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Asymptotics toward rarefaction wave for an inflow problem of the compressible Navier-Stokes-Korteweg equation

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Abstract. In this article, we are concerned with the large-time behavior of solutions to an inflow problem in one-dimensional case for the Navier-Stokes-Korteweg equation, which models compressible fluids with internal capillarity. We first investigate that the asymptotic state is the rarefaction wave under the proper condition of the far fields and boundary values. The asymptotic stability of the rarefaction wave under some smallness conditions is shown. The proof is completed by the energy method with the help of time-decay estimate for the rarefaction wave.

Contents

1. Introduction

In this article, we are interested in the following one dimensional compressible Navier-Stokes-Korteweg equation, which reads as

$$\begin{cases} \rho_t + (\rho u)_{\tilde{x}} = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_{\tilde{x}} = \mu u_{\tilde{x}\tilde{x}} + \kappa \rho \rho_{\tilde{x}\tilde{x}\tilde{x}} \end{cases} \tag{1.1}$$

with the following initial data

$$(\rho, u)(0, \tilde{x}) = (\rho_0, u_0)(\tilde{x}) \text{ for } \tilde{x} > 0, \text{ and } \inf_{\tilde{x} \in \mathbb{R}^+} \rho_0(\tilde{x}) > 0, \tag{1.2}$$

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the boundary condition at the infinity $\tilde{x} = \infty$

$$\lim_{x \to +\infty} (\rho, u)(t, \tilde{x}) = (\rho_+, u_+), \text{ for any } t \geq 0, \tag{1.3}$$

and also the boundary condition at $\tilde{x} = 0$

$$\rho(t, 0) = \rho_-, \ u(t, 0) = u_-, \ \rho_{\tilde{x}}(t, 0) = 0, \text{ for any } t \geq 0. \tag{1.4}$$

Here, ρ and u are unknown functions in t and $\tilde{x}$, which stand for the density and the velocity, respectively. The time and space variables are $t \geq 0$ and $\tilde{x} \in \mathbb{R}^+ := \{\tilde{x} \in \mathbb{R} : \tilde{x} > 0\}$. The function $p(\rho)$ is the pressure defined by $p(\rho) = k\rho^\gamma$, where $k > 0$ and $\gamma \geq 1$ are the gas constants. The positive constants μ, κ denote, respectively, the viscosity and the capillary coefficient, and κ is also called Weber number. ρ_+, ρ_-, u_+ and u_- are constants satisfying $\rho_\pm > 0$ and $u_- > 0$. And $\rho_0(\tilde{x}), u_0(\tilde{x})$ are two given functions.

The model (1.1) considered is supposed to govern the motion of compressible fluids such as liquid vapor mixtures endowed with a variable internal capillarity, and is originated from the works by van der Waals [**42**] and Korteweg [**23**]. Its modern form is actually derived in the 1980s using the second gradient theory (see for instance [**11**]). Recently, Heida and Málek [**17**] also derived the compressible Navier-Stokes-Korteweg equations by the entropy production method which does not require to introduce any new or non-standard concepts such as multipolarity or interstitial working which are used in [**11**]. We point out that special cases of these models have also arisen in other contexts, e.g. in the water waves theory and more recently in quantum hydrodynamics. Finally, one can see easily that when $\kappa = 0$, the system (1.1) is reduced to the classical Navier-Stokes equations for compressible fluids. The mathematical justification from the compressible Navier-Stokes-Kortewg system to the compressible Navier-Stokes equation have shown in [**2**].

Recently, there have been a great number of mathematical studies about the compressible Navier-Stokes-Korteweg equations. About the existence and uniqueness of solutions to the isentropic compressible Navier-Stokes-Korteweg equations, and we can refer to [**1, 5, 9, 10, 13, 14, 15, 16, 22, 24**] and some references therein. In what follows, let us focus on the large-time behavior of solutions to the isentropic compressible Navier-Stokes-Korteweg equations towards the nonlinear wave pattern, which is related to our interest. More precisely, Chen [**3**] and Li and Luo [**29**] discussed asymptotic stability of the rarefaction waves to Cauchy problem for the one-dimensional compressible fluid models of Korteweg type, respectively. Chen, et al. [**4**] also showed asymptotic stability of the rarefaction waves for the one-dimensional compressible Navier-Stokes-Korteweg system with large initial data. Li and Zhu [**32**] further showed asymptotic stability of the rarefaction wave with vacuum for the one-dimensional compressible Navier-Stokes-Korteweg equations. Chen, He and Zhao [**7**] studied nonlinear stability of traveling wave solutions to the Cauchy problem for the one-dimensional compressible Navier-Stokes-Korteweg equations. Li, Chen and Luo, and Li and Luo showed stability of the planar rarefaction wave to two- and three-dimensional compressible Navier-Stokes-Korteweg equations in [**28, 30**], respectively. The stability of stationary solutions of the multi-dimensional isentropic compressible Navier-Stokes-Korteweg equations was studied by Li [**26**], and Wang and Wang [**44**] in the case with a external force, respectively, under the assumption that the states at far fields $\pm\infty$ are equal. Moreover, we also mention that there are some studies about the large-time behavior and the optimal decay rates of the global classical solutions and of the global strong solutions for the isentropic compressible Navier-Stokes-Korteweg equations around the non-vacuum constant states, for example, see [**39, 40, 41, 43**] and some references therein.

In general, it is well known that the large-time behavior of solutions to the compressible Navier-Stokes-Korteweg equations in the half space is much more complicated than that for the corresponding Cauchy problem due to boundary effect. Tsyganov [**38**] discussed the global existence and time-asymptotic behavior of weak solutions for an isothermal model with the viscosity coefficient $\mu(\rho) \equiv 1$, the capillarity coefficient $\kappa(\rho) = \rho^{-5}$ and large initial data on the interval $[0, 1]$. The global existence and exponential decay of strong solutions with small initial data to the Korteweg system in a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ were also obtained by Kotschote in [**25**]. Chen, Li and Sheng [**8**] proved the nonlinear stability of viscous shock wave for an impermeable wall problem of the compressible Navier-Stokes-Korteweg equations with constant viscosity and capillarity coefficients and small initial data. Chen and Li [**6**] discussed the time-asymptotic behavior of strong solutions to the initial-boundary value problem of the compressible Navier-Stokes-Korteweg equations with density-dependent viscosity and capillarity in the half space, and showed the strong solution converges to the rarefaction wave as $t \to \infty$ for the impermeable wall problem under large initial perturbation. Hong [**18**] and Li and Zhu [**33**] showed the existence and stability of stationary solution to an outflow problem of the compressible Navier-Stokes-Korteweg equations with constant viscosity and capillarity coefficients in the half space, respectively. Li, Tang and Yu [**31**] obtained asymptotic stability of rarefaction wave for the outflow problem to the compressible Navier-Stokes-Korteweg equations in the half space. However, to the best of our knowledge, there is little result about the stability of nonlinear wave patterns for the inflow problem on the compressible Navier-Stokes-Korteweg equations which is interest in our paper. Li and Chen [**27**] and Hong [**19**] showed the existence and stability of stationary solution to an inflow problem of the compressible Navier-Stokes-Korteweg equations in the half space, respectively. Hong [**19**] also showed stability of viscous shock wave and the superposition of the stationary wave and the viscous shock wave in the inflow problem for isentropic Navier-Stokes-Korteweg system. Subject to [**19, 27**], we are going to the asymptotic behavior toward rarefaction wave for an inflow problem of the compressible Navier-Stokes-Korteweg equation in the half space.

We now turn back to the inflow problem. As in [**19, 27**], consider the coordinate transformation

$$t = t, \ x = \int_{(0,0)}^{(\tilde{x},t)} \rho d\tilde{x} - \rho u dt,$$

and set $v = \frac{1}{\rho}$, $s_- = -\frac{u_-}{v_-} < 0$. Then the inflow problem (1.1)-(1.4) is transformed into the problem in the Lagrangian coordinate

$$\begin{cases} v_t - u_x = 0, & x > s_-t, \ t > 0, \\ u_t + p(v)_x = \mu\Big(\frac{u_x}{v}\Big)_x + \kappa\Big(\frac{-v_{xx}}{v^5} + \frac{5v_x^2}{2v^6}\Big)_x, & x > s_-t, \ t > 0, \\ (v, u)|_{x=s_-t} = (v_-, u_-), & v_- = \frac{1}{\rho_-}, \ u_- > 0, \\ v_x|_{x=s_-t} = 0, & \\ (v, u)|_{t=0} = (v_0, u_0)(x) \to (v_+, u_+) & \text{as } \ x \to +\infty. \end{cases} \tag{1.5}$$

Further, the corresponding hyperbolic system without viscosity and capillarity is

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0, \end{cases} \tag{1.6}$$

and its characteristic speeds is $\lambda_i = (-1)^i \sqrt{-p'(v)}(i = 1, 2)$. Since the sound speed $c(v)$ is defined by

$$c(v) = v\sqrt{-p'(v)} = \sqrt{k\gamma} v^{-\frac{\gamma-1}{2}},$$

we can decide the (v, u) into three regions:

$$\Omega_{\text{sub}} = \{(u, v) : |u| < c(v), v > 0, u > 0\},$$
$$\Gamma_{\text{trans}} = \{(u, v) : |u| = c(v), v > 0, u > 0\},$$
$$\Omega_{\text{super}} = \{(u, v) : |u| < c(v), v > 0, u > 0\}.$$

We call them the subsonic, transonic, and supersonic regions, respectively. From gas dynamic theory in [**36**], we first review that for given constants $(v_\pm, u_\pm)$ with $v_\pm > 0$, the following Riemann problem of the compressible Euler equations:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0, \quad t > 0, x \in \mathbb{R}, \\ (v(0, x), u(0, x)) = \left(v_0^R(x), u_0^R(x)\right) = \begin{cases} (v_-, u_-), & x < 0, \\ (v_+, u_+), & x > 0 \end{cases} \end{cases} \tag{1.7}$$

admits a weak entropy solution $\left(v_i^R, u_i^R\right)(t, x)$ $(i = 1, 2)$ called the i-rarefaction wave if $(v_+, u_+) \in \mathcal{R}_i(v_-, u_-)$, where

$$\mathcal{R}_i(v_-, u_-) = \left\{(v, u) \in \omega \,\middle|\, u = u_- - \int_{v_-}^{v} \lambda_i(s) ds, u \geq u_-\right\} \tag{1.8}$$

is the i-rarefaction wave curve, and $\left(v_i^R, u_i^R\right)(t, x)$ is expressed by:

$$\left(v_i^R, u_i^R\right)(t, x) = \begin{cases} (v_-, u_-), & -\infty \leq \frac{x}{t} \leq \lambda_i(v_-), \\ \left(\lambda_i^{-1}\left(\frac{x}{t}\right), u_- - \displaystyle\int_{v_-}^{\lambda_i^{-1}(\frac{x}{t})} \lambda_i(s)\, ds\right), & \lambda_i(v_-) \leq \frac{x}{t} \leq \lambda_i(v_+), \\ (v_+, u_+), & \lambda_i(v_+) \leq \frac{x}{t} \leq +\infty. \end{cases}$$

In (1.8), ω is a suitable neighborhood of (v_-, u_-) in $\mathbb{R}^2$.

