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Global existence and asymptotic behavior of solutions for a fractional chemotaxis-Navier-Stokes system

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Abstract. We consider a fractional chemotaxis-Navier-Stokes model in the whole space $\mathbb{R}^N$, $N \geq 2$, with a time-fractional variation in the Caputo sense, a fractional self-diffusion for the physical variables and a fractional dissipation mechanism for the chemoattraction process. We prove the existence and uniqueness of global mild solutions with small initial data in a larger class of critical spaces of Besov-Morrey type. Our result extend the well-posedness ones in the classical (no fractional regime) obtained by Postigo and Ferreira [**16**]. We also prove the long-time asymptotic stability of solutions.

Contents

1. Introduction

Chemotaxis corresponds to the biological mechanism of the movement of living organisms in response to a chemical stimulus, which can be given towards a higher concentration of the chemical concentration (attractive) or towards regions of lower concentration (repulsive). In addition, some experimental studies, as reported in [**17**] and [**28**], have indicated that the interaction between cells and fluids affects

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considerably the dynamics. Some examples of this kind of behaviour have been reported in striking experiments showing spontaneous formation of plume-like aggregates in populations of Bacillus subtilis suspended in sessile water drops [**31**]. This kind of chemotaxis-fluid interaction can be modeled through the following chemotaxis-Navier-Stokes system:

$$\left\{\begin{array}{l} n_t + u\cdot\nabla n = D_n\Delta n - \chi\nabla\cdot(n\nabla c),\\ c_t + u\cdot\nabla c = D_c\Delta c - \rho nc,\\ u_t + (u\cdot\nabla)u = D_u\Delta u - \nabla\pi - n\nabla\phi,\\ \nabla\cdot u = 0, \end{array}\right. \tag{1.1}$$

where $n = n(x,t) \geq 0$, $c = c(x,t) \geq 0$, $\pi(x,t)$ and $u(x,t)$ denote respectively the cell density, the concentration of the chemical signal, the pressure, and the velocity of the fluid at position $x \in \Omega \subseteq \mathbb{R}^N$, $N = 2,3$, and time $t > 0$. In this model it is assumed that the chemical signal is consumed with a rate ρ proportional to the amount of cells. Both the cells and chemical substance are transported by a viscous incompressible fluid, which, in turn, is driven by gravitation-induced force $-n\nabla\phi$, modeling the effects due to density variations caused by cell aggregation. The additional parameters χ, D_n, D_c and D_u are positive constants representing the chemotactic sensitivity, the cell diffusion coefficient, the chemical diffusion coefficient, and the viscosity of fluid, respectively.

Taking into account that the behavior of most biological systems has memory properties, which are neglected when an integer-order time derivative is assumed, a natural consideration is substitute the first order in time derivative in (1.1) by a temporal derivative in a fractional framework, introducing a nonlocal delay in time for the moving population. On the other hand, system (1.1) assumes that the cell diffusion is not affected by the nonlocal spatial behaviour of the organisms. However, recent observations indicate that in certain cases of chemotactic motion, organisms develop alternative search strategies, particularly when chemo-attractants, food, or other targets are sparse or rare. This means that the trajectories of the population of organisms are better described by the so called Lévy flights than Brownian motion (see [**12**] and [**20**]). Lévy flights mechanism has been suggested in several biological contexts, including immune cells, ecology, and human populations (c.f. [**13**] and references therein for a deeper discussion). Therefore, motivated by the previous observations, we propose and analyze theoretically the following chemotaxis-Navier-Stokes system in a fractional setting $\mathbb{R}^N \times (0,\infty)$, $N \geq 2$:

$$\left\{\begin{array}{l} {}^c\mathcal{D}_t^\beta n + u\cdot\nabla n = -D_n\left(-\Delta\right)^{\alpha/2} n - \chi\nabla\cdot(n\nabla((-\Delta)^{-\theta/2}c)),\\ {}^c\mathcal{D}_t^\beta c + u\cdot\nabla c = -D_c\left(-\Delta\right)^{\alpha/2} c - \rho nc,\\ {}^c\mathcal{D}_t^\beta u + (u\cdot\nabla)u = -D_u\left(-\Delta\right)^{\alpha/2} u - \nabla\pi - n\nabla\phi,\\ \nabla\cdot u = 0, \end{array}\right. \tag{1.2}$$

where ${}^c\mathcal{D}_t^\beta$ denotes the Caputo fractional derivative of order $\beta \in (0,1]$. We recall that the Caputo fractional derivative of order β of f is defined by

$${}^c\mathcal{D}_t^\beta f(t) := \frac{d}{dt}\left\{I_t^{1-\beta}[f(t) - f(0)]\right\},$$

for $f \in L^1(0,T;X)$, $T>0$, X a Banach space, and I_t^β denoting the Riemann-Liouville fractional integral of order β of f given by

$$I_t^\beta f(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\, ds, \qquad t \in [0,T].$$

Further, in (1.2), $(-\Delta)^{\alpha/2}$, $\alpha \in (0,2]$, denotes the fractional Laplacian operator of order $\alpha/2$ defined by $(-\Delta)^{\alpha/2} f(x) = \mathcal{F}^{-1}(|\xi|^\alpha \hat{f}(\xi))(x)$, where $\hat{f}(\xi) = \mathcal{F}(f)(\xi)$ and $\mathcal{F}^{-1}(f)(\xi)$ denote the Fourier transform and the inverse Fourier transform of f, respectively. In addition, $\nabla\left((-\Delta)^{-\theta/2} c\right)$, $\theta \in [0,N)$, is also a nonlocal term which can be alternatively represented by $K(x) * c$, $K(x) \sim \frac{x}{|x|^{N-\theta}}$. The case $\beta = 1$, $\alpha = 2$ and $\theta = 0$ formally corresponds to the chemotaxis-Navier-Stokes system (1.1). For simplicity in the notation and without loss of generality we assume that the value of the physical parameters is 1, that is, $D_n = \chi = D_c = D_u = \rho = 1$.

In recent years, the analysis of existence, uniqueness, regularity, and long-time asymptotic behavior solutions of chemotaxis-Navier-Stokes models has attracted the attention of several authors, due to the importance in the biological context and its theoretical challenges. For results on existence, uniqueness, regularity and long time asymptotic behavior of solutions for (1.2) in the case of bounded domains we refer [**2, 3, 9, 23, 18, 26, 10, 29, 30, 31**] and references therein. On the other hand, in the whole space $\mathbb{R}^N$, $N = 2,3$, the local and global-in-time existence of solutions for (1.2) has been analyzed in [**5, 6, 7, 21, 34**], and some references therein. In particular, in [**21**] the authors obtained a class of small global solutions in the framework of weak-L^p spaces including double attraction nonlinear terms in the density equation. A 3D-local well-posedness result for initial data in the non-homogeneous Besov spaces class $B^s_{p,r} \times B^{s+1}_{p,r} \times B^{s+1}_{p,r}$ where $1 < p < \infty$, $1 \le r \le \infty$, and $s > 3/p + 1$, was proved in [**34**] and, in [**7**] the authors obtained a result of existence of local-in-time solutions for large initial data, as well as global-in-time existence for small initial data and some smallness condition on the gravitational term, in critical homogeneous Besov spaces. Later, in [**5**] the authors proved an extension criterion for local-in-time solutions. Even in the 3D case, in [**6**] the existence of global solutions for small initial data in the critical homogeneous Besov space $\dot{B}^{-2+3/p}_{p,1} \times \dot{B}^{3/p}_{p,1} \times \dot{B}^{-1+3/p}_{p,1}$ with $1 \le p < 3$ was proved. More recently, in [**11, 16**], the authors obtained results of existence and asymptotic behavior of small global solutions in different classes of Besov-Morrey spaces type spaces.

Previous results are related to the non-fractional model (1.1). The aim contribution of this paper is to analyze the existence and stability of global solutions for the full fractional system (1.2). To the best of our knowledge, the fractional system (1.2) has not been previously analyzed. However, the time-fractional Navier-Stokes equations (with integer diffusion), which corresponds to a submodel of (1.2), have already been studied in [**4**] in the framework of L^p-spaces. As point out in [**4**], the interest in the analysis of the Navier-Stokes equations was reflected in [**25**] considering smooth solutions and establishing several results about the regularity of their fractional derivative. Subsequently, in [**33**] the author proved the Shinbrot conjecture, which stated that any weak solution of Navier-Stokes equations has any fractional derivative of order less than or equal to 1/2. Some results on the Navier-Stokes system with fractional diffusion were obtained in [**15**]. On the other hand, the fractional Keller-Segel system, even in the case in which the consumption term nc is replaced by $n-c$, also has not been extensively studied. In fact, we only known the references [**1, 8**] where the authors studied the global existence and long-time behaviour of solutions for the time-fractional Keller-Segel system, that is, assuming $\alpha=2,\ \theta=0$ and $\beta\in(0,1)$, and considering the production-degradation regime $n-c$ in the concentration equation. In [**1**], the authors analyzed the global existence and long-time behaviour of solutions considering small initial data in the Besov-Morrey space $\mathcal{N}^{-b}_{r,\lambda,\infty}\times\dot{B}_{\infty,\infty}$ with $N\geq 2$, $0\leq\lambda\leq N-2$, $b=2-\frac{N-\lambda}{r}$ and $\frac{N-\lambda}{2}<r<N-\lambda$. The existence space in [**1**] is based on auxiliary norms like in [**32**]. Some regularity properties of solution for (1.2), assuming $\alpha=2,\ \ \theta=0$ and $\beta\in(0,1)$, and initial data in $L^N\cap L^{N/2}\cap L^\infty\times\dot{B}_{\infty,\infty}$ were obtained in [**8**].

Our aim is to analyze the existence, uniqueness, and asymptotic behaviour of global solutions for the spatio-temporal fractional chemotaxis-Navier-Stokes system (1.2) in the framework of critical Besov-Morrey spaces. Before establishing our main results, briefly we recall some preliminaries results about Morrey and homogeneous Besov-Morrey spaces. For further details see [**19, 22, 24, 27**].

Definition 1.1. *For $1\leq p_1\leq p\leq\infty$ the Morrey space $\mathcal{M}^p_{p_1}=\mathcal{M}^p_{p_1}(\mathbb{R}^N)$ is defined as the set of all measurable functions $u:\mathbb{R}^N\to\mathbb{R}$ such that*

$$\|u\|_{\mathcal{M}^p_{p_1}}=\sup_{x_0\in\mathbb{R}^N}\sup_{R>0}R^{\frac{N}{p}-\frac{N}{p_1}}\|u\|_{L^{p_1}(B(x_0,R))}<\infty,$$

where $B(x_0,R)$ is the closed ball in $\mathbb{R}^N$ with center x_0 and radius R. The space $\mathcal{M}^p_{p_1}$, endowed with the norm $\|\cdot\|_{\mathcal{M}^p_{p_1}}$, is a Banach space. For $1\leq p<\infty$ it holds that $\mathcal{M}^p_p=L^p$. In the case $p=p_1=\infty$, it holds that $\mathcal{M}^\infty_\infty=L^\infty$.

Next we recall the Hölder inequality in the framework of Morrey spaces.

Lemma 1.2. *(Hölder inequality) Let $1\leq p_1\leq p\leq\infty, 1\leq q_1\leq q\leq\infty$ and $1\leq r_1\leq r\leq\infty$. If $\dfrac{1}{r}=\dfrac{1}{p}+\dfrac{1}{q}$ and $\dfrac{1}{r_1}=\dfrac{1}{p_1}+\dfrac{1}{q_1}$, then*

$$\|fg\|_{\mathcal{M}^r_{r_1}}\leq\|f\|_{\mathcal{M}^p_{p_1}}\|g\|_{\mathcal{M}^q_{q_1}}, \tag{1.3}$$

for all $f\in\mathcal{M}^p_{p_1}$ and $g\in\mathcal{M}^q_{q_1}$.

Let us consider the Schwartz class $\mathcal{S}$ and the space of tempered distributions $\mathcal{S}'$. Let $u\in\mathcal{S}'$ and $\varphi(z)$ be a C^∞-function on $[0,\infty)$ such that $0\leq\varphi(z)\leq 1$, $\varphi(z)\equiv 1$ for $z\leq\dfrac{3}{2}$ and *supp* $\varphi\subset[0,5/3)$. Then, for all $j\in\mathbb{Z}$, defining $\psi_j(\xi)=\varphi\left(2^{-j}|\xi|\right)-\varphi\left(2^{1-j}|\xi|\right)$, it holds that $\psi_j(\xi)\in C_0^\infty(\mathbb{R}^N)$ and

$$\sum_{j=-\infty}^{\infty}\psi_j(\xi)=1,\ \text{for all}\ \xi\neq 0.$$

Definition 1.3. *The homogeneous Besov-Morrey space $\mathcal{N}^b_{p,p_1,r}=\mathcal{N}^b_{p,p_1,r}(\mathbb{R}^N)$ is defined as the set of all $u\in\mathcal{S}'/\mathcal{P}$ such that $\varphi_j^\vee*u\in\mathcal{M}^p_{p_1}$ for all j, and*

$$\|u\|_{\mathcal{N}^b_{p,p_1,r}}=\begin{cases}\left(\displaystyle\sum_{j\in\mathbb{Z}}\left(2^{bj}\left\|\psi_j^\vee*u\right\|_{\mathcal{M}^p_{p_1}}\right)^r\right)^{\frac{1}{r}}<\infty, & 1\leq r<\infty,\\ \displaystyle\sup_{j\in\mathbb{Z}}\left(2^{bj}\left\|\psi_j^\vee*u\right\|_{\mathcal{M}^p_{p_1}}\right)<\infty, & r=\infty,\end{cases}$$

for $1 \leq p_1 \leq p \leq \infty$ *and* $b \in \mathbb{R}$, *where* $\mathcal{P}$ *denotes the set of polynomials with* N *variables. The space* $\mathcal{N}^b_{p,p_1,r}$ *is a Banach space with the norm* $\|\cdot\|_{\mathcal{N}^b_{p,p_1,r}}$.

