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Asymptotic behavior of global solutions to some multidimensional quasilinear hyperbolic systems

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Abstract. For the Cauchy problem of multidimensional quasilinear hyperbolic systems of diagonal form without self-interaction, the global existence of classical solutions with small initial data was shown in [**13**]. In this paper, we will first prove that the global solution will scatter to free linear waves in some weighted L^p sense, then based on it, we will study the rigidity aspect of scattering problem for quasilinear waves.

Contents

1. Introduction and main result

In this paper, we will consider the following first order quasilinear hyperbolic system:

$$\partial_t u_i + \lambda_i \cdot \nabla u_i = \sum_{k=1}^{d} \sum_{j \neq i} \lambda_{ijk} u_j \partial_k u_i + \sum_{j \neq i} c_{ij} u_j u_i, \ i = 1, 2, \cdots, n. \tag{1.1}$$

Here $d \geq 1, n \geq 2,\ u = (u_1, \cdots, u_n)$ is the unknown vector-valued function of $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$, and ∇ stands for the spatial gradient in $\mathbb{R}^d$. $\lambda_i \in \mathbb{R}^d$ $(i =$

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$1, 2, \cdots, n)$ are some different constant vectors, without loss of generality, we can assume

$$|\lambda_i - \lambda_j| \geq \delta_0, \ \forall \ i \neq j, \tag{1.2}$$

where δ_0 is a positive constant. λ_{ijk} and c_{ij} $(i, j = 1, 2, \cdots, n, \ k = 1, 2, \cdots, d)$ are some constants.

The global existence for the Cauchy problem of the system (1.1) in some weighted $L^p(\mathbb{R}^d)(1 < p < +\infty)$ framework was proved in [**13**] (see also a one-dimension result in [**11**]). The asymptotically free property of the global solution in the (non-weighted) $L^p(\mathbb{R}^d)$ sense was also considered in [**13**]. Here the asymptotically free property of the global solution means that the global solution will converge to a solution of the homogeneous linearized system, i.e., the quasilinear waves will scatter to free linear waves.

In this paper, based on the global existence result in [**13**], we will first prove that the global solution will scatter to free linear waves in some weighted $L^p(\mathbb{R}^d)$ sense. Then, based on it, we will further study the rigidity aspect of scattering problem for quasilinear waves. That is, we will show that if the scattering field vanishes, then the global solution will also vanish identically. We point out that this work is inspired by [**6**] and [**5**], in which the scattering fields are introduced and the rigidity aspect of scattering problems are studied, for the MHD equations and one-dimension quasilinear wave equations with null conditions, respectively. We also point out that the approach in this paper is different from the corresponding ones in above two works (see Remark 3.2).

The system (1.1) is a d-dimension transport system of diagonal form without self-interaction in the nonlinearity. For one-dimension semilinear wave equations with null conditions, which can be also viewed as some transport system without self-interaction, the global existence of classical solutions with small initial data was proved in [**8**], which strengthens a former result in [**9**]. The result in [**8**] can be viewed as a one dimensional and semilinear analogue of Christodoulou and Klainerman's pioneering works for the global existence of classical solutions for nonlinear wave equations with null conditions in three space dimensions [**2, 4**], and of Alinhac's global existence result for the case of two space dimensions [**1**] (see also some thorough studies in [**3**] and [**10**]). The global existence result in [**8**] was extended to the quasilinear case in [**12**] by some new observation concerning the null structure in the quasilinear part.

Now we give some notations which will be used in the sequel. Following [**8**] and [**13**], for fixed $1 < p < +\infty$ and $\delta > 0$, we denote the weighted energy with positive weight by

$$E(u(t)) = \sum_{i=1}^{n} \left\| \langle x - \lambda_i t \rangle^{1+\delta} u_i(t, x) \right\|^p_{L^p_x(\mathbb{R}^d)},$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$, and the corresponding k-th $(k = 0, 1, 2, \cdots)$ order energy by

$$E_k(u(t)) = \sum_{|\alpha| \leq k} E(\nabla^\alpha u(t)).$$

Inspired by [**1**], [**7**] and [**11**], following [**13**], we also use the following space-time weighted energy

$$\mathcal{E}(u(t)) = \sum_{i=1}^{n}\sum_{j\neq i}\int_0^t \big\|\langle x-\lambda_j s\rangle^{-\frac{1+\delta}{p}}\langle x-\lambda_i s\rangle^{1+\delta}u_i(s,x)\big\|_{L^p_x(\mathbb{R}^d)}^p ds$$

and the corresponding k-th order version

$$\mathcal{E}_k(u(t)) = \sum_{|\alpha|\leq k}\mathcal{E}(\nabla^\alpha u(t)).$$

Consider the Cauchy problem for (1.1) with initial data

$$t=0: u=f(x). \tag{1.3}$$

Assume that u is the global classical solution to the Cauchy problem (1.1)–(1.3), and denote the right hand side of (1.1) by $F=(F_1,\cdots,F_n)$, i.e.,

$$F_i(t,x) = \sum_{k=1}^{d}\sum_{j\neq i}\lambda_{ijk}u_j\partial_k u_i + \sum_{j\neq i}c_{ij}u_ju_i,\ i=1,2,\cdots,n. \tag{1.4}$$