In this paper, we consider the boundary value and far field state satisfying $(v_-, u_-) \in \Omega_{\text{super}}$ and $(v_+, u_+) \in \mathcal{R}_2(v_-, u_-)$. That is, we are going to show the time-asymptotic nonlinear stability of the rarefaction wave $\left(v_2^R, u_2^R\right)(t, x)$. Now the main results are stated as follows.

THEOREM 1.1. *Let $(v_-, u_-) \in \Omega_{\text{super}}$ and $(v_+, u_+) \in \mathcal{R}_2(v_-, u_-)$. Assume that $v_0 - v_0^r \in H_0^2(\mathbb{R}^+)$, $u_0 - u_0^r \in H_0^1(\mathbb{R}^+)$. Then there exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$ and $\|v_0 - v_0^r\|_2 + \|u_0 - u_0^r\|_1 \leq \varepsilon_0$, then there exists a unique strong solution (v, u) of (1.5), which satisfies*

$$v - v^r \in C([0, \infty); H_0^2(\mathbb{R}^+)), \quad u - u^r \in C([0, \infty); H_0^1(\mathbb{R}^+)),$$
$$(v - v^r)_x \in L^2([0, \infty); H^2(\mathbb{R}^+)), \quad (u - u^r)_x \in L^2([0, \infty); H^1(\mathbb{R}^+)).$$

Moreover,

$$\lim_{t\to+\infty}\sup_{x>s_-t}|(v,u)-(v^r,u^r)|=0. \tag{1.9}$$

Here ε, (v^r,u^r) and (v_0^r,u_0^r) are given by (2.3)$_2$, (2.5) and (2.6), respectively.

REMARK 1.2. *For the case $(v_-,u_-)\in\Omega_{\text{super}}$ and $(v_+,u_+)\in\mathcal{R}_1\mathcal{R}_2(v_-,u_-)$, we can find $(\bar v,\bar u)\in\mathcal{R}_1(v_-,u_-)$ and $(v_+,u_+)\in\mathcal{R}_2(\bar v,\bar u)$ such that we can show that the solution (v,u) of (1.5) tends toward the combination of (v_1^R,u_1^R) and (v_2^R,u_2^R) as in* [**37**].

REMARK 1.3. *In this article we only consider the asymptotic behavior of the stationary wave for in-flow problem to one-dimensional compressible Navier-Stokes-Korteweg equations with small initial perturbation, in fact, it is interest and plausible that we can consider the corresponding results for large perturbation as in* [**12**] *for the compressible Navier-Stokes equation. It is expected to be done in the forthcoming papers.*

This article is follow-up study of [**27**]. Now we give main ideas and arguments of the proof for Theorem 1.1. Applying L^2-energy method and some time-decay estimates in L^p-norm of the smoothed rarefaction wave as in [**21, 34, 35, 37**], we prove the asymptotic stability of the rarefaction wave in the case that the initial data are a small perturbation of the rarefaction wave. The key ingredient in the proof of Theorems 1.1 is to deduce the a-priori estimates. The main difficulties are as follows. The first one is the occurrence of the third order dispersion term. The second is how to control the boundary terms in order to establish the dissipation of the density. To overcome the first difficulty, we need more regularities for the density and smooth rarefaction wave, which have made in [**3, 6, 7, 8, 28, 30, 33**]. We also note that the basic energy is obtained with the help of higher order estimates. For the second difficulty, we first have $\varphi(t,0)=\psi(t,0)=\varphi_y(t,0)=0$ from the boundary data (3.6)$_2$ and (iii) of Lemma 2.2. Next, similar as [**37**], we can establish the boundary dissipation of $\psi_y(t,0)$. Finally, we can obtain the boundary dissipation of $\varphi_{yy}(t,0)$ due to the Korteweg term, which is different from the out-flow problem in [**31, 33**]. With these boundary values and the boundary dissipations at hand, we can close the a-priori estimate.

The rest of this paper is organized as follows. After stating some notations, in Section 2, we recall a smooth approximation $(v^r(t,x),u^r(t,x))$ of the rarefaction wave $\left(v_2^R,u_2^R\right)(t,x)$ by (2.5), and list some basic properties of the smooth approximate rarefaction wave $(v^r(t,x),u^r(t,x))$ in this section for later use. Then we reformulate the original problem in terms of the perturbation variables in Section 3. Section 4 is the key part of this article, in which we will establish the *a priori* estimates by the elaborate energy estimates. Finally, we complete the proof of Theorem 1.1 in Section 5.

Notations. Throughout this paper, several generic positive constants are denoted by c and C without any confusion. For function space, $L^p(\Omega)(1\le p\le+\infty)$ is an usual Lebesgue space on $\Omega\subset\mathbb{R}=(-\infty,+\infty)$ with its norm

$$\|f\|_{L^p(\Omega)}=(\int_\Omega|f(x)|^p\mathrm{d}x)^{\frac{1}{p}},1\le p<\infty,\ \|f\|_{L^\infty(\Omega)}=\sup_\Omega|f(x)|.$$

$H^l(\Omega)$ denotes the l-th order Sobolev Space with its norm

$$\|f\|_l = \left(\sum_{i=0}^{l} \left\|\partial_x^i f\right\|^2\right)^{\frac{1}{2}} \quad \text{where } \|\cdot\| := \|\cdot\|_{L^2(\Omega)}.$$

$H_0^l(\Omega)$ is a closure of $C_0^\infty(\Omega)$ with respect to $H^l(\Omega)$-norm, so that $f \in H_0^l(\Omega)$ satisfies $f(\partial\Omega) = 0$. Finally, we denote by $C^0([0,T]; H^k(\Omega))$ (resp. $L^2(0,T; H^k(\Omega))$) the space of continuous (resp. square integrable) functions on $[0,T]$ taking values in the space $H^k(\Omega)$.

2. Smooth approximate rarefaction wave

Since the rarefaction wave $\left(v_2^R, u_2^R\right)\left(\frac{x}{t}\right)$ is not smooth, we need to construct a smooth approximation $\left(v^r, u^r\right)(t,x)$ of the rarefaction wave $\left(v_2^R, u_2^R\right)\left(\frac{x}{t}\right)$. As [**34**], we start with the Riemann problem on $\mathbb{R} = (-\infty, +\infty)$ for the typical Burgers equation:

$$w_t + w w_x = 0, \tag{2.1}$$

with initial data

$$w(0,x) = w_0^R(x) = \begin{cases} w_-, & x < 0 \\ w_+, & x > 0, \end{cases} \tag{2.2}$$

where $w_- < w_+$. The weak solution of (2.1)-(2.2) is a rarefaction wave $w^R(\frac{x}{t})$ connecting w_- and w_+, namely,

$$w^R(\frac{x}{t}) = \begin{cases} w_-, & x < w_- t, \\ \frac{x}{t}, & w_- t \le x \le w_+ t, \\ w_+, & x > w_+ t. \end{cases}$$

From [**36**], it is well known that when $w_- = \lambda_2(v_-) > 0$ and $w_+ = \lambda_2(v_+) > 0$, the centered rarefaction wave $\left(v_2^R, u_2^R\right)\left(\frac{x}{t}\right)$ can be defined by

$$(v^R, u^R)(\frac{x}{t}) = \Big(\lambda_2^{-1}(w^R(\frac{x}{t})), u_- - \int_{v_-}^{\lambda_2^{-1}(w^R(\frac{x}{t}))} \lambda_2(s) ds\Big).$$

It is easy to check that $v_2^R(t,x)$ and $u_2^R(t,x)$ satisfy

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0 \end{cases}$$

with

$$(v,u)(0,x) = (v_0^R, u_0^R) = \begin{cases} (v_-, u_-), & x < 0, \\ (v_+, u_+), & x > 0. \end{cases}$$

Now we approximate the rarefaction wave $w^R(\frac{x}{t})$ by the solution $w(t,x)$ of the following Cauchy problem:

$$\begin{cases} w_t + w w_x = 0, \\ w(0,x) = \begin{cases} w_-, & x < 0, \\ w_- + C_q \tilde{w} \int_0^{\varepsilon x} z^q e^{-z} \mathrm{d}z, & x \ge 0, \end{cases} \end{cases} \tag{2.3}$$

where $\tilde{w} = w_+ - w_-$, $C_q > 0$ is a constant satisfying $C_q \int_0^{+\infty} z^q e^{-z} dz = 1$ with $q \ge 8$ being a positive constant, and $\varepsilon \le 1$ is a positive constant to be determined later. Then the properties of $w(t,x)$ can be summarised in the following lemma.

