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Global regularity of magneto-micropolar equations with logarithmically dissipation

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ABSTRACT. Inspired by a work of Terence Tao on a generalized Navier-Stokes equations (T. Tao, Anal. PDE 2009), we examine in this paper the global regularity of the 3D magneto-micropolar equations with logarithmically hyperdissipative velocity dissipation.

CONTENTS

1. Introduction and main results

Non-Newtonian magneto-micropolar fluid mode is coupled with the incompressible Navier-Stokes flows, micro-rotational effects and magnetic effects. This motion describes the conservation of linear momentum, the conservation of angular momentum and Maxwell's law of the incompressible conducting fluids, respectively. Physically it may represent the fluids consisting of bar-like elements. Certain anisotropic fluids such as liquid crystals which are made up of dumbbell molecules are of this type. The three-dimensional(3D) magneto-micropolar equations are governed by

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the following nonlinear partial differential equations

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - (\nu + \kappa)\Delta u + \nabla p = (b \cdot \nabla)b + 2\kappa \nabla \times w, \\ \partial_t w + (u \cdot \nabla)w - \sigma \Delta w - \mu \nabla \nabla \cdot w + 4\kappa w = 2\kappa \nabla \times u, \\ \partial_t b + (u \cdot \nabla)b - \eta \Delta b = (b \cdot \nabla)u, \\ \nabla \cdot u = \nabla \cdot b = 0, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ denotes the velocity, $w = (w_1, w_2, w_3)$ the microrotation angular velocity and $b = (b_1, b_2, b_3)$ the magnetic field, $p(x, t)$ the scalar pressure, $(x, t) \in \mathbb{R}^3 \times (0, \infty)$. $\nu, \kappa, \sigma, \mu, \eta$ are viscous coefficients.

When $w = 0$, $b = 0$ or $w = b = 0$, the system (1.1) reduces to the classic magnetohydrodynamics(MHD) equations, micropolar equations or the classic Navier-Stokes equations ([**2, 3, 4, 5**]).

Like the 3D Navier-Stokes equations and MHD equations, the problem of global regularity or finite time singularity for the 3D magneto-micropolar equations with large initial data is still open. The main obstacle still lies in the fact that the dissipative mechanism is insufficient to control the nonlinear convection. It is important to understand the balance mechanism between the hyper-dissipation and global regularity of the 3D system. In the pioneer work, Tao([**16**]) firstly investigated the following logarithmically hyperdissipative Navier-Stokes equations,

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \mathcal{D}^2 u + \nabla p = 0, \\ \nabla \cdot u = 0, \end{cases} \tag{1.2}$$

where the Fourier multiplier $\mathcal{D}$ satisfies

$$\widehat{\mathcal{D}u}(\xi) \geq \frac{|\xi|^{\frac{d+2}{4}}}{\mathfrak{g}(|\xi|)} \widehat{u}(\xi), \quad \text{for large } |\xi|$$

and where $\mathfrak{g} : R^+ \rightharpoonup R^+$ is a non-decreasing function satisfying

$$\int_1^\infty \frac{ds}{s\mathfrak{g}^4(s)} = +\infty.$$

He proved that for any smooth divergence-free initial datum, the logarithmically hyperdissipative Navier-Stokes equations possess a global smooth solution with large initial data. We may refer to some further study on well-posedness of the hyperdissipative Navier-Stokes equations ([**15**]).

Motivated by the results of Tao([**16**]), Wu([**17**]) recently examined the global smooth solution of logarithmically hyperdissipative MHD equations. On comparison with the generalized Navier-Stokes equations (1.2), some new regular estimates have been explored due to the special structure of MHD equations. Wang, Wu and Ye([**18**]) recently also examined the global regularity of the three-dimensional micropolar equations with the logarithmically hyperdissipation in velocity fields.

Motivated by the results of Tao([**16**]), Wu([**17**]) and Wang, Wu and Ye([**18**]), the main purpose of this paper is to understand the global smooth solution issue of the 3D logarithmically hyperdissipative magneto-micropolar equations. More precisely, we consider the following 3D magneto-micropolar equations with only logarithmically hyperdissipative velocity dissipation,

$$
\left\{
\begin{aligned}
&\partial_t u + (u\cdot\nabla)u + \mathcal{L}^2 u + \nabla p = (b\cdot\nabla)b + \nabla\times w,\\
&\partial_t w + (u\cdot\nabla)w + 2w = \nabla\times u,\\
&\partial_t b + (u\cdot\nabla)b = (b\cdot\nabla)u,\\
&\nabla\cdot u = \nabla\cdot b = 0,\\
&u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \quad b(x,0) = b_0(x),
\end{aligned}
\right. \tag{1.3}
$$

where the Fourier multiplier $\mathcal{L}$ is defined by

$$
\widehat{\mathcal{L}u}(\xi) = \frac{|\xi|^\alpha}{g(|\xi|)}\widehat{u}(\xi), \quad \alpha \geq \frac{5}{2}
$$

and $g(\tau) \geq 1$ is a radially symmetric, non-decreasing function satisfying

$$
\int_e^\infty \frac{d\tau}{\tau\sqrt{\ln\tau}\,g(\tau)} = \infty. \tag{1.4}
$$

We will show the following global existence results.

