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# Analytic regularity for Navier-Stokes-Korteweg model on pseudo-measure spaces

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Abstract. The purpose of this work is to study the existence and analytic smoothing effect for the compressible Navier-Stokes system with quantum pressure in pseudo-measure spaces. This system has been considered by B. Haspot and an analytic smoothing effect for a Korteweg-type system was considered by F. Charve, R. Danchin and J. Xu, both of them in Besov spaces. Here we give a better lower bound of the radius of analyticity near zero. This work is an opportunity to improve the study of partial differential equations in pseudo-measure spaces by introducing a new functional setting to deal with non-linear terms. The pseudo-measure spaces are well-adapted to obtain a pointwise control of solutions, with a view to study turbulence.

Contents



## 1. Introduction

We are interested by the analytic smoothing properties of the Navier-Stokes-Korteweg system which describes a two-phase compressible and viscous fluids, of density $\rho$ and velocity field $u$. It is generally assumed that the phases are separated by a hypersurface and that the jump in the pressure across the interface is proportional to the curvature. Here we deal with a diffuse interface (DI) model

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that describes fluids when the change of phase corresponds to a thin but regular transition zone for the density and velocity. This type of model differs from the so-called sharp interface (SI) model when the interface between phases corresponds to a discontinuity in the state space. The basic ideas of the DI model considered here, is to add to the classical compressible fluids equation a capillary term, that penalizes high variations of the density. The full derivation of the corresponding equation, that we shall name the compressible Navier-Stokes-Korteweg system, is due to J. E. Dunn and J. Serrin (see [**12**]).

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mathcal{A}u + \nabla \Pi = \operatorname{div}(\mathcal{K}), \\ (\rho, u)|_{t=0} = (\rho_0, u_0), \end{cases} \tag{1.1}$$

where $\Pi := P(\rho)$ is the pressure function, $\mathcal{A}u := \operatorname{div}(2\mu(\rho)D_S(u)) + \nabla(\nu(\rho)\operatorname{div} u))$ is the diffusion operator, $D_S(u) := \frac{1}{2}(\nabla u + {}^t\nabla u)$ is the symmetric gradient and the capillarity tensor is given by

$$\mathcal{K} := \rho\, \operatorname{div}(\kappa(\rho)\nabla\rho) I_{\mathbb{R}^d} + \frac{1}{2}\big(\kappa(\rho) - \rho\kappa'(\rho)\big)|\nabla\rho|^2 I_{\mathbb{R}^d} - \kappa(\rho)\nabla\rho \otimes \nabla\rho.$$

This system is due to J. E. Dunn and J. Serrin in [**12**]. The density-dependent capillarity function $\kappa$ is assumed to be positive. Note that for smooth enough density $\rho$ and capillarity function $\kappa$, we have

$$\operatorname{div}\mathcal{K} = \rho\nabla\Big(\kappa(\rho)\Delta\rho + \frac{1}{2}\kappa'(\rho)|\nabla\rho|^2\Big).$$

The coefficients $\nu = \nu(\rho)$ and $\mu = \mu(\rho)$ designate the bulk and shear viscosity, respectively, and are assumed to satisfy in the neighborhood of some reference constant density $\bar\rho > 0$ the conditions

$$\mu > 0 \qquad \text{and} \qquad \nu + \mu > 0.$$

We shall assume that the functions $\lambda, \mu, \kappa$ and $P$ are real analytic in a neighborhood of $\bar\rho$. To simplify, we set $\bar\rho = 1$. Introducing $a = \rho - 1$ and denoting by $\bar\mu = \mu(1)$, $\bar\nu = \nu(1)$, $\bar\kappa = \kappa(1)$, $\bar\alpha = P'(1)$, the system (1.1) reads

$$\begin{cases} \partial_t a + \operatorname{div}(u) = \tilde f, \\ \partial_t u - \bar{\mathcal{A}}u + \bar\alpha\nabla a - \bar\kappa\nabla\Delta a = \tilde g, \end{cases} \tag{1.2}$$

where $\bar{\mathcal{A}}u = 2\bar\mu\, \operatorname{div}(D_S(u)) + \bar\nu\nabla\, \operatorname{div} u$, $\tilde f = -\operatorname{div}(au)$, $\tilde g = \sum_{i=1}^4 \tilde g_i$ with

$$\begin{cases} \tilde g_1 := -u\cdot\nabla u, \\ \tilde g_2 := (1+a)^{-1}\mathcal{A}u - \bar{\mathcal{A}}u, \\ \tilde g_3 := -(1+a)^{-1}\nabla P(1+a) + \bar\alpha\nabla a, \\ \tilde g_4 := \nabla\Big(\big(\kappa(1+a) - \bar\kappa\big)\Delta a + \frac{1}{2}\kappa'(1+a)|\nabla a|^2\Big). \end{cases}$$

System (1.2) is a hyperbolic/parabolic coupled system, which is common for compressible Navier-Stokes type systems. In contrast with the linearized equation of the classical compressible Navier-Stokes system, it was remarked by F. Charve, R. Danchin, and J. Xu that for the linear part of (1.2), with external forces, both of the density and velocity are smoothed out instantaneously (see Lemma 2.1). In 2018, authors showed in [**8**] a Gevrey analyticity smoothing effect for all the unknowns of the compressible Navier-Stokes-Korteweg system, in Besov spaces by using the method from the works of H. Bae and A. Biswas in [**1**] for dissipative equation with analytic nonlinearity. This is the first related result for a model of compressible fluids. In this paper, we aim to establish this smoothing effect and to estimate the radius of analyticity of the solution, in the pseudo-measure spaces for a particular case presented in the following subsection. Using the method used by J. Y. Chemin, I. Gallagher, and P. Zhang in [**9**] for semi-linear parabolic systems, we give a better estimate on the radius of analyticity near 0, the advantage to work in the pseudo-measure spaces is that we obtained pointwise time-frequency estimate of the decay of the solution, with studying the turbulence as perspective. In the following subsection, we describe a special case of the compressible Navier-Stokes-Korteweg system, so-called the incompressible Navier-Stokes system with quantum pressure, that will be discussed in this paper.

**1.1. Compressible Navier-Stokes system with quantum pressure.** In this note, we consider a special case, which is the so-called compressible Navier-Stokes system with quantum pressure considered by B. Haspot [**15**] which corresponds to the capillarity coefficient

$$\kappa(\rho) := \frac{\kappa}{\rho},$$

where $\kappa$ is a non-negative real number. We also assume that

$$\mu(\rho) := \mu\rho, \quad \nu(\rho) := \nu\rho \quad \text{and} \quad P(\rho) := \alpha\rho,$$

where $\mu > 0, \mu+\nu > 0$ and $\alpha > 0$ are three real constants. Note that the assumption $\nu + \mu > 0$ is little more general that the case of null bulk viscosity (*i.e.* $\nu = 0$), so called shallow water viscosity coefficients regime, traited by B. Haspot in [**15**]. In this paper, we assume the density is non-negative, that is a physically relevant assumption, and we formally perform the following change of variable

$$\rho = e^a,$$

instead $\rho = 1 + a$ as previously. Then, System (1.1) reads

$$\begin{cases} \partial_t a + \operatorname{div}(u) = f(u,a), \\ \partial_t u - \mu \triangle u - (\mu+\nu)\nabla \operatorname{div}(u) + \alpha\nabla a - \kappa\nabla\triangle a = g(u,a), \end{cases} \tag{1.3}$$

where $g := \sum_{j=1}^{3} g_i$ and

$$\begin{cases} f(u,a) := -u\cdot\nabla a, \\ g_1(u,u) := -u\cdot\nabla u, \\ g_2(u,a) := \mu(\nabla u + {}^t\nabla u)\cdot\nabla a + \nu\nabla a \operatorname{div}(u), \\ g_3(a,a) := \frac{\kappa}{2}\nabla(\nabla a\cdot\nabla a). \end{cases} \tag{1.4}$$

We consider the initial value condition

$$(a,u)_{|t=0} = (a_0, u_0). \tag{1.5}$$

In the end, we will produce a solution of the original system with $\rho > 0$. Without the hypothesis $\rho > 0$ the system is quasi-linear due to the presence of a term of type $a\triangle u$ which is delicate to estimate in the functional framework of pseudo-measure spaces.

**1.2. Pseudo-measure spaces.** Let us begin by specifying some notations.

NOTATION 1.1. Throughout the paper, $f \lesssim_{a_1,\dots,a_k} g$ means that there exists a positive constant $C$, which depends on the parameters $a_1, \dots, a_k$ such that $f \le Cg$. We denote by $\widehat{f}$ the Fourier transform with respect to the space variable of the function $f \in \mathcal{C}\left([0,T[;\mathcal{S}'(\mathbb{R}^d)\right)$.

We begin by defining pseudo-measure spaces on the whole space $\mathbb{R}^d$. For all $r \ge 0$, we define the pseudo-measure space of order $r$ by setting

$$PM^r(\mathbb{R}^d) := \left\{ g \in \mathcal{S}'(\mathbb{R}^d) \;\middle|\; \widehat{g} \in L^1_{loc}(\mathbb{R}^d) \text{ and } \|g\|_{PM^r} := \sup_{\xi \in \mathbb{R}^d} \{|\xi|^r |\widehat{g}(\xi)|\} < +\infty \right\}.$$

The pseudo-measure spaces were firstly used for fluids mechanic systems by Y. Le Jan and A. Sznitman in [**16**] for the incompressible Navier-Stokes system, for existence results. Later, the analytic regularity was studied by P. G. Lemarié-Rieusset in [**17**] and W. Deng, M. Paicu and P. Zhang in [**11**] for the global mild solution of incompressible Navier-Stokes system. The introduction of pseudo-measure spaces is motivated by [**6**], related to the theory of turbulences (see also [**7**] and [**3**]). These spaces are particular case of homogeneous Besov spaces constructed over the shift-invariant Banach space of distributions. Here, the so-called shift-invariant Banach space of distributions is the pseudo-measure space $PM^0$ (see [**18**] for more details).