Let us denote by $\{S_\alpha(t)\}_{t\geq 0}$ the fractional heat semigroup, which is defined in Fourier variables as $\widehat{S_\alpha(t)f} = e^{-t|\xi|^\alpha}\widehat{f}$. In this point we observe that if $b<0$ the following equivalence holds (its proof follows in the same way of the proof of Proposition 2.22 in Mazzucato [**24**])):

$$\|u\|_{\mathcal{N}^b_{p,p_1,\infty}} \cong \sup_{t>0} t^{-\frac{b}{\alpha}} \|S_\alpha(t)u\|_{\mathcal{M}^p_{p_1}}. \tag{1.4}$$

In order to eliminate the pressure in (1.2), which can be recovered *a posteriori,* we apply the Leray projector $\mathbb{P} = \{\mathbb{P}_{jk}\}_{j,k=1,2,3}$ onto the velocity equation in (1.2). We recall that $\mathbb{P}$ is defined by $\mathbb{P}_{jk} = \delta_{jk} + R_jR_k$, where $R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}$ corresponds to the Riesz operator. Thus, taking into account the Duhamel principle, system (1.2) is formally equivalent to the following integral formulation:

$$\begin{cases} n(x,t) = \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})n_0 - \displaystyle\int_0^t (t-\tau)^{\beta-1}\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2})(u\cdot\nabla n)\, d\tau \\ \quad - \displaystyle\int_0^t (t-\tau)^{\beta-1}\nabla\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2})(n\nabla((-\Delta)^{-\theta/2}c))\, d\tau, \\ c(x,t) = \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})c_0 - \displaystyle\int_0^t (t-\tau)^{\beta-1}\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2})(u\cdot\nabla c - nc)\, d\tau, \\ u(x,t) = \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})u_0 - \displaystyle\int_0^t (t-\tau)^{\beta-1}\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2})\mathbb{P}(u\cdot\nabla u + n\nabla\phi)\, d\tau, \end{cases} \tag{1.5}$$

where $\{\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})\}_{t\geq 0}$ and $\{\boldsymbol{E}_{\beta,\beta}(-t^\beta(-\Delta)^{\alpha/2})\}_{t\geq 0}$ are the Mittag-Leffler families defined by

$$\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2}) = \int_0^\infty M_\beta(\tau)S_\alpha(\tau t^\beta)d\tau, \tag{1.6}$$

$$\boldsymbol{E}_{\beta,\beta}(-t^\beta(-\Delta)^{\alpha/2}) = \int_0^\infty \beta\tau M_\beta(\tau)S_\alpha(\tau t^\beta)d\tau, \tag{1.7}$$

with $\boldsymbol{M}_\beta : \mathbb{C} \to \mathbb{C}$ being is the Mainardi function which is given by

$$M_\beta(z) = \sum_{N=0}^\infty \frac{(-z)^N}{N!\Gamma(1-\beta(1+N))}.$$

In (1.5) n_0, c_0, u_0, with $\nabla\cdot u_0 = 0$ in $\mathcal{S}'$, are distributions representing the initial data. A triple $[n,c,u]$ satisfying (1.5) is called a *mild solution* of (1.2).

2. Main results

Note that system (1.2) has a scaling property. Indeed, it is not difficult to check that if $[n,c,u]$ is a classical solution of (1.2), then $[n_\lambda, c_\lambda, u_\lambda]$, $\lambda > 0$, defined by

$$[n_\lambda(x,t), c_\lambda(x,t), u_\lambda(x,t)] = [\lambda^\alpha n(\lambda x, \lambda^{\alpha/\beta}t), \lambda^{\alpha+\theta-2}c(\lambda x, \lambda^{\alpha/\beta}t), \lambda^{\alpha-1}u(\lambda x, \lambda^{\alpha/\beta}t)] \tag{2.1}$$

is also solution of (1.2). In this case, the map

$$[n,c,u] \longmapsto [n_\lambda, c_\lambda, u_\lambda]$$

is called the scaling of (1.2), and the solutions invariant by this scaling are called self-similar solutions. Observe that if $[n,c,u]$ is a self-similar solution, the initial data $[n_0, c_0, u_0]$ must be invariant by the scaling, and therefore it must satisfy

$$n_0(x) = \lambda^{\alpha} n_0(\lambda x), \quad c_0(x) = \lambda^{\alpha+\theta-2} c_0(\lambda x), \quad u_0(x,t) = \lambda^{\alpha-1} u_0(\lambda x).$$

Motivated by the above scaling analysis, we consider the following class of critical spaces for the initial data:

(2.2)

$$\begin{cases} n_0 \in \mathcal{N}_{q,q_1,\infty}^{\frac{N}{q}-\alpha}(\mathbb{R}^N), \quad c_0 \in L^{\frac{N}{\alpha+\theta-2}}(\mathbb{R}^N) \ \text{ with } \ \nabla((-\Delta)^{-\theta/2})c_0 \in \mathcal{N}_{r,r_1,\infty}^{\frac{N}{r}-\alpha+1}(\mathbb{R}^N), \\ u_0 \in \mathcal{N}_{p,p_1,\infty}^{\frac{N}{p}-\alpha+1}(\mathbb{R}^N), \end{cases}$$

and the external source ϕ such that

$$t^{\beta-\frac{2\beta}{\alpha}} \nabla\phi(\cdot) \in BC_w((0,\infty); \mathcal{M}_{N_1}^{N}), \tag{2.3}$$

where the exponents p, p_1, q, q_1, r, r_1 and N_1 are as in the following assumption.

ASSUMPTION 1. *Assume that* $N \geq 2$, $\alpha > \frac{3}{2}$, $\alpha + \theta \geq 2$, $\beta > \frac{\alpha}{\alpha+1-\theta}$ *and* $p > r$. *For* $N = 3$, *suppose that the exponents* p, q *and* r *satisfy the conditions* (i), (ii) *or* (iii) *below*

(i) $\frac{N}{2(\alpha-1)} < q < N$, $\quad \frac{N}{\alpha-1} < p < \frac{qN}{N+(1-\alpha)q}$, $\quad \frac{N}{\alpha-1} < r < \frac{qN}{N+(1-\alpha)q}$;

(ii) $q = N$, $\quad \frac{N}{\alpha-1} < p < \frac{N}{2-\alpha}$, $\quad \frac{N}{\alpha-1} < r < \frac{N}{2-\alpha}$;

(iii) $\frac{\beta N}{2\beta\alpha-\beta-\alpha} < q < \frac{2\beta N}{\beta\alpha+\beta-\beta\theta-\alpha}$, $\quad \frac{N}{\alpha-1} < p < \frac{\beta q N}{(2\beta\alpha-\beta-\alpha)q-\beta N}$, $\frac{qN}{N+q(\alpha+\theta-2)} \leq r < \frac{\beta q N}{(2\beta\alpha-\beta-\alpha)q-\beta N}$.

If $N = 2$ *assume that* p, q *and* r *satisfy the condition* (iii) *above. Moreover, we suppose also that* p_1, q_1, r_1 *and* N_1 *satisfy the following conditions:*

(A) $1 \leq p_1 \leq p$, $\quad 1 \leq q_1 \leq q$, $\quad 1 \leq r_1 \leq r$, $\quad 1 \leq N_1 \leq N$;

(B) $\frac{1}{p_1} + \frac{1}{q_1} \leq 1$, $\quad \frac{1}{r_1} + \frac{1}{q_1} \leq 1$, $\quad \frac{1}{p_1} + \frac{\alpha+\theta-2}{N} \leq 1$, $\quad \frac{1}{q_1} + \frac{\alpha+\theta-2}{N} \leq 1$, $\frac{1}{N_1} + \frac{1}{q_1} \leq 1$;

(C) $\frac{p}{p_1} \leq \frac{q}{q_1} = \frac{r}{r_1}$;

(D) $p_1\left(\frac{1}{N_1} + \frac{1}{q_1}\right) \leq p\left(\frac{1}{N} + \frac{1}{q}\right)$.

REMARK 2.1. *Assumption 1 comes from the time decay and nonlinear estimates appearing in Sections 3 and 4 below, which are necessary to use a fixed point argument. If we fix the parameters* β, α *and* θ *fixed as in Assumption 1, then it is possible to find* p_1, q_1, r_1 *and* N_1 *sufficiently close to* p, q, r *and* N, *respectively, such that the Assumption 1 is not empty. Also, if we formally consider* $\beta = 1$, $\alpha = 2$ *and* $\theta = 0$, *our results are in agreement with the results established in* [**16**] *for the non fractional chemotaxis-Navier-Stokes system (1.1).*

We define the functional spaces

$$
\begin{aligned}
&X_1 := \left\{ n : t^{-\frac{\beta N}{\alpha q}+\beta} n(\cdot) \in BC_w((0,\infty); \mathcal{M}^q_{q_1}) \right\}, \\
&X_2 := \Big\{ c : c \in BC_w \left((0,\infty); L^{\frac{N}{\alpha+\theta-2}} \right) \\
&\qquad \text{with } t^{-\frac{\beta N}{\alpha r}-\frac{\beta}{\alpha}+\beta} \nabla((-\Delta)^{-\theta/2} c(\cdot)) \in BC_w \left((0,\infty); \mathcal{M}^r_{r_1} \right) \Big\}, \qquad (2.4) \\
&X_3 := \left\{ u : t^{-\frac{\beta N}{\alpha p}-\frac{\beta}{\alpha}+\beta} \in BC_w \left((0,\infty), \mathcal{M}^p_{p_1} \right) \right\},
\end{aligned}
$$

where, for a general Banach space Z, $BC_w\left((0,\infty), Z\right)$ denotes the class of bounded functions from $(0,\infty)$ to Z that are weakly time continuous in the sense of $\mathcal{S}'$. The spaces X_1, X_2 and X_3 are Banach spaces endowed with the norms

$$
\begin{aligned}
\|n\|_{X_1} &:= \sup_{t>0} t^{-\frac{\beta N}{\alpha q}+\beta} \|n(t)\|_{\mathcal{M}^q_{q_1}}, \\
\|c\|_{X_2} &:= \sup_{t>0} \|c(t)\|_{L^{\frac{N}{\alpha+\theta-2}}} + \sup_{t>0} t^{-\frac{\beta N}{\alpha r}-\frac{\beta}{\alpha}+\beta} \left\| \nabla((-\Delta)^{-\theta/2} c(t)) \right\|_{\mathcal{M}^r_{r_1}}, \\
\|u\|_{X_3} &:= \sup_{t>0} t^{-\frac{\beta N}{\alpha p}-\frac{\beta}{\alpha}+\beta} \|u(t)\|_{\mathcal{M}^p_{p_1}}.
\end{aligned}
$$

Next, let us introduce the product space $\mathcal{X}$

$$
\mathcal{X} := \{[n,c,u] : n \in X_1,\ c \in X_2,\ u \in X_3\} \qquad (2.5)
$$

with the norm

$$
\|[n,c,u]\|_{\mathcal{X}} := \|n\|_{X_1} + \|c\|_{X_2} + \|u\|_{X_3},
$$

and the space of initial data

$$
\mathcal{I} := \{[n_0,c_0,u_0] : n_0, c_0 \text{ and } u_0 \text{ are as in } (2.2)\},
$$

with the norm

(2.6)

$$
\|[n_0,c_0,u_0]\|_{\mathcal{I}} := \|n_0\|_{\mathcal{N}^{\frac{N}{q}-\alpha}_{q,q_1,\infty}} + \|c_0\|_{L^{\frac{N}{\alpha+\theta-2}}} + \left\| \nabla((-\Delta)^{-\theta/2} c_0) \right\|_{\mathcal{N}^{\frac{N}{r}-\alpha+1}_{r,r_1,\infty}} + \|u_0\|_{\mathcal{N}^{\frac{N}{p}-\alpha+1}_{p,p_1,\infty}}.
$$

Now we state our main results.

