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Constant vorticity atmospheric Ekman flows in the modified $\beta-$plane approximation

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Abstract. In this paper, we study the classical problem of the wind in the steady atmospheric Ekman layer with constant eddy viscosity. The full nonlinear governing equations with the general boundary conditions are considered in the sense of modified $\beta-$plane approximation. Under the assumption of a flat surface and constant vorticity vector, we show that the flow has one non-vanishing component, a result that differs from that valid within the framework of the standard $\beta-$plane approximation.

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1. Introduction

Ekman layers, called after the famous Swedish oceanographer who studied the wind-drift in the ocean, are boundary layers in which there is a three-way balance among frictional effects, pressure gradient and the influence of the coriolis force. The atmospheric Ekman layer occurs near the ground and the stress at the ground itself

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is due to the surface wind and its vertical variation [**1, 2, 3**]. The classical Ekman layer assumes a constant vertical eddy viscosity, which is rarely, if ever, observed in the atmosphere or in the ocean, more realistic models with a viscosity that varies with height/depth have been considered. Some results [**4, 5, 6, 7, 8, 9, 10, 11, 12**] have been made on the explicit formula of the standard Ekman equations for non-constant eddy viscosity with the classical boundary conditions, whether in the context of atmospheric flows or regarding wind-generated ocean currents.

Note that the standard Ekman system is an idealized modelling which the inertial acceleration terms and the component of the wind in the direction of z are omitted. Recently, Wang et al. [**13**] considered the following model

$$\begin{cases} \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv - \widehat{f}w + \frac{\partial}{\partial z}(k\frac{\partial u}{\partial z}), \\ \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + \frac{\partial}{\partial z}(k\frac{\partial v}{\partial z}), \\ \frac{Dw}{Dt} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + \widehat{f}u + \frac{\partial}{\partial z}(k\frac{\partial w}{\partial z}), \end{cases} \tag{1.1}$$

with the general boundary conditions

$$p = p_{atm} \ on \ z = 0, \tag{1.2}$$

$$p = 0 \ on \ z = h_0, \tag{1.3}$$

$$w = 0 \ on \ z = 0, \tag{1.4}$$

and a bounded velocity field (u, v, w), which is physically reasonable.

Wang et al. [**13**] used a rectangular coordinate system rotating with the Earth whose origin was fixed at a point on the equator lying on the Earth's surface, and choosed the (x, y, z)-coordinates as: the x-axis was pointing horizontally due east (zonal direction), the y-axis was due north (meridional direction), and the z-axis was pointing vertically upwards and perpendicular to the Earth's surface. $u = u(t, x, y, z)$, $v = v(t, x, y, z)$ and $w = w(t, x, y, z)$ were the components of the wind in the x, y and z directions, respectively, $p = p(t, x, y, z)$ was the atmospheric pressure, ρ was the reference density and $\widehat{f} = 2\Omega\cos\theta$, k denoted the eddy viscosity, $f = 2\Omega\sin\theta$ was the Coriolis parameter at the fixed latitude θ in the Northern Hemisphere, $\Omega \approx 7.29 \times 10^{-5}s^{-1}$ was the angular speed of rotation of the Earth and the θ was the angle of latitude in right-handed rotating spherical coordinates, h_0 was the height of the Ekman boundary layer [**14, 15, 16**].

To mitigate the effects generated by the Coriolis terms, some simple approximate models for geophysical water waves are investigated, such as $\beta-$ and $f-$plane approximation. The $\beta-$plane approximation, which treats the Earth's curved surface approximated (locally) as a tangent plane, only applies in regions within 5^0 latitude of the meridional extent [**17**], the $f-$plane approximation takes a constant Coriolis parameter into account, and this approximation is adequate within a restricted meridional range of approximately 2^0 latitude [**17, 18**]. A great deal of mathematical progress has been made in deriving and analyzing exact solutions to the $\beta-$plane equation [**19**] for simulating equatorially ocean waves propagating to the east. Besides, there exists a considerable literature on the geophysical water waves related to different physical scenarios, including the $\beta-$plane approximation [**20**], the $f-$plane approximation [**21, 22**].