THEOREM 1.1. *Let* $(u_0,\, w_0,\, b_0) \in H^s(\mathbb{R}^3)$ *with* $s > \frac{5}{2}$ *and* $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$*, then the 3D logarithmically hyperdissipative magneto-micropolar equations* (1.3) *admits a unique global regular solution* $(u,\, w,\, b)$ *such that for any given* $T > 0$,

$$
(u,\, w,\, b) \in C([0,T]; H^s(\mathbb{R}^3)), \qquad \mathcal{L}u \in L^2([0,T]; H^s(\mathbb{R}^3)).
$$

2. The proof of Theorem 1.1

In order to prove Theorem 1.1, it suffices to consider $\alpha = \frac{5}{2}$ because the case $\alpha > \frac{5}{2}$ is more easier to deal with. Now we first derive the basic L^2-energy estimates.

LEMMA 2.1. *Under the same condition in Theorem 1.1, the corresponding solution* $(u,\, w,\, b)$ *of* (1.3) *admits the following bound for any* $t > 0$

$$
\begin{aligned}
\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + &\int_0^t (\|\Lambda^{\frac{5-2\theta_1}{2}} u(\tau)\|_{L^2}^2 + \|\mathcal{L}u(\tau)\|_{L^2}^2)\, d\tau\\
&\leq C(t,\, u_0,\, w_0,\, b_0)
\end{aligned} \tag{2.1}
$$

for any $\theta_1 \in (0,\, \frac{3}{2})$.

PROOF OF LEMMA 2.1. Taking the inner product of (1.3) by (u, w, b) as well as adding the resulting equations together, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + \|\mathcal{L}u\|_{L^2}^2 + 2\|w\|_{L^2}^2\\
&= \int_{\mathbb{R}^3} \Big((\nabla\times w)\cdot u + (\nabla\times u)\cdot w\Big)dx.
\end{aligned}
$$

Thanks to the assumptions on g (more precisely, g grows logarithmically), one may conclude that for any given $\vartheta > 0$, there exists $N = N(\vartheta)$ satisfying

$$
g(r) \leq \widetilde{C} r^\vartheta, \quad \text{for any } r \geq N
$$

with the constant $\widetilde{C}=\widetilde{C}(\vartheta)$. Therefore, straightforward computations give for any $\theta_1 \in (0, \frac{5}{2})$ that

$$
\begin{aligned}
\|\mathcal{L}u\|_{L^2}^2 &= \int_{|\xi|<N(\theta_1)} \frac{|\xi|^5}{g^2(|\xi|)}|\widehat{u}(\xi)|^2\,d\xi + \int_{|\xi|\geq N(\theta_1)} \frac{|\xi|^5}{g^2(|\xi|)}|\widehat{u}(\xi)|^2\,d\xi \\
&\geq \int_{|\xi|\geq N(\theta_1)} \frac{|\xi|^5}{\left[\widetilde{C}|\xi|^{\theta_1}\right]^2}|\widehat{u}(\xi)|^2\,d\xi \\
&= \int_{\mathbb{R}^3} \frac{|\xi|^5}{\left[\widetilde{C}|\xi|^{\theta_1}\right]^2}|\widehat{u}(\xi)|^2\,d\xi - \int_{|\xi|<N(\theta_1)} \frac{|\xi|^5}{\left[\widetilde{C}|\xi|^{\theta_1}\right]^2}|\widehat{u}(\xi)|^2\,d\xi \\
&\geq \widetilde{C_1}\|\Lambda^{\frac{5-2\theta_1}{2}}u\|_{L^2}^2 - \widetilde{C_2}\|u\|_{L^2}^2,
\end{aligned}
\tag{2.2}
$$

where $\widetilde{C_1}$ and $\widetilde{C_2}$ depend only on θ_1. Choosing $\theta_1 \in (0, \frac{3}{2})$, we have that by combining all the above estimates

