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Flowbox theorems for a class of Sobolev vector fields

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Abstract. We give sufficient conditions for the flowbox theorem for Sobolev vector fields to be valid not only for the general case but also for the conservative case.

Contents

1. Introduction

Given C^r vector field ($r \geq 1$), which is locally non-null, it is always possible to perform two simple but amazing tasks of analytic-geometric content: first we can integrate the vector field into a 1-dimensional foliation, called a *flow*, displaying rich algebric properties and second, and through a change of coordinates, we are able to straighten out all orbits by means of a *flowbox* in a certain neighborhood. These important results are respectively the Picard-Lindelöf existence and uniqueness theorem and the flowbox theorem. The main ingredients of their proofs are: for the former the Banach fixed point theorem for contractions [**3**, §4] and for the latter the inverse function theorem [**30**, §2]. These theorems completely describe the local behavior of the orbits in a neighborhood of a non-null orbit and shows that, locally, first integrals always exist.

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Despite the fact that a C^r analytic approach is much more user-friendly when compared with a Lipschitz one, existence and uniqueness theorems are also easy to obtain for Lipschitz vector fields, however flowbox theorems are more demanding [**15**]. Furthermore, when the vector field keeps invariant the volume, performing a straighten out by a volume-preserving change of coordinates is somehow more demanding [**8**].

The Lipschitz regularity could be seen as the classic threshold for existence and uniqueness theorems but the remarkable results by DiPerna-Lions in the late 1980's [**20**] showed us that existence and uniqueness go a long way among non-smooth settings. With this in mind, the present work was conceived as an attempt to understand how flowbox theorems can be obtained in even less smooth contexts more particularly for Sobolev vector fields admitting integrability as flows according [**20**]. We hope that our viewpoint sheds some new light on performing a perturbative dynamical systems theory of Sobolev vector fields for both the dissipative case and the conservative one in a parallel with the Lipschitz case treated in [**13, 14**] and in order to keep up with the recent development of the theory in the Sobolev discrete case [**22, 23, 24, 25, 5, 6**]. With respect to the conservative class, one might ask why there is a need to preserve volume in the change of coordinates. The reason is that without it the perturbative theory would not be effective, since we could drop the conservative class after we perform a composition with a dissipative change of coordinates.

It is worth pointing out that there are several results in the literature related with the flowbox theorem in subclasses of smooth vector fields. Regarding with the Hamiltonian vector field context the flowbox theorem goes back to classic books of Abraham and Marsden [**1**] and also of Robinson [**34**] with some approaches revisited by the author and Dias [**12**] and more recently by Cabral [**17**]. Considering the volume-preserving setting the flowbox theorem proof was firstly given by the author in [**10**] (see also the higher dimensional case in [**11**]) and then different proofs were given by Barbarosie [**7**] and by Castro and Oliveira [**18**].

Flows are $\mathbb{R}$-actions and for $\mathbb{R}^n$-actions similar theorems which completely describes the local behavior of the orbits in a neighborhood of a regular orbit can be seen in [**27**, Theorem 18.6]. For conservative actions we have [**9**] and for Frobenius-type theorems for non-smooth distributions we refer [**33**].

The aim of this paper is to prove two flowbox theorems: in Theorem 4.2 we consider the broader case of dissipative flows and in Theorem 5.2 we proceed with a similar study but for divergence-free vector fields. The main difficulties in carrying out the construction of the change of coordinates in Theorem 4.2 are primarily technical issues. Firstly, and related with the integrability of the vector field, we put ourselves in DiPerna and Lions setting [**20**]. Secondly, when dealing with problems related to Sobolev map compositions, we consider some additional hypotheses on the flow according with Kleprlík theorem [**26**]. The proof of Theorem 5.2 is supported in the one of Theorem 4.2 and the crucial main ingredient to achieve the conservativeness of the change of coordinates is Ye's theorem on a certain partial differential equation involving the jacobian in Sobolev contexts [**35**]. It is important to emphasise that in a remarkable paper [**2**] Ambrosio generalized [**20**] to vector fields with BV coefficients and with L^1 divergence. One may ask whether our flowbox theorems are still true in the BV setting possibly with some additional conditions.

Our presentation is therefore organized in the following way: In section 2 we present some basic definitions used along the text and several topics related with existence, uniqueness and continuous dependence on ODE with Sobolev coefficients. Since our change of coordinates is built at the expense of the flow we will try to collect as much information as possible about its regularity. Then, in section 3 we review some important results on the composition of maps in Sobolev spaces and the applicability of the chain rule. Indeed, as our change of coordinates is also built through a composition of maps involving the flow and a reparameterization of the parameter t (the 'first coordinate' of the flowbox) such control on the regularity of the composition is required. The chain rule arises when we inquire about integrability of the derivative of the flow. In section 4 we consider the bounded divergence case proving Theorem 4.2. Notice that our proof is supported on a differential equations approach rather than a standard analytic approach through the use of an implicit function theorem. Finally, in section 5, we consider the divergence-free case proving Theorem 5.2. Here the novelty with relation to Theorem 4.2 is a compensation through the use of Ye's theorem for a control on the possible loss of invariance of the Jacobian made in Theorem 4.2.

