))$, there exists $M > 0$, s.t.

$$\left\|\rho_*(t) - \rho_*\left(\frac{j}{M}\right)\right\|_{L^2} < \varepsilon, \quad \text{for all } t \in \left[\frac{j-1}{M}, \frac{j}{M}\right], \quad j = 1, \cdots, M.$$

From (4.1), there exists $N > 0$, for any $k > N$, for all $t \in [\frac{j-1}{M}, \frac{j}{M}]$, $j = 1, \cdots, M$,

$$\left|\left(\rho_{n_k}(t) - \rho_*(t), \rho_*\left(\frac{j}{M}\right)\right)_{L^2}\right| < \varepsilon.$$

Thus,

$$\begin{aligned}
&|(\rho_{n_k}(t),\rho_*(t))_{L^2}-(\rho_*(t),\rho_*(t))_{L^2}|\\
\le&\left|(\rho_{n_k}(t),\rho_*(t))_{L^2}-\left(\rho_{n_k}(t),\rho_*\left(\frac{j}{M}\right)\right)_{L^2}\right|\\
&+\left|\left(\rho_{n_k}(t),\rho_*\left(\frac{j}{M}\right)\right)_{L^2}-\left(\rho_*(t),\rho_*\left(\frac{j}{M}\right)\right)_{L^2}\right|\\
&+\left|\left(\rho_*(t),\rho_*\left(\frac{j}{M}\right)\right)_{L^2}-(\rho_*(t),\rho_*(t))_{L^2}\right|\\
\le&\|\rho_{n_k}\|_{C([0,1];L^2)}\left\|\rho_*(t)-\rho_*\left(\frac{j}{M}\right)\right\|_{L^2}\\
&+\left|\left(\rho_{n_k}(t)-\rho_*(t),\rho_*\left(\frac{j}{M}\right)\right)_{L^2}\right|\\
&+\|\rho_*\|_{C([0,1];L^2)}\left\|\rho_*(t)-\rho_*\left(\frac{j}{M}\right)\right\|_{L^2}\\
\le&C\varepsilon.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sup_{0\le t\le 1}\|\rho_{n_k}(t)-\rho_*(t)\|_{L^2}^2&\le\|\rho_{n_k}(1)\|_{L^2}^2+\|\rho_*(1)\|_{L^2}^2-2\inf_{0\le t\le 1}\|\rho_*(t)\|_{L^2}^2+2C\varepsilon\\
&=\|\rho_{n_k}(1)\|_{L^2}^2-\|\rho_*(1)\|_{L^2}^2+2C\varepsilon.
\end{aligned}$$

Letting $k\longrightarrow\infty$, we obtain

$$\rho_{n_k}\longrightarrow\rho_*\text{ strongly in }C([0,1];L^2(\Omega)),$$

and then Proposition 4.4 follows from $\underline{\rho}\le\rho_n\le\overline{\rho}$ and the Hölder inequality.

Now we start to show (4.13). By the definition of $U_K^s[E_0,\mathcal{F}](t)$, we can assume that $\rho_n(\tau_n)\in K$, for some $\tau_n\in[0,1]$. Then

$$\rho_{n_k}(\tau_{n_k})\longrightarrow\rho_{**}\text{ strongly in }L^2(\Omega),$$

for some $\rho_{**}\in K\cap L^\infty(\Omega)$. Taking subsequence, we deduce $\lim_{k\to\infty}\tau_{n_k}=\tau_*\in[0,1]$.