LEMMA 2.1. *(See [**6, 21, 34**]) Let $0 < w_- < w_+$, then the Cauchy problem (2.3) admits a unique global smooth solution $w(t,x)$ satisfying:*

(i) $w_- \le w(t,x) \le w_+,\ w_x > 0, \quad x \ge 0,\ t \ge 0.$

(ii) *For any p with $1 \le p \le +\infty$, there exists a constant $C_{p,q} > 0$ such that for $t \ge 0$,*

$$\begin{aligned}
\|w_x(t)\|_{L^p} &\le C_{p,q} \min\left\{\tilde{w}\varepsilon^{1-\frac{1}{p}},\ \tilde{w}^{\frac{1}{p}} t^{-1+\frac{1}{p}}\right\},\\
\|w_{xx}(t)\|_{L^p} &\le C_{p,q} \min\left\{\tilde{w}\varepsilon^{2-\frac{1}{p}},\ \tilde{w}^{\frac{1}{q}} \varepsilon^{1-\frac{1}{p}+\frac{1}{q}} t^{-1+\frac{1}{q}}\right\},\\
\|w_{xxx}(t)\|_{L^p} &\le C_{p,q} \min\left\{\tilde{w}\varepsilon^{3-\frac{1}{p}},\ \tilde{w}^{\frac{2}{q}} \varepsilon^{2-\frac{1}{p}+\frac{2}{q}} t^{-1+\frac{2}{q}}\right\},\\
\|w_{xxxx}(t)\|_{L^p} &\le C_{p,q} \min\left\{\tilde{w}\varepsilon^{4-\frac{1}{p}},\ \tilde{w}^{\frac{3}{q}} \varepsilon^{3-\frac{1}{p}+\frac{3}{q}} t^{-1+\frac{3}{q}}\right\}.
\end{aligned}$$

(iii) *When $x \le w_- t$, it holds that*

$$w(t,x) - w_- = w_x(t,x) = w_{xx}(t,x) = w_{xxx}(t,x) = 0.$$

(iv) $\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \left|w(t,x) - w^R(t,x)\right| = 0.$

Now, we should construct the smooth approximate rarefaction wave $(v^r, u^r)(t,x)$ of $(v_2^R, u_2^R)(t,x)$. As in [**37**], we define $(\tilde{v}^r, \tilde{u}^r)(t,x)$ as follows:

$$(\tilde{v}^r, \tilde{u}^r)(t,x) = \Big(\lambda_2^{-1}(w(t,x)), u_- - \int_{v_-}^{\lambda_2^{-1}(w(t,x))} \lambda_2(s)ds\Big), \tag{2.4}$$

here $w(x,t)$ is the solution of (2.3). Then we set

$$(v^r, u^r)(t,x) = (\tilde{v}^r, \tilde{u}^r)(t,x)\big|_{x \ge s_- t}, \tag{2.5}$$

which together with $(2.3)_2$ and (2.4) implies

$$v_0^r(x) = \lambda_2^{-1}(w_0),\ u_0^r(x) = u_- - \int_{v_-}^{\lambda_2^{-1}(w_0)} \lambda_2(s)ds, \tag{2.6}$$

here

$$w_0(x) = \begin{cases} \lambda_2(v_-), & x < 0,\\ \lambda_2(v_-) + (\lambda_2(v_+) - \lambda_2(v_-))C_q \int_0^{\varepsilon x} z^q e^{-z} \mathrm{d}z, & x \ge 0. \end{cases}$$

It is easy to check from (2.4) and Lemma 2.1 that $(v^r, u^r)(t,x)$ has the following properties:

LEMMA 2.2. *Let $\delta = |v_+ - v_-| + |u_+ - u_-|$, the smooth approximation $(v^r, u^r)(t,x)$ of (v_2^R, u_2^R) has the following properties.*

(i) $u_x^r \ge 0, \quad |u_x^r| \le C\varepsilon, \quad \forall\, t \ge 0,\ x \ge s_- t.$

(ii) *For any p with $1 \le p \le +\infty$, there exists a constant $C_{p,q} > 0$ such that*

$$\begin{aligned}
&\|(v_x^r, u_x^r)(t)\|_{L^p} \le C_{p,q} \min\left\{\delta\varepsilon^{1-\frac{1}{p}},\ \delta^{\frac{1}{p}}(1+t)^{-1+\frac{1}{p}}\right\},\\
&\|(v_{xx}^r, u_{xx}^r)(t)\|_{L^p} \le C_{p,q} \min\left\{\delta\varepsilon^{2-\frac{1}{p}},\ \delta^{\frac{1}{q}}\varepsilon^{1-\frac{1}{p}+\frac{1}{q}}(1+t)^{-1+\frac{1}{q}}\right\},\\
&\|(v_{xxx}^r, u_{xxx}^r)(t)\|_{L^p} \le C_{p,q} \min\left\{\delta\varepsilon^{3-\frac{1}{p}},\ \delta^{\frac{2}{q}}\varepsilon^{2-\frac{1}{p}+\frac{2}{q}}(1+t)^{-1+\frac{2}{q}}\right\},\\
&\|(v_{xxxx}^r, u_{xxxx}^r)(t)\|_{L^p} \le C_{p,q} \min\left\{\delta\varepsilon^{4-\frac{1}{p}},\ \delta^{\frac{3}{q}}\varepsilon^{3-\frac{1}{p}+\frac{3}{q}}(1+t)^{-1+\frac{3}{q}}\right\}.
\end{aligned}$$

(iii) $(v_x^r, u_x^r)\big|_{x\le s_- t} = (v_-, u_-),\ \dfrac{\partial^j}{\partial x^j}(v_x^r, u_x^r)(t,x)\big|_{x\le s_- t} = 0, \quad j = 1,2,3.$

(iv) $\lim\limits_{t\to+\infty} \sup\limits_{x\ge s_- t} \left|(v^r, u^r)(t,x) - \left(v_2^R, u_2^R\right)\left(\frac{x}{t}\right)\right| = 0.$

3. Reformulation of the original problem

Since it is convenient to regard the solution $(v,u)(t,x)$ as the perturbation of $(v^r, u^r)(t,x)$, we are going to reformulate the original problem in terms of the perturbation variables in this section. First of all, we consider the coordinate transformation

$$t = t,\ y = x - s_- t, \tag{3.1}$$

by using (3.1), we rewrite the initial value problem (1.5) as follows

$$\begin{cases} v_t - s_- v_y - u_y = 0, \\ u_t - s_- u_y + p(v)_y = \mu\left(\frac{u_y}{v}\right)_y + k\left(\frac{-v_{yy}}{v^5} + \frac{5v_y^2}{2v^6}\right)_y, \\ (v,u)|_{t=0} = (v_0, u_0)(y) \to (v_+, u_+), \\ (v,u)|_{y=0} = (v_-, u_+), \quad v_y|_{y=0} = 0, \end{cases} \tag{3.2}$$

and, correspondingly (v^r, u^r) satisfies

$$\begin{cases} v_t^r - s_- v_y^r - u_y^r = 0, \\ u_t^r - s_- u_y^r + p(v^r)_y = 0, \\ (v^r, u^r)|_{t=0} = (v_0^r, u_0^r)(y), \\ (v^r, u^r)|_{y=0} = (v_-, u_-),\ v_y^r, v_{yy}^r, v_{yyy}^r|_{y=0} = 0. \end{cases} \tag{3.3}$$

Further, let us define the perturbation $(\varphi, \psi)(t,y)$ by

$$\varphi = v(t,y) - v^r(t,y), \quad \psi = u(t,y) - u^r(t,y). \tag{3.4}$$

From (3.2) and (3.3), one knows at once that $(\varphi,\psi)(t,y)$ satisfies the system in $\mathbb{R}^+ \times \mathbb{R}^+$ below

$$\begin{cases} \varphi_t - s_- \varphi_y - \psi_y = 0, \\ \varphi_t - s_- \psi_y + (p(v^r + \varphi) - p(v^r))y - \mu\left(\frac{\psi_y}{v^r + \varphi}\right)_y \\ = \mu\left(\frac{u_y^r}{v^r+\varphi}\right)_y + K_y, \end{cases} \tag{3.5}$$

and initial boundary values:

$$\begin{cases} (\varphi,\psi)|_{t=0} = (\varphi_0, \psi_0)(y) = (v_0 - v_0^r, u_0 - u_0^r), \\ (\varphi,\psi)|_{y=0} = (0,0), \quad \varphi_y|_{y=0} = 0. \end{cases} \tag{3.6}$$

Here

$$K = \kappa\left(\frac{-\varphi_{yy} - v_{yy}^r}{(v^r+\varphi)^5} + \frac{5(v_y^r + \varphi_y)^2}{2(v^r+\varphi)^6}\right).$$

Therefore, we are now in a position to restate our main results in terms of the perturbed variable $(\varphi,\psi)(t,y)$ as follows.

THEOREM 3.1. *Suppose that all the assumptions of Theorem 1.1 are met. Then there exists a unique global solution $(\varphi,\psi)(t,\xi)$ to problem (3.5)-(3.6), satisfying*

$$\begin{aligned} &\varphi \in C([0,\infty); H_0^2(\mathbb{R}^+)), \psi \in C([0,\infty); H_0^1(\mathbb{R}^+)), \\ &\varphi_y \in L^2([0,\infty); H^2(\mathbb{R}^+)),\ \psi_y \in L^2([0,\infty); H^1(\mathbb{R}^+)), \end{aligned}$$

and

$$\lim_{t\to\infty} \sup_{y\in\mathbb{R}^+} |(\varphi,\psi)(t,y)| = 0. \tag{3.7}$$

To prove this theorem, we shall employ the standard continuation argument based on a local existence theorem in the following Lemma and on a priori estimates stated in the following proposition. First, the local existence of the solution (φ,ψ) to the initial-boundary value problem (3.5)-(3.6) is proved by the standard method, for example, the dual argument and iteration technique. We refer the details to [**15, 16, 25, 38**].

LEMMA 3.2 (Local existence). *Assume that the conditions in Theorem 1.1 hold. Then there exists a positive constant T_0 such that the initial-boundary value problem (3.5)-(3.6) has a unique solution $(\varphi,\psi)(t,y)$ that has the following properties:*

$$\begin{aligned}
&\varphi \in C([0,T_0];H_0^2(\mathbb{R}^+)),\ \ \psi \in C([0,T_0];H_0^1(\mathbb{R}^+)),\\
&\varphi_y \in L^2([0,T_0];H^2(\mathbb{R}^+)),\ \ \psi_y \in L^2([0,T_0];H^1(\mathbb{R}^+)),\\
&\inf_{t,y\in\mathbb{R}^+} v(t,y) > 0.
\end{aligned}$$

Next, we prove the following *a priori* estimates in Sobolev spaces, which are stated in Proposition 3.3.