2. Basic definitions

2.1. Sobolev vector fields, existence and uniqueness of its solutions. Let U be an open bounded subset of $\mathbb{R}^n$ $(n \geq 2)$ with Lipschitz boundary and let $1 \leq p, q \leq \infty$. Recall that a measurable map $f = (f_1, \ldots, f_n)\colon U \to \mathbb{R}^n$ is in the Sobolev class $W^{1,p}(U, \mathbb{R}^n)$ if, for all $i = 1, \ldots, n$, f_i and all its distributional partial derivatives $\partial_{x_j} f_i$ are in $L^p(U)$. Given $f \in W^{1,p}(U, \mathbb{R}^n)$ we consider the norm defined by $\|f\|_{1,p} = \|f\|_p + \|Df\|_p$ where $\|f\|_p = \max_i \|f_i\|_p$ and $\|Df\|_p = \max_{i,j} \left\|\partial_{x_j} f_i\right\|_p$. We shall be interested only on Sobolev maps inside $W^{1,p}(U, \mathbb{R}^n) \cap C^0(\overline{U}, \mathbb{R}^n)$. The standard norm in this space is equivalent to the one defined by $\|f\|_\infty + \|Df\|_p$, since $C^0(\overline{U}, \mathbb{R})$ is compactly included in $L^p(U)$.

Take a vector field $X \in W^{1,1}(U, \mathbb{R}^n) \cap C^0(\overline{U}, \mathbb{R}^n)$. In order to be able to integrate X into a flow $X^t\colon U \to \mathbb{R}^n$ we assume the DiPerna-Lions conditions

$$\operatorname{div} X \in L^\infty(U), \tag{2.1}$$

$$\frac{X}{1 + |x|} \in L^1 + L^\infty, \tag{2.2}$$

and, for $1 \leq p \leq \infty$, by strengthening (2.2), we also consider

$$X \in L^p + (1 + |x|)L^\infty, \tag{2.3}$$

which will be important to obtain L^p integrability of the integral solution of X that will be dealt with now. For such vector field X we now consider the initial value problem:

$$\frac{d}{dt} V(t, x) = X(V(t, x)), \tag{2.4}$$

for $t \in \mathbb{R}$ with initial condition $V(0) = x \in \mathbb{R}^n$. Using an indirect approach via the Cauchy problem for the transport partial differential equations DiPerna and Lions were able to obtain the next important outcome:

Theorem 2.1. ***DiPerna-Lions existence and uniqueness*** [**20**, §III] *If we consider a vector field $X \in W^{1,1}(U,\mathbb{R}^n)$ satisfying* (2.1) *and* (2.2)*, then the ODE* (2.4) *always displays a solution $V(t,x) = X^t(x)$ satisfying the following group property condition*

$$X^{t+s}(x) = X^t(X^s(x)) \tag{2.5}$$

for μ-a.e. $x \in \mathbb{R}^n$ and for all $t, s \in \mathbb{R}$. Moreover, if X satisfies (2.3) *then the time-t map X^t belongs to L^p.*

Fixing the time-parameter $t \in \mathbb{R}$ we have that $X^t \colon U \to \mathbb{R}^n$ is a Sobolev homeomorphism whenever defined. By the last sentence of Theorem 2.1 we infer that among this non-smooth setting it is not *fait accompli* that the regularity of the vector field is carried forward to its corresponding flow. We had to admit the additional hypothesis (2.3) (see also [**16**, pp. 111]). We emphasize that continuous dependence issues are much more delicate when we deal with Sobolev vector fields (see [**19**, Theorem 2.9 and Corollary 2.10] and [**16**, §4]). Since in the sequel we will be considering vector fields X in $W^{1,1}_{loc}(\mathbb{R}^n) \cap C^0(\overline{U},\mathbb{R}^n)$ its correponding flow X^t is also in $W^{1,1}_{loc}(\mathbb{R}^n) \cap C^0(\overline{U},\mathbb{R}^n)$ and we can obtain flow estimates depending on the vector field by using Grönwall's lemma.

Now, on the level of the tangent flow DX^t, by Remark 2) in [**20**, pp. 536] we conclude that DX^t_x is solution of the linear variational equation

$$\frac{d}{dt}\mathcal{V}(t,x) = DX_{X^t(x)} \cdot \mathcal{V}(t,x), \tag{2.6}$$

and so we get

$$\frac{d}{dt}\log\|DX^t_x\| \leq \|DX \cdot DX^t_x\|.$$

Hence, if $DX \in L^p$ for some $1 \leq p \leq \infty$ we obtain for all $t \in \mathbb{R}$ that

$$\big\| \log\|DX^t_x\| \,\big\|_p \leq C|t|.$$

However, we still do not know if $DX^t \in L^p$ and for this reason, we will place this requirement as an additional assumption. This requirement is due to the fact that it is crucial in proving our results to evaluate the L^p integrability of the space derivative of the flow. This can be done by a similar procedure considering (2.3) applied to the vector field DX_x of the ODE (2.6) and then apply Theorem 2.1 or by simply considering $X \in W^{1,1}_{loc}(\mathbb{R}^n) \cap C^0(\overline{U},\mathbb{R}^n)$ such that DX^t is in L^p.