**1.3. Critical space.** We supposed that $d \ge 2$. Here, we want to investigate the existence and regularity for the Cauchy problem associated to (1.3) in critical spaces, related to the invariance by scaling. The invariance by scaling is the main thread for finding some appropriate functional framework. Let us first recall the notion of scaling for System (1.3) (see [**10**] or [**15**] ). If $(a,u)$ solves (1.3), then $(a_\lambda, u_\lambda)$ solves (1.3) with $\lambda^2\alpha$ instead $\alpha$, where

$$a_\lambda := a(\lambda^2\cdot, \lambda\cdot) \quad \text{and} \quad u_\lambda := \lambda u(\lambda^2\cdot, \lambda\cdot),$$

and $\lambda \in \mathbb{R}^*$. This observation leads to the notion of critical spaces. We say that a functional space is a *critical space* for (1.3) if for all positive real numbers $\lambda$, the associated norm is invariant under the transformation

$$(a,u) \longmapsto (a_\lambda, u_\lambda),$$

up to a constant independent of $\lambda$. This suggests to choose initial data $(a_0,u_0)$ in a the space whose norm is invariant for all positive real number $\lambda$ by $(a_0,u_0) \mapsto (a_0(\lambda\cdot), \lambda u_0(\lambda\cdot))$. If we deal with a pseudo-measure space, a natural candidate is the space $PM^d \times PM^{d-1}$. According to the discussion of the critical spaces, the space-time functional space that we investigate in this paper is the following Kato space.

DEFINITION 1.2. Let $p$, $r$ and $T$ be positive real numbers. We define the Kato space $K_T^{p,r}$ as the space of $u \in \mathcal{C}_b(]0,T];PM^{r+\frac{2}{p}})$ such that the quantity

$$\|u\|_{K_T^{p,r}} := \sup_{t \in ]0,T]} \{t^{\frac{1}{p}} \|u(t)\|_{PM^{r+\frac{2}{p}}}\},$$

is finite. We also define the space $K_\infty^{p,r}$ of $u \in \mathcal{C}_b(]0,+\infty[;PM^{r+\frac{2}{p}})$ such that

$$\|u\|_{K_\infty^{p,r}} := \sup_{t \in ]0,+\infty[} \{t^{\frac{1}{p}} \|u(t)\|_{PM^{r+\frac{2}{p}}}\},$$

is finite.

We observe that the space $K^{p,d}_\infty \times K^{p,d-1}_\infty$ verifies the invariance by scaling. The Kato spaces are useful to establish Kato types theorems (see [**2**] and [**9**]), such as theorem 4.1. We use these Kato spaces to establish global existence and regularity results.

**1.4. Radius of analyticity.** If $\Omega$ is an open subset of $\mathbb{C}^d$, we denote by $\mathcal{H}(\Omega)$ the set of holomorphic functions over $\Omega$. Let $r < d$. If $u \in PM^r(\mathbb{R}^d)$, we define the radius of analyticity of $u$ by setting

$$\mathrm{rad}(u) := \sup\left\{\sigma > 0 \;\middle|\; e^{\sigma|D|}u \in PM^r(\mathbb{R}^d)\right\}.$$

If $u = (u_1, u_2, \dots, u_d) \in \left(PM^r(\mathbb{R}^d)\right)^d$ is a vector field, we define this radius by setting $\mathrm{rad}(u) := \min_{k\in[\![1,d]\!]}\{\mathrm{rad}(u_k)\}$. For every $\sigma > 0$, we denote by $S_\sigma$ the open connected set of all $z$ in $\mathbb{C}^d$ such that $|\operatorname{Im}(z)| < \sigma$. The following proposition justifies the denomination "radius of analyticity".

PROPOSITION 1.3. Let $r < d$ and $\sigma > 0$. Let $u$ be in $PM^r(\mathbb{R}^d)$. If $e^{\sigma|D|}u \in PM^r(\mathbb{R}^d)$, then $u$ extends to an unique holomorphic function $U$ in $\mathcal{H}(S_\sigma)$.

This proposition is well known in the mathematical folklore and for the sake of completeness, we give a self contained proof in the appendix. Proposition 1.3 means that we can express $u \in PM^r(\mathbb{R}^d)$, whose Fourier transform have an exponential decay, as the trace on $\mathbb{R}^d$ of a function which is holomorphic on some strip $S_\sigma$.

**1.5. Main results.** We recall that $d \geq 2$. Let's assume that $p > 2$ is such that $d - 3 + \frac{4}{p} > 0$. This condition ensures that nonlinear terms are well defined. We introduce the space $X_T$ of $(a, u) \in (K^{p,d-1}_T \cap K^{p,d}_T) \times K^{p,d-1}_T$, that we equip with the norm defined by

$$\|(a, u)\|_{X_T} := \max\{\|a\|_{K^{p,d-1}_T}, \|a\|_{K^{p,d}_T}\} + \|u\|_{K^{p,d-1}_T}.$$

Using the language of mild solutions of the Navier-Stokes-Korteweg system, as in [**13**], we prove the global existence and regularity of the solution to (1.3) which we state as follows (summing up theorem 3.3 and theorem 3.5).

THEOREM 1.4. *Given an initial data* $(a_0, u_0)$ *in* $\left(PM^{d-1} \cap PM^d\right) \times PM^{d-1}$. *If* $\|(a_0, |D|a_0, u_0)\|_{PM^{d-1}}$ *is small enough, then the Cauchy problem* (1.3)-(1.5) *has a global solution* $(a, u)$ *in the space* $X_\infty$ *which is space analytic at any positive time. Moreover, for any time* $t > 0$, *we have*

$$\mathrm{rad}(a(t), u(t)) \geq c_0\sqrt{t},$$

*for some positive constant* $c_0$ *which depends only on* $\nu$, $\mu$, $\kappa$ *and* $\alpha$.

The first observation is that the lower bound of the radius of analyticity is similar to [**13**] or [**1**] in the case of Besov spaces. Moreover this regularity result holds for critical initial data.
Section 5, we investigate the instantaneous analytic smoothing effect of System (1.3). The following theorem sums up two main results from Section 5.

THEOREM 1.5. *Let* $\delta$ *be in* $]0, \frac{2}{p}]$. *Let* $(a_0, u_0)$ *be in* $\left(PM^{d-1+\delta} \cap PM^{d+\delta}\right) \times PM^{d-1+\delta}$ *an initial data. There exist a positive time* $T$ *and an unique solution* $(a, u)$ *in* $X_T$ *to the Cauchy problem* (1.3)-(1.5). *Moreover, if* $\delta = \frac{2}{p}$, *we have*

$$\liminf_{t\to 0^+} \frac{\mathrm{rad}(a(t), u(t))}{\sqrt{t|\ln(C_1 t)|}} \geq C_2,$$

*for some positive constants $C_1$ and $C_2$.*

The main interest of this theorem is the amelioration of the improvement radius of analyticity near 0, proposed by F. Charve, R. Danchin and J. Xu in [**13**]. This result adapts to our framework the new method of J.-Y. Chemin, I. Gallagher and P. Zhang in [**9**] to estimate the radius of analyticity near 0 of the solution to semi-linear parabolic system. Note that compared with Theorem 1.4, that this theorem contains a local in time existence and uniqueness result for supercritical initial data and holds for arbitrary large initial data. Additionally, we remark that Constant $C_1$ and the existence time interval, depend on the norm of the initial data (see Theorem 4.5 and Theorem 4.1). Furthermore, let us remark that the local solution, given by the previous theorem, belongs in to the same space as the global solution, namely $X_T$, at least until their existence time $T$.

## 2. The linearized system

**2.1. Parabolic estimate for the linearized system.** In this section we investigate the linearized system around $(u, a) = (0, 0)$. This system reads

$$\left\{\begin{array}{l}\partial_t a + \operatorname{div}(u) = F,\\ \partial_t u - \mu \triangle u - (\mu+\nu)\nabla \operatorname{div}(u) + \alpha\nabla a - \kappa\nabla\triangle a = G,\end{array}\right. \tag{2.1}$$

where $F$ and $G$ are externals forces assumed to be smooth enough. For all $\xi \in \mathbb{R}^d$, we define $(d+1)\times(d+1)$-matrix

$$A(\xi) := \begin{pmatrix} 0 & i\xi_1 & \cdots & \cdots & \cdots & i\xi_d \\ i(\alpha\xi_1 + \kappa\xi_1|\xi|^2) & \mu|\xi|^2 + (\mu+\nu)\xi_1^2 & & & & (\mu+\nu)\xi_1\xi_d \\ \vdots & \vdots & \ddots & & \iddots & \vdots \\ \vdots & \vdots & & \cdot & & \vdots \\ \vdots & \vdots & \iddots & & \ddots & \vdots \\ i(\alpha\xi_d + \kappa\xi_d|\xi|^2) & (\mu+\nu)\xi_d\xi_1 & \cdots & \cdots & \cdots & \mu|\xi|^2 + (\mu+\nu)\xi_d^2 \end{pmatrix}$$

The matrix-valued symbol $A$ is the symbol of the space derivative operator of the linearized system. For all $t \geq 0$, we define

$$W(t) := e^{-tA(D)}.$$

The family of Fourier multipliers $(W(t))_{t\geq 0}$ is the semi-group of the linearized system. The key point of our study of the Navier-Stokes-Korteweg system with a quantum pressure, is a pointwise estimate of the semi-group $(W(t))_{t\geq 0}$ (that can be found in [**8**], [**14**] or [**10**]). More precisely, we observe that the linear part of system (1.3) has a parabolic behavior.