PROPOSITION 3.3. *Let (φ,ψ) be a solution to the initial-boundary value problem (3.5)-(3.6) in a time interval $[0,T]$, which has same regularities as in Theorem 3.1. Then there exist constants $\varepsilon_0 > 0$ and $C > 0$ such that if*

$$N(T) := \sup_{t\in[0,T]} \{\|\varphi(t)\|_2 + \|\psi(t)\|_1\} \le \chi, \tag{3.8}$$

and $\varepsilon + \chi \ll \varepsilon_0$, then the following estimate holds for any $t \in [0,T]$

$$\begin{aligned}
&\|\varphi(t)\|_2^2 + \|\psi(t)\|_1^2\\
&+\int_0^t \left(\psi_y(\tau,0)^2 + \varphi_{yy}(\tau,0)^2 + \|\varphi_y(\tau)\|_2^2 + \|\psi_y(\tau)\|_1^2\right) d\tau\\
\le\ & C(\|\varphi_0\|_2^2 + \|\psi_0\|_1^2 + \varepsilon_0^{\frac{1}{6}}).
\end{aligned} \tag{3.9}$$

4. A priori estimates

This section is devoted to the derivation of *a priori* estimates for the unknown function $(\varphi,\psi)(t,y)$ and their derivatives, we then show that Proposition 3.3 is valid. Moreover, in establishing *a priori* estimates, we shall employ a mollifier with respect to time variable t to resolve an insufficiency of regularity of the solution obtained in Proposition 3.3. As this argument is standard, we omit the details and proceed a derivation of those estimates formally. To derive these *a priori* estimates, we assume that there exists a solution $(\varphi,\psi)(t,y)$ to problem (3.5)-(3.6), such that

$$\begin{aligned}
&(\varphi,\psi)(t,y) \in C([0,T];H_0^2(\mathbb{R}^+)) \times C([0,T];H_0^1(\mathbb{R}^+)),\\
&\inf_{(t,y)\in[0,T]\times\mathbb{R}^+} (\varphi + v^r)(t,y) > 0
\end{aligned}$$

for any $T > 0$. From (3.8), one can see easily that there exist two positive constants c and C such that

$$0 < c \le v \le C \text{ for } t \in [0, T], \tag{4.1}$$

since $V \ge c > 0$ for a positive constant c. To this end, we introduce

$$\Phi(v^r, \varphi) = p(v^r)\varphi - \int_{v^r}^{v^r+\varphi} p(\eta)d\eta$$

combining this with (4.1) yields

$$c\varphi^2 \le \Phi(v^r, \varphi) \le C\varphi^2. \tag{4.2}$$

Now let us derive the basic energy estimate. First, utilizing $(3.5)_1$, it is see that

$$\begin{aligned}
K_y\psi &= \kappa\Big\{\Big[\frac{-\varphi_{yy} - v^r_{yy}}{(v^r+\varphi)^5} + \frac{5(v^r_y + \varphi_y)^2}{2(v^r+\varphi)^6}\Big]\psi\Big\}_y \\
&\quad + \frac{\kappa}{(v^r+\varphi)^5}\varphi_{yy}\psi_y + \frac{\kappa}{(v^r+\varphi)^5}v^r_{yy}\psi_y - \frac{5\kappa}{2(v^r+\varphi)^6}(v^r_y+\varphi_y)^2\psi_y \\
&= \kappa\Big\{\Big[\frac{-\varphi_{yy} - v^r_{yy}}{(v^r+\varphi)^5} + \frac{5(v^r_y + \varphi_y)^2}{2(v^r+\varphi)^6}\Big]\psi\Big\}_y + \frac{\kappa}{(v^r+\varphi)^5}\varphi_{yy}(\varphi_t - s_-\varphi_y) \\
&\quad + \frac{\kappa}{(v^r+\varphi)^5}v^r_{yy}\psi_y - \frac{5\kappa}{2(v^r+\varphi)^6}(v^r_y+\varphi_y)^2\psi_y \\
&= \kappa\Big\{\Big[\frac{-\varphi_{yy} - v^r_{yy}}{(v^r+\varphi)^5} + \frac{5(v^r_y + \varphi_y)^2}{2(v^r+\varphi)^6}\Big]\psi\Big\}_y + \frac{\kappa}{(v^r+\varphi)^5}v^r_{yy}\psi_y \\
&\quad + \Big(\frac{\kappa}{(v^r+\varphi)^5}\varphi_y\varphi_t\Big)_y - \Big(\frac{5\kappa s_-}{2(v^r+\varphi)^5}\varphi_y^2\Big)_y - \Big(\frac{\kappa}{2(v^r+\varphi)^5}\varphi_y^2\Big)_t \\
&\quad - \frac{5\kappa}{2(v^r+\varphi)^6}(v^{r2}_y\psi_y + u^r_y\varphi_y^2).
\end{aligned}$$

Further, from (3.5) and using above equality, the straightforward but tedious computations give

$$\Big[\frac{1}{2}\psi^2 + \Phi(v^r,\varphi) + \frac{\kappa}{2(v^r+\varphi)^5}\varphi_y^2\Big]_t + R_{1y} + R_2 = R_3. \tag{4.3}$$

Here

$$\begin{aligned}
R_1 &= -s_-\Big[\frac{1}{2}\psi^2 + \Phi(v^r,\varphi)\Big] + [p(v^r+\varphi) - p(v^r)]\psi - \mu\Big(\frac{u^r_y + \psi_y}{v^r+\varphi} - \frac{u^r_y}{v^r}\Big)\psi \\
&\quad -\kappa\Big[\frac{-\varphi_{yy} - v^r_{yy}}{(v^r+\varphi)^5} + \frac{5(v^r_y+\varphi_y)^2}{2(v^r+\varphi)^6}\Big]\psi - \frac{\kappa}{(v^r+\varphi)^5}\varphi_y\varphi_t + \frac{5\kappa s_-}{2(v^r+\varphi)^5}\varphi_y^2, \\
R_2 &= \mu\frac{\psi_y^2}{v^r+\varphi} - \frac{\mu u^r_y\varphi\psi_y}{v^r(v^r+\varphi)} + [p(v^r+\varphi) - p(v^r) - p'(v^r)\varphi]u^r_y,
\end{aligned}$$

and

$$R_3 = \mu\Big(\frac{u^r_y}{v^r}\Big)_y\psi + \frac{\kappa}{(v^r+\varphi)^5}v^r_{yy}\psi_y - \frac{5\kappa}{2(v^r+\varphi)^6}(v^{r2}_y\psi_y + u^r_y\varphi_y^2).$$

LEMMA 4.1. *Assume that $(\varphi,\psi)(t,y)$ is a solution to (3.5)-(3.6), satisfying the conditions in Proposition 3.3, then the following estimate holds*

$$\begin{aligned}
&\|\varphi(t)\|_1^2+\|\psi(t)\|^2+\int_0^t\left(\|(u_y^r)^{\frac12}\varphi\|^2+\|\psi_y\|^2\right)\mathrm{d}\tau\\
\leq\ & C(\|\varphi_0\|_1+\|\varphi_0\|^2+\varepsilon_0^{\frac16})+C\varepsilon_0\int_0^t\|\varphi_y\|^2\,\mathrm{d}\tau \qquad (4.4)
\end{aligned}$$

for all $t\in[0,T]$.

Proof. Integrating (4.3) with respect to y over $(0,\infty)$ yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty\left[\frac12\psi^2+\Phi(v^r,\varphi)+\frac{\kappa}{2(v^r+\varphi)^2}\varphi_y^2\right]\mathrm{d}y+R_1\Big|_{y=0}+\int_0^\infty R_2\mathrm{d}y\\
=\ &\int_0^\infty R_3dy. \qquad (4.5)
\end{aligned}$$

First, noting (4.1) and using (4.2), we obtain easily

$$\int_0^\infty\left[\frac12\psi^2+\Phi(V,\varphi)+\frac{\kappa}{2(V+\varphi)^2}\varphi_y^2\right]\mathrm{d}y\geq c(\|\varphi\|^2+\|\psi\|^2+\|\varphi_y\|^2). \qquad (4.6)$$

Due to $\varphi(t,0)=\psi(t,0)=\varphi_y(t,0)=0$, it is easy to see

$$R_1|_{y=0}=0 \qquad (4.7)$$

Moreover, in [**37**], the authors have showed

$$R_2\geq c(\|\psi_y\|^2+\|(u_y^r)^{\frac12}\varphi\|^2). \qquad (4.8)$$

Finally, let us estimate the terms of $\int_0^\infty R_3\mathrm{d}y$ one by one. First, from Lemma 2.2, we have

$$\|u_{yy}^r\|_{L^1}^{\frac43}\leq C\|u_{yy}^r\|_{L^1}^{\frac16}\|u_{yy}^r\|_{L^1}^{\frac76}\leq C\varepsilon^{\frac16}(1+t)^{-\frac{49}{48}}, \qquad (4.9)$$

and

$$\|u_y^r\|^{\frac43}\leq C\|u_y^r\|^{\frac16}\|u_y^r\|^{\frac76}\leq C\varepsilon^{\frac1{12}}(1+t)^{-\frac7{12}}. \qquad (4.10)$$

Then it follows from the Hölder inequality, the Sobolev inequality, and the Young inequality and using (4.1), (4.9)-(4.10) that

$$\begin{aligned}
\int_0^{+\infty}\mu\left(\frac{u_y^r}{v^r}\right)_y\psi\,\mathrm{d}y\ \leq\ & C\|\psi\|_{L^\infty}\left(\|u_{yy}^r\|_{L^1}+\|v_y^r\|\|u_y^r\|\right)\\
\leq\ & C\|\psi\|^{\frac12}\|\psi_y\|^{\frac12}\left(\|u_{yy}^r\|_{L^1}+\|v_y^r\|\|u_y^r\|\right)\\
\leq\ & \frac{c}{4}\|\psi_y\|^2+C\varepsilon^{\frac16}\|\psi\|^{\frac23}\left[(1+t)^{-\frac{49}{48}}+(1+t)^{-\frac76}\right]\\
\leq\ & \frac{c}{4}\|\psi_y\|^2+C\varepsilon^{\frac16}[(1+t)^{-\frac{33}{32}}+(1+t)^{-\frac54}]\|\psi\|^2\\
&+C\varepsilon^{\frac16}[(1+t)^{-\frac{65}{64}}+(1+t)^{-\frac98}]. \qquad (4.11)
\end{aligned}$$

Next, utilizing the Young inequality, (4.1) and Lemma 2.2, we have

$$\begin{aligned}
&\int_0^{+\infty}\frac{\kappa}{(v^r+\varphi)^5}v_{yy}^r\psi_y\,\mathrm{d}y\\
\leq\ & \frac{c}{4}\|\psi_y\|^2+C\|v_{yy}^r\|^2\leq\frac{c}{4}\|\psi_y\|^2+C\varepsilon^{\frac14}(1+t)^{-\frac74}, \qquad (4.12)
\end{aligned}$$

and

$$
\begin{aligned}
& -\int_0^{+\infty} \frac{5\kappa}{(v^r+\varphi)^6} v_y^{r2}\psi_y \, \mathrm{d}y \\
\leq\ & \frac{c}{4}\|\psi_y\|^2 + C\|v_y^r\|_{L^4}^{\frac{1}{3}} \|v_y^r\|_{L^4}^{\frac{11}{3}} \leq \frac{c}{4}\|\psi_y\|^2 + C\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{11}{4}}. \qquad (4.13)
\end{aligned}
$$

Finally, from (4.1), the Hölder inequality and Lemma 2.2, it is easy to obtain

$$
-\int_0^{+\infty} \frac{5\kappa}{(v^r+\varphi)^6} u_y^r \varphi_y^2 \, \mathrm{d}y \leq C\varepsilon\|\varphi_y\|^2. \qquad (4.14)
$$

Therefore, combining (4.5), (4.6)-(4.8), (4.11)-(4.13) and (4.14), and integrating the resultant inequality with respect to t, then implies (4.4) provided that $C\varepsilon_0^{\frac{1}{6}} < \frac{1}{4}$. This completes the proof of Lemma 4.1.