Theorem 2.2. *Let $N \geq 2$, the exponents p, p_1, q, q_1, r, r_1 and N_1 be as in Assumption 1. Suppose that the initial data $[n_0, c_0, u_0] \in \mathcal{I}$ and the external force $f \in \mathcal{M}^N_{N_1}(\mathbb{R}^N)$. There exist positive constants ϵ, $\delta(\epsilon)$ and $\mathcal{K}_1$ such that the system (1.2) has a unique global mild solution $[n,c,u] \in \mathcal{X}$ satisfying $\|[n,c,u]\|_{\mathcal{X}} \leq 2\mathcal{K}_1\epsilon$ provided that $\|[n_0,c_0,u_0]\|_{\mathcal{I}} \leq \delta$. Moreover, the data-solution map is locally Lipschitz continuous.*

Theorem 2.3. *Under the hypotheses of Theorem 2.2, assume that $[n,c,u]$ and $[\tilde{n},\tilde{c},\tilde{u}]$ are two solutions given by Theorem 2.2 corresponding to the initial data $[n_0,c_0,u_0]$ and $[\tilde{n}_0,\tilde{c}_0,\tilde{u}_0]$, respectively. Then*

$$
\begin{aligned}
\lim_{t\to\infty} \Bigg(& t^{-\frac{\beta N}{\alpha q}+\beta} \left\| \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})(n_0-\tilde{n}_0) \right\|_{\mathcal{M}^q_{q_1}} + \left\| \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})(c_0-\tilde{c}_0) \right\|_{L^{\frac{N}{\alpha+\theta-2}}} \\
& + t^{-\frac{\beta N}{\alpha r}-\frac{\beta}{\alpha}+\beta} \left\| \nabla(-\Delta)^{-\theta/2} \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})(c_0-\tilde{c}_0) \right\|_{\mathcal{M}^r_{r_1}} \\
& + t^{-\frac{\beta N}{\alpha p}-\frac{\beta}{\alpha}+\beta} \left\| \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})(u_0-\tilde{u}_0) \right\|_{\mathcal{M}^p_{p_1}} \Bigg) = 0, \qquad (2.7)
\end{aligned}
$$

if, and only if,

$$\begin{aligned}
&\lim_{t\to\infty} t^{-\frac{\beta N}{\alpha q}+\beta}\|n(\cdot,t)-\tilde{n}(\cdot,t)\|_{\mathcal{M}^q_{q_1}} = \lim_{t\to\infty}\|c(\cdot,t)-\tilde{c}(\cdot,t)\|_{L^{\frac{N}{\alpha+\theta-2}}}\\
&= \lim_{t\to\infty} t^{-\frac{\beta N}{\alpha r}-\frac{\beta}{\alpha}+\beta}\left\|\nabla(-\Delta)^{-\theta/2}(c(\cdot,t)-\tilde{c}(\cdot,t))\right\|_{\mathcal{M}^r_{r_1}}\\
&= \lim_{t\to\infty} t^{-\frac{\beta N}{\alpha p}-\frac{\beta}{\alpha}+\beta}\|u(\cdot,t)-\tilde{u}(\cdot,t)\|_{\mathcal{M}^p_{p_1}} = 0. \qquad (2.8)
\end{aligned}$$

3. Linear estimates for the Mittag-Leffler operators

We start this section recalling some time-decay estimates of the heat semigroup $\{S_\alpha(t)\}_{t\geq 0}$ on Morrey spaces (e.g. [**14**, Lemma 2.2]).

LEMMA 3.1. *Let k be a multi-index, $1\leq p_1\leq p<\infty$ and $1\leq q_1\leq q<\infty$. If $p\geq q$ and $\frac{p}{p_1}\geq\frac{q}{q_1}$, then there exists a positive constant $C>0$ such that*

$$\|S_\alpha(t)f\|_{\mathcal{M}^p_{p_1}} \leq Ct^{-\frac{N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{\mathcal{M}^q_{q_1}}, \qquad (3.1)$$

$$\left\|\nabla^k S_\alpha(t)f\right\|_{\mathcal{M}^p_{p_1}} \leq Ct^{-\frac{N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{|k|}{\alpha}}\|f\|_{\mathcal{M}^q_{q_1}}, \qquad (3.2)$$

$$\|S_\alpha(t)f\|_{L^\infty} \leq C\|f\|_{L^\infty}, \qquad (3.3)$$

for all $f\in\mathcal{S}'$.

Next lemma is necessary to obtain time-decay estimates for the Mittag–Leffler operators in Besov-Morrey spaces.

LEMMA 3.2. *If $\beta\in(0,1)$ and $-1<l<\infty$, then*

$$\boldsymbol{M}_\beta(t)\geq 0 \text{ for all } t\geq 0, \quad \text{and} \quad \int_0^\infty t^l\boldsymbol{M}_\beta(t)\,dt = \frac{\Gamma(l+1)}{\Gamma(\beta l+1)}.$$

LEMMA 3.3. *Consider $\beta\in(0,1)$ and p_1,p,q_1,q be as in Lemma 3.1 such that*

$$\frac{p}{p_1}\geq\frac{q}{q_1}, \quad 1<q\leq p<\infty, \quad \textit{and} \quad \frac{pN}{\alpha p+N}<q.$$

Then, for any $f\in\mathcal{M}^q_{q_1}$, there exists a constant $C_1>0$ such that

$$\left\|\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})f\right\|_{\mathcal{M}^p_{p_1}} \leq C_1 t^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{\mathcal{M}^q_{q_1}},$$

$$\left\|\boldsymbol{E}_{\beta,\beta}(-t^\beta(-\Delta)^{\alpha/2})f\right\|_{\mathcal{M}^p_{p_1}} \leq C_1 t^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{\mathcal{M}^q_{q_1}}.$$

PROOF. Using (1.6), (3.1) and Lemma 3.2 we obtain

$$\begin{aligned}
\left\|\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})f\right\|_{\mathcal{M}^p_{p_1}} &\leq \int_0^\infty \boldsymbol{M}_\beta(\tau)\left\|S_\alpha(\tau t^\beta)f\right\|_{\mathcal{M}^p_{p_1}}d\tau\\
&\leq \int_0^\infty \boldsymbol{M}_\beta(\tau)\,C\left(\tau t^\beta\right)^{-\frac{N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{\mathcal{M}^q_{q_1}}d\tau\\
&\leq C\int_0^\infty \boldsymbol{M}_\beta(\tau)\,\tau^{-\frac{N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}t^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{\mathcal{M}^q_{q_1}}d\tau\\
&\leq Ct^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\int_0^\infty \tau^{-\frac{N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\boldsymbol{M}_\beta(\tau)\,d\tau\right)\|f\|_{\mathcal{M}^q_{q_1}}\\
&\leq C_1 t^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{\mathcal{M}^q_{q_1}}, \quad t>0.
\end{aligned}$$

On the other hand, using (1.7), (3.1) and Lemma 3.2 we have that

$$\begin{aligned}
\left\|\boldsymbol{E}_{\beta,\beta}(-t^{\beta}(-\Delta)^{\alpha/2})f\right\|_{\mathcal{M}^{p}_{p_1}} &\leq \int_0^\infty \beta\tau\boldsymbol{M}_\beta(\tau)\left\|S_\alpha(\tau t^\beta)f\right\|_{\mathcal{M}^{p}_{p_1}}\,d\tau\\
&\leq \int_0^\infty \beta\tau\boldsymbol{M}_\beta(\tau)\; C\left(\tau t^\beta\right)^{-\frac{N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{\mathcal{M}^{q}_{q_1}}\,d\tau\\
&\leq \beta C\int_0^\infty \boldsymbol{M}_\beta(\tau)\;\tau^{1-\frac{N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}t^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{\mathcal{M}^{q}_{q_1}}\,d\tau\\
&\leq \beta Ct^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\int_0^\infty \tau^{1-\frac{N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\boldsymbol{M}_\beta(\tau)\;d\tau\right)\|f\|_{\mathcal{M}^{q}_{q_1}}\\
&\leq C_1t^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{\mathcal{M}^{q}_{q_1}},\quad t>0.
\end{aligned}$$

Note that if $l_1 := -\frac{N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)$ then by hypotheses $l_1 > -1$, and thus, from Lemma 3.2, the constant $C_1 = C_1(p,q,N,\beta)$ is such that

$$C_1 \geq C\ \max\left\{\frac{\Gamma(l_1+1)}{\Gamma(\beta l_1+1)},\frac{\beta\,\Gamma(l_1+2)}{\Gamma(\beta(l_1+1)+1)}\right\} > 0.$$

□

LEMMA 3.4. *Let k be a multi-index, $\beta\in(0,1)$ and p_1,p,q_1,q be as in Lemma 3.1 such that*

$$\frac{p}{p_1}\geq\frac{q}{q_1}\quad\text{and}\quad\frac{pN}{(\alpha-|k|)\,p+N}<q\leq p<\infty.$$

Then, for any $f\in\mathcal{M}^{q}_{q_1}$, there exists a constant $C_2>0$ such that

$$\left\|\nabla^k\boldsymbol{E}_{\beta}(-t^{\beta}(-\Delta)^{\alpha/2})f\right\|_{\mathcal{M}^{p}_{p_1}} \leq C_2\;t^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{\beta|k|}{\alpha}}\|f\|_{\mathcal{M}^{q}_{q_1}}, \tag{3.4}$$

$$\left\|\nabla^k\boldsymbol{E}_{\beta,\beta}(-t^{\beta}(-\Delta)^{\alpha/2})f\right\|_{\mathcal{M}^{p}_{p_1}} \leq C_2\;t^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{\beta|k|}{\alpha}}\|f\|_{\mathcal{M}^{q}_{q_1}}. \tag{3.5}$$

PROOF. From the Leibnitz rule we have that

$$\begin{aligned}
\nabla^k\boldsymbol{E}_{\beta}(-t^{\beta}(-\Delta)^{\alpha/2})f &= \int_0^\infty \boldsymbol{M}_\beta(\tau)\nabla^k S_\alpha(\tau t^\beta)f\;d\tau,\\
\nabla^k\boldsymbol{E}_{\beta,\beta}(-t^{\beta}(-\Delta)^{\alpha/2})f &= \int_0^\infty \beta\tau\boldsymbol{M}_\beta(\tau)\nabla^k S_\alpha(\tau t^\beta)f\;d\tau.
\end{aligned}$$

Arguing as in the proof of Lemma 3.3, using (3.2) and Lemma 3.2, we can obtain

$$\begin{aligned}
&\left\|\nabla^k\boldsymbol{E}_{\beta}(-t^{\beta}(-\Delta)^{\alpha/2})f\right\|_{\mathcal{M}^{p}_{p_1}} \leq \int_0^\infty \boldsymbol{M}_\beta(\tau)\left\|\nabla^k S_\alpha(\tau t^\beta)f\right\|_{\mathcal{M}^{p}_{p_1}}\,d\tau\\
&\leq \int_0^\infty \boldsymbol{M}_\beta(\tau)\;C\left(\tau t^\beta\right)^{-\frac{N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{|k|}{\alpha}}\|f\|_{\mathcal{M}^{q}_{q_1}}\,d\tau\\
&\leq C\int_0^\infty \boldsymbol{M}_\beta(\tau)\;\tau^{-\frac{N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{|k|}{\alpha}}t^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{\beta|k|}{\alpha}}\|f\|_{\mathcal{M}^{q}_{q_1}}\,d\tau\\
&\leq Ct^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{\beta|k|}{\alpha}}\left(\int_0^\infty \tau^{-\frac{N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{|k|}{\alpha}}\boldsymbol{M}_\beta(\tau)\;d\tau\right)\|f\|_{\mathcal{M}^{q}_{q_1}}\\
&\leq C_2t^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{\beta|k|}{\alpha}}\|f\|_{\mathcal{M}^{q}_{q_1}},\quad t>0.
\end{aligned}$$

On the other hand, by (3.2) and Lemma 3.2, we obtain

$$\begin{aligned}
\left\|\nabla^k \boldsymbol{E}_{\beta,\beta}(-t^\beta(-\Delta)^{\alpha/2})f\right\|_{\mathcal{M}^p_{p_1}} &\leq \int_0^\infty \beta\tau \boldsymbol{M}_\beta(\tau)\left\|\nabla^k S_\alpha(\tau t^\beta)f\right\|_{\mathcal{M}^p_{p_1}} d\tau \\
&\leq \int_0^\infty \beta\tau \boldsymbol{M}_\beta(\tau)\; C\left(\tau t^\beta\right)^{-\frac{N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{|k|}{\alpha}} \|f\|_{\mathcal{M}^q_{q_1}} d\tau \\
&\leq \beta C\int_0^\infty \boldsymbol{M}_\beta(\tau)\; \tau^{-\frac{N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{|k|}{\alpha}+1} t^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{\beta|k|}{\alpha}} \|f\|_{\mathcal{M}^q_{q_1}}\; d\tau \\
&\leq \beta C t^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{\beta|k|}{\alpha}}\left(\int_0^\infty \tau^{1-\frac{N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{|k|}{\alpha}}\boldsymbol{M}_\beta(\tau)\; d\tau\right)\|f\|_{\mathcal{M}^q_{q_1}} \\
&\leq C_2 t^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{\beta|k|}{\alpha}} \|f\|_{\mathcal{M}^q_{q_1}}, \quad t>0.
\end{aligned}$$