For $\varphi\in C_c^\infty(\Omega)$, by (4.9) and (4.10), $\langle\rho_*,\varphi\rangle$ is uniformly (w.r.t. τ) continuous. For any $\varepsilon>0$, there exists $N>0$, such that $|\langle\rho_*(\tau_{n_k})-\rho_*(\tau_*),\varphi\rangle|<\varepsilon$, for all $k>N$. By (4.1),

$$|\langle\rho_{n_k}(\tau_{n_k})-\rho_*(\tau_{n_k}),\varphi\rangle|\le\sup_{\tau\in[0,1]}|\langle\rho_{n_k}(\tau)-\rho_*(\tau),\varphi\rangle|<\varepsilon.$$

Thus,

$$|\langle\rho_{n_k}(\tau_{n_k})-\rho_*(\tau_*),\varphi\rangle|\le|\langle\rho_{n_k}(\tau_{n_k})-\rho_*(\tau_{n_k}),\varphi\rangle|+|\langle\rho_*(\tau_{n_k})-\rho_*(\tau_*),\varphi\rangle|<2\varepsilon.$$

By the density of $C_c^\infty(\Omega)$ in $L^2(\Omega)$, we deduce that $\rho_{n_k}(\tau_{n_k})$ converges to $\rho_*(\tau_*)$ weakly in $L^2(\Omega)$. The uniqueness gives $\rho_{**}=\rho_*(\tau_*)$. Thus $\rho_{n_k}(\tau_{n_k})$ converges to $\rho_*(\tau_*)$ strongly in $L^2(\Omega)$. By Lemma 3.3, $\|\rho_{n_k}(\tau_{n_k})\|_{L^2}=\|\rho_{n_k}(\tau_*)\|_{L^2}$. Then

$$\lim_{k\to\infty}\|\rho_{n_k}(\tau_*)\|_{L^2}=\lim_{k\to\infty}\|\rho_{n_k}(\tau_{n_k})\|_{L^2}=\|\rho_*(\tau_*)\|_{L^2}.$$

By (4.1) again, $\rho_{n_k}(\tau_*)$ converges to $\rho_*(\tau_*)$ weakly in $L^2(\Omega)$. Thus

$$\rho_{n_k}(\tau_*)\longrightarrow\rho_*(\tau_*)\text{ strongly in }L^2(\Omega).$$

Translating τ_* to 1, we finish the proof. □

5. Trajectory attractors

Define

$$\mathcal{A}^s[\mathcal{F}] = \{(\rho, u, b, \theta)(\tau), \tau \in [0,1] \mid (\rho, u, b, \theta) \text{ is a weak solution of (1.1)} \\ \text{in } \Omega \times \mathbb{R} \text{ with } f \in \mathcal{F}^+, \text{ such that (4.6) and (4.7) hold}\}.$$

THEOREM 5.1. *Under the assumptions of Proposition 4.1, $\mathcal{A}^s[\mathcal{F}]$ is compact in $C([0,1]; L^p_{weak}(\Omega)) \times L^2(0,1; H^1(\Omega))$, weakly–$*$ compact in $L^\infty(\Omega \times (0,1)) \times L^\infty(0,1; H^1(\Omega))$ and weakly compact in $L^q(0,1; L^p(\Omega)) \times L^2(0,1; H^2(\Omega))$, for all $1 < p, q < \infty$. Moreover, for any $E_0 > 0$, and $\varphi \in L^{p'}(\Omega)$, $1 < p < \infty$,*

$$\sup_{\Phi \in U^s[E_0,\mathcal{F}](t)} \inf_{\Psi \in \mathcal{A}^s[\mathcal{F}]} \left[\sup_{\tau \in [0,1]} \left| \int_\Omega (\Phi^1 - \Psi^1)\varphi dx \right| + \|\Phi^2 - \Psi^2\|_{L^2(0,1;H^1)} \right] \longrightarrow 0,$$

as $t \longrightarrow +\infty$, where Φ^1 is the ρ–component of Φ, and Φ^2 is the (u, b, θ)–component of Φ.