The $\beta-$plane approximation is generally accepted as reasonable for large ocean currents, nevertheless, from mathematical modelling perspective, because of the flattening out of the Earth's surface, the model approximated by $\beta-$plane approximation partially loses its rigor and structure. The modified $\beta-$plane approximation method is a simple and interesting one for ocean currents, which retains the geometry of the Earth's curvature by way of adding a gravitational-correction term to the standard $\beta-$plane approximation [**23, 24**].

The governing equations of water flows and atmospheric Ekman flows are derived from Navier-Stokes equations [**25, 26, 27, 28**], but the specific forms of the equations are still quite different. In addition, the corresponding boundary conditions are also different due to the different specific backgrounds. Motivated by the above papers, we transfer the approach and theory in water flows to atmospheric Ekman flows and study the atmospheric Ekman flows by the modified $\beta-$plane approximation method.

In this paper, we investigate the full nonlinear governing equations with the general boundary conditions. We show that the Ekman flow is necessarily irrotational, and possess non-vanishing horizontal velocity field if it exhibits a constant vorticity. The constancy of vorticity restrict considerably the nature of the water flow was pioneered in [**29**], but without taking into account Coriolis effects or viscosity. Our results are much different from recent results in [**13**], in which it was proved that the only Ekman flows in the $\beta-$plane approximation exhibiting a constant vorticity vector are those with vanishing velocity field. Moreover, the pressure is $p_{atm} - 2\rho gz$, which is different from [**13**].

2. Preliminaries

In the local cartesian coordinate system, the Earth's surface is approximately regarded as a plane, and the curvature term can be omitted, using the Reynolds averaging theory and the Flux-Gradient theory, thus, the Ekman layer is governed by the following mean equations (see [**1, 30**])

$$\begin{cases} \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv - \widehat{f}w + \frac{\partial}{\partial z}(k\frac{\partial u}{\partial z}) + F_x, \\ \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + \frac{\partial}{\partial z}(k\frac{\partial v}{\partial z}) + F_y, \\ \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \widehat{f}u + \frac{\partial}{\partial z}(k\frac{\partial w}{\partial z}) + F_z, \end{cases}$$

which is the same as (1.1), here k is generally a variable quantity $k = k(x, y, z, t)$, following the traditional simplification in semi-empirical theory, we assume k is a constant and $k \neq 0$, $\overrightarrow{F} = (F_x, F_y, F_z) = (0, 0, -g)$ is the external body force, then we get

$$\begin{cases} \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv - \widehat{f}w + k\frac{\partial^2 u}{\partial z^2} + F_x, \\ \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + k\frac{\partial^2 v}{\partial z^2} + F_y, \\ \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \widehat{f}u + k\frac{\partial^2 w}{\partial z^2} + F_z. \end{cases} \tag{2.1}$$

It is reasonable to approximate a given latitude in the spherical setting by $\theta \approx \frac{y}{R}$, where y is the coordinate due to latitude. We assume that the flow is confined to a strip around the equator, so, to the first-order expansion in terms of θ above zero, the (2.1) becomes

$$\begin{cases} \frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \beta yv - 2\Omega w + k\frac{\partial^2 u}{\partial z^2} + F_x, \\ \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \beta yu + k\frac{\partial^2 v}{\partial z^2} + F_y, \\ \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u + k\frac{\partial^2 w}{\partial z^2} + F_z, \end{cases} \tag{2.2}$$

and velocity fields satisfy the following continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{2.3}$$

where $\beta=\frac{2\Omega}{R}$, the equation (2.2) is a kind of approximation which is known as $\beta-$plane approximation. The region occupied by the fluid is approximated by a tangent plane, which is a valid approach as long as the spatial scale of the motion is moderate enough, but with this simplification the body force given by $\overrightarrow{F}$ is slightly miscalculated when taking the z coordinate of a given point on the tangent plane as its distance from the centre of the Earth, we accommodate a correction term which incorporates the deviation of the tangent plane from the earth's curved surface as follows [**23**]. We consider the point P in Figure 1. Note that its distance from