We (formally) define the scattering field to the Cauchy problem (1.1)–(1.3) by $g=(g_1,\cdots,g_n)$ with

$$g_i(x) = f_i(x) + \int_0^{+\infty}F_i(\tau,x+\lambda_i\tau)d\tau,\ x\in\mathbb{R}^d \tag{1.5}$$

for $i=1,2,\cdots,n$.

The main result in this paper is the following

THEOREM 1.1. *Assume that* (1.2) *is satisfied. For all* $1<p<+\infty,\delta>0$, *there exist positive constants* ε_0 *and* A *such that for any* $0<\varepsilon\leq\varepsilon_0$, *if*

$$\sum_{|\alpha|\leq[\frac{d}{p}]+2}\|\langle x\rangle^{1+\delta}\nabla^\alpha f\|_{L^p_x(\mathbb{R}^d)}\leq\varepsilon,$$

then the Cauchy problem (1.1)–(1.3) *admits a unique global classical solution* u *satisfying the energy bound*

$$\sup_{0\leq s\leq t}E_{[\frac{d}{p}]+2}(u(s)) + \mathcal{E}_{[\frac{d}{p}]+2}(u(t))\leq A^p\varepsilon^p$$

for any $t>0$, *and the corresponding scattering field* g *can be well-defined as in* (1.5) *and satisfies*

$$\lim_{t\to+\infty}\big\|\langle x-\lambda_i t\rangle^{1+\delta}\big(u_i(t,x)-g_i(x-\lambda_i t)\big)\big\|_{L^p_x(\mathbb{R}^d)} = 0,\ i=1,2,\cdots,n.$$

Moreover, if the scattering field g *vanishes, then the global solution itself* u *also vanishes identically.*

2. Preliminaries

Denote

$$\phi(x) = \langle x\rangle^{p+p\delta},\ x\in\mathbb{R}^d. \tag{2.1}$$

It is easy to verify that

$$|\nabla\phi(x)|\leq C\langle x\rangle^{p-1+p\delta}. \tag{2.2}$$

Denote

$$q(\rho) = \int_{-\infty}^{\rho} \langle s\rangle^{-(1+\delta)} ds,\ \ \rho \in \mathbb{R}$$

and

$$\psi_{il}(x) = e^{q((\lambda_l-\lambda_i)\cdot x)},\ \ x \in \mathbb{R}^d.$$

We can verify that

$$(\nabla\psi_{il})(x) = (\lambda_l - \lambda_i)\psi_{il}(x)\langle(\lambda_l - \lambda_i)\cdot x\rangle^{-(1+\delta)}.$$

We note that there exists a positive constant c_1 such that

$$c_1^{-1} \le \psi_{il}(x) \le c_1, \tag{2.3}$$

and noting (1.2), we also have that there exists a positive constant c_2 such that

$$c_2^{-1}\langle x\rangle^{-(1+\delta)} \le (\lambda_l - \lambda_i)\cdot(\nabla\psi_{il})(x)\ \le c_2 \tag{2.4}$$

for any $l \neq i$.

The proofs of the following two lemmas can be found in [**13**].

Lemma 2.1. *For any $\theta \in \mathbb{R}$ and multi-index $\alpha, |\alpha| = k$, we have*

$$\left|\nabla^\alpha(\langle x\rangle^{2\theta})\right| \le C\langle x\rangle^{2\theta}.$$

Lemma 2.2. *Suppose that ϕ_1 and ϕ_2 satisfy*

$$|\nabla^\alpha\phi_1| \le C|\phi_1|,\ \ |\nabla^\alpha\phi_2| \le C|\phi_2|$$

for any multi-index α, and let $1 < p < +\infty$, $|\beta| + |\gamma| \le s, s > d/p, s \in \mathbb{N}$. Then we have

$$\begin{aligned}&\left\|(\phi_1\nabla^\beta f_1)(\phi_2\nabla^\gamma f_2)\right\|_{L^p(\mathbb{R}^d)}\\ &\le C\Big(\sum_{|\alpha|\le s}\left\|\phi_1\nabla^\alpha f_1\right\|_{L^p(\mathbb{R}^d)}\Big)\Big(\sum_{|\alpha|\le s}\left\|\phi_2\nabla^\alpha f_2\right\|_{L^p(\mathbb{R}^d)}\Big).\end{aligned}$$

The following lemma will play a key role in our proof of Theorem 1.1.