Lemma 2.1. *There exists a positive constant $c_0$, depending on $\kappa$, $\mu$ and $\nu$, such that the following inequality holds for all $\xi \in \mathbb{R}^d$ and $t \geq 0$:*

$$\begin{aligned}|(\widehat{a}, |\xi|\widehat{a}, \widehat{u})|(t,\xi) \lesssim_{\kappa,\nu,\mu,d} e^{-c_0 t|\xi|^2}|(\widehat{a}, |\xi|\widehat{a}, \widehat{u})|(0,\xi)\\ + \int_0^t e^{-c_0|\xi|^2(t-\tau)}|(\widehat{F}, |\xi|\widehat{F}, \widehat{G})|(\tau,\xi)\,d\tau.\end{aligned} \tag{2.2}$$

This lemma gives a "parabolic decay" of Fourier modes, in order to obtain the analytic regularisation. This "transfer of parabolicity" is a remarkable property of

Korteweg typs model for compressible fluids.
From here we will omit the dependence on constants $\nu$, $\mu$, $\kappa$ and $\alpha$ in inequalities, in particular on the symbol $\lesssim$.

## 3. Global existence

**3.1. Nonlinear estimates.** In this section we will establish some bilinear estimate, which will be used to control the nonlinear terms of System (1.3). We begin by an elementary lemma where we investigate a convolution inequality.

LEMMA 3.1. *Let $d \geq 2$. Let $r$ and $r'$ be two real numbers such that $r < d$, $r' < d$ and $r + r' > d$. Then, for all $\xi \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \frac{1}{|\xi - \eta|^r} \frac{1}{|\eta|^{r'}} d\eta \lesssim_{r,r',d} \frac{1}{|\xi|^{r+r'-d}}. \tag{3.1}$$

This estimate will be useful when we consider the product of two functions in pseudo-measure spaces.

PROOF. If $\xi = 0$, it is classical that the left-hand side of (3.1) is infinite as the right-hand side this inequality. Suppose that $\xi \neq 0$. We set

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{1}{|\xi - \eta|^r} \frac{1}{|\eta|^{r'}} d\eta = &\underbrace{\int_{B(\xi,|\xi|/2)} \frac{1}{|\xi - \eta|^r} \frac{1}{|\eta|^{r'}} d\eta}_{=:I_1} + \underbrace{\int_{B(0,|\xi|/2)} \frac{1}{|\xi - \eta|^r} \frac{1}{|\eta|^{r'}} d\eta}_{=:I_2} \\ &+ \underbrace{\int_{\mathbb{R}^d \setminus X_\xi} \frac{1}{|\xi - \eta|^r} \frac{1}{|\eta|^{r'}} d\eta}_{=:I_3}.\end{aligned}$$

where $X_\xi := B(0, |\xi|/2) \cup B(\xi, \frac{|\xi|}{2})$ We only need to estimate $I_1$, $I_2$ and $I_3$. For the first one, let use begin by remarking that, if $|\xi - \eta| \leq \frac{|\xi|}{2}$, then, using the inverse triangular inequality, we have $|\eta| \geq \frac{|\xi|}{2}$. Therefore, we get

$$I_1 \leq \int_{B(\xi,|\xi|/2)} \frac{1}{|\xi - \eta|^r} d\eta \left(\frac{2}{|\xi|}\right)^{r'}. \tag{3.2}$$

We aim to estimate the first factor to the right hand side of (3.2). Using the change of variables $\zeta \mapsto \zeta + \xi$, we get

$$\int_{B(\xi,|\xi|/2)} \frac{1}{|\xi - \eta|^r} d\eta = \int_{B(0,|\xi|/2)} \frac{1}{|\zeta|^r} d\zeta.$$

Considering the hypothesis $r < d$, we get, using polar coordinates

$$\int_{B(0,|\xi|/2)} \frac{1}{|\zeta|^r} d\zeta \lesssim_{r,r',d} \frac{1}{|\xi|^{r-d}}. \tag{3.3}$$

Using (3.3) to estimate the first factor of the right of (3.2), we obtain

$$I_1 \lesssim_{r,r',d} \frac{1}{|\xi|^{r+r'-d}}.$$

Observing that $|\eta| \leq \frac{|\xi|}{2}$ implies $|\xi - \eta| \geq \frac{|\xi|}{4}$ and taking into account that $r' < d$, from the inverse triangular inequality and using polar coordinates, we get as the same way

$$I_2 \lesssim_{r,r',d} \int_{B(0,|\xi|/2)} \frac{1}{|\eta|^{r'}} d\eta \left(\frac{2}{|\xi|}\right)^r \lesssim_{r,r',d} \frac{1}{|\xi|^{r+r'-d}}.$$

We decompose the last term, namely $I_3$, in two part

$$I_3 = \int_{B(0,3|\xi|/2)\setminus X_\xi} \frac{1}{|\xi-\eta|^r|\eta|^{r'}} d\eta + \int_{\mathbb{R}^d\setminus B(0,\frac{3|\xi|}{2})} \frac{1}{|\xi-\eta|^r|\eta|^{r'}} d\eta.$$

For the first one, we have

$$\int_{B(0,3|\xi|/2)\setminus X_\xi} \frac{1}{|\xi-\eta|^r|\eta|^{r'}} d\eta \lesssim_{r,r',d} \frac{1}{|\xi|^{r+r'}} \int_{B(0,\frac{3|\xi|}{2})} d\eta \lesssim_{r,r',d} \frac{1}{|\xi|^{r+r'-d}}.$$

Observing that, if $\eta \in \mathbb{R}^d\setminus X_\xi$, then $|\xi-\eta| \geq \frac{|\eta|}{2}$ and using polar coordinates and the hypothesis $r+r' > d$, we obtain

$$\int_{\mathbb{R}^d\setminus B(0,3|\xi|/2)} \frac{1}{|\xi-\eta|^r|\eta|^{r'}} d\eta \lesssim_{r,r',d} \int_{\mathbb{R}^d\setminus B(0,3|\xi|/2)} \frac{1}{|\eta|^{r+r'}} d\eta \lesssim_{r,r',d} \frac{1}{|\xi|^{r+r'-d}},$$

that concludes the proof. □

The lemma above give a pointwise estimate for the decay rate of the convolution, which is the base for considering products in the pseudo-measure spaces. As a consequence of Lemma 3.1, we get following bilinear estimates.

Lemma 3.2. *Let $l$ and $k$ two homogeneous Fourier multipliers of degree 1. Let $c>0$ and let $p>2$. Then, there exists a positive constant $C_k$, that depends on $c$, $p$, $d$ and $k$ such that, for every $u$ and $v$ in $K_T^{p,d-1}$, we have*

$$\|\int_0^t e^{c(t-s)\triangle}\mathbf{k}(u,v)(s)ds\|_{K_T^{p,d-1}} \leq C_k\|u\|_{K_T^{p,d-1}}\|v\|_{K_T^{p,d-1}}, \tag{3.4}$$

*where $\mathbf{k}(u,v) := k(D)(u\cdot v)$. If $p$ additionally satisfies $d-3+\frac{4}{p}>0$, then there exists a positive constant $C_l$, that depends of $c$, $p$, $d$ and $l$ such that, for every $u$ and $v$ in $K_T^{p,d-1}$, we have*

$$\|\int_0^t e^{c(t-s)\triangle}\mathbf{l}(u,v)(s)ds\|_{K_T^{p,d-1}} \leq C_l\|u\|_{K_T^{p,d-1}}\|v\|_{K_T^{p,d-1}}, \tag{3.5}$$

*where $\mathbf{l}(u,v) := u\cdot l(D)v$.*

PROOF. By applying Lemma 3.1 with $r = d-1+\frac{2}{p}$ and $r' = d-2+\frac{2}{p}$

$$
\begin{aligned}
&|\int_0^t e^{-c(t-s)|\xi|^2}\widehat{\mathbf{l}}(u,v)(s,\xi)ds| \\
&\le \int_0^t e^{-c(t-s)|\xi|^2}(|\widehat{u}| \star |\widehat{l(D)v}|)(s,\xi)ds, \\
&\lesssim_l \int_{\mathbb{R}^d}\int_0^t e^{-c(t-s)|\xi|^2}|\widehat{u}(s,\xi-\eta)||\eta||\widehat{v}(s,\eta)|dsd\eta, \\
&\lesssim_l \left(\int_{\mathbb{R}^d}\frac{d\eta}{|\xi-\eta|^{d-1+\frac{2}{p}}|\eta|^{d-2+\frac{2}{p}}}\right)\int_0^t \frac{e^{-c(t-s)|\xi|^2}}{s^{\frac{2}{p}}}ds \\
&\qquad \times \|u\|_{K_t^{p,d-1}}\|v\|_{K_t^{p,d-1}}, \\
&\lesssim_{l,p,d} \frac{1}{|\xi|^{d-3+\frac{4}{p}}}\left(\int_0^t \frac{e^{-c(t-s)|\xi|^2}}{s^{\frac{2}{p}}}ds\right)\|u\|_{K_T^{p,d-1}}\|v\|_{K_T^{p,d-1}}.
\end{aligned}
$$