Note that if $l_2 := -\frac{|k|}{\alpha} - \frac{N}{\alpha}\left(\frac{1}{q}-\frac{1}{p}\right)$ then by hypotheses $l_2 > -1$ and thus the constant $C_2 = C_2(p,q,N,\beta)$ (by Lemma 3.2) is such that

$$C_2 \geq C\, \max\left\{\frac{\Gamma(l_2+1)}{\Gamma(l_2+1)}, \frac{\beta\,\Gamma(l_2+2)}{\Gamma(\beta(l_2+1)+1)}\right\} > 0.$$

□

LEMMA 3.5. *Let k be a multi-index, $1 \leq q_1 \leq q < \infty$, $1 \leq r_1 \leq r < \infty$ and $b < 0$, then there exists a constant $C_3 > 0$ such that*

$$\left\|\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})f\right\|_{\mathcal{M}^q_{q_1}} \leq C_3 t^{\frac{\beta b}{\alpha}} \|f\|_{\mathcal{N}^b_{q,q_1,\infty}},$$

$$\left\|\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})f\right\|_{L^\infty} \leq C_3 \|f\|_{L^\infty},$$

$$\left\|\nabla^k\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})f\right\|_{\mathcal{M}^r_{r_1}} \leq C_3 t^{\frac{\beta b}{\alpha}} \left\|\nabla^k f\right\|_{\mathcal{N}^b_{r,r_1,\infty}}.$$

PROOF. Using the equivalence (1.4) and Lemma 3.2, we get

$$\begin{aligned}
\left\|\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})f\right\|_{\mathcal{M}^q_{q_1}} &\leq \int_0^\infty \boldsymbol{M}_\beta(\tau)\left\|S_\alpha(\tau t^\beta)f\right\|_{\mathcal{M}^q_{q_1}} d\tau \\
&\leq \int_0^\infty \boldsymbol{M}_\beta(\tau)\;\left(\tau t^\beta\right)^{\frac{b}{\alpha}}(\tau t^\beta)^{-\frac{b}{\alpha}}\left\|S(\tau t^\beta)f\right\|_{\mathcal{M}^q_{q_1}} d\tau \\
&\leq \int_0^\infty \boldsymbol{M}_\beta(\tau)\;\tau^{\frac{b}{\alpha}} t^{\frac{\beta b}{\alpha}}\|f\|_{\mathcal{N}^b_{q,q_1,\infty}} d\tau = t^{\frac{\beta b}{\alpha}}\left(\int_0^\infty \boldsymbol{M}_\beta(\tau)\;\tau^{\frac{b}{\alpha}}\; d\tau\right)\|f\|_{\mathcal{N}^b_{q,q_1,\infty}} \\
&= t^{\frac{\beta b}{\alpha}}\frac{\Gamma(\frac{b}{\alpha}+1)}{\Gamma(\beta\frac{b}{\alpha}+1)}\|f\|_{\mathcal{N}^b_{q,q_1,\infty}} \leq C_3 t^{\frac{\beta b}{\alpha}}\|f\|_{\mathcal{N}^b_{q,q_1,\infty}}.
\end{aligned}$$

On the other hand, using (3.3) we obtain

$$\begin{aligned}
\left\|\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})f\right\|_{L^\infty} &\leq \int_0^\infty \boldsymbol{M}_\beta(\tau)\left\|S_\alpha(\tau t^\beta)f\right\|_{L^\infty} d\tau \\
&\leq \int_0^\infty \boldsymbol{M}_\beta(\tau)\; C\,\|f\|_{L^\infty} d\tau \\
&= C\left(\int_0^\infty \boldsymbol{M}_\beta(\tau)\; d\tau\right)\|f\|_{L^\infty} \leq C_3\|f\|_{L^\infty}.
\end{aligned}$$

In addition, using the equivalence (1.4) and Lemma 3.2 we have that

$$\begin{aligned}
\left\|\nabla^k \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})f\right\|_{\mathcal{M}^r_{r_1}} &\le \int_0^\infty \boldsymbol{M}_\beta(\tau)\left\|\nabla^k S_\alpha(\tau t^\beta)f\right\|_{\mathcal{M}^r_{r_1}} d\tau \\
&\le \int_0^\infty \boldsymbol{M}_\beta(\tau)\ \left(\tau t^\beta\right)^{\frac{b}{\alpha}}(\tau t^\beta)^{-\frac{b}{\alpha}}\left\|S_\alpha(\tau t^\beta)\nabla^k f\right\|_{\mathcal{M}^r_{r_1}} d\tau \\
&\le \int_0^\infty \boldsymbol{M}_\beta(\tau)\ \tau^{\frac{b}{\alpha}} t^{\frac{\beta b}{\alpha}}\left\|\nabla^k f\right\|_{\mathcal{N}^b_{r,r_1,\infty}} d\tau \\
&= t^{\frac{\beta b}{\alpha}}\left(\int_0^\infty \boldsymbol{M}_\beta(\tau)\ \tau^{\frac{b}{\alpha}}\ d\tau\right)\left\|\nabla^k f\right\|_{\mathcal{N}^b_{r,r_1,\infty}} \\
&= t^{\frac{\beta b}{\alpha}}\frac{\Gamma(\frac{b}{\alpha}+1)}{\Gamma(\beta\frac{b}{\alpha}+1)}\left\|\nabla^k f\right\|_{\mathcal{N}^b_{r,r_1,\infty}} \\
&\le C_3 t^{\frac{\beta b}{\alpha}}\left\|\nabla^k f\right\|_{\mathcal{N}^b_{r,r_1,\infty}}.
\end{aligned}$$

Thus, we conclude the proof of the lemma. □

4. Global existence and stability

In this section we prove the main results established in Section 3. For that, we prove a set of linear and bilinear estimates in the norms of solution spaces which allow us to apply a fixed point argument. For each triple of initial data $[n_0, c_0, u_0]$ and force ϕ, we consider the mapping $\mathcal{F}([n,c,u]) = [\mathcal{N},\mathcal{C},\mathcal{U}]$, defined by

(4.1)

$$\begin{cases}
\mathcal{N}(t) = \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})n_0 - \displaystyle\int_0^t (t-\tau)^{\beta-1}\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2})(u\cdot\nabla n)(\tau)\ d\tau \\
\quad -\displaystyle\int_0^t (t-\tau)^{\beta-1}\nabla\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2})(n\nabla((-\Delta)^{-\theta/2}c))(\tau)\ d\tau, \\
\quad := \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})n_0 + \mathcal{B}^1_{3,1}(t) + \mathcal{B}^1_{1,2}(t), \\
\\
\mathcal{C}(t) = \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})c_0 - \displaystyle\int_0^t (t-\tau)^{\beta-1}\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2})(u\cdot\nabla c)(\tau)\ d\tau \\
\quad +\displaystyle\int_0^t (t-\tau)^{\beta-1}\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2})(nc)(\tau)\ d\tau, \\
\quad := \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})c_0 + \mathcal{B}^2_{3,2}(t) + \mathcal{B}^2_{1,2}(t), \\
\\
\mathcal{U}(t) = \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})u_0 - \displaystyle\int_0^t (t-\tau)^{\beta-1}\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2})\mathbb{P}(u\cdot\nabla u)(\tau)\ d\tau \\
\quad -\displaystyle\int_0^t (t-\tau)^{\beta-1}\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2})\mathbb{P}(n\nabla\phi)(\tau)\ d\tau \\
\quad := \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})u_0 + \mathcal{B}^3_{3,3}(t) + L_3(t), \qquad 0 < t < \infty.
\end{cases}$$

In (4.1) we use conveniently the notation $\mathcal{B}^j_{i,k}(t)$ where the subscripts i, k indicate the unknowns appearing in the term $\mathcal{B}^j$ (1 for n, 2 for c and 3 for u). Using this notation, we introduce the following lemma, which is key to prove Theorem 2.2. The proof of this lemma can be found in [**16**].

LEMMA 4.1. *For $1 \le i \le 3$, let X_i be a Banach space with the norm $\|\cdot\|_{X_i}$. Consider the Banach space $\mathcal{X} = X_1 \times X_2 \times X_3$ endowed with the norm*

$$\|x\|_{\mathcal{X}} = \|x_1\|_{X_1} + \|x_2\|_{X_2} + \|x_3\|_{X_3},$$

where $x = [x_1, x_2, x_3] \in \mathcal{X}$*. For* $1 \leq i, k, j \leq 3$ *assume that* $\mathcal{B}^j_{i,k} : X_i \times X_k \to X_j$ *is a continuous bilinear map, that is, there is a positive constant* $C^j_{i,k}$ *such that*

$$\left\|\mathcal{B}^j_{i,k}(x_i, x_k)\right\|_{X_j} \leq C^j_{i,k} \left\|x_i\right\|_{X_i} \left\|x_k\right\|_{X_k}, \quad \text{for all } (x_i, x_k) \in X_i \times X_k.$$

Also assume that $L_3 : X_1 \to X_3$ *is a continuous linear map such that* $\|L_3\|_{X_1 \to X_3} \leq \gamma$*. Set the constants*

$$\mathcal{K}_1 := 1 + \gamma \ \text{ and } \ \mathcal{K}_2 := \gamma \sum_{i,k=1}^{3} C^1_{i,k} + \sum_{j,i,k=1}^{3} C^j_{i,k},$$

and let $0 < \epsilon < \dfrac{1}{4\mathcal{K}_1\mathcal{K}_2}$ *and* $\mathscr{B}_\epsilon = \{x \in \mathcal{X} : \|x\|_{\mathcal{X}} \leq 2\mathcal{K}_1\epsilon\}$*. If* $\|y\|_{\mathcal{X}} \leq \epsilon$ *then there exists a unique solution* $x \in \mathscr{B}_\epsilon$ *for the equation* $x = y + \mathcal{B}(x)$*, where* $y = [y_1, y_2, y_3]$*,* $\mathcal{B}(x) = [\mathcal{B}_1(x), \mathcal{B}_2(x), \mathcal{B}_3(x)]$ *and*

$$\begin{aligned} \mathcal{B}_1(x) &= \sum_{i,k=1}^{3} \mathcal{B}^1_{i,k}(x_i, x_k), \qquad \mathcal{B}_2(x) = \sum_{i,k=1}^{3} \mathcal{B}^2_{i,k}(x_i, x_k), \\ \mathcal{B}_3(x) &= \sum_{i,k=1}^{3} \mathcal{B}^3_{i,k}(x_i, x_k) + (L_3 \circ (y_1 + \mathcal{B}_1))(x). \end{aligned}$$

4.1. Bilinear estimates.

LEMMA 4.2. *Under the hypotheses of Theorem 2.2 there exist positive constants* K_1 *and* K_2 *such that*

$$\begin{aligned} \left\|\mathcal{B}^1_{3,1}(u, n)\right\|_{X_1} &\leq K_1 \|u\|_{X_3} \|n\|_{X_1}, \\ \left\|\mathcal{B}^1_{1,2}(n, c)\right\|_{X_1} &\leq K_2 \|n\|_{X_1} \|c\|_{X_2}, \end{aligned}$$

for all $[n, c, u] \in \mathcal{X}$*.*

PROOF. From the conditions (i), (ii) and (iii) in Assumption 1, we have that

$$\beta - \frac{\beta}{\alpha} - \frac{\beta N}{\alpha p} > 0, \qquad 1 - 2\beta + \frac{\beta}{\alpha} + \frac{\beta N}{\alpha p} + \frac{\beta N}{\alpha q} > 0.$$

Taking $s_1 = \frac{p_1 q_1}{p_1 + q_1}$, from (A), (B) and (C) in Assumption 1, it follows that

$$1 \leq s_1 \leq \frac{pq}{p+q} \leq q \ \text{ and } \ \frac{q}{q_1} \geq \frac{pq}{p+q}\frac{1}{s_1},$$

and from Lemma 3.4, we get

$$
\begin{aligned}
(4.2)\quad & \left\|\mathcal{B}^1_{3,1}(u,n)(t)\right\|_{\mathcal{M}^q_{q_1}} \\
& \left\|\int_0^t (t-\tau)^{\beta-1}\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2})(u\cdot\nabla n)(\tau)\ d\tau\right\|_{\mathcal{M}^q_{q_1}} \\
& \le \int_0^t (t-\tau)^{\beta-1}\left\|\nabla\cdot\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2})(un)(\tau)\right\|_{\mathcal{M}^q_{q_1}} d\tau \\
& \le C_2\int_0^t (t-\tau)^{\beta-1}(t-\tau)^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}+\frac{1}{p}-\frac{1}{q}\right)-\frac{\beta}{\alpha}}\left\|(un)(\tau)\right\|_{\mathcal{M}^{\frac{pq}{p+q}}_{s_1}} d\tau \quad \text{(by (3.5))} \\
& \le C_2\int_0^t (t-\tau)^{\beta-1-\frac{\beta N}{\alpha p}-\frac{\beta}{\alpha}}\left\|u(\tau)\right\|_{\mathcal{M}^p_{p_1}}\left\|n(\tau)\right\|_{\mathcal{M}^q_{q_1}} d\tau \quad \text{(by (1.3))} \\
& \le C_2\int_0^t (t-\tau)^{\beta-1-\frac{\beta N}{\alpha p}-\frac{\beta}{\alpha}}\tau^{\frac{\beta N}{\alpha p}+\frac{\beta}{\alpha}-\beta}\tau^{\frac{\beta N}{\alpha q}-\beta}d\tau\left\|u\right\|_{X_3}\left\|n\right\|_{X_1} \\
& = C_2\ t^{\frac{\beta N}{\alpha q}-\beta}\ b\left(\beta-\frac{\beta}{\alpha}-\frac{\beta N}{\alpha p}\ ,\ 1-2\beta+\frac{\beta}{\alpha}+\frac{\beta N}{\alpha p}+\frac{\beta N}{\alpha q}\right)\left\|u\right\|_{X_3}\left\|n\right\|_{X_1} \\
& = K_1\ t^{\frac{\beta N}{\alpha q}-\beta}\left\|u\right\|_{X_3}\left\|n\right\|_{X_1},
\end{aligned}
$$

for all $t>0$, where $K_1=K_1(N,\beta,p,p_1,q,q_1,)$ and $b(\cdot,\cdot)$ denotes the beta function. In addition, from the Assumption 1, we have

$$
\beta-\frac{\beta}{\alpha}-\frac{\beta N}{\alpha r}>0,\qquad 1-2\beta+\frac{\beta}{\alpha}+\frac{\beta N}{\alpha q}+\frac{\beta N}{\alpha r}>0.
$$