LEMMA 4.2. *Assume that $(\varphi,\psi)(t,y)$ is a solution to (3.5)-(3.6), satisfying the conditions in Proposition 3.3, then the following estimate holds*

$$
\|\varphi_y\|^2 + \int_0^t \left(\varphi_y^2 + \varphi_{yy}^2\right) \mathrm{d}\tau \leq C(\|\varphi_0\|_1^2 + \|\psi_0\|_1^2 + \varepsilon_0^{\frac{1}{6}}). \qquad (4.15)
$$

for all $t \in [0,T]$.

Proof. Rewriting equation $(3.5)_2$ as

$$
\begin{aligned}
& \left(\mu\frac{\varphi_y}{v^r+\varphi} - \psi\right)_t - s_-\left(\mu\frac{\varphi_y}{v^r+\varphi} - \psi\right)_y - p'(v^r+\varphi)\varphi_y \\
=\ & \frac{\mu}{(v^r+\varphi)^2} v_y^r \psi_y + [p'(v^r+\varphi) - p'(v^r)]v_y^r \\
& + \frac{\mu}{(v^r+\varphi)^2} v_y^r u_y^r - \mu\frac{u_{yy}^r}{v^r+\varphi} - K_y. \qquad (4.16)
\end{aligned}
$$

Multiplying (4.16) by $\frac{\varphi_y}{v^r+\varphi}$, we obtain

$$
\begin{aligned}
& \left(\frac{\mu\varphi_y^2}{2(v^r+\varphi)^2} - \frac{\psi\varphi_y}{v^r+\varphi}\right)_t + \Big\{\frac{1}{v^r+\varphi}\psi\psi_y - \frac{\mu s_-}{2(v^r+\varphi)^2}\varphi_y^2 + \frac{s_-}{v^r+\varphi}\psi\varphi_y \\
& -\kappa\Big[\frac{\varphi_{yy}\varphi_y}{(v^r+\varphi)^6} + \frac{v_{yy}^r\varphi_y}{(v^r+\varphi)^6} - \frac{5(v_y^r+\varphi_y)^2}{2(v^r+\varphi)^7}\varphi_y\Big]\Big\}_y - \frac{p'(v^r+\varphi)}{v^r+\varphi}\varphi_y^2 \\
& + \frac{\kappa}{(v^r+\varphi)^6}\varphi_{yy}^2 \\
=\ & \frac{1}{v^r+\varphi}\psi_y^2 + \frac{1}{(v^r+\varphi)^2} v_y^r\psi\psi_y - \frac{1}{(v^r+\varphi)^2} u_y^r\psi\varphi_y + \frac{\mu}{(v^r+\varphi)^3} v_y^r\varphi_y\psi_y \\
& + \frac{1}{v^r+\varphi}[p'(v^r+\varphi) - p'(v^r)]v_y^r\varphi_y + \frac{\mu}{(v^r+\varphi)^3} v_y^r u_y^r\varphi_y - \frac{\mu}{(v^r+\varphi)^2} u_{yy}^r\varphi_y \\
& - \frac{\kappa}{(v^r+\varphi)^6} v_{yy}^r\varphi_{yy} + \frac{5\kappa}{2(v^r+\varphi)^7}(v_y^r+\varphi_y)^2\varphi_{yy} \\
& + \frac{\kappa}{(v^r+\varphi)^7}(v_y^r+\varphi_y)\varphi_y\varphi_{yy} + \frac{\kappa}{(v^r+\varphi)^7}(v_y^r+\varphi_y)v_{yy}^r\varphi_y \\
& - \frac{5\kappa}{2(v^r+\varphi)^8}(v_y^r+\varphi_y)^3\varphi_y. \qquad (4.17)
\end{aligned}
$$

Here we used

$$
\begin{aligned}
& K_y \frac{\varphi_y}{v^r+\varphi} \\
= \quad & -\kappa\Big[\frac{\varphi_{yy}}{(v^r+\varphi)^6}\varphi_y + \frac{v^r_{yy}}{(v^r+\varphi)^6}\varphi_y - \frac{5(v^r_y+\varphi_y)^2}{2(v^r+\varphi)^7}\varphi_y\Big]_y \\
& +\frac{\kappa}{(v^r+\varphi)^6}\varphi_{yy}^2 + \frac{k}{(v^r+\varphi)^6}v^r_{yy}\varphi_{yy} - \frac{5\kappa}{2(v^r+\varphi)^7}(v^r_y+\varphi_y)^2\varphi_{yy} \\
& -\frac{\kappa}{(v^r+\varphi)^7}(v^r_y+\varphi_y)\varphi_y\varphi_{yy} - \frac{\kappa}{(v^r+\varphi)^7}(v^r_y+\varphi_y)v^r_{yy}\varphi_y \\
& +\frac{5\kappa}{2(v^r+\varphi)^8}(v^r_y+\varphi_y)^3\varphi_y.
\end{aligned}
$$

Integrating (4.17) with respect to y over $\mathbb{R}^+$ and taking into the boundary condition $(3.6)_2$, we have

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \Big(\frac{\mu\varphi_y^2}{2(v^r+\varphi)^2} - \frac{\psi\varphi_y}{v^r+\varphi}\Big)\,\mathrm{d}y - \int_0^\infty \frac{p'(v^r+\varphi)}{v^r+\varphi}\varphi_y^2\,\mathrm{d}y \\
& +\int_0^\infty \frac{\kappa}{(v^r+\varphi)^6}\varphi_{yy}^2\,\mathrm{d}y \\
= \quad & \int_0^\infty \frac{1}{v^r+\varphi}\psi_y^2\,\mathrm{d}y + \int_0^\infty \frac{1}{(v^r+\varphi)^2}v^r_y\psi\psi_y\,\mathrm{d}y - \int_0^\infty \frac{1}{(v^r+\varphi)^2}u^r_y\psi\varphi_y\,\mathrm{d}y \\
& +\int_0^\infty \frac{\mu}{(v^r+\varphi)^3}v^r_y\varphi_y\psi_y\,\mathrm{d}y + \int_0^\infty \frac{1}{v^r+\varphi}[p'(v^r+\varphi)-p'(v^r)]v^r_y\varphi_y\,\mathrm{d}y \\
& +\int_0^\infty \Big[\frac{\mu}{(v^r+\varphi)^3}v^r_y u^r_y\varphi_y - \frac{\mu}{(v^r+\varphi)^2}u^r_{yy}\varphi_y - \frac{\kappa}{(v^r+\varphi)^6}v^r_{yy}\varphi_{yy}\Big]\,\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
& +\int_0^\infty [\frac{5\kappa}{2(v^r+\varphi)^7}(v^r_y+\varphi_y)^2\varphi_{yy} + \frac{\kappa}{(v^r+\varphi)^7}(v^r_y+\varphi_y)\varphi_y\varphi_{yy} \\
& +\frac{\kappa}{(v^r+\varphi)^7}(v^r_y+\varphi_y)v^r_{yy}\varphi_y - \frac{5\kappa}{2(v^r+\varphi)^8}(v^r_y+\varphi_y)^3\varphi_y]\,\mathrm{d}y,
\end{aligned}
\tag{4.18}
$$

which together with (4.1) yields

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty (\varphi_y^2 - \psi\varphi_y)\,\mathrm{d}y + \int_0^\infty (\varphi_y^2+\varphi_{yy}^2)\,\mathrm{d}y \\
\le \quad & C(\int_0^\infty \psi_y^2\,\mathrm{d}y + H_1 + H_2 + H_3 + H_4),
\end{aligned}
\tag{4.19}
$$

where

$$\begin{aligned} H_1 &= |\int_0^\infty v_y^r\psi\psi_y\,\mathrm{d}y| + |\int_0^\infty u_y^r\psi\varphi_y\,\mathrm{d}y| + |\int_0^\infty v_y^r\varphi\varphi_y\,\mathrm{d}y|, \\ H_2 &= |\int_0^\infty v_y^r u_y^r\varphi_y\,\mathrm{d}y| + |\int_0^\infty u_{yy}^r\varphi_y\,\mathrm{d}y| + |\int_0^\infty v_{yy}^r\varphi_{yy}\,\mathrm{d}y| \\ &\quad + |\int_0^\infty (v_y^r)^2\varphi_{yy}\,\mathrm{d}y| + |\int_0^\infty v_y^r v_{yy}^r\varphi_y\,\mathrm{d}y| + |\int_0^\infty (v_y^r)^3\varphi_y\,\mathrm{d}y|, \\ H_3 &= |\int_0^\infty v_y^r\varphi_y\psi_y\,\mathrm{d}y| + |\int_0^\infty v_y^r\varphi_y\varphi_{yy}\,\mathrm{d}y| + |\int_0^\infty v_{yy}^r\varphi_y^2\,\mathrm{d}y| \\ &\quad + |\int_0^\infty (v_y^r)^2\varphi_y^2\,\mathrm{d}y|, \end{aligned}$$

and

$$H_4 = |\int_0^\infty \varphi_y^2\varphi_{yy}\,\mathrm{d}y| + |\int_0^\infty v_y^r\varphi_y^3\,\mathrm{d}y| + |\int_0^\infty \varphi_y^4\,\mathrm{d}y|.$$

Now let us estimate the terms on the right-hand side of (4.19). First, using Lemma 2.2, Hölder inequality and Young inequality, we have

$$\begin{aligned} H_1 &\le C\|\psi_y\|^2 + \frac{1}{8}\|\varphi_y\|^2 + C(\|v_y^r\|_{L^\infty}^2 + \|u_y^r\|_{L^\infty}^2)(\|\varphi\|^2 + \|\psi\|^2) \\ &\le C\|\psi_y\|^2 + \frac{1}{8}\|\varphi_y\|^2 + C\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{11}{6}}(\|\varphi\|^2 + \|\psi\|^2). \end{aligned} \tag{4.20}$$