We observe that when X is a divergence-free vector field its corresponding flow X^t is a volume-preserving homeomorphism (cf. [**20**, Equation (51)]).

2.2. From $\mathbb{R}^n$ to manifolds. Wee can define similar spaces and differential equations on smooth manifolds. Indeed, let M be a connected, closed and C^∞ Riemannian manifold of dimension $n \geq 2$. Since along this paper we deal with divergence-free vector fields we assume that M is also a volume-manifold with a volume form $\mathcal{V} \colon TM^n \to \mathbb{R}$ where TM stands for the tangent bundle. Furthermore, we equip M with an atlas $\mathcal{A} = \{(\varphi_i, U_i)_i\}$ of M (cf. [**28**]), such that $(\varphi_i)_*\mathcal{V} = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$, where x_i are the canonical coordinates in the Euclidian space, $\varphi_i \colon U_i \to \mathbb{R}^n$ a local C^∞ diffeomorphism and U_i an open subset of M. The fact that M is compact guarantees that $\mathcal{A}$ can be taken finite, say $\mathcal{A} = \{(\varphi_i, U_i)\}_{i=1}^k$.

We call *Lebesgue measure* the measure associated to $\mathcal{V}$ and denote it by μ. More precisely, we let

$$\mu(\mathcal{B}) = \mu_{\mathcal{V}}(\mathcal{B}) := \int_{\varphi(\mathcal{B})} \mathcal{V}_{\varphi^{-1}(x)}(D\varphi_1^{-1}\cdot e_1, \dots, D\varphi_n^{-1}\cdot e_n)\, dx_1 \dots dx_n,$$

for some Borelian $\mathcal{B} \subset M$ where $\{e_1, \dots, e_n\}$ is the canonical base of $\mathbb{R}^n$. Accordingly, despite Theorem 4.2 and Theorem 5.2 will be stated in the Euclidian space equivalent statements for vector fields in M follows straightforwardly.

3. Composition of maps and the chain rule

One main difficulty in the study of dynamics of maps in $W^{1,p}$ Sobolev classes is that they are not closed under composition. Furthemore, the chain rule does not hold in general. In fact there exists a homeomorphism f in the 3-dimensional cube $]0,1[^3$ such that f and f^{-1} are of class $W^{1,2}$ but both f and f^{-1} have zero Jacobian matrix at Lebesgue almost every point [**29**]. In [**26**] Kleprlík considered the problem related to the Sobolev regularity of the composition. More precisely, if f is a homeomorphism and u belongs to a certain Sobolev space we want to know when the composition $u \circ f$ belongs to other Sobolev space. In order to present the main result in [**26**] we begin by considering some definitions. Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be open subsets and consider the map $f\colon \Omega_1 \to \Omega_2$. For $x \in \Omega_1$ we say that the operator T_f defined by:

$$\begin{array}{rccc} T_f\colon & W^{1,q}_{loc}(\Omega_2) & \longrightarrow & W^{1,p}_{loc}(\Omega_1) \\ & u & \longmapsto & (T_f u)(x) = u(f(x)), \end{array} \tag{3.1}$$

is *continuous* if $T_f u \in W^{1,p}_{loc}(\Omega_1)$ for all functions $u \in W^{1,q}_{loc}(\Omega_2)$ and there exists a constant C (independent of u) such that

$$\|DT_f u\|_{L^p(\Omega_1)} \le C\, \|Du\|_{L^q(\Omega_1)}. \tag{3.2}$$

We consider an analogous definition for the operator

$$\begin{array}{rccc} T_f\colon & W^{1,q}_{loc}(\Omega_2)\cap C(\Omega_2) & \longrightarrow & W^{1,p}_{loc}(\Omega_1) \\ & u & \longmapsto & (T_f u)(x) = u(f(x)), \end{array}$$

saying that it is *continuous* if $(T_f u)(x) \in W^{1,p}_{loc}(\Omega_1)$ for all functions $u \in W^{1,q}_{loc}(\Omega_2) \cap C(\Omega_2)$ and there exists a constant C (independent of u) such that (3.2) holds.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$. We say that f has *finite distortion* if

- $\det Df \in L^1_{loc}(\Omega)$ and
- $|Df_x| = 0$ for μ-a.e. $x \in \Omega$ such that $\det Df_x = 0$.

Given $1 \le q < \infty$ let

$$\kappa_q(x) := \begin{cases} \frac{|Df_x|^q}{|\det Df_x|}, & \text{if } \det Df_x \ne 0 \\ 0, & \text{otherwise} \end{cases} \tag{3.3}$$

Theorem 3.1. [**26**, Theorem 1.3] *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be open subsets, $1 \le p \le q < \infty$ and let $f \in W^{1,1}_{loc}(\Omega_1, \Omega_2)$ be a homeomorphism of finite distortion such that $\kappa_q(x) \in L^{\frac{p}{q-p}}(\Omega_1)$. Then*

(i) *if $1 \le q \le n$, then the operator T_f in (3.1) is continuous from $W^{1,q}_{loc}(\Omega_2)$ into $W^{1,p}_{loc}(\Omega_1)$ and*

(ii) *if* $n < q < \infty$, *then the operator* T_f *in* (3.1) *is continuous from* $W^{1,q}_{loc}(\Omega_2) \cap C(\Omega_2)$ *into* $W^{1,p}_{loc}(\Omega_1)$.