PROOF. For $\Psi_n = (\rho_n, u_n, b_n, \theta_n) \in \mathcal{A}^s[\mathcal{F}]$, there exists a weak solution $\overline{\Psi}_n$ of (1.1) in $\Omega \times \mathbb{R}$ with $f_n \in \mathcal{F}^+$, such that (4.6) and (4.7) hold and $\Psi_n = \overline{\Psi}_n|_{[0,1]}$. Let $\widetilde{\Psi}_n = \mathcal{T}(-t_n)\overline{\Psi}_n$, then $\widetilde{\Psi}_n$ is a weak solution of (1.1) in $\Omega \times \mathbb{R}$ with $\mathcal{T}(-t_n)f \in \mathcal{F}^+$ satisfying (4.6) and (4.7). Noticing that $\Psi_n(\cdot) = \widetilde{\Psi}_n(\cdot + t_n)$, then the compactness of $\mathcal{A}^s[\mathcal{F}]$ follows from Proposition 4.1.

Fixing φ, we set

$$J(t, \Phi, \Psi) = \sup_{\tau \in [0,1]} \left| \int_\Omega (\Phi^1 - \Psi^1)\varphi dx \right| + \|\Phi^2 - \Psi^2\|_{L^2(0,1;H^1)},$$

for $\Phi \in U^s[E_0, \mathcal{F}](t)$ and $\Psi \in \mathcal{A}^s[\mathcal{F}]$. Define

$$L(t) = \sup_{\Phi \in U^s[E_0,\mathcal{F}](t)} \inf_{\Psi \in \mathcal{A}^s[\mathcal{F}]} J(t, \Phi, \Psi).$$

By Proposition 4.1, for $t > t_0$, $J(t, \Phi, \Psi)$ and $L(t)$ is bounded. Then, there exists a sequence $t_n \longrightarrow +\infty$, such that

$$\lim_{n\to\infty} L(t_n) = \limsup_{t\to+\infty} L(t).$$

For any $\varepsilon > 0$, there exists $\Phi_n \in U^s[E_0, \mathcal{F}](t_n)$, s.t.

$$L(t_n) \le \varepsilon + \inf_{\Psi \in \mathcal{A}^s[\mathcal{F}]} J(t_n, \Phi_n, \Psi).$$

Using Proposition 4.1 again, we can take a subsequence $\{\Phi_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\Phi^1_{n_k} \longrightarrow \Phi^1_* \text{ in } C([0,1]; L^p_{\text{weak}}(\Omega)),$$
$$\Phi^2_{n_k} \longrightarrow \Phi^2_* \text{ strongly in } L^2(0,1; H^1(\Omega)),$$

for some $\Phi_* \in \mathcal{A}^s[\mathcal{F}]$. Then

$$\lim_{k\to\infty} J(t, \Phi_{n_k}, \Phi_*) = 0.$$

Noticing that

$$L(t_n) \le \varepsilon + \inf_{\Psi \in \mathcal{A}^s[\mathcal{F}]} J(t_n, \Phi_n, \Psi) \le \varepsilon + J(t_n, \Phi_n, \Phi_*).$$

Then

$$\lim_{k\to\infty} L(t_{n_k}) \le \varepsilon.$$

By the arbitrariness of ε, we deduce

$$\limsup_{t\to+\infty} L(t) = \lim_{k\to\infty} L(t_{n_k}) = 0.$$

□

Theorem 5.1 says that $\mathcal{A}^s[\mathcal{F}]$ attracts $U^s[E_0,\mathcal{F}](t)$. In other words, the short trajectories of weak solutions with finite initial data always get close to the set consisting of complete bounded trajectories.

To consider the behavior of states, we define

$$U[E_0,\mathcal{F}](t) = \{\Phi(0) \mid \Phi \in U^s[E_0,\mathcal{F}](t)\},$$

and

$$\mathcal{A}[\mathcal{F}] = \{\Phi(0) \mid \Phi \in \mathcal{A}^s[\mathcal{F}]\}.$$

Theorem 5.2. *Under the assumptions of Proposition 4.1, $\mathcal{A}[\mathcal{F}]$ is compact in $L^\infty_{weak-*}(\Omega)\times L^r(\Omega)$, for all $1\le r<\infty$. Moreover, for any $E_0>0$, and $\varphi\in L^1(\Omega)$,*

$$\sup_{\Phi\in U[E_0,\mathcal{F}](t)} \inf_{\Psi\in\mathcal{A}[\mathcal{F}]} \left[\left|\int_\Omega (\Phi^1-\Psi^1)\varphi dx\right| + \|\Phi^2-\Psi^2\|_{L^r}\right] \longrightarrow 0,$$

as $t\longrightarrow+\infty$.