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + \frac{1}{2}\|\mathcal{L}u\|_{L^2}^2 + \widetilde{C_1}\|\Lambda^{\frac{5-2\theta_1}{2}}u\|_{L^2}^2 \\
\leq\ & \widetilde{C_2}\|u\|_{L^2}^2 + \int_{\mathbb{R}^3}\Big((\nabla\times w)\cdot u + (\nabla\times u)\cdot w\Big)dx \\
\leq\ & \widetilde{C_2}\|u\|_{L^2}^2 + 2\|\nabla u\|_{L^2}\|w\|_{L^2} \\
\leq\ & \widetilde{C_2}\|u\|_{L^2}^2 + 2\left(\|u\|_{L^2}^{\frac{3-2\theta_1}{5-2\theta_1}}\|\Lambda^{\frac{5-2\theta_1}{2}}u\|_{L^2}^{\frac{2}{5-2\theta_1}}\right)\|w\|_{L^2} \\
\leq\ & \frac{\widetilde{C_1}}{2}\|\Lambda^{\frac{5-2\theta_1}{2}}u\|_{L^2}^2 + C\left(\|u\|_{L^2}^2 + \|w\|_{L^2}^2\right).
\end{aligned}
$$

Consequence, it implies

$$
\begin{aligned}
&\frac{d}{dt}\left(\|u(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2\right) + \|\mathcal{L}u\|_{L^2}^2 + \|\Lambda^{\frac{5-2\theta_1}{2}}u\|_{L^2}^2 \\
&\qquad\qquad \leq C\left(\|u\|_{L^2}^2 + \|w\|_{L^2}^2 + \|b\|_{L^2}^2\right).
\end{aligned}
$$

Making use of the Gronwall inequality, it directly yields (2.1). This completes the proof of Lemma 2.1. □

The following regular estimates are crucial for the proof of Theorem 1.1.

LEMMA 2.2. *Assume* (u_0, w_0, b_0) *satisfies the assumptions stated in Theorem 1.1. Then the corresponding solution* (u, w, b) *of* (1.3) *admits the following bound for any* $t>0$ *and for any* $\widetilde{\sigma} \in (0, \frac{3-2\theta_1}{2})$

$$
\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 + \|\Lambda^{\widetilde{\sigma}}w(t)\|_{L^2}^2 + \int_0^t (\|\mathcal{L}\nabla u(\tau)\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}-\theta_2}\nabla u(\tau)\|_{L^2}^2)\,d\tau \\
&\qquad\qquad \leq C(t, u_0, w_0, b_0),
\end{aligned}
\tag{2.3}
$$

where $\theta_2 \in (0, \frac{1}{2}]$. *In particular, it implies*

$$
\int_0^t \|\nabla u(\tau)\|_{L^\infty}^2\,d\tau \leq C(t, u_0, w_0, b_0).
\tag{2.4}
$$

PROOF OF LEMMA 2.2. Applying $\Lambda^{\widetilde{\sigma}}$ to $(1.3)_2$ and taking the inner product with $\Lambda^{\widetilde{\sigma}}w$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^{\widetilde{\sigma}}w(t)\|_{L^2}^2+2\|\Lambda^{\widetilde{\sigma}}w\|_{L^2}^2\\
=\ &\int_{\mathbb{R}^3}\Lambda^{\widetilde{\sigma}}(\nabla\times u)\Lambda^{\widetilde{\sigma}}w\,dx-\int_{\mathbb{R}^3}[\Lambda^{\widetilde{\sigma}},u\cdot\nabla]w\Lambda^{\widetilde{\sigma}}w\,dx\\
:=\ &K_1+K_2.
\end{aligned}$$

For K_1, Gagliardo-Nirenberg inequality implies, for $\widetilde{\sigma}\in(0,\ \frac{3-2\theta_1}{2})$, we have

$$\begin{aligned}
K_1\ &\leq\ C\|\Lambda^{\widetilde{\sigma}+1}u\|_{L^2}\|\Lambda^{\widetilde{\sigma}}w\|_{L^2}\\
&\leq\ \|\Lambda^{\widetilde{\sigma}}w\|_{L^2}^2+C\|u\|_{H^{\frac{5}{2}-\theta_1}}^2.
\end{aligned}$$