(iii) *Moreover the chain rule holds, that is for* μ*-a.e.* $x \in \Omega_1$ *we have the equality* $D(u \circ f)_x = Du_{f(x)}\, Df_x$.

When we make compositions of homeomorphisms where one of them is a diffeomorphism pre-composition and post-composition become more treatable as next result shows.

Theorem 3.2. [**23**, Lemma 3.1] *Let* $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ *be open subsets,* $n \geq 3$, $p > n-1$, $f \in W^{1,p}(\Omega_1, \mathbb{R}^n)$ *be a homeomorphism and let* $g \in Diff^1(\Omega_2, \mathbb{R}^n)$. *Then*

(i) *if* $g(\Omega_2) \subset \Omega_1$, *then* $f \circ g \in W^{1,p}(\Omega_2, \mathbb{R}^n)$, $f \circ g$ *is differentiable Lebesgue a.e. and* $D(f \circ g)_x = Df_{g(x)}\, Dg_x$ *for Lebesgue a.e.* $x \in \Omega_2$;

(ii) *if* $f(\Omega_1) \subset \Omega_2$, *then* $g \circ f \in W^{1,p}(\Omega_1, \mathbb{R}^n)$, $g \circ f$ *is differentiable Lebesgue a.e. and* $D(g \circ f)_x = Dg_{f(x)}\, Df_x$ *for Lebesgue a.e.* $x \in \Omega_1$.

4. The bounded divergence case

We say that two vector fields $X_1 \colon U_1 \to \mathbb{R}^n$ and $X_2 \colon U_2 \to \mathbb{R}^n$ in $W^{1,1}_{loc}(\mathbb{R}^n) \cap C^0(\overline{U}, \mathbb{R}^n)$ are *locally* $W^{1,1}$ *conjugate near* $p_1 \in U_1$ and $p_2 \in U_2$ if there exist two open neighborhoods $O_i \ni p_i$ $(i = 1, 2)$ and a $W^{1,1}$ mapping $\phi \colon O_1 \to O_2$ with $\phi(p_1) = p_2$ displaying inverse ϕ^{-1} also in $W^{1,1}$ and such that for any $x \in O_1$ and a small interval I containing 0 the integral curve $X_1^{(\cdot)}(x) \colon I \to O_1$ defined by $X_1^0(x) = x$ and $\frac{d}{dt} X_1^t(x)\Big|_{t=s} = X_1(X_1^s(x))$ for all $s \in I$ is a solution associated to X_1 if and only if the integral curve $\phi(X_1^t(x)) \colon I \to O_2$ is a solution associated to X_2. That is, we have the conjugacy equation

$$\phi(X_1^t(x)) = X_2^t(\phi(x)), \tag{4.1}$$

for all $x \in O_1$ and $|t|$ sufficiently small.

Remark 4.1. Usualy in the literature the flowbox theorem follows from a direct application of the implicit function theorem. Indeed, regular points imposes a non-degeneracy condition which allows us to use the implicit function theorem and get the change of coordinates straightly. Despite some Sobolev versions of this theorem are available (see [**31, 36**]) we are not totally clear if such versions using Clarke jacobians could be useful to get the flowbox theorem.

As already mentioned we will obtain the change of coordinates of the flowbox theorem using the foliation structure given by the flow. It is time to state our first main result.

Theorem 4.2. ***Flowbox theorem for Sobolev bounded divergence vector fields***
Let be given a vector field $X \colon U \to \mathbb{R}^n$ *in* $W^{1,1}_{loc}(\mathbb{R}^n) \cap C^0(\overline{U}, \mathbb{R}^n)$ *satisfying* (2.1) *and* (2.3) *and generating a flow* X^t *which:*

(i) *has an* L^p*-integrable tangent flow;*

(ii) *is without fixed points in* U;

(iii) *is of finite distortion for any fixed time* t *with* κ_q *on* (3.3) *in* $L^\infty(U)$ *and*

(iv) *is a homeomorphism* $X^t \colon \overline{U} \to \mathbb{R}^n$, *for fixed* t, *whenever the map* X^t *is properly defined.*

Let be given a point $p_1 \in U$ and the trivial vector field defined by:

$$T(x_1, x_2, \dots, x_n) = (1, 0, \dots, 0). \tag{4.2}$$

Then, X and T are locally $W^{1,1}$ conjugate near p_1 and $p_2 = \vec{0}$.