Lemma 2.3. *Consider the following linear hyperbolic system:*

$$\partial_t v_i + \lambda_i\cdot\nabla v_i = \sum_{k=1}^{d}\sum_{j\neq i}\lambda_{ijk}u_j\partial_k v_i + G_i,\ \ i = 1, 2, \cdots, n,$$

where $u = (u_1, \cdots, u_n)$ and $G = (G_1, \cdots, G_n)$ are some given vector-valued functions of $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$. Assume that (1.2) is satisfied and

$$\varepsilon_1 \equiv \sup_{0\le s\le t} E^{1/p}_{[\frac{d}{p}]+2}(u(s))$$

is sufficiently small. Then it holds that

$$\begin{aligned}&\sup_{0\le s\le t} E(v(s)) + \mathcal{E}(v(t))\\ &\le CE(u(0)) + C\sum_{i=1}^{n}\int_0^t \left\|\langle x - \lambda_i s\rangle^{p+p\delta}|v_i|^{p-1}G_i\right\|_{L^1_x(\mathbb{R}^d)} ds \end{aligned}\tag{2.5}$$

and

$$E(v(0)) + \mathcal{E}(v(t)) \le CE(u(t)) + C\sum_{i=1}^{n}\int_0^t \big\|\langle x-\lambda_i s\rangle^{p+p\delta}|v_i|^{p-1}G_i\big\|_{L^1_x(\mathbb{R}^d)}ds. \tag{2.6}$$

PROOF. (2.5) has been shown in [**13**]. Here we only give the proof of (2.6). Consider the i-th equation

$$\partial_t v_i + \lambda_i\cdot\nabla v_i = \sum_{k=1}^{d}\sum_{j\neq i}\lambda_{ijk}u_j\partial_k v_i + G_i. \tag{2.7}$$

For $1\le l\le n, l\neq i$, multiply $\psi_{il}(\lambda_l t-x)\phi(x-\lambda_i t)p\mathrm{sgn}(v_i)|v_i|^{p-1}$ on both sides of (2.7). Then Leibniz's rule gives

$$\begin{aligned}
&(\partial_t+\lambda_i\cdot\nabla)\big(\psi_{il}(\lambda_l t-x)\phi(x-\lambda_i t)|v_i|^p\big) - (\lambda_l-\lambda_i)\cdot(\nabla\psi_{il})(\lambda_l t-x)\phi(x-\lambda_i t)|v_i|^p\\
&=\sum_{k=1}^{d}\sum_{j\neq i}\partial_k\big(\psi_{il}(\lambda_l t-x)\phi(x-\lambda_i t)\lambda_{ijk}u_j|v_i|^p\big)\\
&-\sum_{k=1}^{d}\sum_{j\neq i}\psi_{il}(\lambda_l t-x)\phi(x-\lambda_i t)\lambda_{ijk}(\partial_k u_j)|v_i|^p\\
&+\sum_{k=1}^{d}\sum_{j\neq i}\big((\partial_k\psi_{il})(\lambda_l t-x)\phi(x-\lambda_i t)-\psi_{il}(\lambda_l t-x)(\partial_k\phi)(x-\lambda_i t)\big)\lambda_{ijk}u_j|v_i|^p\\
&+\psi_{il}(\lambda_l t-x)\phi(x-\lambda_i t)p\mathrm{sgn}(v_i)|v_i|^{p-1}G_i.
\end{aligned}$$

Then by the above differential equality, the divergence theorem, (2.1), (2.2), (2.3) and (2.4), we get

$$\begin{aligned}
&\int_{\mathbb{R}^d}\langle x\rangle^{p+p\delta}|v_i(0,x)|^p dx + \int_0^t\int_{\mathbb{R}^d}\langle x-\lambda_l s\rangle^{-(1+\delta)}\langle x-\lambda_i s\rangle^{p+p\delta}|v_i|^p\ dxds\\
&\le C\int_{\mathbb{R}^d}\langle x-\lambda_i t\rangle^{p+p\delta}|v_i(t,x)|^p dx\\
&+C\sum_{j\neq i}\int_0^t\int_{\mathbb{R}^d}\langle x-\lambda_i s\rangle^{p+p\delta}(|\nabla u_j|+|u_j|)|v_i|^p\ dxds\\
&+C\int_0^t\int_{\mathbb{R}^d}\langle x-\lambda_i s\rangle^{p+p\delta}|v_i|^{p-1}|G_i|\ dxds.
\end{aligned} \tag{2.8}$$