For K_2, employing the classic Kato-Ponce commutator estimates([**12**]) yields

$$\begin{aligned}
K_2\ &=\ -\int_{\mathbb{R}^3}[\Lambda^{\widetilde{\sigma}}\partial_{x_i},u_i]w\Lambda^{\widetilde{\sigma}}w\,dx\\
&\leq\ \|[\Lambda^{\widetilde{\sigma}}\partial_{x_i},u_i]w\|_{L^2}\|\Lambda^{\widetilde{\sigma}}w\|_{L^2}\\
&\leq\ C\left(\|\nabla u\|_{L^\infty}\|\Lambda^{\widetilde{\sigma}}w\|_{L^2}+\|\Lambda^{\widetilde{\sigma}+1}u\|_{L^{\frac{3}{\widetilde{\sigma}}}}\|w\|_{L^{\frac{6}{3-2\widetilde{\sigma}}}}\right)\|\Lambda^{\widetilde{\sigma}}w\|_{L^2}\\
&\leq\ C\left(\|\nabla u\|_{L^\infty}\|\Lambda^{\widetilde{\sigma}}w\|_{L^2}+\|\Lambda^{\frac{5}{2}}u\|_{L^2}\|\Lambda^{\widetilde{\sigma}}w\|_{L^2}\right)\|\Lambda^{\widetilde{\sigma}}w\|_{L^2}\\
&\leq\ C\left(\|\nabla u\|_{L^\infty}+\|\Lambda^{\frac{5}{2}}u\|_{L^2}\right)\|\Lambda^{\widetilde{\sigma}}w\|_{L^2}^2.
\end{aligned}$$

Collecting the above estimates, we have for any $\widetilde{\sigma}\in(0,\ \frac{3-2\theta_1}{2})$, such that

$$\frac{d}{dt}\|\Lambda^{\widetilde{\sigma}}w(t)\|_{L^2}^2\leq C\left(1+\|\nabla u\|_{L^\infty}+\|\Lambda^{\frac{5}{2}}u\|_{L^2}\right)\|\Lambda^{\widetilde{\sigma}}w\|_{L^2}^2+C\|u\|_{H^{\frac{5-2\theta_1}{2}}}^2. \tag{2.5}$$

Applying the gradient operator ∇ to the equations of $(1.3)_1$ and $(1.3)_3$, multiplying them by ∇u and ∇b, respectively, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2\right)+\|\mathcal{L}\nabla u\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}\nabla\nabla\times w\cdot\nabla u\,dx-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla u)\cdot\nabla u\,dx-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla b)\cdot\nabla b\,dx\\
&\quad+\int_{\mathbb{R}^3}\Big(\nabla(b\cdot\nabla u)\cdot\nabla b+\nabla(b\cdot\nabla b)\cdot\nabla u\Big)\,dx\\
&:=\sum_{k=1}^{4}A_k.
\end{aligned}\tag{2.6}$$

With the similar derivation of (2.2), we can establish the following estimate for any $\theta_2\in(0,\ \frac{5}{2})$

$$\|\mathcal{L}\nabla u\|_{L^2}^2\geq C_1\|\Lambda^{\frac{5}{2}-\theta_2}\nabla u\|_{L^2}^2-C_2\|\nabla u\|_{L^2}^2. \tag{2.7}$$

Making use of (2.7) for any $\theta_2\in(0,\ \frac{1}{2}]$, we deduce that

$$\begin{aligned}
A_1\ &\leq\ C\|w\|_{L^2}\|\Lambda^2\nabla u\|_{L^2}\\
&\leq\ C\|w\|_{L^2}(\|\mathcal{L}\nabla u\|_{L^2}+\|\nabla u\|_{L^2})\\
&\leq\ \frac{1}{16}\|\mathcal{L}\nabla u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+C\|w\|_{L^2}^2.
\end{aligned}$$

For the remainder three terms, it is not difficult to deduce

$$\begin{aligned} A_2 &\le C\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^2}^2, \\ A_3 &\le C\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^2}^2, \\ A_4 &\le C\|\nabla u\|_{L^\infty}\|\nabla b\|_{L^2}^2. \end{aligned}$$

Putting together the previous estimates, it follows that

$$\begin{aligned} &\frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 + \|\Lambda^{\widetilde{\sigma}} w(t)\|_{L^2}^2) + \|\mathcal{L}\nabla u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}-\theta_2}\nabla u\|_{L^2}^2 \\ &\le C(1 + \|\nabla u\|_{L^\infty} + \|\Lambda^{\frac{5}{2}} u\|_{L^2})(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \|\Lambda^{\widetilde{\sigma}} w\|_{L^2}^2) \\ &\quad + C\|u\|_{H^{\frac{5-2\theta_1}{2}}}^2 + C\|w\|_{L^2}^2. \end{aligned} \tag{2.8}$$

Letting

$$\begin{aligned} \widetilde{A}(t) &:= \|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 + \|\Lambda^{\widetilde{\sigma}} w(t)\|_{L^2}^2, \\ \widetilde{B}(t) &:= \|\mathcal{L}\nabla u(t)\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}-\theta_2}\nabla u(t)\|_{L^2}^2, \\ \widetilde{h}(t) &:= C\|u(t)\|_{H^{\frac{5-2\theta_1}{2}}}^2 + C\|w(t)\|_{L^2}^2, \end{aligned}$$

we may rewrite (2.8)

$$\frac{d}{dt}\widetilde{A}(t) + \widetilde{B}(t) \le C(1 + \|\nabla u\|_{L^\infty} + \|\Lambda^{\frac{5}{2}} u\|_{L^2})\widetilde{A}(t) + \widetilde{h}(t). \tag{2.9}$$