Using Lemma 3.1 with $r = r' = d-1+\frac{2}{p}$, we obtain by the same approach

$$
\begin{aligned}
&|\int_0^t e^{-c(t-s)|\xi|^2}\widehat{\mathbf{k}}(u,v)(s,\xi)ds| \\
&\le \int_0^t e^{-c(t-s)|\xi|^2}|k(\xi)|(|\widehat{u}| \star |\widehat{v}|)(s,\xi)ds \\
&\lesssim_l |\xi|\int_{\mathbb{R}^d}\int_0^t e^{-c(t-s)|\xi|^2}|\widehat{u}(s,\xi-\eta)||\widehat{v}(s,\eta)|dsd\eta \\
&\lesssim_l \left(|\xi|\int_{\mathbb{R}^d}\frac{d\eta}{|\xi-\eta|^{d-1+\frac{2}{p}}|\eta|^{d-1+\frac{2}{p}}}\right)\int_0^t \frac{e^{-cc(t-s)|\xi|^2}}{s^{\frac{2}{p}}}ds \\
&\qquad \times \|u\|_{K_T^{p,d-1}}\|v\|_{K_T^{p,d-1}} \\
&\lesssim_{l,p,d} \frac{1}{|\xi|^{d-3+\frac{4}{p}}}\left(\int_0^t \frac{e^{-c(t-s)|\xi|^2}}{s^{\frac{2}{p}}}ds\right)\|u\|_{K_T^{p,d-1}}\|v\|_{K_T^{p,d-1}}.
\end{aligned}
$$

Finally, using that the function $y \in \mathbb{R}_+ \mapsto e^{-\delta y}y^{1-\frac{1}{p}}$ is bounded $\left(\frac{1-\frac{1}{p}}{ec}\right)^{1-\frac{1}{p}}$ and the change of variable $\sigma \mapsto t\sigma$ to make appear (taking into account the hypothesis $p > 2$) the first Euler integral (see [**4**] Proposition 4 page 312), we have

$$
\begin{aligned}
\frac{1}{|\xi|^{d-3+\frac{4}{p}}}\left(\int_0^t \frac{e^{-c(t-s)|\xi|^2}}{s^{\frac{2}{p}}}ds\right) &= \frac{1}{|\xi|^{d-1+\frac{2}{p}}}\left(\int_0^t \frac{e^{-c(t-s)|\xi|^2}(c(t-s)|\xi|^2)^{1-\frac{1}{p}}}{\delta^{1-\frac{1}{p}}(t-s)^{1-\frac{1}{p}}s^{\frac{2}{p}}}ds\right) \\
&\lesssim_{p,d} \frac{1}{c^{1-\frac{1}{p}}|\xi|^{d-1+\frac{2}{p}}}\left(\int_0^t \frac{1}{(t-s)^{1-\frac{1}{p}}s^{\frac{2}{p}}}ds\right) \\
&\lesssim_{p,d} \frac{t^{-\frac{1}{p}}}{c^{1-\frac{1}{p}}|\xi|^{d-1+\frac{2}{p}}}\left(\int_0^1 \frac{1}{(1-\sigma)^{1-\frac{1}{p}}\sigma^{\frac{2}{p}}}d\sigma\right) \\
&\lesssim_{p,d} \frac{t^{-\frac{1}{p}}}{c^{1-\frac{1}{p}}|\xi|^{d-1+\frac{2}{p}}}.
\end{aligned}
$$

This concludes the proof of the lemma. □

For the remainder of this paper, we supposed that $d \geq 2$ and $p > 2$ is such that $d - 3 + \frac{4}{p} > 0$.

**3.2. Global existence theorem.** In this section we study the global existence of the solution to System (1.3) for critical initial data. The main result of this section is the following theorem.

THEOREM 3.3. *There exists $h > 0$ and $R > 0$ such that, for every $(a_0, u_0)$ in $(PM^{d-1} \cap PM^d) \times PM^{d-1}$ satisfying*

$$\|(a_0, |D|a_0, u_0)\|_{PM^{d-1}} \leq h,$$

*there exists a unique solution $(a, u) \in X_\infty$ of the Cauchy problem* (1.3)-(1.5), *such that*

$$\|(a, u)\|_{X_\infty} \leq R.$$

The proof is based on the Banach fixed-point theorem.

PROOF. First observe that for all $v \in PM^{d-1}$, since the function $y \in \mathbb{R}_+ \mapsto e^{-c_0 y} y^{\frac{1}{p}}$ is bounded by $(ec_0 p)^{-\frac{1}{p}}$ where $c_0$ is the constant that appears in Lemma 2.1. Then, we have

$$e^{-c_0 t|\xi|^2} t^{\frac{1}{p}} |\widehat{v}(\xi)||\xi|^{d-1+\frac{2}{p}} = e^{-c_0 t|\xi|^2 (t|\xi|^2)^{\frac{1}{p}} |\widehat{v}(\xi)||\xi|^{d-1}} \leq \frac{1}{(c_0 pe)^{\frac{1}{p}}} \|v\|_{PM^{d-1}},$$

hence

$$\|e^{c_0 t \triangle} v\|_{K^{p,d-1}_\infty} \leq \frac{1}{(c_0 pe)^{\frac{1}{p}}} \|v\|_{PM^{d-1}}. \tag{3.6}$$

It follows from Lemma 2.1 and (3.6) that, for all $(a_0, u_0) \in (PM^{d-1} \cap PM^d) \times PM^{d-1}$, we have

$$W(\cdot)(a_0, u_0) \in K^{p,d-1}_\infty$$

and

$$\|W(t)(a_0, u_0)\|_{X_\infty} \leq \tilde{C} \|(a_0, |D|a_0, u_0)\|_{PM^{d-1}}. \tag{3.7}$$

where $\tilde{C}$ is a positive constant, that depends on constants $c_0$ from Lemma 2.1, $p$ and $d$. Combining Lemma 2.1 and 3.2, we get the following estimates : for every $a$ and $b$ in $K^{p,d-1}_T \cap K^{p,d}_T$ and for all $u$ and $v$ in $K^{p,d-1}_T$,

$$\| \int_0^t W(t-s) f(u, a)(s) ds \|_{K^{p,d-1}_\infty} \leq C_f \|u\|_{K^{p,d-1}_\infty} \|a\|_{K^{p,d-1}_\infty}, \tag{3.8}$$

$$\| \int_0^t W(t-s) |D| f(u, a)(s) ds \|_{K^{p,d-1}_\infty} \leq \tilde{C}_f \|u\|_{K^{p,d-1}_\infty} \|a\|_{K^{p,d}_\infty}, \tag{3.9}$$

$$\| \int_0^t W(t-s) g_1(u, v)(s) ds \|_{K^{p,d-1}_\infty} \leq C_{g_1} \|u\|_{K^{p,d-1}_\infty} \|v\|_{K^{p,d-1}_\infty}, \tag{3.10}$$

$$\| \int_0^t W(t-s) g_2(u, a)(s) ds \|_{K^{p,d-1}_\infty} \leq C_{g_2} \|u\|_{K^{p,d-1}_\infty} \|a\|_{K^{p,d}_\infty}, \tag{3.11}$$

$$\| \int_0^t W(t-s) g_3(a, b)(s) ds \|_{K^{p,d-1}_\infty} \leq C_{g_3} \|a\|_{K^{p,d}_T} \|b\|_{K^{p,d}_\infty}, \tag{3.12}$$

where positive constants $C_f$, $\tilde{C}_f$, $C_{g_1}$, $C_{g_2}$ and $C_{g_3}$ only depends of $d$, $p$ and $c_0$. For any positive real numbers $R$, we denote by $B(0,R)$ the ball of center 0 and radius $R$ in $X_T$.

$$\begin{array}{rcl}\Phi : X_\infty & \to & X_\infty \\ (a,u) & \mapsto & W(\cdot)(a_0,u_0) + \int_0^{\cdot} W(\cdot - s)(f(u,a)(s), g(u,a)(s))ds,\end{array}$$

where $(a_0,u_0) \in (PM^{d-1} \cap PM^d) \times K^{d-1}$ are the initial data. The goal is to prove the existence of a fixed-point for $\Phi$. We assume that

$$\|(a_0, |D|a_0, u_0)\|_{PM^{d-1}} < h, \tag{3.13}$$

for $h > 0$, small enough, which we will be fixed later. We begin by proving that for some radius $R > 0$, small enough, the ball $B(0,R)$ is stable by $\Phi$. If $(a,u)$ is in $B(0,R)$, then, we deduce from (**??**), (3.8), (3.9), (3.10), (3.11), (3.12) and (3.13) that

$$\|\Phi(a,u)\|_{X_\infty} \le \tilde{C}h + 5K_\Phi R^2. \tag{3.14}$$

where $K_\Phi := \max\{C_f, \tilde{C}_f, C_{g_1}, C_{g_2}, C_{g_3}\}$. Now, we assume that $R$ satisfies

$$5K_\Phi R < \frac{1}{2}, \tag{3.15}$$

and we set

$$h := \frac{R}{2\tilde{C}}. \tag{3.16}$$

For $R > 0$ satisfying (3.15) and for this choice of $h$, using (3.14) and (3.7), we get for all $(a,u)$ in $B(0,R)$

$$\|\Phi(a,u)\|_{X_\infty} < R,$$

which means that $B(0,R)$ is stable by $\Phi$. Let $(a,u)$ and $(b,v)$ be in $X_\infty$. We have

$$\begin{aligned} f(u,a) - f(v,b) &= f(u, a-b) + f(u-v, b), \\ g_1(u,u) - g_1(v,v) &= g_1(u-v, u) + g_1(v, u-v), \\ g_2(u,a) - g_2(v,b) &= g_2(u, a-b) + g_2(u-v, b), \\ g_3(a,a) - g_3(b,b) &= g_3(a-b, a) + g_3(b, a-b). \end{aligned}$$

Thus , applying (3.8), (3.9), (3.10), (3.11) and (3.12), we deduce, from the triangular inequality, that

$$\begin{aligned} \|\Phi(a,u) - \Phi(b,v)\|_{X_\infty} &\le K_\Phi(\|u-v\|_{X_\infty}(\|b\|_{X_\infty} + \|u\|_{X_\infty} + \|v\|_{X_\infty} + 2\||D|b\|_{X_\infty}) \\ &\quad + \|a-b\|_{X_\infty}\|u\|_{X_\infty} \\ &\quad + \||D|(a-b)\|_{X_\infty}(2\|u\|_{X_\infty} + \||D|a\|_{X_\infty} + \||D|b\|_{X_\infty})) \\ &\le 8K_\Phi\|(a,u) - (b,v)\|_{X_\infty}(\|(a,u)\|_{X_\infty} + \|(b,v)\|_{X_\infty}). \end{aligned}$$

Now, if we take $(a,u)$ and $(b,v)$ in the ball $B(0,R)$, from previous inequalities, it follows that

$$\|\Phi(a,u) - \Phi(b,v)\|_{X_\infty} \le 16K_\Phi R\|(a,u) - (b,v)\|_{X_\infty}.$$

However, from (3.15), we get

$$16RK_\Phi < 1. \tag{3.17}$$

Now, assume that $R$ satisfies (3.17). Since, (3.17) implies (3.15) , then, for $h$ given by (3.16), $\Phi$ is a strict contractive map of $B(0,R)$ into itself. We conclude with Banach fixed-point theorem (see [**5**], Theorem 5.7). $\square$

As a by-product of the proof of this theorem, we obtain the following result.