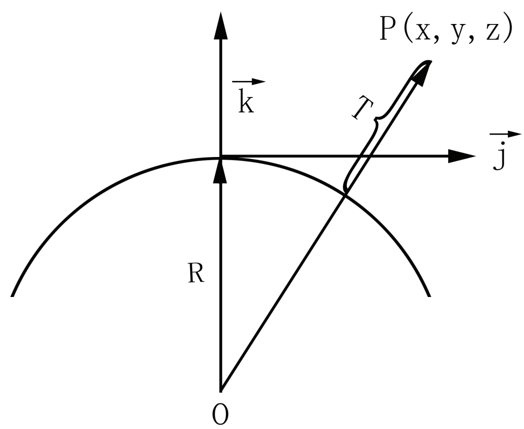

FIGURE 1. Schematic of tangent plane approximation.

the Earth's centre O is $OP=R+T=\sqrt{(R+z)^2+y^2}$, we can approximate the gravitational potential at P by

$$\Lambda(x,y,z)=Tg,$$

where

$$T=\sqrt{(R+z)^2+y^2}-R=\frac{z^2+2Rz+y^2}{\sqrt{(R+z)^2+y^2}+R}\approx z+\frac{y^2}{2R},$$

as R is significantly larger than either y or z. Hence, the body force of the modified equatorial $\beta-$plane equations $\overrightarrow{F}$ is given by

$$\overrightarrow{F}=-\nabla\Lambda=(0,-\frac{y}{R}g,-g),$$

then from (2.2), we obtain

$$\begin{cases}\frac{Du}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\beta yv-2\Omega w+k\frac{\partial^2 u}{\partial z^2},\\ \frac{Dv}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-\beta yu+k\frac{\partial^2 v}{\partial z^2}-\frac{g}{R}y,\\ \frac{Dw}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+2\Omega u+k\frac{\partial^2 w}{\partial z^2}-g.\end{cases} \tag{2.4}$$

We defined the vorticity vector $\overrightarrow{\Omega}$ as the curl of the velocity field, that is,

$$\overrightarrow{\Omega} = (\Omega_1, \Omega_2, \Omega_3) = curl\overrightarrow{U} = (w_y - v_z, u_z - w_x, v_x - u_y),$$

where $\overrightarrow{U} = (u, v, w)$. Thus, (2.4) can be written as

$$\overrightarrow{U_t} + (\overrightarrow{U} \cdot \nabla)\overrightarrow{U} + 2\Omega(w, 0, -u) - \beta y(v, -u, 0) - k(u_{zz}, v_{zz}, w_{zz}) = -\nabla\frac{1}{\rho}p + \overrightarrow{F},$$

as

$$(\overrightarrow{U} \cdot \nabla)\overrightarrow{U} = \nabla(\frac{1}{2}\overrightarrow{U} \cdot \overrightarrow{U}) + (\nabla \times \overrightarrow{U}) \times \overrightarrow{U},$$

so we have

$$\begin{aligned}&\overrightarrow{U_t} + (\nabla \times \overrightarrow{U}) \times \overrightarrow{U} + 2\Omega(w, 0, -u) - \beta y(v, -u, 0) - k(u_{zz}, v_{zz}, w_{zz})\\ &= -\nabla(\frac{1}{\rho}p + \frac{1}{2}\overrightarrow{U} \cdot \overrightarrow{U}) + \overrightarrow{F},\end{aligned}$$

for the above formula, we take the curl and get

$$\overrightarrow{\Omega_t} + \nabla \times (\overrightarrow{\Omega} \times \overrightarrow{U}) + \nabla \times [2\Omega(w, 0, -u) + \beta y(-v, u, 0)] - k\overrightarrow{\Omega_{zz}} = 0,$$

by direct calculation, we have

$$\overrightarrow{\Omega_t} + \nabla \times (\overrightarrow{\Omega} \times \overrightarrow{U}) - 2\Omega(u_y, v_y, w_y) - \beta y(u_z, v_z, w_z) + \beta(0, 0, v) - k\overrightarrow{\Omega_{zz}} = 0,$$

so we obtain the vorticity equation

$$\overrightarrow{\Omega_t} + (\overrightarrow{U} \cdot \nabla)\overrightarrow{\Omega} - 2\Omega(u_y, v_y, w_y) - \beta y(u_z, v_z, w_z) + \beta(0, 0, v) - k\overrightarrow{\Omega_{zz}} = (\overrightarrow{\Omega} \cdot \nabla)\overrightarrow{U}. \tag{2.5}$$