Now we estimate the second term on the right hand side of (2.8). For $j\neq i$, it follows from Sobolev embedding $W^{[\frac{d}{p}]+1,p}(\mathbb{R}^d)\hookrightarrow L^\infty(\mathbb{R}^d)$ and Lemma 2.1 that

$$\begin{aligned}
&\|\langle x-\lambda_i s\rangle^{p+p\delta}(|\nabla u_j|+|u_j|)|v_i|^p\|_{L^1_{s,x}}\\
&\le\|\langle x-\lambda_j s\rangle^{-\frac{1+\delta}{p}}\langle x-\lambda_i s\rangle^{1+\delta}v_i\|^p_{L^p_{s,x}}\|\langle x-\lambda_j s\rangle^{1+\delta}(|\nabla u_j|+|u_j|)\|_{L^\infty_{s,x}}\\
&\le C\sup_{0\le s\le t}E^{1/p}_{[\frac{d}{p}]+2}(u(s))\mathcal{E}(v(t)).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\langle x\rangle^{p+p\delta}|v_i(0,x)|^p dx+\sum_{l\neq i}\int_0^t\int_{\mathbb{R}^d}\langle x-\lambda_l s\rangle^{-(1+\delta)}\langle x-\lambda_i s\rangle^{p+p\delta}|v_i|^p\ dxds\\
&\leq C\int_{\mathbb{R}^d}\langle x-\lambda_i t\rangle^{p+p\delta}|v_i(t,x)|^p dx+C\sup_{0\leq s\leq t}E^{1/p}_{[\frac{d}{p}]+2}(u(s))\mathcal{E}(v(t))\\
&+C\sum_{i=1}^n\int_0^t\int_{\mathbb{R}^d}\langle x-\lambda_i s\rangle^{p+p\delta}|v_i|^{p-1}|G_i|\ dxds,
\end{aligned}$$

which implies

$$\begin{aligned}
&E(v(0))+\mathcal{E}(v(t))\\
&\leq CE(v(t))+C\varepsilon_1\mathcal{E}(v(t))+C\sum_{i=1}^n\int_0^t\left\|\langle x-\lambda_i s\rangle^{p+p\delta}|v_i|^{p-1}G_i\right\|_{L^1_x(\mathbb{R}^d)}ds.
\end{aligned}$$

If ε_1 is sufficiently small, we can obtain (2.5). □

3. Proof of Theorem 1.1

Now we will prove Theorem 1.1. The global existence part in Theorem 1.1 has been shown in [**13**]. Because it is closely related to the asymptotic behavior part (scattering and rigidity), we will give a sketch of the proof for it. Assume that u is a classical solution to the Cauchy problem (1.1)–(1.3). We will first prove that there exist positive constants ε_0 and A such that

$$\sup_{0\leq s\leq t}E_{[\frac{d}{p}]+2}(u(s))+\mathcal{E}_{[\frac{d}{p}]+2}(u(t))\leq A^p\varepsilon^p \tag{3.1}$$

under the assumption

$$\sup_{0\leq s\leq t}E_{[\frac{d}{p}]+2}(u(s))+\mathcal{E}_{[\frac{d}{p}]+2}(u(t))\leq 2^pA^p\varepsilon^p, \tag{3.2}$$

where $0<\varepsilon\leq\varepsilon_0$. Then we will give the proof of asymptotic behavior part of Theorem 1.1, based on (3.1) and other related estimates.

3.1. Energy estimates. For $|\alpha|\leq[\frac{d}{p}]+2$, applying of Leibniz's rule results in

$$\partial_t(\nabla^\alpha u_i)+\lambda_i\cdot\nabla(\nabla^\alpha u_i)=\sum_{k=1}^d\sum_{j\neq i}\lambda_{ijk}u_j\partial_k(\nabla^\alpha u_i)+G_{i\alpha},$$

where

$$\begin{aligned}
G_{i\alpha}=&\sum_{k=1}^d\sum_{j\neq i}\sum_{\substack{\beta+\gamma=\alpha\\ \beta\neq 0}}\lambda_{ijk\alpha\beta\gamma}(\nabla^\beta u_j)(\nabla^\gamma\partial_k u_i)\\
&+\sum_{j\neq i}\sum_{\beta+\gamma=\alpha}c_{ij\alpha\beta\gamma}(\nabla^\beta u_j)(\nabla^\gamma u_i),
\end{aligned}\tag{3.3}$$

$\lambda_{ijk\alpha\beta\gamma}$ and $c_{ij\alpha\beta\gamma}$ are some constants.