COROLLARY 3.4. Let $T>0$. Let $(a_0,u_0)$ be in $(PM^{d-1}\cap PM^d)\times PM^{d-1}$.There exists two positive constants $C_1$ and $C_2$ that depends only of $\mu$,$\nu$, $p$ and $d$ such that, for all solutions $(a,u)$ of (1.3) in $X_T$, we have

$$\|(a,u)\|_{X_T} \le C_1\|(a_0,|D|a_0,u_0)\|_{PM^{d-1}} + C_2\|(a,u)\|^2_{X_T}.$$

**3.3. Analyticity for global solution.** The purpose of this section is to prove the analyticity of the global solution constructed in the previous section. Furthermore, we give an lower bound for the radius of analyticity. This result holds for small enough critical initial data. We investigate later the case of supercritical data.

THEOREM 3.5. *Let $h$ and $R$ as in Theorem 3.3. For every $(a_0,u_0)$ in the space $(PM^{d-1}\cap PM^d)\times PM^{d-1}$ such that*

$$\|(a_0,|D|a_0,u_0)\|_{PM^{d-1}} \le \frac{h}{2e^{2c_0}},$$

*the solution of the Cauchy problem* (1.3)-(1.5) *given by Theorem 3.3 is analytic in space for every time $t>0$ with a radius of analyticity bounded below by $c_0\sqrt{t}$.*

The proof of the global existence for analytic solutions follows the main scheme than the proof of global existence. The difference is the choice of the functional space where we look for the fixed-point. The idea is to consider a weighted norm of the form $e^{\delta\sqrt{t}|D|}$, where $\delta\sqrt{t}$ gives a radius of analyticity for the solution at any positive time $t$. This method is well-known (see [**8**] for this system and [**1**] for some general nonlinear dissipative system for example). We begin by a version of Lemma 3.2, with analytic norm. We recall that $d\ge 2$ and $p>2$ is such that $d-3+\frac{4}{p}>0$.

LEMMA 3.6. *Let $T$ be in $]0,+\infty]$. Let $l$ and $k$ two homogeneous Fourier multipliers of degree 1. Then for every $u$ and $v$ in the Kato space $K^{p,d-1}_T$, we have*

$$\begin{aligned} &\Big\|\int_0^t e^{c_0(t-s)\,\triangle}\mathcal{L}_t(u,v)(s)ds\Big\|_{K^{p,d-1}_T} \\ &\le 2^{1-\frac{1}{p}}e^{2c_0}C_l\|e^{c_0\sqrt{t}|D|}u\|_{K^{p,d-1}_T}\|e^{c_0\sqrt{t}|D|}v\|_{K^{p,d-1}_T}, \end{aligned} \tag{3.18}$$

$$\begin{aligned} &\Big\|\int_0^t e^{c_0(t-s)\,\triangle}\mathcal{K}_t(u,v)(s)ds\Big\|_{K^{p,d-1}_T} \\ &\le 2^{1-\frac{1}{p}}e^{2c_0}C_k\|e^{c_0\sqrt{t}|D|}u\|_{K^{p,d-1}_T}\|e^{c_0\sqrt{t}|D|}v\|_{K^{p,d-1}_T}, \end{aligned} \tag{3.19}$$

*where, for all $t>0$,*

$$\mathcal{L}_t(u,v) := e^{c_0\sqrt{t}|D|}\left(u\cdot l(D)v\right),$$

*and*

$$\mathcal{K}_t(u,v) := e^{c_0\sqrt{t}|D|}\left(k(D)(u\cdot v)\right).$$

PROOF. We adapt the proof of Lemma 3.2. The additional key point that we need here is the inequality

$$e^{-\frac{c_0}{2}(t-s)|\xi|^2}e^{-c_0\sqrt{s}|\xi-\eta|}e^{-c_0\sqrt{s}|\eta|} \le e^{2c_0}e^{-c_0\sqrt{t}|\xi|}. \tag{3.20}$$

From the inverse triangular inequality, it follows that $-\sqrt{s}|\xi-\eta|-\sqrt{s}|\eta|\leq-\sqrt{s}|\xi|$. Hence, to establish (3.20), it is enough to prove that

$$I := (\sqrt{t}-\sqrt{s})|\xi|(1-(\sqrt{t}+\sqrt{s})\frac{|\xi|}{2}) \leq 2.$$

If $\sqrt{t}|\xi| \geq 2$, we deduce that $I \leq 0 \leq 2$ and if $\sqrt{t}|\xi| < 2$, then $I \leq \sqrt{t}|\xi| \leq 2$. Then we obtain the expected upper bound for $I$, that concludes the proof of the lemma. □

In Lemma 3.6, constants $C_l$ and $C_k$ are the same as in Lemma 3.2. We can also notice the presence of the factor $2^{1-\frac{1}{p}}e^{2c_0}$ unlike the non analytic version, which come from the presence of the weight $e^{c_0\sqrt{t}|D|}$ in nonlinear estimates. This coefficient has the effect of restraining the set of initial data (that need to be smaller than in Theorem 3.3) which can selected in order to get an unique analytic solution. To deal with the analytic setting, we introduce the analytic space $Y_T$ of $(a,u)$ in $X_T$ such that $e^{c_0\sqrt{t}|D|}(a,u)$ belong in $X_T$, that we equip with the norm defined by

$$\|(a,u)\|_{Y_T} := \|(e^{c_0\sqrt{t}|D|}a, e^{c_0\sqrt{t}|D|}u)\|_{X_T}.$$

Proof of Theorem 3.5. First, we remark that for every $\xi \in \mathbb{R}^d$ and positive time $t$, we have

(3.21)
$$e^{-c_0t|\xi|^2}(t|\xi|^2)^{\frac{1}{p}}e^{c_0\sqrt{t}|\xi|} = \left(\frac{2}{c_0}\right)^{\frac{1}{p}} \times \left(e^{-\frac{c_0}{2}t|\xi|^2}\left(\frac{c_0}{2}t|\xi|^2\right)^{\frac{1}{p}}\right) \times \left(e^{c_0\sqrt{t}|\xi|}e^{-\frac{c_0}{2}t|\xi|^2}\right).$$

The second factor of the right-hand side of the previous identity is bounded by $(ep)^{-\frac{1}{p}}$. Using Inequality (3.20), the third factor to the right-hand of (3.21) is bounded by $e^{2c_0}$. Hence, for all $v \in PM^{d-1}$, we have

$$e^{-c_0t|\xi|^2}t^{\frac{1}{p}}e^{c_0\sqrt{t}|\xi|}|\widehat{v}(\xi)||\xi|^{d-1+\frac{2}{p}} \leq \frac{2^{\frac{1}{p}}e^{2c_0}}{(c_0pe)^p}\|v\|_{PM^{d-1}}.$$

We suppose that the initial data $(a_0,u_0) \in (PM^{d-1}\cap PM^d)\times PM^{d-1}$ satisfy

$$\|(a_0,|D|a_0,u_0)\|_{PM^{d-1}} < \tilde{h}, \tag{3.22}$$

for some positive real number $\tilde{h}$, small enough. Using Lemma 3.6, we deduce by the same way of the proof of global existence, that for a radius $\tilde{R} := \frac{R}{2^{1-\frac{1}{p}}e^{2c_0}}$ and for $\tilde{h} := \frac{h}{2e^{2c_0}}$. The map

$$\begin{array}{rcl} \Psi : Y_\infty & \to & Y_\infty \\ (a,u) & \mapsto & W(\cdot)(a_0,u_0) + \int_0^{\cdot} W(\cdot - s)(f(u,a)(s), g(u,a)(s))ds, \end{array}$$

has a unique fixed-point $(a,u)$ in the ball of center 0 and radius $\tilde{R}$ in $Y_\infty$.