Employing the classic Littlewood-Paley decomposition (refer to [**1**] for details), one shows that

$$\|\nabla u\|_{L^\infty} \le \|\Delta_{-1}\nabla u\|_{L^\infty} + \sum_{l=0}^{N-1}\|\Delta_l\nabla u\|_{L^\infty} + \sum_{l=N}^{\infty}\|\Delta_l\nabla u\|_{L^\infty},$$

where Δ_l ($l = -1, 0, 1, \cdots$) denote the nonhomogeneous dyadic blocks. Applying Bernstein inequality obeys

$$\|\Delta_{-1}\nabla u\|_{L^\infty} \le C\|u\|_{L^2},$$

$$\sum_{l=N}^{\infty}\|\Delta_l\nabla u\|_{L^\infty} \le C\sum_{l=N}^{\infty}2^{l(\theta_2-1)}\|\Delta_l\Lambda^{\frac{5}{2}-\theta_2}\nabla u\|_{L^2} \le C2^{N(\theta_2-1)}\|\Lambda^{\frac{5}{2}-\theta_2}\nabla u\|_{L^2},$$

due to $\theta_2 \in (0, \frac{1}{2}]$.

Similarly,

$$\begin{aligned}\sum_{l=0}^{N-1}\|\Delta_l\nabla u\|_{L^\infty} &\le C\sum_{l=0}^{N-1}2^{\frac{5l}{2}}\|\Delta_l u\|_{L^2}\\
&\le C\sum_{l=0}^{N-1}\Big\|\Delta_l\Big(g(\Lambda)\frac{\Lambda^{\frac52}}{g(\Lambda)}u\Big)\Big\|_{L^2}\\
&\le C\sum_{l=0}^{N-1}\Big\|\varphi(2^{-l}\xi)g(|\xi|)\frac{|\xi|^{\frac52}}{g(|\xi|)}\widehat{u}(\xi)\Big\|_{L^2}\\
&\le C\sum_{l=0}^{N-1}g(2^l)\Big\|\frac{|\xi|^{\frac52}}{g(|\xi|)}\widehat{\Delta_l u}(\xi)\Big\|_{L^2}\\
&\le C\left(\sum_{l=0}^{N-1}g^2(2^l)\right)^{\frac12}\left(\sum_{l=0}^{N-1}\Big\|\frac{|\xi|^{\frac52}}{g(|\xi|)}\widehat{\Delta_l u}(\xi)\Big\|_{L^2}^2\right)^{\frac12}\\
&\le Cg(2^N)\left(\sum_{l=1}^{N-1}1\right)^{\frac12}\left\|\frac{\Lambda^{\frac52}}{g(\Lambda)}u\right\|_{L^2}\\
&\le Cg(2^N)\sqrt{N}\,\|\mathcal{L}u\|_{L^2},\end{aligned}$$

where we have used the fact that g is a non-decreasing function.

Putting the above estimates altogether implies

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\le C\|u\|_{L^2}+Cg(2^N)\sqrt{N}\,\|\mathcal{L}u\|_{L^2}\\ &\quad +C2^{N(\theta_2-1)}\|\Lambda^{\frac52-\theta_2}\nabla u\|_{L^2}.\end{aligned}\tag{2.10}$$

By the same argument, it also gives

$$\begin{aligned}\|\Lambda^{\frac52}u\|_{L^2} &\le C\|u\|_{L^2}+Cg(2^N)\sqrt{N}\,\|\mathcal{L}u\|_{L^2}\\ &\quad +C2^{N(\theta_2-1)}\|\Lambda^{\frac52-\theta_2}\nabla u\|_{L^2}.\end{aligned}\tag{2.11}$$

It thus follows from (2.9) that

$$\frac{d}{dt}\widetilde{A}(t)+\widetilde{B}(t)\le C\Big(1+g(2^N)\sqrt{N}\,\|\mathcal{L}u\|_{L^2}+2^{N(\theta_2-1)}\|\Lambda^{\frac52-\theta_2}\nabla u\|_{L^2}\Big)\widetilde{A}(t)+\widetilde{h}(t).$$

Noting that if we further take $\theta_2\in(0,\ \frac12]$, it also shows that

$$\frac{d}{dt}\widetilde{A}(t)+\widetilde{B}(t)\le C\Big(1+g(2^N)\sqrt{N}\,\|\mathcal{L}u\|_{L^2}+2^{-\frac{N}{2}}\|\Lambda^{\frac52-\theta_2}\nabla u\|_{L^2}\Big)\widetilde{A}(t)+\widetilde{h}(t).$$