By Lemma 2.3 we obtain

$$\begin{aligned}&\sup_{0\le s\le t} E_{[\frac{d}{p}]+2}(u(s))+\mathcal{E}_{[\frac{d}{p}]+2}(u(t))\\&\le CE_{[\frac{d}{p}]+2}(u(0))+C\sum_{i=1}^{n}\sum_{|\alpha|\le[\frac{d}{p}]+2}\int_0^t\|\langle x-\lambda_i s\rangle^{p+p\delta}|\nabla^\alpha u_i|^{p-1}G_{i\alpha}\|_{L^1_x(\mathbb{R}^d)}ds\end{aligned}$$

and

$$\begin{aligned}&E(u(0))+\mathcal{E}(u(t))\\&\le CE(u(t))+C\sum_{i=1}^{n}\int_0^t\|\langle x-\lambda_i s\rangle^{p+p\delta}|u_i|^{p-1}G_{i0}\|_{L^1_x(\mathbb{R}^d)}ds.\end{aligned}$$

(3.3) implies

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{|\alpha|\le[\frac{d}{p}]+2}\int_0^t\|\langle x-\lambda_i s\rangle^{p+p\delta}|\nabla^\alpha u_i|^{p-1}G_{i\alpha}\|_{L^1_x(\mathbb{R}^d)}ds\\&\le C\sum_{i=1}^{n}\sum_{j\ne i}\sum_{|\alpha|\le[\frac{d}{p}]+2}\sum_{\substack{\beta+\gamma=\alpha\\ \beta\ne0}}\|\langle x-\lambda_i s\rangle^{p+p\delta}|\nabla^\alpha u_i|^{p-1}(\nabla^\beta u_j)(\nabla^{\gamma+1}u_i)\|_{L^1_{s,x}}\\&+C\sum_{i=1}^{n}\sum_{j\ne i}\sum_{|\alpha|\le[\frac{d}{p}]+2}\sum_{\beta+\gamma=\alpha}\|\langle x-\lambda_i s\rangle^{p+p\delta}|\nabla^\alpha u_i|^{p-1}(\nabla^\beta u_j)(\nabla^{\gamma}u_i)\|_{L^1_{s,x}}\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}&\sum_{i=1}^{n}\int_0^t\|\langle x-\lambda_i s\rangle^{p+p\delta}|u_i|^{p-1}G_{i0}\|_{L^1_x(\mathbb{R}^d)}ds\\&\le C\sum_{i=1}^{n}\sum_{j\ne i}\int_0^t\|\langle x-\lambda_i s\rangle^{p+p\delta}|u_i|^{p}u_j\|_{L^1_x(\mathbb{R}^d)}ds.\end{aligned}\tag{3.5}$$

For $i=1,2,\cdots,n, j\ne i, |\alpha|\le[\frac{d}{p}]+2, \beta+\gamma=\alpha, \beta\ne0$, it follows from Hölder inequality that

$$\begin{aligned}&\|\langle x-\lambda_i s\rangle^{p+p\delta}|\nabla^\alpha u_i|^{p-1}(\nabla^\beta u_j)(\nabla^{\gamma+1}u_i)\|_{L^1_{s,x}}\\&\le\|\langle x-\lambda_j s\rangle^{-\frac{1+\delta}{p}}\langle x-\lambda_i s\rangle^{1+\delta}|\nabla^\alpha u_i|\|^{p-1}_{L^p_{s,x}}\\&\cdot\|\langle x-\lambda_j s\rangle^{1+\delta}(\nabla^{\beta-1}\nabla u_j)\langle x-\lambda_j s\rangle^{-\frac{1+\delta}{p}}\langle x-\lambda_i s\rangle^{1+\delta}(\nabla^\gamma\nabla u_i)\|_{L^p_{s,x}}.\end{aligned}\tag{3.6}$$

It is obvious that Lemma 2.1 implies

$$\left|\nabla^\alpha(\langle x-\lambda_j s\rangle^{1+\delta})\right|\le C\langle x-\lambda_j s\rangle^{1+\delta}$$

and

$$\left|\nabla^\alpha(\langle x-\lambda_j s\rangle^{-\frac{1+\delta}{p}}\langle x-\lambda_i s\rangle^{1+\delta})\right|\le C\langle x-\lambda_j s\rangle^{-\frac{1+\delta}{p}}\langle x-\lambda_i s\rangle^{1+\delta}.$$

Noting $|\beta|-1+|\gamma|\leq[\frac{d}{p}]+1$, by Lemma 2.2 we get

$$\begin{aligned}&\|\langle x-\lambda_j s\rangle^{1+\delta}(\nabla^{\beta-1}\nabla u_j)\langle x-\lambda_j s\rangle^{-\frac{1+\delta}{p}}\langle x-\lambda_i s\rangle^{1+\delta}(\nabla^{\gamma}\nabla u_i)\|_{L^p_{s,x}}\\&\leq C\Big(\sum_{|\alpha|\leq[\frac{d}{p}]+1}\sup_{0\leq s\leq t}\big\|\langle x-\lambda_j s\rangle^{1+\delta}\nabla^{\alpha}\nabla u_j\big\|_{L^p_x(\mathbb{R}^d)}\Big)\\&\cdot\Big(\sum_{|\alpha|\leq[\frac{d}{p}]+1}\big\|\langle x-\lambda_j s\rangle^{-\frac{1+\delta}{p}}\langle x-\lambda_i s\rangle^{1+\delta}\nabla^{\alpha}\nabla u_i\big\|_{L^p_{s,x}}\Big).\end{aligned}\tag{3.7}$$

Thus the combination of (3.6) and (3.7) gives the following upper bound for the first part on the right hand side of (3.4)

$$\mathcal{E}_{[\frac{d}{p}]+2}(u(t))\sup_{0\leq s\leq t}E^{1/p}_{[\frac{d}{p}]+2}(u(s)).$$

The second part on the right hand side of (3.4) can be estimated by the same way and admits the same bound.