Furthermore, if the initial data $(a_0,u_0)$ satisfies (3.22), using $\tilde{h} < h$, we deduce the existence of a global solution, constructed as the unique[1] fixed-point of $\Phi$ in the ball $B(0,R)$ of $X_\infty$. Moreover the fixed-point of $\Psi$ found previously, namely $(a,u)$, is in the ball $B(0,R)$ of the space $X_\infty$ and is the unique fixed-point of $\Phi$ in this ball. Indeed, it is enough to observe that

$$\|\Phi(a,u)\|_{X_\infty} \leq \|\Psi(a,u)\|_{Y_\infty} \leq \tilde{R} < R,$$

[1] We recall that the fixed-point of the Banach fixed-point, like in [**5**] Theorem 5.7, is unique

keeping in mind that $\tilde{h} < h$. In particular, we conclude that, if the initial data $(a_0, u_0)$ satisfies (3.22), the solution from Theorem 3.3 is analytic and, at any positive time $t$, its radius of analyticity is bounded below by $c_0\sqrt{t}$. □

## 4. Estimate near 0

Now, we turn our attention to the case of supercritical initial data. First we give a local in time existence and uniqueness theorem (in Subsection 5.1), so-called Kato type theorem. This result holds for supercritical initial data, which we will pick arbitrarily large. In Subsection 5.2, we establish the analyticity of the solution with supercritical initial data and give the same lower bound as in the case of critical initial data. To end this section, we investigate in Subsection 5.3 the instantaneous smoothing effect, giving a better estimate of the radius of analyticity near 0. This last result constitutes the main novelty of this paper, related to the study of the radius of analyticity for Navier-Stokes-Korteweg type system.

**4.1. Kato type theorem for local existence with supercritical data.** In order to give a better estimate of the radius of analyticity in the neighbourhood of 0, we need to obtain a local existence results for supercritical initial data, namely in the space $\left(PM^{d-1+\delta} \cap PM^{d+\delta}\right) \times PM^{d-1+\delta}$. Our goal here is to prove a Kato type theorem to obtain the local existence which will be proved by a Banach fixed-point argument.

THEOREM 4.1. *Let $\delta$ in $]0, \frac{2}{p}]$. For any initial data*

$$(a_0, u_0) \in \left(PM^{d-1+\delta} \cap PM^{d+\delta}\right) \times PM^{d-1+\delta},$$

*there exists a positive real number $T$ such that the Cauchy problem* (1.3)-(1.5) *has a unique solution $(a, u)$ in the space $X_T$. Moreover, there exists a positive constant $c_{\delta,p,c_0}$, that does not depend on the initial data $(a_0, u_0)$, such that*

$$T \geq c_{\delta,p,c_0} \|(a_0, |D|a_0, u_0)\|_{PM^{d-1+\delta}}^{-\frac{2}{\delta}}.$$

Compared with the global existence (Theorem 3.3) this theorem give existence and uniqueness of the solution only for small times in the case of supercritical initial data. Furthermore, we don't need any smallness assumptions on the initial data, but the existence interval gets smaller as the norm of the initial data gets bigger.

PROOF. Let $(a_0, u_0)$ be in $\left(PM^{d-1+\delta} \cap PM^{d+\delta}\right) \times PM^{d-1+\delta}$ and $T$ a positive time will be chosen later. We deduce from Lemma 2.1 that

$$\|W(t)(a_0, u_0)\|_{K_T^{p,d-1}} \lesssim_d \|e^{c_0 t \triangle}(a_0, u_0)\|_{K_T^{p,d-1}}. \tag{4.1}$$

Furthermore, since the function $y \in \mathbb{R}_+ \mapsto e^{-c_0 y} y^{\frac{1}{p} - \frac{\delta}{2}}$ is bounded, for every $t \in [0, T]$ and $\xi \in \mathbb{R}^d$ we have

$$e^{-c_0 t|\xi|^2} t^{\frac{1}{p}} |(\widehat{a}_0, |\xi|\widehat{a}_0, \widehat{u}_0)(\xi)| |\xi|^{d-1+\frac{2}{p}} \lesssim_{c_0,p,\delta} T^{\frac{\delta}{2}} \|(a_0, |D|a_0, u_0)\|_{PM^{d-1+\delta}},$$

hence

$$\|e^{c_0 t \triangle}(a_0, |D|a_0, u_0)\|_{PM^{d-1+\frac{2}{p}}} \lesssim_{c_0,p,\delta} T^{\frac{\delta}{2}} \|(a_0, |D|a_0, u_0)\|_{PM^{d-1+\delta}}.$$

From (4.1) and the estimate above, we conclude that there exist a positive constant $\tilde{C}_\delta$, such that

$$\|W(t)(a_0, u_0)\|_{X_T} \leq \tilde{C}_\delta T^{\frac{\delta}{2}} \|(a_0, |D|a_0, u_0)\|_{PM^{d-1+\delta}}. \tag{4.2}$$

Now, we consider the map

$$\begin{array}{rcl} \Phi : X_T & \to & X_T \\ (a,u) & \mapsto & W(\cdot)(a_0,u_0) + \int_0^{\cdot} W(\cdot - s)(f(u,a)(s), g(u,a)(s))ds. \end{array}$$

As in the proof of the global existence theorem, for some positive radius $R$, if $(a,u)$ and $(b,v)$ in $B(0,R)$ we have

$$\|\Phi(a,u) - W(\cdot)(a_0,u_0)\|_{X_T} \leq 5K_\Phi R^2, \tag{4.3}$$

$$\|\Phi(a,u) - \Phi(b,v)\|_{X_T} \leq 16K_\Phi R\|(a,u)-(b,v)\|_{X_T}. \tag{4.4}$$

Assume that the radius $R$ and the time $T$ satisfy

$$16RK_\Phi < 1 \quad \text{and} \quad T = c_{\delta,p,c_0}\|(a_0,|D|a_0,u_0)\|_{PM^{d-1+\delta}}^{-\frac{2}{\delta}}$$

where $c_{\delta,p,c_0} := \left(\frac{R}{2\tilde{C}_\delta}\right)^{\frac{\delta}{2}}$. Then, using (4.2), (4.3) and (4.4), we observe that $\Phi$ is an contraction of the ball $B(0,R)$ of $X_T$ into itself. We conclude the proof by applying the Banach fixed-point theorem. $\square$

As in the previous section, we can get the analyticity of solutions with an estimate of the radius of analyticity.

THEOREM 4.2. *Let $\delta$ in $]0, \frac{2}{p}]$. For any initial data*

$$(a_0,u_0) \in \left(PM^{d-1+\delta} \cap PM^{d+\delta}\right) \times PM^{d-1+\delta},$$

*there exists a positive real number $T$ such that the Cauchy problem* (1.3)-(1.5) *has unique solution $(a,u)$ in the space $X_T$, such that for all $t \in ]0,T]$, $(a(t),u(t))$ is real analytic with*

$$\mathrm{rad}((a(t),u(t))) \geq c_0\sqrt{t}.$$

*Moreover, there exists a positive constant $d_{\delta,p,c_0}$, that does not depend on the initial data $(a_0,u_0)$, such that $T \geq d_{\delta,p,c_0}\|(a_0,|D|a_0,u_0)\|_{PM^{d-1+\delta}}^{-\frac{2}{\delta}}$.*

PROOF. Let $(a_0,u_0)$ be in $\left(PM^{d-1+\delta} \cap PM^{d+\delta}\right) \cap PM^{d-1+\delta}$ and $T$ a positive time that will be chosen later. Now, we pick $v$ in $PM^{d-1+\delta}$. Lemma 2.1 provides

$$\|e^{c_0\sqrt{t}|D|}W(t)v\|_{K_T^{p,d-1}} \lesssim_d \|e^{c_0 t\triangle}e^{c_0\sqrt{t}|D|}v\|_{K_T^{p,d-1}}. \tag{4.5}$$

Moreover, for all $t \in [0,T]$, we have

$$\begin{aligned} e^{-c_0t|\xi|^2}t^{\frac{1}{p}}e^{c_0\sqrt{t}|\xi|}|\widehat{v}(\xi)||\xi|^{d-1+\frac{2}{p}} &= t^{\frac{\delta}{2}}\left(e^{-\frac{c_0}{2}t|\xi|^2}(t|\xi|)^{\frac{1}{p}-\frac{\delta}{2}}\right) \times \left(|\widehat{v}(\xi)||\xi|^{d-1+\delta}\right) \\ &\quad \times \left(e^{-\frac{c_0}{2}t|\xi|^2}e^{c_0\sqrt{t}|\xi|}\right) \\ &\leq T^{\frac{\delta}{2}}\left(\frac{(\frac{1}{p}-\frac{\delta}{2})}{2ec_0}\right)^{\frac{1}{p}-\frac{\delta}{2}} e^{2c_0}\|v\|_{PM^{d-1+\delta}}. \end{aligned}$$

Combining the last estimate and (4.5), we get

$$\|e^{c_0\sqrt{t}|D|}W(t)v\|_{K_T^{p,d-1}} \lesssim_{\delta,p,c_0} T^{\frac{\delta}{2}}\|v\|_{PM^{d-1+\delta}}.$$

Hence, there exists a positive constant $D_{\delta,p,c_0}$ such that

$$\|W(t)(a_0,u_0)\|_{Y_T} \leq D_{\delta,p,c_0}T^{\delta}\|e^{c_0\sqrt{t}|D|}(a_0,|D|a_0,u_0)\|_{PM^{d-1+\delta}}. \tag{4.6}$$

Now we consider the map

$$\begin{array}{rcl}\Phi : Y_T & \to & Y_T \\ (a,u) & \mapsto & W(\cdot)(a_0,u_0)+\int_0^{\cdot} W(\cdot - s)(f(u,a)(s),g(u,a)(s))ds.\end{array}$$

Let $R$ be a positive radius that will be chosen later. From Lemma 3.6, we deduce that there exists a positive constant $K_\Phi$ such that, for all $(a,u)$ and $(b,v)$ in the ball $B(0,R)$ of $Y_T$, we have

$$\|\Phi(a,u) - W(\cdot)(a_0,u_0)\|_{Y_T} \leq 5K_\Phi R^2$$

and

$$\|\Phi(a,u) - \Phi(b,v)\|_{Y_T} \leq 16K_\Phi R\|(a,u)-(b,v)\|_{Y_T}.$$

For $R>0$ and $T$ such that

$$16RK_\Phi < 1 \quad \text{and} \qquad T = d_{\delta,p,c_0}\|(a_0,|D|a_0,u_0)\|_{PM^{d-1+\delta}}^{-\frac{2}{\delta}},$$

where $d_{\delta,p,c_0} := \left(\frac{R}{2D_{\delta,p,c_0}}\right)^{\frac{\delta}{2}}$, as in the proof of Theorem 4.1, from the Banach fixed-point theorem we conclude that there exists an unique fixed-point $(a,u)\in Y_T$ of $\Phi$ which solve (1.3)-(1.5). Since $(a,u)\in Y_T$, for all $t$ in $]0,T]$, we get $\operatorname{rad}((a(t),u(t))) \leq c_0\sqrt{t}$, which completes the proof.