Proof. The compactness follows from Corollary 4.2. The asymptotic behavior can be obtained by similar technique used in the proof of Theorem 5.1. □

For $U^s_K[E_0,\mathcal{F}](t)$, we can also obtain similar properties. Define

$$\mathcal{A}^s_K[\mathcal{F}] = \{(\rho,u,b,\theta)\in\mathcal{A}^s[\mathcal{F}] \mid \text{there exists } \tau\in[0,1] \text{ such that } \rho(\tau)\in K\},$$

and

$$U_K[E_0,\mathcal{F}](t) = \{\Phi(0) \mid \Phi\in U^s_K[E_0,\mathcal{F}](t)\},$$
$$\mathcal{A}_K[\mathcal{F}] = \{\Phi(0) \mid \Phi\in\mathcal{A}^s_K[\mathcal{F}]\}.$$

Theorem 5.3. *Under the assumptions of Proposition 4.1,*

(i) $\mathcal{A}^s_K[\mathcal{F}]$ is compact in $C([0,1];L^p(\Omega))\times L^2(0,1;H^1(\Omega))$, weakly–$*$ compact in $L^\infty(\Omega\times(0,1))\times L^\infty(0,1;H^1(\Omega))$ and weakly compact in $L^q(0,1;L^p(\Omega))\times L^2(0,1;H^2(\Omega))$, for all $1<p,q<\infty$. Moreover, for any $E_0>0$, $1<p<\infty$,

$$\sup_{\Phi\in U^s_K[E_0,\mathcal{F}](t)} \inf_{\Psi\in\mathcal{A}^s_K[\mathcal{F}]} [\|\Phi^1-\Psi^1\|_{C([0,1];L^p)} + \|\Phi^2-\Psi^2\|_{L^2(0,1;H^1)}] \longrightarrow 0,$$

as $t\longrightarrow+\infty$;

(ii) $\mathcal{A}_K[\mathcal{F}]$ is compact in $L^p(\Omega)\times L^r(\Omega)$, for all $1\le p,r<\infty$. Moreover, for any $E_0>0$,

$$\sup_{\Phi\in U_K[E_0,\mathcal{F}](t)} \inf_{\Psi\in\mathcal{A}_K[\mathcal{F}]} [\|\Phi^1-\Psi^1\|_{L^p} + \|\Phi^2-\Psi^2\|_{L^r}] \longrightarrow 0,$$

as $t\longrightarrow+\infty$.

The proof of Theorem 5.3 is analogous to those of Theorem 5.1 and Theorem 5.2. We omit the details here.

References

[1] H. O. Bae, B. J. Jin, Temporal and spatial decays for the Navier-Stokes equations, *Proc. Roy. Soc. Edinburgh Sect. A* **135, no.3** (2005), 461-477.

[2] H. O. Bae, B. J. Jin, Upper and lower bounds of temporal and spatial decays for the Navier-Stokes equations, *J. Differential Equations* **209, no.2** (2005), 365-391.

[3] L. Brandolese, Space-time decay of Navier-Stokes flows invariant under rotations, *Math. Ann.* **329, no.4** (2004), 685-706.

[4] L. Caffarelli, R. Kohn, L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations, *Comm. Pure Appl. Math.* **35, no.6** (1982), 771-831.

[5] C. Cao, J. Wu, Two regularity criteria for the 3D MHD equations, *J. Differential Equations* **248, no.9** (2010), 2263-2274.

[6] V. V. Chepyzhov, M. I. Vishik, Evolution equations and their trajectory attractors, *J. Math. Pures Appl.* **76, no.10** (1997), 913-964.

[7] V. V. Chepyzhov, M. I. Vishik, Attractors for equations of mathematical physics, American Mathematical Society, Providence, RI, 2002.