While for the right hand side of (3.5), for $i=1,2,\cdots,n, j\neq i$, by Sobolev embedding $W^{[\frac{d}{p}]+1,p}(\mathbb{R}^d)\hookrightarrow L^{\infty}(\mathbb{R}^d)$ and Lemma 2.1 we get

$$\begin{aligned}&\|\langle x-\lambda_i s\rangle^{p+p\delta}|u_i|^p u_j\|_{L^1_{s,x}}\\&\leq\|\langle x-\lambda_j s\rangle^{-\frac{1+\delta}{p}}\langle x-\lambda_i s\rangle^{1+\delta}u_i\|^p_{L^p_{s,x}}\|\langle x-\lambda_j s\rangle^{1+\delta}u_j\|_{L^\infty_{s,x}}\\&\leq C\sup_{0\leq s\leq t}E^{1/p}_{[\frac{d}{p}]+2}(u(s))\mathcal{E}(u(t)).\end{aligned}$$

Finally, we have

$$\begin{aligned}&\sup_{0\leq s\leq t}E_{[\frac{d}{p}]+2}(u(s))+\mathcal{E}_{[\frac{d}{p}]+2}(u(t))\\&\leq C_1E_{[\frac{d}{p}]+2}(u(0))+C_2\mathcal{E}_{[\frac{d}{p}]+2}(u(t))\sup_{0\leq s\leq t}E^{1/p}_{[\frac{d}{p}]+2}(u(s))\end{aligned}\tag{3.8}$$

and

$$E(u(0))+\mathcal{E}(u(t))\leq C_1E(u(t))+C_2\sup_{0\leq s\leq t}E^{1/p}_{[\frac{d}{p}]+2}(u(s))\mathcal{E}(u(t)).\tag{3.9}$$

3.2. Global existence. Under the assumption (3.2), by (3.8) we have

$$\sup_{0\leq s\leq t}E_{[\frac{d}{p}]+2}(u(s))+\mathcal{E}_{[\frac{d}{p}]+2}(u(t))\leq C_1\varepsilon^p+2^{p+1}C_2A^{p+1}\varepsilon^{p+1}.$$

Taking $A^p=2C_1$ and ε_0 so small that

$$2^{p+2}C_2A\varepsilon_0\leq 1,\tag{3.10}$$

for any ε with $0<\varepsilon\leq\varepsilon_0$, we have

$$\sup_{0\leq s\leq t}E_{[\frac{d}{p}]+2}(u(s))+\mathcal{E}_{[\frac{d}{p}]+2}(u(t))\leq A^p\varepsilon^p.\tag{3.11}$$

This completes the proof of global existence part of Theorem 1.1.

3.3. Scattering. It follows from (1.5) that

$$g_i(x-\lambda_i t)=f_i(x-\lambda_i t)+\int_0^{+\infty}F_i(\tau,x-\lambda_i t+\lambda_i\tau)d\tau.$$

We also have

$$u_i(t,x)=f_i(x-\lambda_i t)+\int_0^{t}F_i(\tau,x-\lambda_i t+\mu_i\tau)d\tau.$$

Thus it holds that

$$\begin{aligned}&\langle x-\lambda_i t\rangle^{1+\delta}\big(u_i(t,x)-g_i(x-\lambda_i t)\big)\\&=-\int_t^{+\infty}\langle x-\lambda_i t\rangle^{1+\delta}F_i(\tau,x-\lambda_i t+\lambda_i\tau)d\tau.\end{aligned}$$

Then we have

$$\begin{aligned}&\big\|\langle x-\lambda_i t\rangle^{1+\delta}\big(u_i(t,x)-g_i(x-\lambda_i t)\big)\big\|_{L^p_x(\mathbb{R}^d)}\\&\leq\Big\|\int_t^{+\infty}\langle x\rangle^{1+\delta}|F_i(\tau,x+\lambda_i\tau)|d\tau\Big\|_{L^p_x(\mathbb{R}^d)}.\end{aligned}$$