Taking $s_2=\frac{r_1q_1}{r_1+q_1}$ it follows that

$$
1\le s_2\le\frac{rq}{r+q}\le q\ \text{ and }\ \frac{q}{q_1}\ge\frac{rq}{r+q}\frac{1}{s_2},
$$

and applying Lemma 3.4 we get

$$
\begin{aligned}
(4.3)\quad & \left\|\mathcal{B}^1_{1,2}(n,c)(t)\right\|_{\mathcal{M}^q_{q_1}} \\
& = \left\|\int_0^t (t-\tau)^{\beta-1}\nabla\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2})(n\nabla((-\Delta)^{-\theta/2}c))(\tau)\ d\tau\right\|_{\mathcal{M}^q_{q_1}} \\
& \le \int_0^t (t-\tau)^{\beta-1}\left\|\nabla\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2})(n\nabla((-\Delta)^{-\theta/2}c))(\tau)\right\|_{\mathcal{M}^q_{q_1}} d\tau \\
& \le C_2\int_0^t (t-\tau)^{\beta-1}(t-\tau)^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}+\frac{1}{r}-\frac{1}{q}\right)-\frac{\beta}{\alpha}}\left\|(n\nabla((-\Delta)^{-\theta/2}c))(\tau)\right\|_{\mathcal{M}^{\frac{rq}{r+q}}_{s_2}} d\tau \\
& \le C_2\int_0^t (t-\tau)^{\beta-1-\frac{\beta N}{\alpha r}-\frac{\beta}{\alpha}}\left\|n(\tau)\right\|_{\mathcal{M}^q_{q_1}}\left\|\nabla((-\Delta)^{-\theta/2}c)(\tau)\right\|_{\mathcal{M}^r_{r_1}} d\tau \quad \text{(by (1.3))} \\
& \le C_2\int_0^t (t-\tau)^{\beta-1-\frac{\beta N}{\alpha r}-\frac{\beta}{\alpha}}\tau^{\frac{\beta N}{\alpha q}-\beta}\tau^{\frac{\beta N}{\alpha r}+\frac{\beta}{\alpha}-\beta}d\tau\left\|n\right\|_{X_1}\left\|c\right\|_{X_2} \\
& = C_2\ t^{\frac{\beta N}{\alpha q}-\beta}\ b\left(\beta-\frac{\beta}{\alpha}-\frac{\beta N}{\alpha r}\ ,\ 1-2\beta+\frac{\beta}{\alpha}+\frac{\beta N}{\alpha q}+\frac{\beta N}{\alpha r}\right)\left\|n\right\|_{X_1}\left\|c\right\|_{X_2} \\
& = K_2\ t^{\frac{\beta N}{\alpha q}-\beta}\left\|n\right\|_{X_1}\left\|c\right\|_{X_2},
\end{aligned}
$$

for all $t>0$, where $K_2=K_2(N,\beta,q,q_1,r,r_1)$. Thus, from (4.2) and (4.3), we conclude the proof of the lemma. □

Lemma 4.3. *Under the hypotheses of Theorem 2.2 there exist positive constants* K_3, K_4, K_5 *and* K_6 *such that*

$$\left\|\mathcal{B}^2_{3,2}(u,c)\right\|_{X_2} \leq (K_3+K_4)\left\|u\right\|_{X_3}\left\|c\right\|_{X_2}, \tag{4.4}$$

$$\left\|\mathcal{B}^2_{1,2}(n,c)\right\|_{X_2} \leq (K_5+K_6)\left\|n\right\|_{X_1}\left\|c\right\|_{X_2}, \tag{4.5}$$

for all $[n,c,u] \in \mathcal{X}$.

Proof. From the conditions (i), (ii) and (iii) in Assumption 1, we have that

$$\beta - \frac{\beta}{\alpha} - \frac{\beta N}{\alpha p} > 0, \quad 1-\beta+\frac{\beta}{\alpha}+\frac{\beta N}{\alpha p} > 0 \quad \text{and} \quad \frac{\beta N}{\alpha r} - \frac{\beta N}{\alpha p} > 0.$$

Taking $s_3 = \frac{p_1 N}{N+p_1(\alpha+\theta-2)}$, from (A), (B) and (C) in Assumption 1 we get

$$1 \leq s_3 \leq \frac{pN}{N+p(\alpha+\theta-2)} \leq \frac{N}{\alpha+\theta-2} \;\text{ and }\; 1 \geq \frac{pN}{N+p(\alpha+\theta-2)}\frac{1}{s_3},$$

and

$$1 \leq s_3 \leq \frac{pN}{N+p(\alpha+\theta-2)} \leq r \;\text{ and }\; \frac{r}{r_1} \geq \frac{pN}{N+p(\alpha+\theta-2)}\frac{1}{s_3}.$$

Hence, from Lemma 3.4 we have the following bound for $\mathcal{B}^2_{3,2}$ and $\nabla((-\Delta)^{-\theta/2}\mathcal{B}^2_{3,2})$:

(4.6)

$$\begin{aligned}
&\left\|\mathcal{B}^2_{3,2}(u,c)(t)\right\|_{L^{\frac{N}{\alpha+\theta-2}}} \\
&= \left\|\int_0^t (t-\tau)^{\beta-1}\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^{\beta}(-\Delta)^{\alpha/2})(u\cdot\nabla c)(\tau)d\tau\right\|_{L^{\frac{N}{\alpha+\theta-2}}} \\
&\leq \int_0^t (t-\tau)^{\beta-1}\left\|\nabla\cdot\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^{\beta}(-\Delta)^{\alpha/2})(uc)(\tau)\right\|_{L^{\frac{N}{\alpha+\theta-2}}} d\tau \\
&\leq C_2\int_0^t (t-\tau)^{\beta-1}(t-\tau)^{-\frac{\beta N}{\alpha}\left(\frac{\alpha+\theta-2}{N}+\frac{1}{p}-\frac{\alpha+\theta-2}{N}\right)-\frac{\beta}{\alpha}}\left\|(uc)(\tau)\right\|_{\mathcal{M}_{s_3}^{\frac{pN}{N+p(\alpha+\theta-2)}}} d\tau \\
&\leq C_2\int_0^t (t-\tau)^{\beta-1-\frac{\beta N}{\alpha p}-\frac{\beta}{\alpha}}\left\|u(\tau)\right\|_{\mathcal{M}^p_{p_1}}\left\|c(\tau)\right\|_{L^{\frac{N}{\alpha+\theta-2}}} d\tau \quad (\text{by}(1.3)) \\
&\leq C_2\int_0^t (t-\tau)^{\beta-1-\frac{\beta N}{\alpha p}-\frac{\beta}{\alpha}}\tau^{\frac{\beta N}{\alpha p}+\frac{\beta}{\alpha}-\beta}d\tau\left\|u\right\|_{X_3}\left\|c\right\|_{X_2} \\
&= C_2\; b\left(\beta-\frac{\beta}{\alpha}-\frac{\beta N}{\alpha p}\,,\; 1-\beta+\frac{\beta}{\alpha}+\frac{\beta N}{\alpha p}\right)\left\|u\right\|_{X_3}\left\|c\right\|_{X_2} \\
&= K_3\;\left\|u\right\|_{X_3}\left\|c\right\|_{X_2},
\end{aligned}$$

and

(4.7)

$$
\begin{aligned}
&\left\| \nabla((-\Delta)^{-\theta/2}(B^2_{3,2}(u,c))(t) \right\|_{\mathcal{M}^r_{r_1}} \\
&= \left\| \nabla(-\Delta)^{-\theta/2} \int_0^t (t-\tau)^{\beta-1} \boldsymbol{E}_{\beta,\beta}(-(t-\tau)^{\beta}(-\Delta)^{\alpha/2})(u\cdot\nabla c)(\tau)\, d\tau \right\|_{\mathcal{M}^r_{r_1}} \\
&\le \int_0^t (t-\tau)^{\beta-1} \left\| \nabla^{2-\theta}\cdot \boldsymbol{E}_{\beta,\beta}(-(t-\tau)^{\beta}(-\Delta)^{\alpha/2})(uc)(\tau) \right\|_{\mathcal{M}^r_{r_1}} d\tau \\
&\le C_2 \int_0^t (t-\tau)^{\beta-1} (t-\tau)^{-\frac{\beta N}{\alpha}\left(\frac{1}{p}+\frac{\alpha+\theta-2}{N}-\frac{1}{r}\right)-\frac{\beta(2-\theta)}{\alpha}} \left\|(uc)(\tau)\right\|_{\mathcal{M}_{s_3}^{\frac{pN}{N+p(\alpha+\theta-2)}}} d\tau \\
&\le C_2 \int_0^t (t-\tau)^{\frac{\beta N}{\alpha r}-\frac{\beta N}{\alpha p}-1} \left\|u(\tau)\right\|_{\mathcal{M}^p_{p_1}} \left\|c(\tau)\right\|_{L^{\frac{N}{\alpha+\theta-2}}} d\tau \\
&\le C_2 \int_0^t (t-\tau)^{\frac{\beta N}{\alpha r}-\frac{\beta N}{\alpha p}-1} \tau^{\frac{\beta N}{\alpha p}+\frac{\beta}{\alpha}-\beta} d\tau\, \|u\|_{X_3} \|c\|_{X_2} \\
&= C_2\, t^{\frac{\beta N}{\alpha r}+\frac{\beta}{\alpha}-\beta}\, b\left(\frac{\beta N}{\alpha r}-\frac{\beta N}{\alpha p}\,,\ 1-\beta+\frac{\beta}{\alpha}+\frac{\beta N}{\alpha p}\right) \|u\|_{X_3} \|c\|_{X_2} \\
&= K_4\, t^{\frac{\beta N}{\alpha r}+\frac{\beta}{\alpha}-\beta} \|u\|_{X_3} \|c\|_{X_2}\,.
\end{aligned}
$$

Analogously, from the conditions (i), (ii) and (iii) in Assumption 1, we have that

$$
\beta-\frac{\beta N}{\alpha q}>0, \quad 1-\beta+\frac{\beta N}{\alpha q}>0, \quad \text{and} \quad \frac{\beta}{\alpha}-\frac{\beta N}{\alpha q}+\frac{\beta N}{\alpha r}>0.
$$

Taking $s_4=\frac{q_1 N}{N+q_1(\alpha+\theta-2)}$, from (A), (B) and (C) in Assumption 1, we obtain

$$
1\le s_4\le \frac{qN}{N+q(\alpha+\theta-2)} \le \frac{N}{\alpha+\theta-2} \quad \text{and} \quad 1\ge \frac{qN}{N+q(\alpha+\theta-2)}\frac{1}{s_4},
$$

and

$$
1\le s_4\le \frac{qN}{N+q(\alpha+\theta-2)} \le r \ \text{ and } \ \frac{r}{r_1}\ge \frac{qN}{N+q(\alpha+\theta-2)}\frac{1}{s_4}.
$$