By taking N such that

$$2^N\approx e+\widetilde{A}(t),$$

we thus deduce

$$\begin{aligned}\frac{d}{dt}\widetilde{A}(t)+\widetilde{B}(t) &\le C\left(1+\|\mathcal{L}u\|_{L^2}\right)g\big(e+\widetilde{A}(t)\big)\sqrt{\ln\big(e+\widetilde{A}(t)\big)}\big(e+\widetilde{A}(t)\big)\\
&\quad +C\widetilde{B}^{\frac12}(t)\big(e+\widetilde{A}(t)\big)^{\frac12}+\widetilde{h}(t)\\
&\le C\left(1+\|\mathcal{L}u\|_{L^2}\right)g\big(e+\widetilde{A}(t)\big)\sqrt{\ln\big(e+\widetilde{A}(t)\big)}\big(e+\widetilde{A}(t)\big)\\
&\quad +\frac12\widetilde{B}(t)+C\big(e+\widetilde{A}(t)\big)+\widetilde{h}(t),\end{aligned}$$

or

$$
\begin{aligned}
&\frac{d}{dt}\widetilde{A}(t)+\widetilde{B}(t)\le C\left(1+\|\mathcal{L}u\|_{L^2}\right)\\
&g\big(e+\widetilde{A}(t)\big)\sqrt{\ln\left(e+\widetilde{A}(t)\right)}\big(e+\widetilde{A}(t)\big)+C\widetilde{h}(t). \qquad (2.12)
\end{aligned}
$$

Due to

$$
g\big(e+\widetilde{A}(t)\big)\sqrt{\ln\left(e+\widetilde{A}(t)\right)}\big(e+\widetilde{A}(t)\big)\ge 1,
$$

it follows from (2.12) that

$$
\int_{e+\widetilde{A}(0)}^{e+\widetilde{A}(t)}\frac{d\tau}{\tau\sqrt{\ln\tau}\,g(\tau)}\le C\int_0^t\left(1+\widetilde{h}(\tau)+\|\mathcal{L}u(\tau)\|_{L^2}\right)d\tau.
$$

Notice that the condition

$$
\int_e^\infty\frac{d\tau}{\tau\sqrt{\ln\tau}\,g(\tau)}=+\infty
$$

and the following bound due to (2.1)

$$
\int_0^t\left(1+\widetilde{h}(\tau)+\|\mathcal{L}u(\tau)\|_{L^2}\right)d\tau\le C(t,\,u_0,\,w_0,\,b_0),
$$

we have

$$
\widetilde{A}(t)\le C(t,\,u_0,\,w_0,\,b_0),
$$

and

$$
\int_0^t\widetilde{B}(\tau)\,d\tau\le C(t,\,u_0,\,w_0,\,b_0).
$$

That is

$$
\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2+\|\Lambda^{\widetilde{\sigma}}w(t)\|_{L^2}^2\\
+\;&\int_0^t\left(\|\mathcal{L}\nabla u(\tau)\|_{L^2}^2+\|\Lambda^{\frac52-\theta_2}\nabla u(\tau)\|_{L^2}^2\right)d\tau\\
\le\;&C(t,\,u_0,\,w_0,\,b_0).
\end{aligned}
$$

In particular,

$$
\|\nabla u\|_{L^\infty}\le C\|\Lambda^{\frac52-\theta_2}\nabla u\|_{L^2}^2+C\|u\|_{L^2}^2,\quad \theta_2<1.
$$

This completes the proof of Lemma 2.2. □

The rest of this section is devoted to proving the global H^s-estimate. Applying Λ^s with $s>\frac52$ to the system (1.3) and taking the L^2 inner product with $\Lambda^s u$, $\Lambda^s b$

and $\Lambda^s w$ respectively, adding them up, we can get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s w\|_{L^2}^2+\|\Lambda^s b\|_{L^2}^2\right)+\|\mathcal{L}\Lambda^s u\|_{L^2}^2+2\|\Lambda^s w\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}\Big(\Lambda^s(\nabla\times u)\cdot\Lambda^s w+\Lambda^s(\nabla\times w)\cdot\ \Lambda^s u\Big)\,dx-\int_{\mathbb{R}^3}[\Lambda^s,u\cdot\nabla]u\ \cdot\Lambda^s u\,dx\\
&\quad-\int_{\mathbb{R}^3}[\Lambda^s,u\cdot\nabla]w\ \cdot\Lambda^s w\,dx+\int_{\mathbb{R}^3}[\Lambda^s,b\cdot\nabla]b\ \cdot\Lambda^s u\,dx\\
&\quad+\int_{\mathbb{R}^3}[\Lambda^s,b\cdot\nabla]u\ \cdot\Lambda^s b\,dx-\int_{\mathbb{R}^3}[\Lambda^s,u\cdot\nabla]b\ \cdot\Lambda^s b\,dx\\
&:=\sum_{k=1}^{6}J_k.
\end{aligned}\tag{2.13}$$