Noting (1.4), we get

$$\begin{aligned}&\int_t^{+\infty}\langle x\rangle^{1+\delta}|F_i(\tau,x+\lambda_i\tau)|d\tau\\&\leq C\sum_{j\neq i}\int_t^{+\infty}\langle x\rangle^{1+\delta}|u_j(\tau,x+\lambda_i\tau)||\nabla u_i(\tau,x+\lambda_i\tau)|d\tau\\&+C\sum_{j\neq i}\int_t^{+\infty}\langle x\rangle^{1+\delta}|u_j(\tau,x+\lambda_i\tau)||u_i(\tau,x+\lambda_i\tau)|d\tau.\end{aligned}$$

Take p' such that $\frac{1}{p}+\frac{1}{p'}=1$. For $j\neq i$, by Hölder inequality we have

$$\begin{aligned}&\int_t^{+\infty}\langle x\rangle^{1+\delta}|u_j(\tau,x+\lambda_i\tau)||\nabla u_i(\tau,x+\lambda_i\tau)|d\tau\\&\leq\Big(\int_t^{+\infty}\big|\langle x+(\lambda_i-\lambda_j)\tau\rangle^{\frac{1+\delta}{p'}}|u_j(\tau,x+\lambda_i\tau)|\langle x\rangle^{1+\delta}|\nabla u_i(\tau,x+\lambda_i\tau)|\big|^p d\tau\Big)^{\frac{1}{p}}\\&\cdot\Big(\int_t^{+\infty}\langle x+(\lambda_i-\lambda_j)\tau\rangle^{-1-\delta}d\tau\Big)^{\frac{1}{p'}}\\&\leq C\Big(\int_t^{+\infty}\big|\langle x+(\lambda_i-\lambda_j)\tau\rangle^{\frac{1+\delta}{p'}}|u_j(\tau,x+\lambda_i\tau)|\langle x\rangle^{1+\delta}|\nabla u_i(\tau,x+\lambda_i\tau)|\big|^p d\tau\Big)^{\frac{1}{p}}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_t^{+\infty}\langle x\rangle^{1+\delta}|u_j(\tau,x+\lambda_i\tau)||u_i(\tau,x+\lambda_i\tau)|d\tau\\&\leq C\Big(\int_t^{+\infty}\big|\langle x+(\lambda_i-\lambda_j)\tau\rangle^{\frac{1+\delta}{p'}}|u_j(\tau,x+\lambda_i\tau)|\langle x\rangle^{1+\delta}|u_i(\tau,x+\lambda_i\tau)|\big|^p d\tau\Big)^{\frac{1}{p}}.\end{aligned}$$

Thus we get

$$\begin{aligned}
&\big\|\langle x-\lambda_i t\rangle^{1+\delta}\big(u_i(t,x)-g_i(x-\lambda_i t)\big)\big\|^p_{L^p_x(\mathbb{R}^d)}\\
&\leq C\sum_{j\neq i}\int_t^{+\infty}\int_{\mathbb{R}^d}\big|\langle x+(\lambda_i-\lambda_j)\tau\rangle^{\frac{1+\delta}{p'}}|u_j(\tau,x+\lambda_i\tau)|\langle x\rangle^{1+\delta}|\nabla u_i(\tau,x+\lambda_i\tau)|\big|^p dxd\tau\\
&+C\sum_{j\neq i}\int_t^{+\infty}\int_{\mathbb{R}^d}\big|\langle x+(\lambda_i-\lambda_j)\tau\rangle^{\frac{1+\delta}{p'}}|u_j(\tau,x+\lambda_i\tau)|\langle x\rangle^{1+\delta}|u_i(\tau,x+\lambda_i\tau)|\big|^p dxd\tau.
\end{aligned}$$

Thus, in order to prove

$$\lim_{t\to+\infty}\big\|\langle x-\lambda_i t\rangle^{1+\delta}\big(u_i(t,x)-g_i(x-\lambda_i t)\big)\big\|_{L^p_x(\mathbb{R}^d)}=0,\ i=1,2,\cdots,n, \tag{3.12}$$

we need to show that for $i=1,2,\cdots,n, j\neq i$,

$$\begin{aligned}
&\big\|\langle x+(\lambda_i-\lambda_j)t\rangle^{\frac{1+\delta}{p'}}|u_j(t,x+\lambda_i t)|\langle x\rangle^{1+\delta}|\nabla u_i(t,x+\lambda_i t)|\big\|_{L^p_{t,x}(\mathbb{R}^+\times\mathbb{R}^d)}\\
&\qquad<+\infty
\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}
&\big\|\langle x+(\lambda_i-\lambda_j)t\rangle^{\frac{1+\delta}{p'}}|u_j(t,x+\lambda_i t)|\langle x\rangle^{1+\delta}|u_i(t,x+\lambda_i t)|\big\|_{L^p_{t,x}(\mathbb{R}^+\times\mathbb{R}^d)}\\
&\qquad<+\infty.
\end{aligned}\tag{3.14}$$