PROOF. We must prove that there exist two open neighborhoods $U \supset O_i \ni p_i$ $(i = 1, 2)$ and a $W^{1,1}$ map $\phi\colon O_1 \to O_2$ with $\phi(p_1) = p_2$ displaying inverse ϕ^{-1} also in $W^{1,1}$ and such that for any $x \in O_1$ and a small interval I containing 0 the integral curve $X^{(\cdot)}(x)\colon I \to O_1$ defined by $X^0(x) = x$ and $\frac{d}{dt}X^t(x)\Big|_{t=s} = X(X^s(x))$ for all $s \in I$ is a solution associated to X if and only if the integral curve $\phi(X^t(x))\colon I \to O_2$ is a solution associated to T. In overall, (4.1) is valid, that is, we have $\phi(X^t(x)) = T^t(\phi(x))$ where $T^t(x) = (t + x_1, x_2, \dots, x_n)$ if the flow associated to (4.2).

To simplify the presentation we assume that $U = B_r(\vec{0})$, $x_1 = 0$, $p_1, p_2 = \vec{0}$ and $X(\vec{0}) = (1, 0, \dots, 0)$. Define the projection in the first coordinate by:

$$\begin{array}{llcl} \Pi_1\colon & \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & (x_1, x_2, \dots, x_n) & \longmapsto & x_1 \end{array} \tag{4.3}$$

Since $X \in C^0(\overline{U}, \mathbb{R}^n)$ that exists $r > 0$ such that for all $x \in B_r(\vec{0})$ we have

$$\Pi_1(X(x)) > \frac{1}{2} \quad \text{and} \quad \|X(x)\| < 2. \tag{4.4}$$

Let $\Sigma := X(\vec{0})^\perp = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n\colon x_1 = 0\}$ be the transversal section at $\vec{0}$ normal to $X(\vec{0})$ say $\Sigma = \ker(\Pi_1)$. By (ii) a small neighborhood of $\vec{0}$ is foliated by flow orbits in a sense that for all x with $\Pi_1(x) \geq 0$ belonging to this neighborhood we can choose $t_x \geq 0$ such that $X^{-t_x}(x) \in \Sigma$. In other words we consider the quotient of the foliation by Σ and so for each x in a sufficiently small neighborhood there is a single representative on Σ of the small segment of orbit through x. Write

$$X^{-t_x}(x) = (0, u_2, u_3, \dots, u_n) \in \Sigma. \tag{4.5}$$

Hence we define:

$$\phi(x) = X^{-t_x}(x) + t_x X(\vec{0}) = X^{-t_x}(x) + (t_x, 0, \dots, 0), \tag{4.6}$$

and, as $\Pi_1(u) = \Pi_1(u_1, u_2, \dots, u_n) = u_1$ we also define

$$\psi(u) = X^{u_1}(0, u_2, \dots, u_n). \tag{4.7}$$

Clearly, $\psi \circ \phi(x) = x$ and $\phi \circ \psi(u) = u$ and so $\psi = \phi^{-1}$. Observing that

$$t_{X^t(x)} = t + t_x, \tag{4.8}$$

whenever it makes sense we get

$$
\begin{aligned}
\phi(X^t(x)) &\overset{(4.6)}{=} X^{-t_{X^t(x)}}(X^t(x)) + t_{X^t(x)}X(\vec{0})\\
&\overset{(2.5)}{=} X^{t-t_{X^t(x)}}(x) + (t_{X^t(x)}, 0, \dots, 0)\\
&\overset{(4.8)}{=} X^{-t_x}(x) + (t+t_x, 0, \dots, 0)\\
&\overset{(4.5)}{=} (0, u_2, \dots, u_n) + (t+t_x, 0, \dots, 0)\\
&= (t+t_x, u_2, \dots, u_n)\\
&= T^t((t_x, u_2, \dots, u_n))\\
&\overset{(4.5)}{=} T^t(X^{-t_x}(x) + (t_x, 0, \dots, 0))\\
&\overset{(4.6)}{=} T^t(\phi(x)).
\end{aligned}
$$

Claim: If $X \in W^{1,p}_{loc}(\mathbb{R}^n) \cap C^0(\overline{U}, \mathbb{R}^n)$ and has an L^p-integrable tangent flow, then the function $t\colon B_r(\vec{0}) \longrightarrow \mathbb{R}$ defined by $t(x) = t_x$ is in $W^{1,p}$.

As $t_x \in L^p$ we will see that $\partial_{x_i} t_x \in L^p$ for all $i = 1, \dots, n$ and this integrability depends on the L^p integrability hypothesis (i) of DX^t. For $y = x + h = x + se_i$ where $s \in \mathbb{R}$, e_i is the i vector of the canonic basis of $\mathbb{R}^n$ and since $\Pi_1(X^{-t_w}(w)) = 0$ we get, considering the region of integration $B_r(\vec{0})$ and abbreviating $d\mu(x) = d\mu$, that:

$$
\begin{aligned}
& \int |\partial_{x_i} t_x|^p \, d\mu\\
= & \int \left(\lim_{s\to 0} \frac{|t_{x+se_i} - t_x|}{|s|} \right)^p d\mu\\
\overset{(4.4)}{\leq} & \; 2^p \int \left(\lim_{\|h\|\to 0} \frac{\|\Pi_1(X^{-t_x}(x)) - \Pi_1(X^{-t_{x+h}}(x))\|}{\|h\|} \right)^p d\mu\\
\leq & \; 2^p \int \left(\lim_{\|h\|\to 0} \frac{\|\Pi_1(X^{-t_x}(x)) - \Pi_1(X^{-t_{x+h}}(x+h))\|}{\|h\|} \right.\\
+ & \left. \frac{\|\Pi_1(X^{-t_{x+h}}(x+h)) - \Pi_1(X^{-t_{x+h}}(x))\|}{\|h\|} \right)^p d\mu\\
\leq & \; 2^p \int \left(\lim_{\|h\|\to 0} \frac{\|\Pi_1(X^{-t_{x+h}}(x+h)) - \Pi_1(X^{-t_{x+h}}(x))\|}{\|h\|} \right)^p d\mu\\
\leq & \; 2^p \int \left(\lim_{\|h\|\to 0} \frac{\|X^{-t_{x+h}}(x+h) - X^{-t_{x+h}}(x) \pm DX_x^{-t_{x+h}} \cdot h\|}{\|h\|} \right)^p d\mu\\
\leq & \; 2^p \int \left(\lim_{\|h\|\to 0} \frac{\|X^{-t_{x+h}}(x+h) - X^{-t_{x+h}}(x) - DX_x^{-t_{x+h}} \cdot h\|}{\|h\|} \right.\\
+ & \left. \frac{\|DX_x^{-t_{x+h}}\| \, \|h\|}{\|h\|} \right)^p d\mu\\
= & \; 4^p \int \|DX_x^{-t_x}\|^p \, d\mu < \infty,
\end{aligned}
$$

and the claim is proved.

We are now in a position to show that ϕ defined in (4.6) and ϕ^{-1} defined in (4.7) are in $W^{1,1}$. Let $x = (x_1, x_2, \ldots, x_n)$ and so the map $X^{-t_x}(x)$ is described by the composition

$$\begin{array}{ccccc} \mathbb{R}^n & \longrightarrow & \mathbb{R} \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ x & \longmapsto & (t_x, x) & \longmapsto & X^{-t_x}(x) \end{array}$$

As, from (iii) and (iv), the flow X^t is a homeomorphism of finite distortion with $\kappa_q \in L^\infty(U)$ by Theorem 3.1 we can use the chain rule and get

$$D\phi(x) = -\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial t_x}{\partial x_1} & \frac{\partial t_x}{\partial x_2} & \ldots & \frac{\partial t_x}{\partial x_n} \end{pmatrix} + DX_x^{-t_x} + \begin{pmatrix} \frac{\partial t_x}{\partial x_1} & \frac{\partial t_x}{\partial x_2} & \ldots & \frac{\partial t_x}{\partial x_n} \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix},$$

concluding that $D\phi \in L^1$. Now we will see that $\phi^{-1} \in W^{1,1}$. As the 'first coordinate' of the flowbox gain regularity[1] we can use Lemma 3.2 (see also [**4**]) and observing that

$$\begin{array}{ccccc} \mathbb{R}^n & \longrightarrow & \mathbb{R} \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ u & \longmapsto & (u_1, (0, u_2, \ldots, u_n)) & \longmapsto & X^{u_1}(0, u_2, \ldots, u_n) \end{array}$$

we apply the chain rule obtaining

$$D\psi_u = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} + \begin{pmatrix} 0 & \partial_{u_2} X_1^{u_1} & \ldots & \partial_{u_n} X_1^{u_1} \\ 0 & \partial_{u_2} X_2^{u_1} & \ldots & \partial_{u_n} X_2^{u_1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \partial_{u_2} X_n^{u_1} & \ldots & \partial_{u_n} X_n^{u_1} \end{pmatrix}, \tag{4.9}$$

that is

$$D\psi_u = \begin{pmatrix} X_1 & \partial_{u_2} X_1^{u_1} & \ldots & \partial_{u_n} X_1^{u_1} \\ X_2 & \partial_{u_2} X_2^{u_1} & \ldots & \partial_{u_n} X_2^{u_1} \\ \vdots & \vdots & \ddots & \vdots \\ X_n & \partial_{u_2} X_n^{u_1} & \ldots & \partial_{u_n} X_n^{u_1} \end{pmatrix}, \tag{4.10}$$

and so $\phi^{-1} \in W^{1,1}$. □

5. The divergence-free case

Until further notice we use [**20**, §III.1], that is, the vector field X is in $W^{1,1}(\Omega)$ and $\nabla X(x) = 0$ for μ-a.e. $x \in \Omega$. We also assume condition (2.2). We use a very useful result by Ye, generalizing [**21**], to obtain the proof of Theorem 5.2, i.e. a conservative local change of coordinates which trivialize the action of the flow. Define:

$$W^{m,p,+}(\Omega) := \left\{ f \in W^{m,p}(\Omega) \colon \exists\, c > 0 \colon \inf_{x \in \Omega} f(x) \geq c \right\}.$$

[1] Recalling Pugh's makeshift solution (see [**32**]) we note that the orbits of a vector field gain more regularity than the one of the vector field itself. Indeed, despite the fact that we know that vector fields with a certain regularity generate C^1 flows with the same regularity, a C^1 flow, for example, can be generated by a C^0 vector field.