Due to the proof of (2.2), we have that for any $\theta_3\in(0,\ \frac{5}{2})$, such that

$$\|\mathcal{L}\Lambda^s u\|_{L^2}^2\geq C_1\|\Lambda^{\frac{5}{2}-\theta_3}\Lambda^s u\|_{L^2}^2-C_2\|\Lambda^s u\|_{L^2}^2.$$

Now we thus deduce that for $\theta_3\in(0,\ \frac{3}{2})$,

$$\begin{aligned}
J_1\ &\leq\ C\|\Lambda^s w\|_{L^2}\|\Lambda^s\nabla u\|_{L^2}\\
&\leq\ C\|\Lambda^s w\|_{L^2}(\|\mathcal{L}\Lambda^s u\|_{L^2}+\|\Lambda^s u\|_{L^2})\\
&\leq\ \frac{1}{16}\|\mathcal{L}\Lambda^s u\|_{L^2}^2+C(\|\Lambda^s w\|_{L^2}^2+\|\Lambda^s u\|_{L^2}^2).
\end{aligned}$$

Using the Kato-Ponce inequality([**12**]), we immediately get

$$\begin{aligned}
J_2\ &\leq\ C\|[\Lambda^s,u\cdot\nabla]u\|_{L^2}\|\Lambda^s u\|_{L^2}\\
&\leq\ C\|\nabla u\|_{L^\infty}\|\Lambda^s u\|_{L^2}^2.
\end{aligned}$$

It follows from Galiardo-Nirenberg inequality that

$$\begin{aligned}
J_3=&-\int_{\mathbb{R}^3}[\Lambda^s\partial_{x_i},u_i]w\Lambda^s w\,dx\\
\leq&\|[\Lambda^s\partial_{x_i},u_i]w\|_{L^2}\|\Lambda^s w\|_{L^2}\\
\leq&C(\|\nabla u\|_{L^\infty}\|\Lambda^s w\|_{L^2}+\|\Lambda^{s+1}u\|_{L^{\frac{3}{\widetilde{\sigma}}}}\|w\|_{L^{\frac{6}{3-2\widetilde{\sigma}}}})\|\Lambda^s w\|_{L^2}\\
\leq&C(\|\nabla u\|_{L^\infty}\|\Lambda^s w\|_{L^2}+\|u\|_{L^2}^{1-\frac{2s+5-2\widetilde{\sigma}}{2s+5-2\theta_3}}\\
&\quad\|\Lambda^{\frac{5}{2}-\theta_3}\Lambda^s u\|_{L^2}^{\frac{2s+5-2\widetilde{\sigma}}{2s+5-2\theta_3}}\|\Lambda^{\widetilde{\sigma}}w\|_{L^2})\|\Lambda^s w\|_{L^2}\\
\leq&C(\|\nabla u\|_{L^\infty}\|\Lambda^s w\|_{L^2}+\|u\|_{L^2}^{1-\frac{2s+5-2\widetilde{\sigma}}{2s+5-2\theta_3}}(\|\mathcal{L}\Lambda^s u\|_{L^2}\\
&\quad+\|\Lambda^s u\|_{L^2})^{\frac{2s+5-2\widetilde{\sigma}}{2s+5-2\theta_3}}\|\Lambda^{\widetilde{\sigma}}w\|_{L^2})\|\Lambda^s w\|_{L^2}\\
&\quad\leq\frac{1}{16}\|\mathcal{L}\Lambda^s u\|_{L^2}^2+C\|\nabla u\|_{L^\infty}\|\Lambda^s w\|_{L^2}^2\\
&\quad+C(1+\|u\|_{L^2}^2+\|\Lambda^{\widetilde{\sigma}}w\|_{L^2}^2)(1+\|\Lambda^s u\|_{L^2}+\|\Lambda^s w\|_{L^2}^2),
\end{aligned}\tag{2.14}$$

where $\widetilde{\sigma}>\theta_3$. Now take $\widetilde{p}>2$ satisfying

$$\frac{1}{\widetilde{p}}\geq\frac{19-10\theta_3-4(1-\theta_3)s}{6(5-2\theta_3)},$$