REMARK 2.1. In the following discussion, we will assume that $\overrightarrow{\Omega}$ is constant, in addition, Ω_3, Ω_2 satisfies

$$\Omega_3 + f_0 \neq 0, \Omega_2 + f_0 \neq 0 \tag{2.6}$$

we know that this is a reasonable assumption, since $\Omega \approx 0.73 \times 10^{-4}\ rads^{-1}$ while $\Omega_3 \approx 10^{-3}s^{-1}, \Omega_2 \approx 10^{-3}s^{-1}$ (see [**1**]).

3. Main results

THEOREM 3.1. *Suppose* $\overrightarrow{\Omega}$ *is constant throughout the flow and also* (2.6)*,* w *satisfies* (1.4)*, then the vorticity vector vanishes.*

PROOF. Since $\overrightarrow{\Omega}$ is constant throughout the flow, we get $\overrightarrow{\Omega_t} = 0$, $\overrightarrow{\Omega_{zz}} = 0$ and $(\overrightarrow{U} \cdot \nabla)\overrightarrow{\Omega} = 0$, then (2.5) becomes

$$(\overrightarrow{\Omega} \cdot \nabla)\overrightarrow{U} + 2\Omega(u_y, v_y, w_y) + \beta y(u_z, v_z, w_z) - \beta(0, 0, v) = 0,$$

that is

$$\begin{cases}\Omega_1 u_x + (\Omega_2 + 2\Omega)u_y + (\Omega_3 + \beta y)u_z = 0,\\ \Omega_1 v_x + (\Omega_2 + 2\Omega)v_y + (\Omega_3 + \beta y)v_z = 0,\\ \Omega_1 w_x + (\Omega_2 + 2\Omega)w_y + (\Omega_3 + \beta y)w_z - \beta v = 0.\end{cases} \tag{3.1}$$

We will prove the theorem in three steps.

Step 1, we assert that $w_y \equiv 0$.

From the first formula of (3.1), we get

$$\beta y u_z = -\Omega_1 u_x - (\Omega_2 + 2\Omega)u_y - \Omega_3 u_z,$$

so

$$\triangle(\beta y u_z) = -\Omega_1 \triangle u_x - (\Omega_2 + 2\Omega)\triangle u_y - \Omega_3 \triangle u_z = 0,$$

then

$$y\triangle u_z + 2u_{zy} = 0.$$

Note that the u, v, w and u_x, u_y, u_z are harmonic functions within the fluid domain [**20**], using the harmonic of u_z, we conclude that

$$u_{zy} = 0,$$

similarity, using the second and the third formula of (3.1), we infer that

$$v_{zy} = 0,\ w_{zy} = 0. \tag{3.2}$$

From the definition of Ω_1 and Ω_2, we get

$$w_{yy} = v_{zy} = 0, \tag{3.3}$$

and

$$w_{xy} = u_{zy} = 0, \tag{3.4}$$

due to (3.2), (3.3) and (3.4), we conclude that $w_y = f(t)$ for some function $f(t)$, in view of (1.4) and the constancy of w_y, we infer that

$$w_y \equiv 0, \tag{3.5}$$

throughout the flow, which implies

$$v_z = -\Omega_1. \tag{3.6}$$

Step 2, we assert that $w_x \equiv 0$.