□

**4.2. Estimate of the radius of analyticity.** In this section we show an improvement of the estimate of the radius of analyticity near 0. We will adapt the method used by J.-Y. Chemin, I. Gallagher and P. Zhang in [**9**] to our context, in order to obtain an estimate of the radius of analyticity when the initial data is in the space $(PM^{p,d-1+\frac{2}{p}}\cap PM^{p,d+\frac{2}{p}})\times PM^{p,d-1+\frac{2}{p}}$.
For all $T,\lambda>0$ and for each $t\in[0,T]$ and $\varepsilon>0$, we define the Fourier multiplier

$$\theta_\lambda(t,D,\varepsilon) := -\frac{\lambda^2}{4(1-\varepsilon)c_0}\frac{t}{T}+\lambda\frac{t}{\sqrt{T}}|D|.$$

For every $f\in L^1_{loc}\left([0,T];\mathcal{S}'(\mathbb{R}^d)\right)$, we set

$$\underline{f}(t) := e^{\theta_\lambda(t,D,\varepsilon)}f(t) \qquad (t\in[0,T]) \tag{4.7}$$

In order to study the radius of analyticity, we begin to establish some nonlinear estimate in the new analytic norm provided by $e^{\theta_\lambda(\cdot,D,\varepsilon)}$.

Lemma 4.3. *Let $\varepsilon$, $T$ and $\lambda$ three non-negative real number. For $l$ and $\mathbf{l}$, $k$ and $\mathbf{k}$, $u$ and $v$ as in Lemma 3.2, we have the following inequalities*

$$\|\int_0^t e^{c_0(t-s)\triangle}\underline{\mathbf{l}}_t(u,v)(s)ds\|_{K_T^{p,d-1}} \lesssim_{\varepsilon,c_0,p,l,d} e^{\frac{\lambda^2}{4(1-\varepsilon)c_0}}\|\underline{u}\|_{K_T^{p,d-1}}\|\underline{v}\|_{K_T^{p,d-1}}, \tag{4.8}$$

$$\|\int_0^t e^{c_0(t-s)\triangle}\underline{\mathbf{k}}_t(u,v)(s)ds\|_{K_T^{p,d-1}} \lesssim_{\varepsilon,c_0,p,l,d} e^{\frac{\lambda^2}{4(1-\varepsilon)c_0}}\|\underline{u}\|_{K_T^{p,d-1}}\|\underline{v}\|_{K_T^{p,d-1}}, \tag{4.9}$$

*where $\underline{\mathbf{l}}_t := e^{\theta_\lambda(t,D,\varepsilon)}\mathbf{l}$ and $\underline{\mathbf{k}}_t := e^{\theta_\lambda(t,D,\varepsilon)}\mathbf{k}$.*

Remark that the constant in (4.8) and (4.9) does not depend of $\lambda$ and $T$.

Proof. First, we recall some properties of symbols of $\theta_\lambda(t,D,\varepsilon)$. For every $t$ and $s$ in $[0,T]$ and for all $\xi$ and $\eta$ in $\mathbb{R}^d$, we have

$$\theta_\lambda(t,\xi,\varepsilon)=\theta_\lambda(t-s,\xi,\varepsilon)+\theta_\lambda(s,\xi,\varepsilon), \tag{4.10}$$

$$\theta_\lambda(t,\xi,\varepsilon)-c_0t|\xi|^2=-\frac{\lambda^2}{4(1-\varepsilon)c_0}\frac{t}{T}+\lambda\frac{t}{\sqrt{T}}|\xi|-c_0t|\xi|^2\le-\varepsilon c_0t|\xi|^2, \tag{4.11}$$

$$\theta_\lambda(t,\xi,\varepsilon)\le\theta_\lambda(t,\xi-\eta,\varepsilon)+\theta_\lambda(t,\eta,\varepsilon)+\frac{\lambda^2}{4(1-\varepsilon)c_0}\frac{t}{T}. \tag{4.12}$$

Let $\xi\in\mathbb{R}^d$. Using (4.10) and (4.11), for all $t\in[0,T]$, we get

$$\begin{aligned}
&|\int_0^t e^{-c_0(t-s)|\xi|^2}\widehat{\underline{\mathbf{l}}}_t(u,v)(s,\xi)ds|\\
&=|\int_0^t e^{-c_0(t-s)|\xi|^2+\theta_\lambda(t-s,\xi,\varepsilon)}\left(e^{\theta_\lambda(s,\xi,\varepsilon)}\widehat{\mathbf{l}(u,v)}(s,\xi)\right)ds|,\\
&\le\int_0^t e^{-\varepsilon c_0(t-s)|\xi|^2}\left(e^{\theta_\lambda(s,\xi,\varepsilon)}|\widehat{\mathbf{l}(u,v)}(s,\xi)|\right)ds.
\end{aligned}$$

Therefore, we deduce from (4.12) that, for all $s\in[0,T]$, we have

$$\begin{aligned}
e^{\theta_\lambda(s,\xi,\varepsilon)}|\widehat{\mathbf{l}}(u,v)(s,\xi)|&\lesssim_{l,d}\int e^{\theta_\lambda(s,\xi,\varepsilon)}|\widehat{u}(s,\xi-\eta)||\eta||\widehat{v}(s,\eta)|d\eta,\\
&\lesssim_{l,d}\int e^{\frac{\lambda^2}{4(1-\varepsilon)c_0}\frac{s}{T}}|\widehat{\underline{u}}(s,\xi-\eta)||\eta||\widehat{\underline{v}}(s,\eta)|d\eta,\\
&\lesssim_{l,d}e^{\frac{\lambda^2}{4(1-\varepsilon)c_0}}\int|\widehat{\underline{u}}(s,\xi-\eta)||\eta||\widehat{\underline{v}}(s,\eta)|d\eta,\\
&\lesssim_{l,d}\frac{e^{\frac{\lambda^2}{4(1-\varepsilon)c_0}}}{s^{\frac{2}{p}}}\int\frac{d\eta}{|\xi-\eta|^{d-1+\frac{2}{p}}|\eta|^{d-2+\frac{2}{p}}}\|\underline{u}\|_{K_T^{p,d-1}}\|\underline{v}\|_{K_T^{p,d-1}}.
\end{aligned}$$

Using Lemma 3.1, we obtain

$$e^{\theta_\lambda(s,\xi,\varepsilon)}|\widehat{\mathbf{l}(u,v)}(s,\xi)|\lesssim_{l,d,p}\frac{e^{\frac{\lambda^2}{4(1-\varepsilon)c_0}}}{|\xi|^{d-3+\frac{4}{p}}s^{\frac{2}{p}}}\|\underline{u}\|_{K_T^{p,d-1}}\|\underline{v}\|_{K_T^{p,d-1}}. \tag{4.13}$$

Similarly to the end of the proof of Lemma 3.2, we have

$$\int_0^t\frac{e^{-\varepsilon c_0(t-s)|\xi|^2}}{s^{\frac{2}{p}}}ds\lesssim_{p,d}\left(\frac{1-\frac{1}{p}}{ec_0\varepsilon}\right)^{\frac{1}{p}}\frac{1}{|\xi|^{2-\frac{2}{p}}}.$$

Thus, we deduce from (4.13), that

$$\begin{aligned}
&|\int_0^t e^{-c_0(t-s)|\xi|^2}\widehat{\underline{\mathbf{l}}}_t(u,v)(s,\xi)ds|\\
&\lesssim_{l,d,p}e^{\frac{\lambda^2}{4(1-\varepsilon)c_0}}\int_0^t\frac{e^{-\varepsilon c_0(t-s)|\xi|^2}}{s^{\frac{2}{p}}|\xi|^{d-3+\frac{4}{p}}}ds\|\underline{u}\|_{K_T^{p,d-1}}\|\underline{v}\|_{K_T^{p,d-1}}\\
&\lesssim_{l,d,p,\varepsilon,c_0}\frac{e^{\frac{\lambda^2}{4(1-\varepsilon)c_0}}}{t^{\frac{1}{p}}|\xi|^{d-1+\frac{2}{p}}}\|\underline{u}\|_{K_T^{p,d-1}}\|\underline{v}\|_{K_T^{p,d-1}}.
\end{aligned}$$

The first inequality follows. The proof of the second inequality is similar, with some modifications in the same manner as the proof of (3.4). $\square$

As in the proof of Theorem 3.3, using Lemma 4.3, we can prove the following estimate.