Hence, using Lemma 3.3 we can estimate $\mathcal{B}^2_{1,2}$ and $\nabla((-\Delta)^{-\theta/2}\mathcal{B}^2_{1,2})$ as follows:

$$
\begin{aligned}
(4.8)\quad & \left\|\mathcal{B}^2_{1,2}(n,c)(t)\right\|_{L^{\frac{N}{\alpha+\theta-2}}} \\
&= \left\|\int_0^t (t-\tau)^{\beta-1}\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^{\beta}(-\Delta)^{\alpha/2})(nc)(\tau)d\tau\right\|_{L^{\frac{N}{\alpha+\theta-2}}} \\
&\le \int_0^t (t-\tau)^{\beta-1}\left\|\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^{\beta}(-\Delta)^{\alpha/2})(nc)(\tau)\right\|_{L^{\frac{N}{\alpha+\theta-2}}} d\tau \\
&\le C_1\int_0^t (t-\tau)^{\beta-1}(t-\tau)^{-\frac{\beta N}{\alpha}\left(\frac{\alpha+\theta-2}{N}+\frac{1}{q}-\frac{\alpha+\theta-2}{N}\right)}\left\|(nc)(\tau)\right\|_{\mathcal{M}_{s_4}^{\frac{qN}{N+q(\alpha+\theta-2)}}} d\tau \\
&\le C_1\int_0^t (t-\tau)^{\beta-1-\frac{\beta N}{\alpha q}}\left\|n(\tau)\right\|_{\mathcal{M}^q_{q_1}}\left\|c(\tau)\right\|_{L^{\frac{N}{\alpha+\theta-2}}} d\tau \quad \text{(by (1.3))} \\
&\le C_1\int_0^t (t-\tau)^{\beta-1-\frac{\beta N}{\alpha q}}\tau^{\frac{\beta N}{\alpha q}-\beta}d\tau\,\|n\|_{X_3}\|c\|_{X_2} \\
&= C_1\; b\left(\beta-\frac{\beta N}{\alpha q}\;,\;1-\beta+\frac{\beta N}{\alpha q}\right)\|n\|_{X_3}\|c\|_{X_2} \\
&= K_5\;\|n\|_{X_3}\|c\|_{X_2},
\end{aligned}
$$

and

(4.9)

$$
\begin{aligned}
&\left\|\nabla((-\Delta)^{-\theta/2}(B^2_{1,2}(n,c))(t)\right\|_{\mathcal{M}^r_{r_1}} \\
&= \left\|\nabla(-\Delta)^{-\theta/2}\int_0^t (t-\tau)^{\beta-1}\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^{\beta}(-\Delta)^{\alpha/2})(nc)(\tau)\;d\tau\right\|_{\mathcal{M}^r_{r_1}} \\
&\le \int_0^t (t-\tau)^{\beta-1}\left\|\nabla^{1-\theta}\cdot\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^{\beta}(-\Delta)^{\alpha/2})(nc)(\tau)\right\|_{\mathcal{M}^r_{r_1}} d\tau \\
&\le C_2\int_0^t (t-\tau)^{\beta-1}(t-\tau)^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}+\frac{\alpha+\theta-2}{N}-\frac{1}{r}\right)-\frac{\beta(1-\theta)}{\alpha}}\left\|(nc)(\tau)\right\|_{\mathcal{M}_{s_4}^{\frac{qN}{N+q(\alpha+\theta-2)}}} d\tau \\
&\le C_2\int_0^t (t-\tau)^{\frac{\beta N}{\alpha r}-\frac{\beta N}{\alpha q}+\frac{\beta}{\alpha}-1}\left\|n(\tau)\right\|_{\mathcal{M}^q_{q_1}}\left\|c(\tau)\right\|_{L^{\frac{N}{\alpha+\theta-2}}} d\tau \\
&\le C_2\int_0^t (t-\tau)^{\frac{\beta N}{\alpha r}-\frac{\beta N}{\alpha q}+\frac{\beta}{\alpha}-1}\tau^{\frac{\beta N}{\alpha q}-\beta}d\tau\,\|n\|_{X_3}\|c\|_{X_2} \\
&= C_2\;t^{\frac{\beta N}{\alpha r}+\frac{\beta}{\alpha}-\beta}\;b\left(\frac{\beta}{\alpha}-\frac{\beta N}{\alpha q}+\frac{\beta N}{\alpha r},\;1-\beta+\frac{\beta N}{\alpha q}\right)\|n\|_{X_3}\|c\|_{X_2} \\
&= K_6\;t^{\frac{\beta N}{\alpha r}+\frac{\beta}{\alpha}-\beta}\|n\|_{X_3}\|c\|_{X_2},
\end{aligned}
$$

for all $t > 0$, where $K_3 = K_3(N,\beta,p,p_1)$, $K_4 = K_4(N,\beta,p,p_1,r,r_1)$, $K_5 = K_5(N,\beta,q,q_1)$, $K_6 = K_6(N,\beta,q,q_1,r,r_1)$. Thus, from (4.6)-(4.9), we conclude the proof of the lemma. □

Lemma 4.4. *Under the hypotheses of Theorem 2.2 there exists a positive constant K_7 such that for all $u \in X_3$,*

$$\left\|\mathcal{B}^3_{3,3}(u,u)\right\|_{X_3} \;\le\; K_7\,\|u\|^2_{X_3}.$$

Proof. From the conditions (i), (ii) and (iii) in Assumption 1, we have that

$$\beta - \frac{\beta}{\alpha} - \frac{\beta N}{\alpha p} > 0, \qquad 1 - 2\beta + \frac{2\beta}{\alpha} + \frac{2\beta N}{\alpha p} > 0,$$

and since the projector operator $\mathbb{P}$ is bounded in $\mathcal{M}^p_{p_1}$, we get

$$\begin{aligned}
&\left\|\mathcal{B}^3_{3,3}(u,u)(t)\right\|_{\mathcal{M}^p_{p_1}} = \left\|\int_0^t (t-\tau)^{\beta-1}\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^{\beta}(-\Delta)^{\alpha/2}))\mathbb{P}(u\cdot\nabla u)(\tau)d\tau\right\|_{\mathcal{M}^p_{p_1}} \\
&\le \int_0^t (t-\tau)^{\beta-1}\left\|\mathbb{P}\nabla\cdot\boldsymbol{E}_{\beta,\beta}((t-\tau)^{\beta}(-\Delta)^{\alpha/2}))(u\otimes u)(\tau)\right\|_{\mathcal{M}^p_{p_1}} d\tau \\
&\le \int_0^t (t-\tau)^{\beta-1}\left\|\nabla\cdot\boldsymbol{E}_{\beta,\beta}((t-\tau)^{\beta}(-\Delta)^{\alpha/2}))(u\otimes u)(\tau)\right\|_{\mathcal{M}^p_{p_1}} d\tau \\
&\le C_2\int_0^t (t-\tau)^{\beta-1}(t-\tau)^{-\frac{\beta N}{\alpha}\left(\frac{2}{p}-\frac{1}{p}\right)-\frac{\beta}{\alpha}}\left\|(u\otimes u)(\tau)\right\|_{\mathcal{M}^{\frac{p}{2}}_{\frac{p_1}{2}}} d\tau \quad \text{(by (3.5))} \\
&\le C_2\int_0^t (t-\tau)^{\beta-1-\frac{\beta N}{\alpha p}-\frac{\beta}{\alpha}}\left\|u(\tau)\right\|_{\mathcal{M}^p_{p_1}}\left\|u(\tau)\right\|_{\mathcal{M}^p_{p_1}} d\tau \quad \text{(by (1.3))} \\
&= C_2\int_0^t (t-\tau)^{\beta-1-\frac{\beta N}{\alpha p}-\frac{\beta}{\alpha}}\left\|u(\tau)\right\|^2_{\mathcal{M}^p_{p_1}} d\tau \\
&\le C_2 \int_0^t (t-\tau)^{\beta-1-\frac{\beta N}{\alpha p}-\frac{\beta}{\alpha}}\tau^{\frac{2\beta N}{\alpha p}+\frac{2\beta}{\alpha}-2\beta}d\tau\left\|u(\tau)\right\|^2_{X_3} \\
&= C_2\, t^{\frac{\beta N}{\alpha p}+\frac{\beta}{\alpha}-\beta}\, b\left(\beta-\frac{\beta}{\alpha}-\frac{\beta N}{\alpha p}\,,\ 1-2\beta+\frac{2\beta}{\alpha}+\frac{2\beta N}{\alpha p}\right)\left\|u(\tau)\right\|^2_{X_3} \\
&= K_7\, t^{\frac{\beta N}{\alpha p}+\frac{\beta}{\alpha}-\beta}\left\|u\right\|^2_{X_3},
\end{aligned} \tag{4.10}$$

for all $t > 0$, where $K_7 = K_7(N, \beta, p, p_1)$.

□

Lemma 4.5. *There exists a constant $\gamma > 0$ such that for all $n \in X_1$,*

$$\left\|L_3(n)(t)\right\|_{X_3} \le \gamma\left\|n\right\|_{X_1}.$$

Proof. From the conditions (i), (ii) and (iii) in Assumption 1, we have that

$$\beta - \frac{\beta}{\alpha} - \frac{\beta N}{\alpha q} + \frac{\beta N}{\alpha p} > 0, \qquad 1 - 2\beta + \frac{2\beta}{\alpha} + \frac{\beta N}{\alpha q} > 0.$$

Taking $s_5 = \frac{N_1 q_1}{N_1 + q_1}$, from (A), (B) and (C) in Assumption 1, we get

$$1 \le s_5 \le \frac{Nq}{N+q} \le p \ \text{ and } \ \frac{p}{p_1} \ge \frac{Nq}{N+q}\frac{1}{s_5}.$$

Thus $L_3(n)$ can be estimated as follows:

(4.11)

$$\begin{aligned}
\|L_3(n)(t)\|_{\mathcal{M}^p_{p_1}} &= \left\| \int_0^t (t-\tau)^{\beta-1} \boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2}) \mathbb{P}(n\nabla\phi)(\tau)\, d\tau \right\|_{\mathcal{M}^p_{p_1}} \\
&\leq \int_0^t (t-\tau)^{\beta-1} \left\| \mathbb{P}\boldsymbol{E}_{\beta,\beta}((t-\tau)^\beta(-\Delta)^{\alpha/2}))(n\nabla\phi)(\tau) \right\|_{\mathcal{M}^p_{p_1}} d\tau \\
&\leq C_1 \int_0^t (t-\tau)^{\beta-1}(t-\tau)^{-\frac{\beta N}{\alpha}\left(\frac{1}{q}+\frac{1}{N}-\frac{1}{p}\right)} \|(n\nabla\phi)(\tau)\|_{\mathcal{M}^{\frac{Nq}{N+q}}_{s_5}} d\tau \\
&\leq C_1 \int_0^t (t-\tau)^{\beta-1-\frac{\beta N}{\alpha q}-\frac{\beta}{\alpha}+\frac{\beta N}{\alpha p}} \|\nabla\phi(\tau)\|_{\mathcal{M}^N_{N_1}} \|n(\tau)\|_{\mathcal{M}^q_{q_1}} d\tau \quad \text{(by (1.3))} \\
&\leq C_\phi \int_0^t (t-\tau)^{\beta-1-\frac{\beta N}{\alpha q}-\frac{\beta}{\alpha}+\frac{\beta N}{\alpha p}} \tau^{-\beta+\frac{2\beta}{\alpha}} \tau^{\frac{\beta N}{\alpha q}-\beta} d\tau\, \|n(\tau)\|_{X_1} \quad \text{(by (2.3))} \\
&\leq C_\phi\, t^{\frac{\beta N}{\alpha q}+\frac{\beta}{\alpha}-\beta}\, b\left(\beta-\frac{\beta}{\alpha}-\frac{\beta N}{\alpha q}+\frac{\beta N}{\alpha p},\ 1-2\beta+\frac{2\beta}{\alpha}+\frac{\beta N}{\alpha q}\right) \|n(\tau)\|_{X_1} \\
&= \gamma\, t^{\frac{\beta N}{\alpha q}+\frac{\beta}{\alpha}-\beta} \|n\|_{X_1},
\end{aligned}$$

for all $t > 0$, where $\gamma = \gamma(N, \beta, N_1, p, p_1, q, q_1, \phi)$. □

4.2. Proof of Theorem 2.2. Consider X_1, X_2, X_3 as in (2.4) and let

$$y = \left[\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})n_0,\ \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})c_0,\ \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})u_0\right].$$

For $x = [n, c, u] \in \mathcal{X} = X_1 \times X_2 \times X_3$, we denote $\mathcal{B}(x) = [\mathcal{B}_1(x),\ \mathcal{B}_2(x),\ \mathcal{B}_3(x)]$, where

$$\begin{aligned}
\mathcal{B}_1(x) &= \mathcal{B}^1_{3,1}(u,n)(x) + \mathcal{B}^1_{1,2}(n,c)(x), \\
\mathcal{B}_2(x) &= \mathcal{B}^2_{3,2}(u,c)(x) + \mathcal{B}^2_{1,2}(n,c)(x), \\
\mathcal{B}_3(x) &= \mathcal{B}^3_{3,3}(u,u)(x) + L_3 \circ \left(\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})n_0 + \mathcal{B}_1\right)(x).
\end{aligned} \tag{4.12}$$