Differentiating with respect to y of the third formula of (3.1), we have

$$\Omega_1 w_{xy} + (\Omega_2 + 2\Omega)w_{yy} + (\Omega_3 + \beta y)w_{yz} + \beta w_z - \beta v_y = 0,$$

which leads to

$$w_z = v_y, \tag{3.7}$$

as we have used that $w_{xy} = w_{yy} = w_{yz} = 0$, then

$$w_{zz} = v_{yz} = (-\Omega_1)_y = 0, \tag{3.8}$$

and

$$v_{yy} = w_{yz} = 0,$$

(3.6) delivers

$$v_{zz} = (-\Omega_1)_z = 0,$$

we infer that

$$v_{xx} = 0,\ w_{xx} = 0,$$

from the harmonicity of u and w. We summarize these conclusions as

$$w_{xx} = w_{yy} = w_{zz} = 0, \tag{3.9}$$

and

$$v_{xx} = v_{yy} = v_{zz} = 0.$$

Differentiating with respect to z in (2.3) we obtain

$$u_{xz} + v_{zy} + w_{zz} = 0,$$

which, by (3.8) and (3.9) becomes

$$u_{xz} = 0.$$

Similarly, differentiating with respect to y in (2.3), we obtain

$$u_{xy} = 0.$$

After differentiation with respect to z in the first formula of (3.1), we get

$$\Omega_1 u_{xz} + (\Omega_2 + 2\Omega)u_{yz} + (\Omega_3 + \beta y)u_{zz} = 0,$$

which leads to

$$(\Omega_3 + \beta y)u_{zz} = 0,$$

this implies that $u_{zz} = 0$ for all (x, y, z) with $\Omega_3 + \beta y \neq 0$, from the continuity of u_{zz}, it follows that

$$u_{zz} = 0,$$

by the definition of Ω_2, we get

$$w_{xz} = u_{zz} = 0,$$

recall that we already know that $w_{xx} = w_{yx} = 0$, then w_x is constant through the domain, since $w = 0$ on the bed $z = 0$, then $w_x = 0$ on $z = 0$, we conclude that

$$w_x = 0, \tag{3.10}$$

which leads to $u_z = \Omega_2$.

Step 3, we assert that $\Omega_1 = \Omega_2 = \Omega_3 = 0$.

(3.10) and (3.5) yield that

$$w_{zx} = w_{zy} = 0, \tag{3.11}$$

(3.11) together with $w_{zz} = 0$ yield that

$$w_z = \delta(z)$$

for some function $\delta(z)$. Differentiating with respect to x of the second formula of (3.1), we have

$$\Omega_1 v_{xx} + (\Omega_2 + 2\Omega)v_{yx} + (\Omega_3 + \beta y)v_{xz} = 0,$$

which leads to

$$(\Omega_2 + 2\Omega)v_{yx} = 0,$$

where we have used that $v_{xx} = 0$ and $v_{zx} = (-\Omega_1)_x = 0$, thus we get

$$v_{yx} = 0, \tag{3.12}$$

the definition of Ω_3 and (3.12) imply that

$$u_{yy} = 0,$$

since $u_{zz} = 0$ we infer from the harmonicity of u that also $u_{xx} = 0$, recall that we already know that $u_{xz} = u_{xy} = 0$, these findings say that

$$u_x = \xi(t)$$

for some function $\xi(t)$, Similarly, we obtain

$$u_y = \zeta(t)$$

for some function $\zeta(t)$. In order for the first and the second formula of (3.1) to hold throughout the fluid domain, it is necessary that

$$u_z = v_z = 0, \tag{3.13}$$

which lead to

$$\Omega_1 = \Omega_2 = 0. \tag{3.14}$$

We note first that from (3.1) and using (3.13) and (3.14), we obtain

$$\begin{cases} 2\Omega u_y = 0, \\ 2\Omega v_y = 0, \end{cases}$$

which yields

$$u_y = v_y = 0,$$

from (3.7), we get $w_z = 0$, recall that we already know that $w_x = w_y = 0$, the third formula of (3.1) becomes

$$(\Omega_3 + \beta y)w_z = \beta v,$$

which shows $v = 0$. we conclude that $\Omega_3 = 0$ by using the definition of Ω_3. From the previous facts which show $w_x = w_y = w_z = 0$ and from the kinematic condition (1.4), we conclude that $w = 0$. The proof is completed. □