In fact, for $i=1,2,\cdots,n, j\neq i$, by Sobolev embedding $W^{[\frac{d}{p}]+1,p}(\mathbb{R}^d)\hookrightarrow L^\infty(\mathbb{R}^d)$ and Lemma 2.1 we get

$$\begin{aligned}
&\big\|\langle x+(\lambda_i-\lambda_j)t\rangle^{\frac{1+\delta}{p'}}|u_j(t,x+\lambda_i t)|\langle x\rangle^{1+\delta}|\nabla u_i(t,x+\lambda_i t)|\big\|_{L^p_{t,x}(\mathbb{R}^+\times\mathbb{R}^d)}\\
&=\big\|\langle x-\lambda_j t\rangle^{\frac{1+\delta}{p'}}|u_j(t,x)|\langle x-\lambda_i t\rangle^{1+\delta}|\nabla u_i(t,x)|\big\|_{L^p_{t,x}(\mathbb{R}^+\times\mathbb{R}^d)}\\
&=\big\|\langle x-\lambda_j t\rangle^{1+\delta}|u_j(t,x)|\langle x-\lambda_j t\rangle^{-\frac{1+\delta}{p}}\langle x-\lambda_i t\rangle^{1+\delta}|\nabla u_i(t,x)|\big\|_{L^p_{t,x}(\mathbb{R}^+\times\mathbb{R}^d)}\\
&\leq\big\|\langle x-\lambda_j t\rangle^{-\frac{1+\delta}{p}}\langle x-\lambda_i t\rangle^{1+\delta}\nabla u_i(t,x)\big\|_{L^p_{t,x}(\mathbb{R}^+\times\mathbb{R}^d)}\big\|\langle x-\lambda_j t\rangle^{1+\delta}u_j(t,x)\big\|_{L^\infty_{t,x}(\mathbb{R}^+\times\mathbb{R}^d)}\\
&\leq\sup_{0\leq t<+\infty}\mathcal{E}^{1/p}_{[\frac{d}{p}]+2}(u(t))\sup_{0\leq t<+\infty}E^{1/p}_{[\frac{d}{p}]+2}(u(t))\leq A^2\varepsilon^2.
\end{aligned}$$

Thus (3.13) holds. (3.14) can be shown similarly. Now we have shown (3.12).

Remark 3.1. By some similar way, we can also show

$$\lim_{t\to+\infty}\sum_{|\alpha|\leq[\frac{d}{p}]+1}\big\|\langle x-\lambda_i t\rangle^{1+\delta}\nabla^\alpha\big(u_i(t,x)-g_i(x-\lambda_i t)\big)\big\|_{L^p_x(\mathbb{R}^d)}=0,\ i=1,2,\cdots,n.$$

3.4. Rigidity. Now assume that the scattering filed g vanishes. Then by (3.12), we get

$$\lim_{t\to+\infty}\big\|\langle x-\lambda_i t\rangle^{1+\delta}u_i(t,x)\big\|_{L^p_x(\mathbb{R}^d)}=0,\ i=1,2,\cdots,n,$$

i.e.,

$$\lim_{t\to+\infty}E(u(t))=0.$$

Thus for any $\overline{\varepsilon}>0$, there exists sufficiently large $t_1>0$ such that

$$E(u(t_1))\leq\overline{\varepsilon}. \tag{3.15}$$

In view of (3.9), (3.15), (3.11) and (3.10), we have

$$E(u(0))+\mathcal{E}(u(t_1))\leq C_1E(u(t_1))+C_2\sup_{0\leq s\leq t_1}E^{1/p}_{[\frac{d}{p}]+2}(u(s))\mathcal{E}(u(t_1))$$

$$\leq C_1\overline{\varepsilon}+C_2A\varepsilon\mathcal{E}(u(t_1))\leq C_1\overline{\varepsilon}+C_2A\varepsilon_0\mathcal{E}(u(t_1))\leq C_1\overline{\varepsilon}+\frac{1}{2}\mathcal{E}(u(t_1)),$$

which implies

$$E(u(0))+\frac{1}{2}\mathcal{E}(u(t_1))\leq C_1\overline{\varepsilon}.$$

By the arbitrariness of $\overline{\varepsilon}>0$, we have

$$E(u(0))=0,$$

which gives

$$u(0,x)=f(x)=0,\ \ x\in\mathbb{R}^d.$$

By the uniqueness of global classical solution to the Cauchy problem (1.1)–(1.3), we get

$$u(t,x)=0,\ \ t\geq 0,\ \ x\in\mathbb{R}^d.$$

REMARK 3.2. From the above proof, we can see that our rigidity result follows from the energy estimate (2.6), which relies on the time reversal invariance of the system (1.1). Thus we do not need to introduce some position parameters, which are used in [**6**] and [**5**].

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