Next, similar to (4.12) and (4.13), one deal with the terms of H_2 as follows:

$$|\int_0^\infty v_y^r u_y^r\varphi_y\,\mathrm{d}y| + |\int_0^\infty u_{yy}^r\varphi_y\,\mathrm{d}y| \le \frac{1}{8}\|\varphi_y\|^2 + C\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{7}{4}},$$

$$|\int_0^\infty v_{yy}^r\varphi_{yy}\,\mathrm{d}y| + |\int_0^\infty (v_y^r)^2\varphi_{yy}\,\mathrm{d}y| \le \frac{1}{8}\|\varphi_{yy}\|^2 + C\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{7}{4}},$$

and

$$\begin{aligned} &|\int_0^\infty v_y^r v_{yy}^r\varphi_y\,\mathrm{d}y| + |\int_0^\infty (v_y^r)^3\varphi_y\,\mathrm{d}y| \\ \le\ & \frac{1}{8}\|\varphi_y\|^2 + C\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{11}{4}} + C\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{15}{4}}. \end{aligned}$$

Then we have

$$\begin{aligned} H_2 &\le \frac{1}{8}(\|\varphi_y\|^2 + \|\varphi_{yy}\|^2) + C\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{7}{4}} \\ &\quad + C\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{11}{4}} + C\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{15}{4}}. \end{aligned} \tag{4.21}$$

Moreover, similar to (4.14), we can obtain

$$H_3 \le C\varepsilon\left(\|\varphi_y\|^2 + \|\varphi_{yy}\|^2\right), \tag{4.22}$$

Finally, using Cauchy inequality, Sobolev inequality and the assumption (3.8), we get

$$\begin{aligned} H_4 &\le C\|\varphi_y\|_{L^\infty}(\|\varphi_y\|^2 + \|\varphi_{yy}\|^2) \\ &\quad + C\|v_y^r\|_{L^\infty}\|\varphi_y\|_{L^\infty}\|\varphi_y\|^2 + C\|\varphi_y\|_{L^\infty}^2\|\varphi_y\|^2 \\ &\le C(\chi+\varepsilon)(\|\varphi_y\|^2 + \|\varphi_{yy}\|^2). \end{aligned} \tag{4.23}$$

Hence, combining (4.19), (4.20)-(4.22) and (4.23), then integrating the resultant inequality with respect to t and using (4.4), we can obtain (4.15) provided that $C\varepsilon_0^{\frac{1}{6}} < \frac{1}{4}$. This completes the proof of Lemma 4.2.

With Lemmas 4.1 and 4.2 in hand, we can show the fundamental energy estimate.

COROLLARY 4.3. *Assume that $(\varphi, \psi)(t, y)$ is a solution to (3.5)- (3.6), satisfying the conditions in Proposition 3., then it holds that*

$$\begin{aligned}&\|\varphi(t)\|_1^2 + \|\psi(t)\|^2 + \int_0^t \left[\|\varphi_y(\tau)\|_1^2 + \|\psi_y(\tau)\|^2\right] \mathrm{d}\tau \\ &\leq C\left(\|\varphi_0\|_1^2 + \|\psi_0\|^2 + \varepsilon_0^{\frac{1}{6}}\right)\end{aligned} \tag{4.24}$$

for any $t \in [0, T]$.

LEMMA 4.4. *Assume that $(\varphi, \psi)(t, y)$ is a solution to (3.5)-(3.6), satisfying the conditions in Proposition 3.3, then the following estimate holds*

$$\begin{aligned}&\|\psi_y(t)\|^2 + \|\varphi_{yy}(t)\|^2 + \int_0^t \left(\|\psi_{yy}\|^2 + \psi_y(\tau,0)^2 + \varphi_{yy}(\tau,0)^2\right) \mathrm{d}\tau \\ &\leq C\left(\|\psi_0\|_1^2 + \|\varphi_0\|_2^2 + \varepsilon_0^{\frac{1}{4}}\right)\end{aligned} \tag{4.25}$$

for all $t \in [0, T]$.

Proof: Multiplying $(3.5)_2$ by $-\psi_{yy}$, one have

$$\begin{aligned}&\left(\frac{1}{2}\psi_y^2 + \frac{\kappa}{2(v^r+\varphi)^5}\varphi_{yy}^2\right)_t \\ &-\left(\psi_t\psi_y - \frac{s_-}{2}\psi_y^2 + \frac{\kappa}{(v^r+\varphi)^5}\varphi_{yy}\psi_{yy} + \frac{\kappa s_-}{2(v^r+\varphi)^5}\varphi_{yy}^2\right)_y + \frac{\mu}{v^r+\varphi}\psi_{yy}^2 \\ = &-\frac{\mu}{v^r+\varphi}u_{yy}^r\psi_{yy} + \frac{\mu}{(v^r+\varphi)^2}u_y^r(v_y^r+\varphi_y)\psi_{yy} + \frac{\mu}{(v^r+\varphi)^2}(v_y^r+\varphi_y)\psi_y\psi_{yy} \\ &+[p(v^r+\varphi)-p(v^r)]_y\psi_{yy} - \frac{5\kappa}{(v^r+\varphi)^6}(v_y^r+\varphi_y)\varphi_{yy}\psi_{yy} \\ &-\frac{10\kappa}{(v^r+\varphi)^6}(v_y^r+\varphi_y)v_{yy}^r\psi_{yy} + \frac{\kappa}{(v^r+\varphi)^5}v_{yyy}^r\psi_{yy} \\ &+\frac{15\kappa}{2(v^r+\varphi)^7}(v_y^r+\varphi_y)^3\psi_{yy} - \frac{5\kappa}{2(v^r+\varphi)^6}(u_y^r+\psi_y)\varphi_{yy}^2.\end{aligned} \tag{4.26}$$

Here we used

$$-\psi_t\psi_{yy} = -(\psi_t\psi_y)_y + \left(\frac{1}{2}\psi_y^2\right)_t,$$

and

$$\begin{aligned}\left(\frac{\kappa\varphi_{yy}}{(v^r+\varphi)^5}\right)_y\psi_{yy} &= \left(\frac{\kappa}{(v^r+\varphi)^5}\varphi_{yy}\psi_{yy}\right)_y - \frac{\kappa}{(v^r+\varphi)^5}\varphi_{yy}\psi_{yyy} \\ &= \left(\frac{\kappa}{(v^r+\varphi)^5}\varphi_{yy}\psi_{yy}\right)_y - \frac{\kappa}{(v^r+\varphi)^5}\varphi_{yy}(\varphi_{tyy} - s_-\varphi_{yyy}) \\ &= \left(\frac{\kappa}{(v^r+\varphi)^5}\varphi_{yy}\psi_{yy}\right)_y - \left(\frac{\kappa}{2(v^r+\varphi)^5}\varphi_{yy}^2\right)_t \\ &\quad+\left(\frac{\kappa s_-}{2(v^r+\varphi)^5}\varphi_{yy}^2\right)_y - \frac{5\kappa}{2(v^r+\varphi)^6}(u_y^r+\psi_y)\varphi_{yy}^2\end{aligned}$$

with the help of $\varphi_{tyy} - s_-\varphi_{yyy} - \psi_{yyy} = 0$. Moreover, from $\varphi_{ty} - s_-\varphi_{yy} - \psi_{yy} = 0$ and $\varphi_{ty}(0) = 0$, it is easy to see

$$\psi_{yy}(0) = -s_-\varphi_{yy}(0). \tag{4.27}$$

Then integrating the equality (4.26) with respect to y over $\mathbb{R}^+$ and taking into the boundary condition $(3.6)_2$ and (4.27), and (4.1), we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty (\psi_y^2 + \varphi_{yy}^2)\,\mathrm{d}y - \frac{s_-}{2}\psi_y(t,0)^2 - \frac{\kappa s_-}{2v_-^5}\varphi_{yy}(t,0)^2 + \int_0^\infty \psi_{yy}^2\,\mathrm{d}y \\ \le\ & C(I_1 + I_2 + I_3 + I_4 + I_5). \end{aligned} \tag{4.28}$$

Here

$$\begin{aligned} I_1 &= |\int_0^\infty u^r_{yy}\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty v^r_{yyy}\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty v^r_y u^r_y\psi_{yy}\,\mathrm{d}y| \\ &\quad + |\int_0^\infty v^r_y v^r_{yy}\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty (v^r_y)^3\psi_{yy}\,\mathrm{d}y|, \\ I_2 &= |\int_0^\infty v^r_y\varphi\psi_{yy}\,\mathrm{d}y|,\ \ I_3 = |\int_0^\infty \varphi_y\psi_{yy}\,\mathrm{d}y|, \\ I_4 &= |\int_0^\infty v^r_y\psi_y\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty u^r_y\varphi_y\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty v^r_{yy}\varphi_y\psi_{yy}\,\mathrm{d}y| \\ &\quad + |\int_0^\infty u^r_y\varphi_{yy}^2\,\mathrm{d}y| + |\int_0^\infty v^r_y\varphi_{yy}\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty (v^r_y)^2\varphi_y\psi_{yy}\,\mathrm{d}y|, \end{aligned}$$

and

$$\begin{aligned} I_5 &= |\int_0^\infty \varphi_y\psi_y\psi_{yy}\,\mathrm{d}y| + |\int_0^\infty \psi_y\varphi_{yy}^2\,\mathrm{d}y| + |\int_0^\infty \varphi_y^3\psi_{yy}\,\mathrm{d}y| \\ &\quad + |\int_0^\infty v^r_y\varphi_y^2\psi_{yy}\,\mathrm{d}y|. \end{aligned}$$

Now let us estimate the terms on the right-hand side of (4.19). First, similar as (4.12)-(4.13) and (4.20), we have

$$\begin{aligned} I_1 &\le \frac{1}{8}\|\psi_{yy}\|^2 + C\varepsilon^{\frac14}(1+t)^{-\frac74} + C\varepsilon^{\frac14}(1+t)^{-\frac{11}{4}} \\ &\quad + C\varepsilon^{\frac14}(1+t)^{-\frac{15}{4}} + C\varepsilon^{\frac12}(1+t)^{-\frac32}, \end{aligned} \tag{4.29}$$

and

$$I_2 \le \frac{1}{8}\|\psi_{yy}\|^2 + C\varepsilon^{\frac16}(1+t)^{-\frac{11}{6}}\|\varphi\|^2. \tag{4.30}$$

Next, from the Cauchy inequality and the inequality, it is easy to obtain

$$I_3 \le \frac{1}{8}\|\psi_{yy}\|^2 + C\|\varphi_y\|^2. \tag{4.31}$$

Finally, similar to (4.14) and (4.23), we obtain

$$I_4 \le C\varepsilon\left(\|\varphi_y\|^2 + \|\psi_y\|^2 + \|\varphi_{yy}\|^2 + \|\psi_{yy}\|^2\right), \tag{4.32}$$

and

$$I_5 \le C(\chi+\varepsilon)\left(\|\varphi_y\|^2 + \|\psi_y\|^2 + \|\varphi_{yy}\|^2 + \|\psi_{yy}\|^2\right). \tag{4.33}$$

Hence, insertion (4.29)-(4.33) into (4.28), then integrating the resultant inequality with respect to t and using (4.24), we can obtain (4.25). This completes the proof of Lemma 4.4.

