Applications of Nijenhuis Geometry IV: multicomponent KdV and Camassa-Holm equations

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ABSTRACT. We construct a new series of multi-component integrable PDE systems that contains as particular examples (with appropriately chosen parameters) and generalises many famous integrable systems including KdV, coupled KdV [1], Harry Dym, coupled Harry Dym [2], Camassa-Holm, multi-component Camassa-Holm [14], Dullin-Gottwald-Holm and Kaup-Boussinesq systems. The series also contains integrable systems with no low-component analogues.

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1. Introduction

In our paper we consider two types of PDE systems. The first one is an n-component evolutionary system of PDEs:

$$(1.1) \qquad u_t^i = \xi^i(u), \quad i = 1, \ldots, n,$$

where $u^i = u^i(x, t)$ are unknown functions and $\xi^i(u)$ is a differential polynomial in $u^1, \ldots, u^n$, that is a polynomial in derivatives $u_x^j, u_{xx}^j, u_{xxx}^j, \ldots, j = 1, \ldots, n$, whose coefficients are functions of $u^1, \ldots, u^n$. For $n = 1$, an example of such a system is the famous KdV equation

$$u_t = \frac{1}{2} u_{xxx} + \frac{3}{2} u u_x.$$

A well known two-component case is the Kaup-Boussinesq system, see e.g. [**22**, eqn. (4)]:

$$u_t^1 = u_x^2 - \frac{3}{2} u^1 u_x^1 \,,$$

$$u_t^2 = \frac{m}{2} u_{xxx}^1 - u^2 u_x^1 - \frac{1}{2} u^1 u_x^2 \,.$$

where m is an arbitrary constant.

We will also consider evolutionary systems of PDEs *with differential constraints*. They are given by

$$(1.2) \qquad \begin{aligned} u_t^i &= \xi^i(u, q), \quad i = 1, \ldots, n, \\ 0 &= p(u, q). \end{aligned}$$

Here, unknown functions are $u(x, t) = (u^1(x, t), \ldots, u^n(x, t))^\top$ and $q(x, t)$. Each $\xi^i(u, q)$ is a differential polynomial in $u^1, \ldots, u^n$ and q, whereas the *differential constraint* $p(u, q)$ is a differential polynomial in q and a function in u (but not in derivatives of u). In most cases we consider just one constraint; the only exception is Example 3.5.

In all the systems that we construct and analyse, the differential constraint p can be explicitly resolved w.r.t. at least one of u^i's. Solving $p(u, q) = 0$ with respect to u^i and substituting $u^i = \tilde{p}(u^1, \ldots, u^{i-1}, u^{i+1}, \ldots, u^n, q)$ in the first equation of (1.2) allows us to think of (1.2) as a system of $n - 1$ evolutionary PDEs and one non-evolutionary PDE on n unknown functions $u^1, \ldots, u^{i-1}, u^{i+1}, \ldots, u^n$ and q. As an example, consider the Dullin-Gottwald-Holm equation [**11**] (in this case, $n = 1$):

$$(1.3) \qquad \begin{aligned} u_t &= \frac{\gamma}{2} q_{xxx} + u q_x + \frac{1}{2} q u_x, \\ u &= q + \frac{m}{2} q_{xx}. \end{aligned}$$

Substituting the expression for u into the first equation gives

$$q_t + \frac{m}{2} q_{xxt} = \frac{\gamma}{2} q_{xxx} + \frac{3}{2} q q_x + \frac{m}{2} q_{xx} q_x + \frac{1}{2} q q_{xxx},$$

which is a single non-evolutionary PDE on $q(x, t)$.

We will also deal with PDE systems given by (1.1) with the right hand side ξ^i being a formal differential series (i.e., infinite sum of monomials in derivative variables $u_x^j, u_{xx}^j, u_{xxx}^j, \ldots$ with coefficients being functions of $u^1, \ldots, u^n$). We refer to such systems as *formal* evolutionary PDEs (systems of type (1.1) with ξ^i being a differential polynomial will be called *non-formal*).

Let us explain, using the Dullin-Gottwald-Holm equation (1.3) as an example, the relation between systems of type (1.2) and formal evolutionary PDEs. Observe that the second equation of (1.3) can be formally solved with respect to q:

$$q = u - \frac{m}{2}u_{xx} + \frac{m^2}{4}u_{xxxx} - \dots$$

(we discuss neither convergence of this differential series nor boundary or other conditions). Substituting this formal expression into the first equation of (1.3) leads to the formal evolutionary PDE

$$u_t = \frac{3}{2}uu_x + \frac{\gamma}{2}u_{xxx} - \frac{m}{2}\left(\frac{\gamma}{2}u_{xxxxx} - uu_{xxx} - \frac{1}{2}u_{xx}u_x\right) + \dots,$$

depending on m as a parameter.