Theorem 5.1. [**35**, Theorem 1] *Let* $\Omega = \overline{B(x,r)} \subset \mathbb{R}^n$*, let* $m \geq 1$ *be an integer,* $p \in \left]\max\left\{1, \frac{n}{m}\right\}, +\infty\right[$ *and* $f \in W^{m,p,+}(\Omega)$*. Then, there exists* $\varphi\colon \Omega \to \mathbb{R}^n$ *such that* $\varphi, \varphi^{-1} \in W^{m+1,p}(\Omega, \mathbb{R}^n)$ *and satisfies*

$$\det D\varphi_x = \lambda\, f(x), \tag{5.1}$$

for all $x \in \Omega$ *where* $\lambda = vol(\Omega)/\int_\Omega f$*. We also have* $\varphi = Id$ *at* $\partial\Omega$.

Theorem 5.2. ***Flowbox theorem for Sobolev divergence-free vector fields*** *Let be given a divergence-free vector field* $X\colon U \to \mathbb{R}^n$ *in* $W^{1,p}_{loc}(\mathbb{R}^n) \cap C^0(\overline{U}, \mathbb{R}^n)$*, for* $p > n$*, satisfying* (2.3) *and generating a flow* X^t *which:*

(i) *has an* L^p*-integrable tangent flow;*
(ii) *is without fixed points in* U*;*
(iii) *is of finite distortion for any fixed time* t *with* κ_q *on* (3.3) *in* $L^\infty(U)$ *and*
(iv) *is a homeomorphism* $X^t\colon \overline{U} \to \mathbb{R}^n$*, for fixed* t*, whenever the map* X^t *is properly defined.*

Let be given a point $p_1 \in U$ *and the trivial vector field* T *of* $\mathbb{R}^n$*. Then,* X *and* T *are locally* $W^{1,p}$ *conjugate near* p_1 *and* $p_2 = \vec{0}$ *via a volume-preserving invertible map* Φ *with inverse also in* $W^{1,p}$*.*

Proof. By Theorem 4.2 the homeomorphism ϕ which gives the conjugacy is a $W^{1,1}$ Sobolev homeomorphism where $\phi\colon O_1 \ni p_1 \to O_2 \ni 0$. Assume that X evolves in $\mathbb{R}^n$ with coordinates $(x_1, \dots, x_n)$, $p_1 = \vec{0}$ and $X(p_1) \in \{(\alpha, 0, \dots, 0) \in \mathbb{R}^n \colon \alpha \in \mathbb{R}\}$. As in Theorem 4.2 we let $\Sigma = X(p_1)^\perp$. Take $r > 0$ sufficiently small such that $\Omega := B_r(\vec{0}) \subset \Sigma$ and $\Omega \subset O_1$. Using the same notation as in Theorem 4.2 for each $x \in O_1$, there exists a unique $t_x \in]-T, T[$ such that $X^{-t_x}(x)$ is in a very small $(n-1)$-dimensional ball centered in 0 inside Σ. The map ϕ was defined in (4.6) by $\phi(x) = X^{-t_x}(x) + t_x X(\vec{0}) = X^{-t_x}(x) + (t_x, 0, \dots, 0)$ and so, we let $\phi(\Omega) = \Omega$. We have by (4.7) that $\phi^{-1}(u_1, u_2, \dots, u_n) = X^{u_1}(0, u_2, \dots, u_n)$.

Recalling that $\Pi_1(X)$ is the projection into the first coordinate of X defined in (4.3) and since by (4.4) we have $\Pi_1(X)$ bounded away from 0 we define the $W^{1,p,+}$ function

$$\begin{array}{cccc} f\colon & \Omega & \longrightarrow & \mathbb{R} \\ & (x_2, \dots, x_n) & \longmapsto & \frac{1}{\Pi_1(X(0, x_2, \dots, x_n))}. \end{array} \tag{5.2}$$

Indeed, as the projection is linear and the composition of a function in $W^{1,p,+}$ with a differentiable function is in $W^{1,p,+}$ (see Lemma 3.2) we get $f \in W^{1,p,+}$. Hence, since the function f has the regularity required in Theorem 5.1 we apply this theorem to $\Omega = B(0,r) \subseteq \mathbb{R}^{n-1}$ and so there exists $\varphi \in W^{2,p}(\overline{\Omega}, \mathbb{R}^n)$ such that

$$\begin{array}{cccc} \varphi\colon & \Omega & \longrightarrow & \varphi(\Omega) = \Omega \subseteq \mathbb{R}^{n-1} \\ & (x_2, \dots, x_n) & \longmapsto & (\varphi_1(x_2, \dots, x_n), \dots, \varphi_{n-1}(x_2, \dots, x_n)) \end{array}$$

satisfying (5.1) i.e. the partial differential equation

$$\det D\varphi_y = \lambda\, f(y), \tag{5.3}$$

for all $y := (x_2, \dots, x_n) \in \Omega$ where $\lambda = \mathrm{vol}(\Omega)/\int_\Omega f$, and $\varphi|_{\partial\Omega}$ is the identity. Now, we define the change of coordinates by:

$$\begin{array}{cccc} \Psi\colon & O_2 \subset \mathbb{R} \times \Omega & \longrightarrow & O_1 \\ & x = (x_1, y) & \longmapsto & X^{\lambda^{-1} x_1}((0, \varphi(y)) \end{array}$$

where

$$X^{\lambda^{-1}x_1}((0,\varphi(y)) = (X_1^{\lambda^{-1}x_1}(0,\varphi(y)),\dots,X_n^{\lambda^{-1}x_1}(0,\varphi(y))).$$