Finally, we are going to establish the dissipation for φ_{yyy}.

LEMMA 4.5. *Let $(\varphi,\psi)(t,y)$ be a solution to (3.5)-(3.6), satisfying the conditions in Proposition 3.3, then it holds that*

$$\int_0^t \|\varphi_{yyy}(\tau)\|^2 \mathrm{d}\tau \le C(\|\varphi_0\|_2^2+\|\psi_0\|_1^2+\varepsilon_0^{\frac14}) \tag{4.34}$$

for an arbitrary $t\in[0,T]$.

Proof. We first recall that

$$\varphi_{ty}-s_-\varphi_{yy}-\psi_{yy}=0, \tag{4.35}$$

and

$$\varphi_{tyy}-s_-\varphi_{yyy}-\psi_{yyy}=0, \tag{4.36}$$

further, we have

$$(\psi_t-s_-\psi_y)\varphi_{yyy}=(\psi_t\varphi_{yy})_y-(\psi_y\varphi_{yy})_t+(\psi_y\psi_{yy})_y-\psi_{yy}^2, \tag{4.37}$$

and

$$\begin{aligned}
&\frac{\mu}{v^r+\varphi}\psi_{yy}\varphi_{yyy}\\
=\ &\frac{\mu}{v^r+\varphi}(\varphi_{ty}-s_-\varphi_{yy})\varphi_{yyy}\\
=\ &\Big(\frac{\mu}{v^r+\varphi}\varphi_{ty}\varphi_{yy}-\frac{\mu s_-}{2(v^r+\varphi)}\varphi_{yy}^2\Big)_y-\Big(\frac{\mu}{2(v^r+\varphi)}\varphi_{yy}^2\Big)_t\\
&+\frac{\mu}{(v^r+\varphi)^2}(v_y^r+\varphi_y)\varphi_{yy}(s_-\varphi_{yy}+\psi_{yy})-\frac{\mu}{2(v^r+\varphi)^2}(u_y^r+\psi_y)\varphi_{yy}^2\\
&-\frac{\mu s_-}{(v^r+\varphi)^2}(v_y^r+\varphi_y)\varphi_{yy}^2.
\end{aligned} \tag{4.38}$$

Then multiply $(3.5)_2$ by φ_{yyy} and use (4.37)-(4.38) to obtain

$$\begin{aligned}
&\Big[\frac{\mu}{2(v^r+\varphi)}\varphi_{yy}^2-\psi_y\varphi_{yy}\Big]_t\\
&+\Big[p'(v^r+\varphi)\varphi_y\varphi_{yy}+\frac{\mu s_-}{2(v^r+\varphi)}\varphi_{yy}^2-\frac{\mu}{v^r+\varphi}\varphi_{ty}\varphi_{yy}+\psi_y\psi_{yy}\Big]_y\\
&-p'(v^r+\varphi)\varphi_{yy}^2+\frac{\kappa}{(v^r+\varphi)^5}\varphi_{yyy}^2\\
=\ &\psi_{yy}^2+\frac{\mu u_{yy}^r}{v^r+\varphi}\varphi_{yyy}-\frac{\kappa}{(v^r+\varphi)^5}v_{yyy}^r\varphi_{yyy}-[p'(v^r+\varphi)-p'(v^r)]v_y^r\varphi_{yyy}\\
&+p''(v^r+\varphi)(v_y^r+\varphi_y)\varphi_y\varphi_{yy}+\frac{\mu}{(v^r+\varphi)^2}(v_y^r+\varphi_y)(s_-\varphi_{yy}+\psi_{yy})\varphi_{yy}\\
&-\frac{\mu}{2(v^r+\varphi)^2}(u_y^r+\psi_y)\varphi_{yy}^2-\frac{\mu}{(v^r+\varphi)^2}(u_y^r+\psi_y)(v_y^r+\varphi_y)\varphi_{yyy}\\
&-\frac{\mu s_-}{(v^r+\varphi)^2}(v_y^r+\varphi_y)\varphi_{yy}^2+\frac{10\kappa}{(v^r+\varphi)^6}(v_y^r+\varphi_y)(v_{yy}^r+\varphi_{yy})\varphi_{yyy}\\
&-\frac{15\kappa(v_y^r+\varphi_y)^3}{(v^r+\varphi)^7}\varphi_{yyy}.
\end{aligned}$$

Integrating the above equality with respect to y over $\mathbb{R}^+$ and taking into the boundary condition $(3.6)_2$ and (4.27), and (4.1), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty (\varphi_{yy}^2 - \psi_y\varphi_{yy})\,\mathrm{d}y - \frac{\mu s_-}{4v_-}\varphi_{yy}(t,0)^2 + \int_0^\infty \varphi_{yy}^2\,\mathrm{d}y + \int_0^\infty \varphi_{yyy}^2\,\mathrm{d}y$$

$$\leq C(\int_0^\infty \psi_{yy}^2\,\mathrm{d}y + \psi_y(t,0)^2 + J_1 + J_2 + J_3 + J_4). \tag{4.39}$$

Here

$$\begin{aligned} J_1 &= |\int_0^\infty v_y^r\varphi\varphi_{yyy}\,\mathrm{d}y|,\\ J_2 &= |\int_0^\infty u_{yy}^r\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty v_{yyy}^r\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty v_y^r u_y^r\varphi_{yyy}\,\mathrm{d}y|\\ &\quad +|\int_0^\infty v_y^r v_{yy}^r\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty (v_y^r)^3\varphi_{yyy}\,\mathrm{d}y|,\\ J_3 &= |\int_0^\infty v_y^r\psi_y\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty u_y^r\varphi_y\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty v_y^r\varphi_y\varphi_{yy}\,\mathrm{d}y|\\ &\quad +|\int_0^\infty v_y^r\varphi_{yy}^2\,\mathrm{d}y| + |\int_0^\infty u_y^r\varphi_{yy}^2\,\mathrm{d}y| + |\int_0^\infty v_y^r\varphi_{yy}\varphi_{yyy}\,\mathrm{d}y|\\ &\quad +|\int_0^\infty v_{yy}^r\varphi_y\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty (v_y^r)^2\varphi_y\varphi_{yyy}\,\mathrm{d}y|, \end{aligned}$$

and

$$\begin{aligned} J_4 &= |\int_0^\infty \varphi_y\psi_y\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty \psi_y\varphi_{yy}^2\,\mathrm{d}y| + |\int_0^\infty \varphi_y\varphi_{yy}^2\,\mathrm{d}y|\\ &\quad +|\int_0^\infty \varphi_y\varphi_{yy}\varphi_{yyy}\,\mathrm{d}y| + |\int_0^\infty \varphi_y^2\varphi_{yy}\,\mathrm{d}y| + |\int_0^\infty \varphi_y^3\varphi_{yyy}\,\mathrm{d}y|\\ &\quad +|\int_0^\infty v_y^r\varphi_y^2\varphi_{yyy}\,\mathrm{d}y|. \end{aligned}$$

Now let us deal with the terms on the right-hand side of (4.19). First, similar as (4.20), we have

$$J_1 \leq \frac{1}{8}\|\varphi_{yyy}\|^2 + C\varepsilon^{\frac{1}{6}}(1+t)^{-\frac{11}{6}}\|\varphi\|^2. \tag{4.40}$$

Next, similar to (4.12)-(4.13), one get

$$\begin{aligned} J_2 &\leq \frac{1}{8}\|\varphi_{yyy}\|^2 + C\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{7}{4}} + C\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{11}{4}} + C\varepsilon^{\frac{1}{4}}(1+t)^{-\frac{15}{4}}\\ &\quad +C\varepsilon^{\frac{1}{2}}(1+t)^{-\frac{3}{2}}. \end{aligned} \tag{4.41}$$

Moreover, similar to (4.14) and (4.23), we obtain

$$J_3 \leq C\varepsilon\left(\|\varphi_y\|^2 + \|\psi_y\|^2 + \|\varphi_{yy}\|^2 + \|\varphi_{yyy}\|^2\right), \tag{4.42}$$

and

$$J_4 \leq C(\chi+\varepsilon)\left(\|\varphi_y\|^2 + \|\psi_y\|^2 + \|\varphi_{yy}\|^2 + \|\varphi_{yyy}\|^2\right). \tag{4.43}$$

Therefore, insertion (4.40)-(4.43) into (4.39) yields

$$\begin{aligned} &\frac{d}{dt}\int_0^\infty (\varphi_{yy}^2 - \psi_y\varphi_{yy})dy + \varphi_{yy}(t,0)^2 + \|\varphi_{yy}(t)\|^2 + \|\varphi_{yyy}(t)\|^2\\ \leq\ & C\|\psi_{yy}(t)\|^2 + C\psi_y(t,0)^2 + C(\chi+\varepsilon)\left(\|\varphi_y\|^2 + \|\psi_y\|^2\right), \end{aligned}$$

further, integrating the above inequality with respect to t, and using (4.24) and (4.25), we obtain (4.34). This completes the proof.

Proof of Proposition 3.2. Summing up the estimates (4.24), (4.25) and (4.34), we immediately have (3.9).

5. The proof of Theorem 1.1

This section is concerned with the proof of our main theorem. To prove Theorem 1.1, we employ the standard continuation argument based on a local existence theorem and the *a priori* estimates. Therefore, to complete the proof of Theorem 1.1, we need only to investigate the large-time behavior of the solution $(v, u)(t, x - s_-t)$ to the initial boundary value problem (1.5) as time tends to infinity.