Note that from Lemmas 4.2, 4.3 and 4.4 the operators $\mathcal{B}^k_{i,j}$ in (4.12) are bilinear and continuous; also, from Lemma 4.5 it holds that L_3 is a continuous linear map. Now, let

$$\mathcal{K}_1 = 1 + \gamma \quad \textit{and} \quad \mathcal{K}_2 = \gamma(K_1 + K_2) + \sum_{i=1}^{7} K_i, \tag{4.13}$$

where $K_1, K_2, \cdots, K_7$ and γ are the constants given in the previous lemmas. Then, computing the norm $\mathcal{X}$ of y, we get

$$\begin{aligned}
\|y\|_{\mathcal{X}} &= \left\|\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})n_0\right\|_{X_1} + \left\|\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})c_0\right\|_{X_2} + \left\|\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})u_0\right\|_{X_3} \\
&= \sup_{t>0} t^{-\frac{\beta N}{\alpha q}+\beta} \left\|\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})n_0\right\|_{\mathcal{M}^q_{q_1}} + \sup_{t>0} \left\|\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})c_0\right\|_{L^{\frac{N}{\alpha+\theta-2}}} \\
&\quad + \sup_{t>0} t^{-\frac{\beta N}{\alpha r}-\frac{\beta}{\alpha}+\beta} \left\|\nabla(-\Delta)^{-\theta/2}\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})c_0\right\|_{\mathcal{M}^r_{r_1}} \\
&\quad + \sup_{t>0} t^{-\frac{\beta N}{\alpha p}-\frac{\beta}{\alpha}+\beta} \left\|\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})u_0\right\|_{\mathcal{M}^p_{p_1}}.
\end{aligned}$$

Recalling that the initial data satisfies (2.2), applying Lemma 3.5 and taking into account the definition of the norm $\mathcal{X}$ in (2.6) we obtain

$$
\begin{aligned}
\|y\|_{\mathcal{X}} &= \sup_{t>0} t^{-\frac{\beta N}{\alpha q}+\beta}\, \mathcal{C}_0\, t^{\frac{\frac{\beta N}{q}-\beta\alpha}{\alpha}}\, \|n_0\|_{\mathcal{N}^{\frac{N}{q}-\alpha}_{q,q_1,\infty}} + \mathcal{C}_0\, \|c_0\|_{L^{\frac{N}{\alpha+\theta-2}}} \\
&\quad + \sup_{t>0} t^{-\frac{\beta N}{\alpha r}-\frac{\beta}{\alpha}+\beta}\, \mathcal{C}_0\, t^{\frac{\frac{\beta N}{r}-\beta\alpha+\beta}{\alpha}} \left\|\nabla((-\Delta)^{-\theta/2}c_0)\right\|_{\mathcal{N}^{\frac{N}{r}-\alpha+1}_{r,r_1,\infty}} \\
&\quad + \sup_{t>0} t^{-\frac{\beta N}{\alpha p}-\frac{\beta}{\alpha}+\beta}\, \mathcal{C}_0\, t^{\frac{\frac{\beta N}{p}-\beta\alpha+\beta}{\alpha}}\, \|u_0\|_{\mathcal{N}^{\frac{N}{p}-\alpha+1}_{p,p_1,\infty}} \\
&= \mathcal{C}_0 \sup_{t>0} t^{-\frac{\beta N}{\alpha q}+\beta}\, t^{\frac{\beta N}{\alpha q}-\beta}\, \|n_0\|_{\mathcal{N}^{\frac{N}{q}-\alpha}_{q,q_1,\infty}} + \mathcal{C}_0\, \|c_0\|_{L^{\frac{N}{\alpha+\theta-2}}} \\
&\quad + \mathcal{C}_0 \sup_{t>0} t^{-\frac{\beta N}{\alpha r}-\frac{\beta}{\alpha}+\beta}\, t^{\frac{\beta N}{\alpha r}+\frac{\beta}{\alpha}-\beta} \left\|\nabla((-\Delta)^{-\theta/2}c_0)\right\|_{\mathcal{N}^{\frac{N}{r}-\alpha+1}_{r,r_1,\infty}} \\
&\quad + \mathcal{C}_0 \sup_{t>0} t^{-\frac{\beta N}{\alpha p}-\frac{\beta}{\alpha}+\beta}\, t^{\frac{\beta N}{\alpha p}+\frac{\beta}{\alpha}-\beta}\, \|u_0\|_{\mathcal{N}^{\frac{N}{p}-\alpha+1}_{p,p_1,\infty}} \\
&\le \mathcal{C}_0 \left(\|n_0\|_{\mathcal{N}^{\frac{N}{q}-\alpha}_{q,q_1,\infty}} + \|c_0\|_{L^{\frac{N}{\alpha+\theta-2}}} + \left\|\nabla((-\Delta)^{-\theta/2}c_0)\right\|_{\mathcal{N}^{\frac{N}{r}-\alpha+1}_{r,r_1,\infty}} + \|u_0\|_{\mathcal{N}^{\frac{N}{p}-\alpha+1}_{p,p_1,\infty}} \right) \\
&\le \mathcal{C}_0\, \|(n_0, c_0, u_0)\|_{\mathcal{I}}\, .
\end{aligned}
$$

Then $\|y\|_{\mathcal{X}} \le \epsilon$ provided that $\|(n_0, c_0, u_0)\|_{\mathcal{I}} \le \delta$, where $\delta = \frac{\epsilon}{\mathcal{C}_0}$. Hence, from Lemma 4.1 we have that, if $0 < \epsilon < \dfrac{1}{4\mathcal{K}_1\mathcal{K}_2}$, there exists a unique solution $x \in \mathcal{X}$ for the equation $x = y + \mathcal{B}(x)$, and thus, there exists a unique solution $[n, c, u]$ for the system (4.1) such that $\|[n, c, u]\|_{\mathcal{X}} \le 2\mathcal{K}_1\epsilon$. ■

4.3. Proof from Theorem 2.3. We first show that (2.7) implies (2.8). Let $[n, c, u]$ and $[\tilde{n}, \tilde{c}, \tilde{u}]$ be two solutions provided by Theorem 2.2 and let

$$
l_q = -\frac{\beta N}{\alpha q} + \beta, \quad \mu_r = -\frac{\beta N}{\alpha r} - \frac{\beta}{\alpha} + \beta \quad \text{and} \quad \mu_p = -\frac{\beta N}{\alpha p} - \frac{\beta}{\alpha} + \beta.
$$

Taking the difference $n - \tilde{n}$ and computing the norm $\|\cdot\|_{X_1}$, we get

(4.14)

$$
\begin{aligned}
&t^{l_q}\, \|n(t) - \tilde{n}(t)\|_{\mathcal{M}^q_{q_1}} \\
&\le t^{l_q} \left\|\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})(n_0 - \tilde{n}_0)\right\|_{\mathcal{M}^q_{q_1}} \\
&+ t^{l_q} \int_0^t (t-\tau)^{\beta-1} \left\|\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2})\,(u\cdot\nabla n - \tilde{u}\cdot\nabla\tilde{n})\right\|_{\mathcal{M}^q_{q_1}} d\tau \\
&+ t^{l_q} \int_0^t (t-\tau)^{\beta-1} \left\|\nabla\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2}) \left(n\nabla((-\Delta)^{-\theta/2}c) - \tilde{n}\nabla((-\Delta)^{-\theta/2}\tilde{c})\right)\right\|_{\mathcal{M}^q_{q_1}} \\
&:= t^{l_q} \left\|\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})(n_0 - \tilde{n}_0)\right\|_{\mathcal{M}^q_{q_1}} + J_1(t) + J_2(t).
\end{aligned}
$$

Using that $u\cdot\nabla n-\tilde{u}\cdot\nabla\tilde{n}=(u-\tilde{u})\cdot\nabla n+\tilde{u}\cdot\nabla(n-\tilde{n})$ and working as in (4.2), the integral J_1 is bounded as follows:

(4.15)
$$\begin{aligned}
J_1(t)\leq&\tilde{C}_1\ t^{l_q}\int_0^t(t-\tau)^{\mu_p-1}\left(\|(u-\tilde{u})(\tau)\|_{\mathcal{M}^p_{p_1}}\|n(\tau)\|_{\mathcal{M}^q_{q_1}}+\|\tilde{u}(\tau)\|_{\mathcal{M}^p_{p_1}}\|(n-\tilde{n})(\tau)\|_{\mathcal{M}^q_{q_1}}\right)\\
&\leq\tilde{C}_1\ t^{l_q}\int_0^t(t-\tau)^{\mu_p-1}\tau^{-\mu_p-l_q}\tau^{\mu_p}\|(u-\tilde{u})(\tau)\|_{\mathcal{M}^p_{p_1}}\|n\|_{X_1}\,d\tau\\
&\quad+\tilde{C}_1\ t^{l_q}\int_0^t(t-\tau)^{\mu_p-1}\tau^{-\mu_p-l_q}\tau^{l_q}\|\tilde{u}\|_{X_4}\|(n-\tilde{n})(\tau)\|_{\mathcal{M}^q_{q_1}}\,d\tau\\
&\leq\tilde{C}_1\int_0^1(1-z)^{\mu_p-1}z^{-\mu_p-l_q}(tz)^{\mu_p}\|(u-\tilde{u})(tz)\|_{\mathcal{M}^p_{p_1}}\|n\|_{X_1}\,dz\\
&\quad+\tilde{C}_1\int_0^1(1-z)^{\mu_p-1}z^{-\mu_p-l_q}(tz)^{l_q}\|\tilde{u}\|_{X_4}\|(n-\tilde{n})(tz)\|_{\mathcal{M}^q_{q_1}}\,dz\quad(\text{taking }\tau=tz)\\
&\leq\tilde{C}_1\int_0^1(1-z)^{\mu_p-1}z^{-\mu_p-l_q}\\
&\qquad\left((tz)^{\mu_p}\|(u-\tilde{u})(tz)\|_{\mathcal{M}^p_{p_1}}\|n\|_{X_1}+(tz)^{l_q}\|\tilde{u}\|_{X_4}\|(n-\tilde{n})(tz)\|_{\mathcal{M}^q_{q_1}}\right)dz.
\end{aligned}$$

Working in analogous way to the last inequality, but taking into account the reasoning used in (4.3) we have that

(4.16)
$$\begin{aligned}
J_2(t)\leq&\tilde{C}_2\ t^{l_q}\int_0^t(t-\tau)^{\mu_r-1}\tau^{-l_q-\mu_r}\tau^{l_q}\|(n-\tilde{n})(\tau)\|_{\mathcal{M}^q_{q_1}}\|c(\tau)\|_{X_2}\,d\tau\\
&\quad+\tilde{C}_2\ t^{l_q}\int_0^t(t-\tau)^{\mu_r-1}\tau^{-l_q-\mu_r}\tau^{\mu_r}\|\tilde{n}(\tau)\|_{X_1}\left\|\nabla((-\Delta)^{-\theta/2}(c-\tilde{c}))(\tau)\right\|_{\mathcal{M}^r_{r_1}}d\tau\\
&\leq\tilde{C}_2\int_0^1(1-z)^{\mu_r-1}z^{-l_q-\mu_r}(tz)^{l_q}\|(n-\tilde{n})(tz)\|_{\mathcal{M}^q_{q_1}}\|c\|_{X_2}\,dz\\
&\quad+\tilde{C}_2\int_0^1(1-z)^{\mu_r-1}z^{-l_q-\mu_r}(tz)^{\mu_r}\|\tilde{n}\|_{X_1}\left\|\nabla((-\Delta)^{-\theta/2}(c-\tilde{c}))(tz)\right\|_{\mathcal{M}^r_{r_1}}dz\\
&\leq\tilde{C}_2\int_0^1(1-z)^{\mu_r-1}z^{-l_q-\mu_r}\\
&\quad\left((tz)^{l_q}\|(n-\tilde{n})(tz)\|_{\mathcal{M}^q_{q_1}}\|c\|_{X_2}+(tz)^{\mu_r}\left\|\nabla((-\Delta)^{-\theta/2}(c-\tilde{c}))(tz)\right\|_{\mathcal{M}^r_{r_1}}\|\tilde{n}\|_{X_1}\right)dz.
\end{aligned}$$