Theorem 3.2. *Suppose $\overrightarrow{\Omega}$ is constant throughout the flow and also* (2.6)*, the only bounded solution of the problem* (2.2), (2.4) *with the boundary conditions* (1.2), (1.3), (1.4) *is uniform and azimuthal, and pressure $p(x, y, z, t) = -2\rho g z + p_{atm}$.*

Proof. From the Theorem 3.1, we know that $w = 0$ and $v = 0$, the (2.3) implies $u_x = 0$, so we have $u(x, y, z, t) = u(t)$ for some function $u(t)$. Using the fact of $(u, v, w) = (u(t), 0, 0)$, the (2.4) will become

$$\begin{cases} \frac{\partial p}{\partial x} = -\rho u'(t), \\ \frac{\partial p}{\partial y} = -\rho(\beta y u(t) + \frac{g}{R} y), \\ \frac{\partial p}{\partial z} = \rho(2\Omega u(t) - g), \end{cases}$$

therefore, the pressure can be given as

$$p(x, y, z, t) = -\rho u'(t)x - \rho(\beta u(t) + \frac{g}{R})\frac{y^2}{2} + \rho(2\Omega u(t) - g)z + h(t)$$

for some function $h(t)$. The boundedness of $p(x, y, z, t)$ implies

$$u(t) = -\frac{g}{2\Omega},$$

the condition (1.2) implies $h(t) = p_{atm}$, the dynamic boundary condition (1.3) can be written

$$h_0 = \frac{p_{atm}}{2\rho g},$$

which mean the height of the Ekman layer, then $p(x, y, z, t) = p_{atm} - 2\rho g z$.

□

Remark 3.3. If we change the dynamic boundary condition (1.2), the kinematic boundary condition (1.4) to

$$p = p_{atm} \ on \ z = \eta(x - ct), \tag{3.15}$$

and

$$w = (u - c)\eta_x \ on \ z = \eta(x - ct), \tag{3.16}$$

respectively [**13**], then the flow domain will be as the following Figure 2, the (u, v, w, p, η) which is given by

$$\begin{cases} u(x,y,z,t) = -\frac{g}{\beta R}, v(x,y,z,t) = 0, w(x,y,z,t) = 0, \\ p(x,y,z,t) = p_{atm} - 2\rho g(z - \eta_0), \\ \eta(x,t) = \eta_0, \end{cases}$$

satisfies (2.3) and (2.4) with the boundary conditions (1.3), (3.15) and (3.16), here η_0 is a constant.

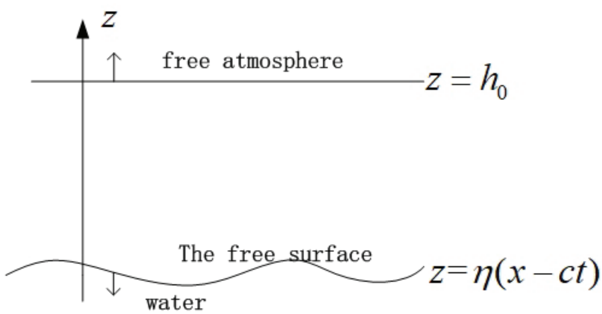

FIGURE 2. The flow domain.

REMARK 3.4. Our results are much different from the flows using the standard $\beta-$plane approximation. In fact, in [**13**] Wang et al. have shown that the only Ekman flow satisfying the governing equations in the $\beta-$plane approximation is the stationary flow. By using the modified $\beta-$plane approximation, we show that the flow has one non-vanishing component, see Table 1.

TABLE 1. A comparison between the modified $\beta-$plane and the $\beta-$plane approximation

Reference	plane type	The velocity	The pressure	The height of the Ekman layer
[**13**]	$\beta-$plane	$(0,0,0)$	$p_{atm} - \rho$gz	$\frac{p_{atm}}{\rho g}$
this paper	modified $\beta-$plane	$(-\frac{g}{2\Omega}, 0, 0)$	$p_{atm} - 2\rho$gz	$\frac{p_{atm}}{2\rho g}$

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