LEMMA 4.4. *Let $T$, $\lambda$ and $\varepsilon$ three non-negative real number and let $\delta$ be in $[0, \frac{2}{p}]$. Let $(a_0, u_0) \in (PM^{d-1+\delta} \cap PM^{d+\delta}) \times PM^{d-1+\delta}$ an initial data and $(a, u) \in X_T$ a solution of the Cauchy problem* (1.3)-(1.5). *There exists two positive constants $C$ and $C_\varepsilon$ such that*

$$\|(\underline{a}, \underline{u})\|_{X_T} \leq C\left(\|e^{\varepsilon c_0 t \triangle}(a_0, u_0)\|_{X_T} + C_\varepsilon e^{\frac{\lambda^2}{4(1-\varepsilon)c_0}} \|(\underline{a}, \underline{u})\|^2_{X_T}\right). \tag{4.14}$$

*Moreover $C$ depend of $p, d, \delta, c_0, \nu, \mu, \alpha, \kappa$ and $C_\varepsilon$ depend of $p, d, \delta, c_0, \nu, \mu, \alpha, \kappa$ and $\varepsilon$ and both of them are independent of $\lambda$ and $T$.*

Let us remark that in (4.11), the term $e^{-\varepsilon c_0 t|\xi|^2}$ give an "residual decay", which appear when we compensate the loss of decay due to the weight $e^{\theta_\lambda(t,\varepsilon,\xi)}$ by $e^{-c_0 t|\xi|^2}$, while to keeping an "parabolic decay" that is $e^{-\varepsilon c_0 t|\xi|^2}$. This is why in (4.14) we get $e^{\varepsilon c_0 t \triangle}(a_0, u_0)$ instead $e^{c_0 t \triangle}(a_0, u_0)$. The lemma above combined with a bootstrap argument is the key point to prove the following theorem, that is the main result of this section.

THEOREM 4.5. *Let $(a_0, u_0) \in (PM^{d-1+\frac{2}{p}} \cap PM^{d+\frac{2}{p}}) \times PM^{d-1+\frac{2}{p}}$. If there exists a solution $(a, u) \in X_{T^\star}$ of* (1.3)-(1.5) *for some positive times $T^\star$, then*

$$\liminf_{T \longrightarrow 0^+} \frac{\operatorname{rad}(a(t), u(t))}{\sqrt{T\left|\ln\left(T\|(a_0, |D|a_0, u_0)\|^p_{PM^{d-1+\frac{2}{p}}}\right)\right|}} \geq \sqrt{\frac{4c_0}{p}}, \tag{4.15}$$

*where $c_0$ is the constants that appear in Lemma 2.1.*

PROOF. We use a bootstrap argument. Let $\varepsilon > 0$. For every $T \in [0, T^\star]$, we denote by $H(T)$ the following induction hypothesis,

$$\|(\underline{a}, \underline{u})\|_{X_T} \leq D_\varepsilon e^{-\frac{\lambda_T^2}{4(1-\varepsilon)c_0}}, \tag{4.16}$$

where the positive real number $\lambda_T$ will be chosen later and

$$D_\varepsilon := \frac{1}{2CC_\varepsilon}.$$

If $H(T)$ is satisfied, we deduce from Lemma 4.4 that

$$\|(\underline{a}, \underline{u})\|_{X_T} \leq C\|e^{\varepsilon c_0 t \triangle}(a_0, u_0)\|_{X_T} + \frac{1}{2}\|(\underline{a}, \underline{u})\|_{X_T},$$

that is

$$\|(\underline{a}, \underline{u})\|_{X_T} \leq 2\|e^{\varepsilon c_0 t \triangle}(a_0, u_0)\|_{X_T}. \tag{4.17}$$

Then, for all $T \in [0, T^\star]$, we have

$$\|e^{\varepsilon c_0 t \triangle}(a_0, u_0)\|_{X_T} \leq D_{p,\varepsilon} T^{\frac{1}{p}} \|(a_0, |D|a_0, u_0)\|_{PM^{d-1+\frac{2}{p}}},$$

where $D_{\varepsilon,p} := (epc_0\varepsilon)^{-\frac{1}{p}}$. Let us define

$$T_\varepsilon := \eta_\varepsilon \|(a_0, |D|a_0, u_0)\|^{-p}_{PM^{d-1+\frac{2}{p}}},$$

where

$$\eta_\varepsilon := \left(\frac{D_\varepsilon}{4CD_{p,\varepsilon}}\right)^p.$$

Thus, for every $T \in [0, T_\varepsilon]$, we have

$$T^{\frac{1}{p}}\|(a_0,|D|a_0,u_0)\|_{PM^{d-1+\frac{2}{p}}} \leq \frac{D_\varepsilon}{4CD_{p,\varepsilon}} = \eta_\varepsilon^{\frac{1}{p}}. \tag{4.18}$$

Then, for all $T \in ]0, T^\star]$, we define the following non-negative real number $\lambda_T$ by setting

$$\lambda_T^2 := \frac{4(1-\varepsilon)c_0}{p}\left|\ln\left(\frac{\eta_\varepsilon}{T\|(a_0,|D|a_0,u_0)\|^p_{PM^{d-1+\frac{2}{p}}}}\right)\right|. \tag{4.19}$$

Note that the positivity of the right-hand side follows from (4.18). Moreover, for every $T \in [0, T_\varepsilon]$, such that $H(T)$ holds, we deduce from (4.17) and (4.19) that

$$\begin{aligned}
\|(\underline{a},\underline{u})\|_{X_T} &\leq 2CD_{p,\varepsilon}T^{\frac{1}{p}}\|(a_0,|D|a_0,u_0)\|_{PM^{d-1+\frac{2}{p}}},\\
&< 4CD_{p,\varepsilon}T^{\frac{1}{p}}\|(a_0,|D|a_0,u_0)\|_{PM^{d-1+\frac{2}{p}}},\\
&= D_\varepsilon e^{-\frac{\lambda_T^2}{4(1-\varepsilon)c_0}}.
\end{aligned}$$

This in turn shows that $H(T)$ holds for every $T \in [0, T_\varepsilon]$. Moreover, for all $T \in [0, T_\varepsilon]$, it follows from (4.19) that

$$T^{\frac{1}{p}}\|e^{\lambda_T\sqrt{T}|D|}(a(T),|D|a(T),u(T))\|_{PM^{d-1+\frac{2}{p}}} \leq D_\varepsilon.$$

Hence, for every $T \in [0, T_\varepsilon]$, we have

$$R(T) \geq \sqrt{\frac{4(1-\varepsilon)c_0}{p}T\left|\ln\left(\frac{\eta_\varepsilon}{T\|(a_0,|D|a_0,u_0)\|^p_{PM^{d-1+\frac{2}{p}}}}\right)\right|},$$

where $R(T) := \mathrm{rad}(a(t),u(t))$. This shows that

$$\liminf_{T\to 0^+}\frac{R(T)}{\sqrt{T\left|\ln\left(T\|(a_0,|D|a_0,u_0)\|^p_{PM^{d-1+\frac{2}{p}}}\right)\right|}} \geq \sqrt{\frac{4(1-\varepsilon)c_0}{p}}. \tag{4.20}$$

Since (4.20) holds for $\varepsilon > 0$ chosen arbitrarily, the theorem is proved. □

## Appendix A. Characterization of analyticity with Fourier transform

In this appendix we prove Proposition 1.3.

Proposition A.1. Let $r < d$ and $\sigma > 0$. Let $u$ be in $PM^r(\mathbb{R}^d)$. If $e^{\sigma|D|}u \in PM^r(\mathbb{R}^d)$, then $u$ extends to a unique holomorphic function $U$ in $\mathcal{H}(S_\sigma)$.

Proof. Since $\widehat{u} \in L^1_{loc}(\mathbb{R}^d)$, we deduce that $\widehat{u}$ is integrable on a neighborhood of 0 and using that $e^{\sigma|D|}u \in PM^r(\mathbb{R}^d)$, we deduce that $\widehat{u} \in L^1(\mathbb{R}^d)$. Then, using that the Fourier transform is an isomorphism of $\mathcal{S}'(\mathbb{R}^d)$ and that $\widehat{u}$ belong to $L^1(\mathbb{R}^d)$, the following identity

$$u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\widehat{u}(\xi)d\xi, \tag{A.1}$$

holds for almost every $x \in \mathbb{R}^d$. We denote by $v(x)$ the right-hand side of (A.1). It is sufficient to prove that the function $v : x \in \mathbb{R}^d \mapsto v(x)$ extends to an holomorphic function on $S_\sigma$. Let $\tilde{\sigma} \in ]0, \sigma[$. Then for all $z \in S_{\tilde{\sigma}}$, we have

$$\begin{aligned}|e^{iz\cdot\xi}\widehat{u}(\xi)| &\leq e^{|\operatorname{Im}(z)||\xi|}|\widehat{u}(\xi)| \\ &\leq e^{\tilde{\sigma}|\xi|}|\widehat{u}(\xi)| \\ &\leq \frac{e^{-(\sigma-\tilde{\sigma})|\xi|}}{|\xi|^r}\|e^{\sigma|D|}u\|_{PM^r}.\end{aligned}$$

Using the hypothesis $r < d$, we deduce by a classical argument that the function $\xi \mapsto e^{ix\cdot\xi}\widehat{u}(\xi)$ is in $L^1(\mathbb{R}^d)$. This legitimate, for every $z \in S_\sigma$, the definition of the quantities

$$U(z) := \frac{1}{(2\pi)^d}\int e^{iz\cdot\xi}\widehat{u}(\xi)d\xi.$$

From

$$|e^{iz\cdot\xi}\widehat{u}(\xi)| \leq \frac{e^{-(\sigma-\tilde{\sigma})|\xi|}}{|\xi|^r}\|e^{\sigma|D|}u\|_{PM^r}, \tag{A.2}$$

that holds for each $z \in S_{\tilde{\sigma}}$ and $\xi \in \mathbb{R}^d$, and observing that the right-hand side of (A.2) defines a $L^1(\mathbb{R}^d)$ function that does not depend on $z \in S_{\tilde{\sigma}}$, we deduce that $U \in \mathcal{H}(S_{\tilde{\sigma}})$. Since $\tilde{\sigma}$ is arbitrarily chosen in $]0, \sigma[$, we deduce that $U$ is holomorphic over $S_\sigma$. □

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