On the other hand,

(4.17)
$$\begin{aligned}
&\|(c-\tilde{c})(t)\|_{L^{\frac{N}{\alpha+\theta-2}}}\leq\left\|\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})(c_0-\tilde{c}_0)\right\|_{L^{\frac{N}{\alpha+\theta-2}}}\\
&\quad+\int_0^t(t-\tau)^{\beta-1}\left\|\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\alpha/2})\left(u\cdot\nabla c-nc-\tilde{u}\cdot\nabla\tilde{c}-\tilde{n}\tilde{c}\right)(\tau)\right\|_{L^{\frac{N}{\alpha+\theta-2}}}\\
&:=\left\|\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})(c_0-\tilde{c}_0)\right\|_{L^{\frac{N}{\alpha+\theta-2}}}+J_3(t),
\end{aligned}$$

$$
\begin{aligned}
(4.18)\qquad & t^{\mu_r}\left\|\nabla(-\Delta)^{-\theta/2}(c-\tilde{c})(t)\right\|_{\mathcal{M}^r_{r_1}}\\
&\le t^{\mu_r}\left\|\nabla(-\Delta)^{-\theta/2}\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^{\beta}(-\Delta)^{\alpha/2})(c_0-\tilde{c}_0)\right\|_{\mathcal{M}^r_{r_1}}\\
&+t^{\mu_r}\int_0^t(t-\tau)^{\beta-1}\\
&\left\|\nabla(-\Delta)^{-\theta/2}\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^{\beta}(-\Delta)^{\alpha/2})\left(u\cdot\nabla c-nc-\tilde{u}\cdot\nabla\tilde{c}-\tilde{n}\tilde{c}\right)(\tau)\right\|_{\mathcal{M}^r_{r_1}}\\
&:=t^{\mu_r}\left\|\nabla(-\Delta)^{-\theta/2}\boldsymbol{E}_{\beta,\beta}(-(t-\tau)^{\beta}(-\Delta)^{\alpha/2})(c_0-\tilde{c}_0)\right\|_{\mathcal{M}^r_{r_1}}+J_4(t),
\end{aligned}
$$

and

$$
\begin{aligned}
(4.19)\qquad & t^{\mu_p}\left\|(u-\tilde{u})(t)\right\|_{\mathcal{M}^p_{p_1}}\le t^{\mu_p}\left\|\boldsymbol{E}_{\beta}(-t^{\beta}(-\Delta)^{\alpha/2})(u_0-\tilde{u}_0)\right\|_{\mathcal{M}^p_{p_1}}\\
&+t^{\mu_p}\int_0^t(t-\tau)^{\beta-1}\left\|\boldsymbol{E}_{\beta,\beta}((t-\tau)^{\beta}(-\Delta)^{\alpha/2})\mathbb{P}(u\cdot\nabla u-\tilde{u}\cdot\nabla\tilde{u})(\tau)\right\|_{\mathcal{M}^p_{p_1}}d\tau\\
&+t^{\mu_p}\int_0^t(t-\tau)^{\beta-1}\left\|\boldsymbol{E}_{\beta,\beta}((t-\tau)^{\beta}(-\Delta)^{\alpha/2})(n\nabla\phi-\tilde{n}\nabla\tilde{\phi})(\tau)\right\|_{\mathcal{M}^p_{p_1}}d\tau\\
&:=t^{\mu_p}\left\|\boldsymbol{E}_{\beta}(-t^{\beta}(-\Delta)^{\alpha/2})(u_0-\tilde{u}_0)\right\|_{\mathcal{M}^p_{p_1}}+J_5(t)+J_6(t).
\end{aligned}
$$

On the other hand, from the estimates (4.6),(4.7),(4.8),(4.9),(4.10) and (4.11) we have

$$
\begin{aligned}
(4.20)\qquad & J_3(t)\\
&\le\tilde{C}_3\int_0^1(1-z)^{\mu_p-1}z^{-\mu_p}\\
&\left((tz)^{\mu_p}\left\|(u-\tilde{u})(tz)\right\|_{\mathcal{M}^p_{p_1}}\|c\|_{X_2}+\|\tilde{u}\|_{X_4}\left\|(c-\tilde{c})(tz)\right\|_{L^{\frac{N}{\alpha+\theta-2}}}\right)dz\\
&+\tilde{C}_4\int_0^1(1-z)^{l_q-1}z^{-l_q}\\
&\left((tz)^{l_q}\left\|(n-\tilde{n})(tz)\right\|_{\mathcal{M}^q_{q_1}}\|c\|_{X_2}+\|\tilde{n}\|_{X_4}\left\|(c-\tilde{c})(tz)\right\|_{L^{\frac{N}{\alpha+\theta-2}}}\right)dz,
\end{aligned}
$$

$$
\begin{aligned}
J_4(t)&\le\tilde{C}_5\int_0^1(1-z)^{\mu_p-1}z^{-\mu_p-\mu_r}\\
&\left((tz)^{\mu_p}\left\|(u-\tilde{u})(tz)\right\|_{\mathcal{M}^p_{p_1}}\|c\|_{X_2}+\|\tilde{u}\|_{X_4}(tz)^{\mu_r}\left\|\nabla(-\Delta)^{-\theta/2}(c-\tilde{c})(tz)\right\|_{\mathcal{M}^r_{r_1}}\right)dz\\
&+\tilde{C}_6\int_0^1(1-z)^{l_q-\mu_r-1}z^{-l_q}\\
&\left((tz)^{\mu_p}\left\|(u-\tilde{u})(tz)\right\|_{\mathcal{M}^p_{p_1}}\|c\|_{X_2}+\|\tilde{u}\|_{X_4}\left\|(c-\tilde{c})(tz)\right\|_{L^{\frac{N}{\alpha+\theta-2}}}\right)dz,
\end{aligned}
$$

$$
\begin{aligned}
J_5(t)&\le\tilde{C}_7\int_0^1(1-z)^{\mu_p-1}z^{-2\mu_p}\\
&\left((tz)^{\mu_p}\left\|(u-\tilde{u})(tz)\right\|_{\mathcal{M}^p_{p_1}}\|u\|_{X_4}+\|\tilde{u}\|_{X_4}(tz)^{\mu_p}\left\|(u-\tilde{u})(tz)\right\|_{\mathcal{M}^p_{p_1}}\right)dz,
\end{aligned}
$$

and

$$
(4.21)\qquad J_6(t)\le\tilde{\gamma}\int_0^1(1-z)^{l_q-\mu_p-1}z^{-l_q}(tz)^{l_q}\left\|(n-\tilde{n})(tz)\right\|_{\mathcal{M}^q_{q_1}}dz.
$$

Now we define

$$A_1 := \limsup_{t\to\infty} t^{l_q} \|n(\cdot,t) - \tilde{n}(\cdot,t)\|_{\mathcal{M}^q_{q_1}}, \quad A_2 := \limsup_{t\to\infty} \|c(\cdot,t) - \tilde{c}(\cdot,t)\|_{L^{\frac{N}{\alpha+\theta-2}}},$$
$$A_3 := \limsup_{t\to\infty} t^{\mu_r} \left\|\nabla(-\Delta)^{-\theta/2}(c(\cdot,t) - \tilde{c}(\cdot,t))\right\|_{\mathcal{M}^r_{r_1}},$$
$$A_4 := \limsup_{t\to\infty} t^{\mu_p} \|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{\mathcal{M}^p_{p_1}}.$$

Since $\|[n,c,u]\|_{\mathcal{X}}, \|[\tilde{n},\tilde{c},\tilde{u}]\| \leq 2\mathcal{K}_1\epsilon$, we have that $A_1, A_2, A_3, A_4 < \infty$. Taking the $\limsup_{t\to\infty}$ in (4.14),(4.17), (4.18), (4.19), and using (4.15), (4.16) and (4.20)-(4.21), we obtain

$$\begin{aligned}
&A_1 \leq 0 + \tilde{C}_1\ 2\mathcal{K}_1\epsilon \int_0^1 (1-z)^{\mu_p-1} z^{-\mu_p-l_q}\ dz\ (A_4 + A_1) \\
&+ \tilde{C}_2\ 2\mathcal{K}_1\epsilon \int_0^1 (1-z)^{\mu_r-1} z^{-l_q-\mu_r}\ dz\ (A_1 + A_3) \\
&\leq 2\mathcal{K}_1\epsilon\left[K_1(A_1+A_4) + K_2(A_1+A_3)\right], \\
&A_2 \leq 0 + \tilde{C}_3 2\mathcal{K}_1\epsilon \int_0^1 (1-z)^{\mu_p-1} z^{-\mu_p}\ dz\ (A_4 + A_2) \\
&+ \tilde{C}_4\ 2\mathcal{K}_1\epsilon \int_0^1 (1-z)^{\mu_r-1} z^{-l_q}\ dz\ (A_1 + A_2) \\
&\leq 2\mathcal{K}_1\epsilon\left[K_3(A_2+A_4) + K_4(A_1+A_2)\right], \\
&A_3 \leq 0 + \tilde{C}_5\ 2\mathcal{K}_1\epsilon \int_0^1 (1-z)^{\mu_p-1} z^{-\mu_p-\mu_r}\ dz\ (A_4 + A_3) \\
&+ \tilde{C}_6\ 2\mathcal{K}_1\epsilon \int_0^1 (1-z)^{l_q-\mu_r-1} z^{-l_q}\ dz\ (A_1 + A_2) \\
&\leq 2\mathcal{K}_1\epsilon\left[K_5(A_3+A_4) + K_6(A_1+A_2)\right], \\
&A_4 \leq 0 + \tilde{C}_7 2\mathcal{K}_1\epsilon \int_0^1 (1-z)^{\mu_p-1} z^{-2\mu_p}\ dz\ (A_4 + A_4) + \\
&\tilde{\gamma} \int_0^1 (1-z)^{l_q-\mu_p-1} z^{-l_q}\ dz\ A_1 \\
&\quad \leq 2\mathcal{K}_1\epsilon\left[K_7\ 2A_4\right] + \gamma A_1,
\end{aligned}$$

where $K_1, K_2, K_3, K_4, K_5, K_6, K_7$ and γ are as in Lemmas 4.2,4.3, 4.4 and 4.5, respectively. Recalling that $\mathcal{K}_1 = 1+\gamma$ and $\mathcal{K}_2 = \gamma(K_1+K_2) + \sum_{i=1}^7 K_i$ (see (4.13)) and summing all A_i's, we arrive at

$$\begin{aligned}
&A_1 + A_2 + A_3 + A_4 \\
&\leq 2\mathcal{K}_1\epsilon\left[A_1(K_1+K_2+K_4+K_6)\right] + A_2(K_3+K_4+K_6) + A_3(K_2+K_5) \\
&\qquad + A_4(K_1+K_3+K_5+2K_7) + \gamma A_1] \\
&\leq 2\mathcal{K}_1\epsilon\left[A_1(K_1+K_2+K_4+K_6) + \gamma(K_1+K_2) + A_2(K_3+K_4+K_6)\right. \\
&\qquad \left. + A_3(K_2+K_5+\gamma K_2) + A_4(K_1+K_3+2K_7+\gamma K_1)\right] \\
&\leq 2\mathcal{K}_1\epsilon\left[A_1\left((K_1+K_2+K_4+K_6) + \gamma(K_1+K_2)\right) + A_2(K_3+K_4+K_6)\right. \\
&\qquad\qquad \left. + A_3(K_2+K_5+\gamma K_2) + A_4(K_1+K_3+2K_7+\gamma K_1)\right] \\
&\leq 2\mathcal{K}_1\epsilon(A_1+A_2+A_3+A_4) \times \left[(k_1+k_2+K_3+K_4+K_5+K_6+2K_7) + \gamma(K_1+K_2)\right],
\end{aligned}$$

Note that $K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + 2K_7 + \gamma(K_1 + K_2) \leq 2\mathcal{K}_2$, and then

$$A_1 + A_2 + A_3 + A_4 \leq 4\mathcal{K}_1\mathcal{K}_2\epsilon\ (A_1 + A_2 + A_3 + A_4),$$

but observing that $4\mathcal{K}_1\mathcal{K}_2\epsilon < 1$, we conclude that

$$A_1 = A_2 = A_3 = A_4 = 0.$$

Now we turn to show that (2.8) implies (2.7). We proceed as in the estimates (4.14), (4.17)-(4.19) and using the hypotheses $A_1 = A_2 = A_3 = A_4 = 0$ in order to obtain

$$\limsup_{t\to\infty} t^{l_q} \left\| \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})(n_0 - \tilde{n}_0) \right\|_{\mathcal{M}^q_{q_1}} \leq A_1 + \limsup_{t\to\infty} (J_1(t) + J_2(t))$$
$$\leq A_1 + 2\mathcal{K}_1\epsilon K_1(A_1 + A_4) + 2\mathcal{K}_1\epsilon K_2(A_1 + A_3) \leq 0,$$

$$\limsup_{t\to\infty} \left\| \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})(c_0 - \tilde{c}_0) \right\|_{L^{\frac{N}{\alpha+\theta-2}}} \leq A_2 + \limsup_{t\to\infty} J_3(t)$$
$$\leq A_2 + 2\mathcal{K}_1\epsilon K_3(A_2 + A_4) + 2\mathcal{K}_1\epsilon K_4(A_1 + A_2) \leq 0,$$

$$\limsup_{t\to\infty} t^{\mu_r} \left\| \nabla(-\Delta)^{-\theta/2}\boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})(c_0 - \tilde{c}_0) \right\|_{\mathcal{M}^r_{r_1}} \leq A_3 + \limsup_{t\to\infty} J_4(t)$$
$$\leq A_3 + 2\mathcal{K}_1\epsilon K_5(A_3 + A_4) + 2\mathcal{K}_1\epsilon K_6(A_1 + A_2) \leq 0,$$

and

$$\limsup_{t\to\infty} t^{\mu_p} \left\| \boldsymbol{E}_\beta(-t^\beta(-\Delta)^{\alpha/2})(u_0 - \tilde{u}_0) \right\|_{\mathcal{M}^p_{p_1}} \leq A_4 + \limsup_{t\to\infty}(J_5(t) + J_6(t))$$
$$\leq A_4 + 2\mathcal{K}_1\epsilon K_7 2A_4 + \gamma A_1 \leq 0$$

and thus, the proof is finished. ∎

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