[8] P. Constantin, C. Foias, Global Lyapunov exponents, Kaplan-Yorke formulas and the dimension of the attractors for 2D Navier-Stokes equations, *Comm. Pure Appl. Math.* **38, no.1** (1985), 1-27.

[9] B. Desjardins, C. L. Bris, Remarks on a nonhomogeneous model of magnetohydrodynamics, *Differential Integral Equations* **11, no.3** (1998), 377-394.

[10] G. Duvaut, J. L. Lions, Inéquations en thermoélasticité et magnétohydrodynamique, *Arch. Rational Mech. Anal.* **46** (1972), 241-279.

[11] E. Feireisl, Propagation of oscillations, complete trajectories and attractors for compressible flows, *NoDEA Nonlinear Differential Equations Appl.* **10, no.1** (2003), 33-55.

[12] E. Feireisl, H. Petzeltová, Asymptotic compactness of global trajectories generated by the Navier-Stokes equations of a compressible fluid, *J. Differential Equations* **173, no.2** (2001), 390-409.

[13] G. P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems, 2nd ed., Springer, 2011.

[14] C. He, D. Zhou, Existence and asymptotic behavior for an incompressible Newtonian flow with intrinsic degree of freedom, *Math. Methods Appl. Sci.* **37, no.8** (2014), 1191-1205.

[15] O. A. Ladyžhenskaya, A dynamical system that is generated by the Navier-Stokes equations, *Dokl. Akad. Nauk SSSR.* **205** (1972), 318-320.

[16] P. L. Lions, Mathematical topics in fluid mechanics, Vol. 1, Incompressible models, The Clarendon Press, Oxford University Press, 1996.

[17] J. Málek, J. Nečas, A finite-dimensional attractor for three-dimensional flow of incompressible fluids, *J. Differential Equations* **127, no.2** (1996), 498-518.

[18] M. E. Schonbek, L^2 decay for weak solutions of the Navier-Stokes equations, *Arch. Rational Mech. Anal.* **88, no.3** (1985), 209-222.

[19] G. R. Sell, Global attractors for the three-dimensional Navier-Stokes equations, *J. Dynam. Differential Equations* **8, no.1** (1996), 1-33.

[20] M. Sermange, R. Temam, Some mathematical questions related to the MHD equations, *Comm. Pure Appl. Math.* **36, no.5** (1983), 635-664.

[21] Y. Wang, G. Wu, D. Zhou, Refined regularity class of suitable weak solutions to the magnetohydrodynamic equations with an application, *Z. Angew. Math. Phys.* **67, no.6** (2016), Art. 136, 22pp.

[22] W. Wang, H. Yu, P. Zhang, Global strong solutions for 3D viscous incompressible heat conducting Navier-Stokes flows with the general external force, *Math. Methods Appl. Sci.* **41, no.12** (2018), 4589-4601.

[23] H. Wu, Strong solutions to the incompressible magnetohydrodynamic equations with vacuum, *Comput. Math. Appl.* **61, no.9** (2011), 2742-2753.

[24] J. Wu, Generalized MHD equations, *J. Differential Equations* **195, no.2** (2003), 284-312.

[25] J. Wu, Y. Wu, Global small solutions to the compressible 2D magnetohydrodynamic system without magnetic diffusion, *Adv. Math.* **310** (2017), 759-888.

[26] W. Yang, Q. Jiu, J. Wu, The 3D incompressible magnetohydrodynamic equations with fractional partial dissipation, *J. Differential Equations* **266, no.1** (2019), 630-652.

[27] X. Zhong, Global existence and large time behavior of strong solutions for 3D nonhomogeneous heat-conducting magnetohydrodynamic equations, *J. Geom. Anal.* **31, no.11** (2021), 10648-10678.

[28] X. Zhong, Global well-posedness to the 2D Cauchy problem of nonhomogeneous heat conducting magnetohydrodynamic equations with large initial data and vacuum, *Calc. Var. Partial Differential Equations* **60, no.2** (2021), paper No. 64, 24pp.

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