then it is not hard to check that

$$\begin{aligned}
J_4 &\leq C\|[\Lambda^s, b\cdot\nabla]b\|_{L^{\frac{2\widetilde{p}}{\widetilde{p}+2}}}\|\Lambda^s u\|_{L^{\frac{2\widetilde{p}}{\widetilde{p}-2}}} \\
&\leq C\|\nabla b\|_{L^{\widetilde{p}}}\|\Lambda^s b\|_{L^2}\|\Lambda^s u\|_{L^{\frac{2\widetilde{p}}{\widetilde{p}-2}}} \\
&\leq C(\|\nabla b\|_{L^2}^{1-\frac{3(\widetilde{p}-2)}{2(s-1)\widetilde{p}}}\|\Lambda^s b\|_{L^2}^{\frac{3(\widetilde{p}-2)}{2(s-1)\widetilde{p}}}) \\
&\quad \|\Lambda^s b\|_{L^2}(\|\nabla u\|_{L^2}^{1-\frac{2(s-1)\widetilde{p}+6}{(2s+3-2\theta_3)\widetilde{p}}}\|\Lambda^{\frac{5}{2}-\theta_3}\Lambda^s u\|_{L^2}^{\frac{2(s-1)\widetilde{p}+6}{(2s+3-2\theta_3)\widetilde{p}}}) \\
&\leq C(\|\nabla b\|_{L^2}^{1-\frac{3(\widetilde{p}-2)}{2(s-1)\widetilde{p}}}\|\Lambda^s b\|_{L^2}^{\frac{3(\widetilde{p}-2)}{2(s-1)\widetilde{p}}}) \\
&\quad \|\Lambda^s b\|_{L^2}\|\nabla u\|_{L^2}^{1-\frac{2(s-1)\widetilde{p}+6}{(2s+3-2\theta_3)\widetilde{p}}}(\|\mathcal{L}\Lambda^s u\|_{L^2}+\|\Lambda^s u\|_{L^2})^{\frac{2(s-1)\widetilde{p}+6}{(2s+3-2\theta_3)\widetilde{p}}} \\
&\leq \frac{1}{16}\|\mathcal{L}\Lambda^s u\|_{L^2}^2 + H(\|\nabla u\|_{L^2},\, \|\nabla b\|_{L^2})(\|\Lambda^s u\|_{L^2}+\|\Lambda^s b\|_{L^2}^2),
\end{aligned}$$

where $H(x, y)$ is a smooth increasing function with respect to variable x and y. Similar arguments also all us to show

$$\begin{aligned}
J_5,\, J_6 &\leq C(\|[\Lambda^s, u\cdot\nabla]b\|_{L^2} + \|[\Lambda^s, b\cdot\nabla]u\|_{L^2})\|\Lambda^s b\|_{L^2} \\
&\leq C(\|\nabla u\|_{L^\infty}\|\Lambda^s b\|_{L^2} + \|\nabla b\|_{L^{\widetilde{p}}}\|\Lambda^s u\|_{L^{\frac{2\widetilde{p}}{\widetilde{p}-2}}})\|\Lambda^s b\|_{L^2} \\
&\leq \frac{1}{16}\|\mathcal{L}\Lambda^s u\|_{L^2}^2 + H(\|\nabla u\|_{L^2},\, \|\nabla b\|_{L^2})(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s b\|_{L^2}^2) \\
&\quad + C\|\nabla u\|_{L^\infty}\|\Lambda^s b\|_{L^2}^2.
\end{aligned}$$

Collecting all the above estimates, one may derive that

$$\frac{d}{dt}Z(t) + \|\mathcal{L}\Lambda^s u\|_{L^2}^2 \leq CV(t)\big(e + Z(t)\big),$$

where $Z(t)$ and $V(t)$ are given by

$$\begin{aligned}
Z(t) &:= \|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s w(t)\|_{L^2}^2 + \|\Lambda^s b(t)\|_{L^2}^2, \\
V(t) &:= \Big(1 + \|\nabla u\|_{L^\infty} + \|\Lambda^{\widetilde{\sigma}} w\|_{L^2}^2 + H(\|\nabla u\|_{L^2},\, \|\nabla b\|_{L^2})\Big)(t).
\end{aligned}$$

It follows from (2.1) and (2.4) that

$$\int_0^t V(\tau)\, d\tau \leq C(t,\, u_0,\, w_0,\, b_0).$$

Now noticing the above bound and using the Gronwall inequality, we eventually obtain

$$\|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s w(t)\|_{L^2}^2 + \|\Lambda^s b(t)\|_{L^2}^2 + \int_0^t \|\mathcal{L}\Lambda^s u(\tau)\|_{L^2}^2\, d\tau \leq C(t,\, u_0,\, w_0,\, b_0).$$

We therefore complete the proof of Theorem 1.1.

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