The completion of the proof of Theorem 1.1. Based upon the energy estimates derived in the previous sections, we will complete the proof of Theorem 1.1. To this end, we first prove that

$$\sup_{x \geq s_-t} |(v - v^r, u - u^r)(t, x)| \to 0, \tag{5.1}$$

namely,

$$\sup_{y \in \mathbb{R}^+} |(\varphi, \psi)(t, y)| \to 0, \tag{5.2}$$

as $t \to \infty$.

This is obvious suppose that we have proved the following assertion

$$\lim_{t \to +\infty} \|(\varphi_y, \psi_y)(t)\| = 0. \tag{5.3}$$

As a matter of fact, if it were true, we infer from the Sobolev inequality that

$$\|(\varphi, \psi)\|_{L^\infty} \to 0, \text{ as } t \to +\infty. \tag{5.4}$$

Hence, it remains to show (5.3). To this end, from the relations (4.18) and (4.28), and Corollary 4.3, Lemmas 4.4 and 4.5, one can show that

$$\int_0^\infty \left(\|\varphi_y\|^2 + \|\psi_y\|^2\right) + \psi_y(t, 0)^2 \tau < +\infty, \tag{5.5}$$

and that

$$\int_0^\infty \left|\frac{d}{dt}\|\varphi_y\|^2\right| \mathrm{d}\tau < +\infty, \quad \int_0^\infty \left|\frac{d}{dt}\|\psi_y\|^2\right| \mathrm{d}\tau < +\infty. \tag{5.6}$$

Then (5.3) follows from inequalities (5.5)-(5.6). Consequently, from (5.1), we prove (1.9) and complete the proof of Theorem 1.1.

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References

[1] D. Bresch, B. Desjardins, C. Lin, On some compressible fluid models: Korteweg, lubrication and shallow water systems, *Comm. Partial Differential Equations* **28** (2003), 843-868.

[2] D. Bian, L. Yao, C. Zhu, Vanishing capillarity limit of the compressible fluid models of Korteweg type to the Navier-Stokes equations, *SIAM J. Math. Anal.* **46** (2014), 1633-1650.

[3] Z. Chen, Asymptotic stability of strong rarefaction waves for the compressible fluid models of Korteweg type, *J. Math. Anal. Appl.* **394** (2012), 438-448.

[4] Z. Chen, X. Chai, B. Dong, H. Zhao, Global classical solutions to the one-dimensional compressible fluid models of Korteweg type with large initial data, *J. Differential Equations* **259** (2015), 4376-4411.

[5] F. Charve, B. Haspot, Existence of global strong solution and vanishing capillarity-viscosity limit in one dimension for the Korteweg system, *SIMA J. Math. Anal.* **45** (2014), 469-494.

[6] Z. Chen, Y. Li, Asymptotic behavior of solutions to an impermeable wall problem of the compressible fluid models of Korteweg type with density-dependent viscosity and capillarity, *SIAM J. Math.Anal.* **53** (2021), 1434-1473.

[7] Z. Chen, L. He, H. Zhao, Nonlinear stability of traveling wave solutions for the compressible fluid models of Korteweg type, *J. Math. Anal. Appl.* **422** (2015), 1213-1234.

[8] Z. Chen, Y. Li, M. Sheng, Asymptotic stability of viscous shock profiles for the 1D compressible Navier-Stokes-Korteweg system with boundary effect, *Dyn. Partial Differential Equations* **16** (2019), 225-251.

[9] R. Danchin, B. Desjardins, Existence of solutions for compressible fluid models of Korteweg type, *Ann. Inst. Henri Poincaré Anal. Non linéaire* **18** (2001), 97-133.

[10] P. Germain, P. LeFloch, Finite energy method for compressible fluids: the Navier-Stokes-Korteweg model, *Comm. Pure Appl. Math.* **69** (2016), 3-61.

[11] J. Dunn, J. Serrin, On the thermodynamics of interstital working, *Arch. Rational Mech. Anal.* **88** (1985), 95-133.

[12] L. Fan, H. Liu, T. Wang, H. Zhao, Inflow problem for the one-dimensional compressible Navier-Stokes equations under large initial perturbation, *J. Differential Equations* **257** (2014), 3521-3553.

[13] B. Haspot, Existence of global weak solution for compressible fluid models of Korteweg type, *J. Math. Fluid Mech.* **13** (2011), 223-249.

[14] B. Haspot, Existence of global strong solution for Korteweg system with large infinite energy initial data, *J. Math. Anal. Appl.* **438** (2016), 395-443.

[15] H. Hattori, D. Li, Solutions for two dimensional system for materials of Korteweg type, *SIAM J. Math. Anal.* **25** (1994), 85-98.

[16] H. Hattori, D. Li, Golobal solutions of a high dimensional system for Korteweg materials, *J. Math. Anal. Appl.* **198** (1996), 84-97.

[17] M. Heida, J. Málek, On compressible Korteweg fluid-like materials, *Internat. J. Engrg. Sci.* **48** (2010), 1313-1324.

[18] H. Hong, Stationary solutions to outflow problem for 1-D compressible fluid models of Korteweg type: Existence, stability and convergence rate, *Nonlinear Anal. Real World Appl.* **53** (2020), 103055.

[19] H. Hong, Stability of stationary solutions and viscous shock wave in the inflow problem for isentropic Navier-Stokes-Korteweg system, *J. Differential Equations* **314** (2022), 518-573.

[20] F. Huang, A. Matsumura, X. Shi, Viscous shock wave and boundary layer solution to an inflow problem for compressible viscous gas, *Comm. Math. Phys.* **239** (2003), 261-285.

[21] F. Huang, A. Matsumura, X. Shi, A gas-solid free boundary problem for a compressible viscous gas, *SIAM J. Math. Anal.* **34** (2003), 1331-1355.

[22] X. Hou, H. Peng, C. Zhu, Global classical solutions to the 3D Navier-Stokes-Korteweg equations with small initial energy, *Anal. Appl.* **16** (2018), 55-84.

[23] D. Korteweg, Sur la forme que prennent les équations des mouvement des fluids si l'on tient comple des forces capillaries par des variations de densité, *Arch. Neerl. Sci. Exactes Nat. Ser. II* **6** (1901), 1-24.

[24] M. Kotschote, Strong solutions for a compressible fluid model of Korteweg type, *Ann. Inst. Henri Poincaré Anal. Non linéaire* **25** (2008), 679-696.

[25] M. Kotschote, Existence and time-asymptotics of global strong solutions to dynamic Korteweg models, *Indiana Univ. Math. J.* **63** (2014), 21-51.

[26] Y. Li, Global existence and optimal decay rate of the compressible Navier-Stokes-Korteweg equations with external force, *J. Math. Anal. Appl.* **388** (2012), 1218-1232.
[27] Y. Li, Z. Chen, Large-time behavior of solutions to an inflow problem for the compressible Navier-Stokes-Korteweg equation in the half space, Prepint, 2021.
[28] Y. Li, Z. Chen, Z. Luo, Stability of the planar rarefaction wave to two dimensional Navier-Stokes-Korteweg equations of compressible fluids, *Math. Meth. Appl. Sci.* **43** (2020), 3307-3330.
[29] Y. Li, Z. Luo, Zero-capillarity-viscosity limit to rarefaction waves for the one-dimensional compressible Navier-Stokes-Korteweg equations, *Math. Meth. Appl. Sci.* **39** (2016), 5513-5528.
[30] Y. Li, Z. Luo, Stability of the planar rarefaction wave to three-dimensional Navier-Stokes-Korteweg equations of compressible fluids, *Nonlinearity* **34** (2021), 2689-2714.
[31] Y. Li, J. Tang, S. Yu, Asymptotic stability of rarefaction wave for the compressible Navier-Stokes-Korteweg equations in the half space, To be appeared in *Proc. Royal Soc. Edinburgh A-Mathematics*, 2022.
[32] Y. Li, P. Zhu, Zero-viscosity-capillarity limit to rarefaction wave with vacuum for the compressible Navier-Stokes-Korteweg equations, *J. Math. Phy.* **61** (2020), 111501.
[33] Y. Li, P. Zhu, Asymptotic stability of the stationary solution to the Navier-Stokes-Korteweg equations of compressible fluids, *Nonlinear Anal.: Real World Applications* **57** (2021), 103193.
[34] A. Matsumura, K. Nishihara, Global asymptotics towards the rarefaction waves for the solutions of viscous p-system with boundary effect, *Quart. Appl. Math.* **58**(2000), 69-83.
[35] A. Matsumura, K. Nishihara, Large-time behavior of solutions to an inflow problem in the half space for a one-dimensional isentropic model system for compressible viscous gas, *Comm. Math. Phys.* **222** (2001), 449-474.
[36] J. Smoller, Shock waves and reaction-diffusion equation, Springer Verlag, New York, Berlin, 1983.
[37] X. Shi, On the stability of rarefaction wave solutions for viscous p-system with boundary effect, *Acta. Math. Appl. Sin. Engl. Ser.* **19** (2003), 341-352.
[38] E. Tsyganov, Global existence and asymptotic convergence of weak solutions for the one-dimensional Navier-Stokes equations with capillarity and nonmonotonic pressure, *J. Differential Equations* **245** (2008), 3936-3955.
[39] Z. Tan, Y. Wang, Large time behavior of solutions to the isentropic compressible fluid models of Korteweg type in $\mathbb{R}^3$, *Comm. Math. Sci.* **10** (2012), 1207-1223.
[40] Z. Tan, H. Wang, J. Xu, Global existence and optimal L^2 decay rate for the strong solutions to the compressible fluid models of Korteweg type, *J. Math. Anal. Appl.* **390** (2012), 181-187.
[41] Z. Tan, R. Zhang, Optimal decay rates of the compressible fluid models of Korteweg type, *Z. Angew. Math. Phys.* **65** (2014), 279-300.
[42] J. Van der Waals, Thermodynamische Theorie der Kapillarität unter Voraussetzung stetiger Dichteänderung, *Z. Phys. Chem.* **13** (1894), 657-725.
[43] Y. Wang, Z. Tan, Optimal decay rates for the compressible fluid models of Korteweg type, *J. Math. Anal. Appl.* **379**(2011), 256-271.
[44] W. Wang, W. Wang, Decay rate of the compressible Navier-Stokes-Korteweg equations with potential force, *Dis. Contin. Dyn. Syst.* **35** (2015), 513-536.

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