Recall that a (formal) evolutionary PDE-system $u_\tau^i = \eta^i(u)$ is a *(formal) symmetry* of a (formal) evolutionary PDE-system $u_t^i = \xi^i(u)$ (or, equivalently, these two PDE-systems *commute*) if the following *commutator* vanishes

$$(1.4) \qquad \sum_{j=0}^{\infty}\sum_{\alpha=1}^{n}\left(\frac{\partial\xi^i}{\partial u_{x^j}^\alpha}D^j(\eta^\alpha) - \frac{\partial\eta^i}{\partial u_{x^j}^\alpha}D^j(\xi^\alpha)\right) = 0.$$

Here and below $u_{x^j} = u_{\underbrace{xx\dots x}_{j \text{ times}}}$, $D = \frac{d}{dx}$ is the *total derivative* in x (for example, $D(u^\alpha u_x^\alpha) = (u_x^\alpha)^2 + u^\alpha u_{xx}^\alpha$) and D^j stands for the j-th power of D (for example $D^3(u^\alpha) = u_{xxx}^\alpha = u_{x^3}^\alpha$). In the case of usual (= non-formal) PDEs, $u_\tau^i = \eta^i(u)$ is a symmetry of $u_t^i = \xi^i(u)$ if an only if (at least in the analytic category) both systems can be solved simultaneously, that is, there exist n functions $u^i(x,t,\tau)$, which satisfy all $2n$ PDEs $u_\tau^i = \eta^i(u)$ and $u_t^i = \xi^i(u)$.

A (formal) differential series v is said to be a *conservation law density* of the (formal) evolutionary equation $u_t^i = \xi^i(u)$ if

$$(1.5) \qquad v_t = \sum_{j=0}^{\infty}\sum_{\alpha=1}^{n}\frac{\partial v}{\partial u_{x^j}^\alpha}D^j(\xi^\alpha) = Dw$$

for some formal differential series w. Such v is defined up to addition of an arbitrary total derivative $D\tilde{v}$. In the non-formal case, v should be understood as the *density* of a conservation law: in this case, for any fast decaying or periodic solution $u(t,x)$, the integral $\int v(u,u_x,\dots)\,dx$ is independent of t for any solution $u(x,t)$ of (1.1). We notice that both formulas (1.4) and (1.5) 'respect' the degree of differential monomials so that they are well defined for formal differential series.

The main result of our paper is a construction of a family of multi-component integrable PDE systems of the form (1.1) and (1.2). There are several different notions of integrability in this context in the literature. In the case of systems (1.1), we construct infinite hierarchies of (non-formal) conservation laws and of (non-formal) pairwise commuting symmetries. The differential degrees of the conservations laws and symmetries grow within each hierarchy.

By integrability of systems (1.2), we understand the existence of infinitely many independent (possibly, formal) commuting symmetries and conservations laws of the formal evolutionary equation obtained from (1.2) by the method demonstrated in the example above; this can be done for all systems we construct.

The paper is organised as follows. In Section 2 we describe our main examples. They are parameterised by certain discrete and continuous parameters and are of four types. The equations of Type II and Type IV are evolutionary, whereas those of

Types I and III are evolutionary with differential constrains. All of them, however, can be considered as different representatives of one single integrable system understood as a commutative algebra of evolutionary flows, which includes both formal and non-formal flows. We give explicit recursive formulas for common (non-formal) symmetries and conservation laws for all these PDE systems, see Theorem 1.

All the famous integrable systems listed in the abstract correspond to certain choices of parameters. In Section 3.3, we specify those parameters for each of them. We also obtain other known integrable systems, e.g., the Marvan-Pavlov system [20], which we essentially generalise. The generalisations are given by explicit formulas and have no low-component analogues.

Theorem 1 follows directly from a more general construction described in Section 3, see Theorems 2, 3 and 4. Namely, Theorem 2 constructs a family of evolutionary PDEs with differential constrains, as well as formal commuting symmetries and conservations laws for them. Next, Theorem 3 explains how one can 'cook' non-formal evolutionary PDEs, non-formal symmetries and non-formal conservation laws starting from those constructed in Theorem 2. The constructions in Theorems 2, 3 depend on a solution of a certain, possibly infinite, system of PDEs on a Nijenhuis manifold (M^n, L). In Theorem 4, we solve this system under the additional assumption that L is *differentially non-degenerate* and, hence, come to the integrable systems from Theorem 1.