Observe that O_1 can diminish due to the parameter λ (we keep the notation O_1). Since Ψ is the composition of a $W^{1,p}$ homeomorphism of finite distortion for any fixed time $\lambda^{-1}x_1$ with κ_q on Definition 3.3 in $L^\infty(U)$ and a $W^{2,p}$ map we have, from Theorem 3.1, that Ψ is a $W^{1,p}$ map. Thus we may now apply the chain rule for μ-a.e. $x \in \Omega$. We begin by claiming that:

$$\det D\Psi_{(0,x_2,\dots,x_n)} = 1 \text{ for } \mu\text{-a.e. } (0,x_2,\dots,x_n) \in O_2. \tag{5.4}$$

Note that, taking $x_1 = 0$ for $j = 2,\dots,n$ and $i = 2,\dots,n$, we have the partial derivative at $(0,x_2,\dots,x_n)$ given by:

$$\frac{\partial}{\partial x_i} X_j^{\lambda^{-1}x_1}(0,\varphi(y)) = \frac{\partial \varphi_j}{\partial x_i}(y), \tag{5.5}$$

and for $j = 1$ and $i = 2,\dots,n$ we have the partial derivative at $(0,x_2,\dots,x_n)$ given by:

$$\frac{\partial}{\partial x_i} X_1^{\lambda^{-1}x_1}(0,\varphi(y)) = 0. \tag{5.6}$$

Let us compute the derivatives when $t = x_1 = 0$. Taking into account that the first column is the time-derivative of a flow i.e. the vector field, and also (5.5) and (5.6), we obtain, by an analog computation as in (4.9) and (4.10), that

$$D\Psi_{(0,x_2,\dots,x_n)} = \begin{pmatrix} \lambda^{-1}X_1(X^0((0,\varphi(y))) & 0 & \dots & 0 \\ \lambda^{-1}X_2(X^0((0,\varphi(y))) & \frac{\partial\varphi_1}{\partial x_2}|_y & \dots & \frac{\partial\varphi_1}{\partial x_n}|_y \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{-1}X_n(X^0((0,\varphi(y))) & \frac{\partial\varphi_{n-1}}{\partial x_2}|_y & \dots & \frac{\partial\varphi_{n-1}}{\partial x_n}|_y \end{pmatrix}.$$

Using (5.3) and Laplace expansion along the first line a simple computation gives,

$$\begin{aligned} \det D\Psi_{(0,x_2,\dots,x_n)} &= \lambda^{-1}X_1((0,\varphi(y)))\det D\varphi_y \\ &\overset{(5.2)}{=} \lambda^{-1}\frac{1}{f(y)}\det D\varphi_y \\ &\overset{(5.3)}{=} 1, \end{aligned}$$

therefore (5.4) is proved. What remains is to show that $\det D\Psi_{(x_1^0,x_2^0,\dots,x_n^0)} = 1$ for μ-a.e. $(x_1^0,x_2^0,\dots,x_n^0) \in O_2$. Notice that,

$$\begin{aligned} \Psi(x_1,x_2,\dots,x_n) &= X^{\lambda^{-1}x_1^0}[X^{\lambda^{-1}(x_1-x_1^0)}((0,\varphi(x_2,\dots,x_n)))] \\ &\overset{(2.5)}{=} X^{\lambda^{-1}x_1^0}[\Psi(x_1-x_1^0,x_2,\dots,x_n)], \end{aligned}$$

so, for μ-a.e. point, we have (modulo a fiber identification) that

$$D\Psi_{(x_1,x_2,\dots,x_n)} = DX^{\lambda^{-1}x_1^0}_{\Psi(x_1-x_1^0,x_2,\dots,x_n)}\, D\Psi_{(x_1-x_1^0,x_2,\dots,x_n)}.$$

Evaluating $D\Psi_{(x_1,x_2,\dots,x_n)}$ at $x_1 = x_1^0$ we get:

$$D\Psi_{(x_1^0,x_2,\dots,x_n)} = DX^{\lambda^{-1}x_1^0}_{\Psi(0,x_2,\dots,x_n)}\, D\Psi_{(0,x_2,\dots,x_n)}.$$

Since X^t is volume-preserving by (5.4) we conclude that $\det D\Psi_{(x_1^0,x_2^0,\dots,x_n^0)} = 1$ for μ-a.e.. Finally, take $\Phi = \Psi^{-1}$. □

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