Let us now comment on the circle of ideas which led us to these results. The construction was developed within the *Nijenhuis Geomery* programme [8]. Its main ingredient is a Nijenhuis operator L on a manifold M^n, that is a (1,1)-tensor field $L = L^i_j$ such that its Nijenhuis torsion vanishes, i.e.,

$$L^2[\nu, \eta] - L[L\nu, \eta] - L[\nu, L\eta] + [L\nu, L\eta] = 0$$

for arbitrary vector fields ν, η. In our recent paper [7], we constructed all non-degenerate pencils of compatible ∞-dimensional Poisson structures of type $\mathscr{P}_3 + \mathscr{P}_1$, where the Poisson structure $\mathscr{P}_1$ has order 1 and $\mathscr{P}_3$ is a Darboux-Poisson structure of order 3. Magri-Lenard scheme applied to these pencils leads to certain integrable bi-Hamiltonian systems. Translating them to the language of Nijenhuis Geometry allowed us to generalise our construction further and obtain integrable systems which are not necessarily Hamiltonian. We view Nijenhuis Geometry as the most natural framework for them and expect that the systems and their properties can and should be understood in the context of Nijenhuis operators, with no other geometric structure involved.

2. Explicit formulas for new integrable systems, their symmetries and conservation laws

2.1. Four types of equations. A Nijenhuis operator L on M^n is called differentially non-degenerate, if the differentials of the coefficients of its characteristic

polynomial are linearly independent at every point. Typical examples of differentially non-degenerate Nijenhuis operators are as follows:
(2.1)
$$
L_{\mathsf{comp}} = \begin{pmatrix} u^1 & 1 & 0 & \ldots & 0 \\ u^2 & 0 & 1 & \ldots & 0 \\ \vdots & & & \ddots & \\ u^{n-1} & 0 & 0 & \ldots & 1 \\ u^n & 0 & 0 & \ldots & 0 \end{pmatrix} \quad \text{and} \quad L_{\mathsf{diag}} = \begin{pmatrix} x^1 & 0 & 0 & \ldots & 0 \\ 0 & x^2 & 0 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & \ldots & x^{n-1} & 0 \\ 0 & 0 & \ldots & \ldots & x^n \end{pmatrix}.
$$

Here $u^1, ..., u^n$ and $x^1, ..., x^n$ are local coordinate charts on M^n. Moreover, in the case of L_{diag} we assume that x^i are all different. In fact, these two operators are locally isomorphic: if we rewrite L_{diag} in the coordinates $u^1(x), ..., u^n(x)$ that are coefficients of its characteristic polynomial (more precisely, we set $\det(t\,\mathrm{Id} - L_{\mathsf{diag}}) = t^n - u^1 t^{n-1} - \cdots - u^n$), then it transforms into L_{comp}. Note also that every differentially non-degenerate Nijenhuis operator L reduces to the companion form L_{comp} by an appropriate coordinate transform, and to the diagonal form L_{diag} near those points where L has n distinct real eigenvalues, see e.g. [**8**].

Choose $N \geq 0$ and consider natural numbers $n_0, n_1, \ldots, n_N$ and $\ell_1, ..., \ell_N$ with conditions $n_0 + n_1 + \cdots + n_N = n = \dim \mathrm{M}$ and $n_0 - \ell_1 n_1 - \cdots - \ell_N n_N = d \geq 0$, and fix a polynomial

$$
(2.2) \qquad\qquad m(\lambda) = m_0 + m_1 \lambda + \cdots + m_d \lambda^d
$$

of degree $\leq d$. These are the parameters of our construction: $2N + 1$ natural numbers and $d + 1$ coefficients $m_0, \ldots, m_d$.

Next, take the direct product $\mathrm{M}^n = U_0 \times \cdots \times U_N$ of $N + 1$ discs $U_0, ..., U_N$ of dimensions $n_0, n_1, ..., n_N$ equipped with differentially non-degenerate Nijenhuis operators $L_0, ..., L_N$. The operator field L on M^n is defined as

$$
(2.3) \qquad\qquad L = L_0 \oplus \cdots \oplus L_N.
$$

Consider the following family of functions $\sigma(\lambda)$ on M^n, depending on λ as a parameter (in general $\lambda \in \mathbb{C}$, so the functions might be complex-valued):

$$
(2.4) \qquad\qquad \sigma(\lambda) = \frac{\det(L_0 - \lambda\,\mathrm{Id})}{\det(L_1 - \lambda\,\mathrm{Id})^{\ell_1} \ldots \det(L_N - \lambda\,\mathrm{Id})^{\ell_N}}
$$

Here in each expression $\det(L_i - \lambda\,\mathrm{Id})^{\ell_i}$, the identity matrix Id is of the same size as L_i, i.e., $n_i \times n_i$ and ℓ_i denotes the power.

Next, consider the vector field ζ_0 on M^n uniquely defined by the relations

$$
\mathcal{L}_{\zeta_0} \det(L_0 - \lambda\,\mathrm{Id}) = 1, \quad \mathcal{L}_{\zeta_0} \det(L_i - \lambda\,\mathrm{Id}) = 0, \quad i = 1, \ldots, N,
$$

where $\mathcal{L}_{\zeta_0}$ denotes the Lie derivative, and define another vector field ζ by setting

$$
(2.5) \qquad\qquad \zeta = p(L)\zeta_0.
$$

where $p(t) = \det(L_1 - \lambda\,\mathrm{Id})^{\ell_1} \ldots \det(L_N - \lambda\,\mathrm{Id})^{\ell_N} m(t)$ is a polynomial in t with coefficients being functions on M^n.

Based on these settings, we finally introduce four types of equations. As unknown functions, we consider $u^1(x,t), \ldots, u^n(x,t), q(x,t)$, where $(u^1, \ldots, u^n)$ should be understood as coordinates on M^n and q as an additional function.

Type I. For each real or complex number λ consider the equations

(2.6)
$$u_t = q_{xxx}(L - \lambda \operatorname{Id})^{-1}\zeta + q(L - \lambda \operatorname{Id})^{-1}u_x,$$
$$0 = m(\lambda)q_{xx}q - \frac{1}{2}m(\lambda)(q_x)^2 + \sigma(\lambda)q^2 - 1.$$

This is a system of the form (1.2), i.e., an n-component system with a differential constraint.

Type II. If λ_i is a root of $m(\lambda)$, then the differential constraint in Type I becomes degenerate and takes the form $\sigma(\lambda_i)q^2 = 1$. Thus, we get an evolutionary PDE

(2.7)
$$u_t = \left(\frac{1}{\sqrt{\sigma(\lambda_i)}}\right)_{xxx}(L - \lambda_i \operatorname{Id})^{-1}\zeta + \left(\frac{1}{\sqrt{\sigma(\lambda_i)}}\right)(L - \lambda_i \operatorname{Id})^{-1}u_x.$$

Type III. Consider the equation

(2.8)
$$u_t = q_{xxx}\zeta + (L + q \operatorname{Id})u_x,$$
$$\frac{1}{2}\left(\operatorname{tr} L_0 - \sum_{j=1}^{N}\ell_j \operatorname{tr} L_j\right) = q + (-1)^d\frac{m_d}{2}q_{xx},$$

where m_d is the highest coefficients of the polynomial $m(t) = m_0 + m_1 t + \cdots + m_d t^d$. This is, again, a system of of the form (1.2), i.e., an n-component system with a differential constraint.

Type IV. Assume now $m_d = 0$ (in this case we say that $m(t)$ has a root at infinity, the terminology will be clarified later). Then (2.8) takes the form

(2.9)
$$u_t = \frac{1}{2}\left(\operatorname{tr} L_0 - \sum_{j=1}^{N}\ell_j \operatorname{tr} L_j\right)_{xxx}\zeta + \left(L + \frac{1}{2}\left(\operatorname{tr} L_0 - \sum_{j=1}^{N}\ell_j \operatorname{tr} L_j\right)\operatorname{Id}\right)u_x.$$

2.2. Commuting flows and conservation laws for the equations of Types I − IV. Here we describe an explicit procedure that generates commuting symmetries and conservation laws for the above four types of equations.

Step 1. In the one-component case, consider the relation

(2.10)
$$\sigma = \frac{1}{2}u^2 + u_x.$$

and its formal solution $u = u_1 + u_2 + \ldots$ as a differential series in σ.

The recursion formula for the components of u from (2.10) was essentially discovered by Kruskal and Miura. In the form we need (up to notation) it appeared e.g. in [**23**, eqns. 2.16–2.19]:

(2.11)
$$u_1 = \sqrt{2\sigma}, \quad u_{i+1} = -\frac{1}{u_1}\left(\frac{1}{2}\sum_{j=2}^{i}u_j u_{i+2-j} + (u_i)_x\right), \quad i \geq 1.$$

In the expansion $u = \sum u_i$, we are interested in the odd terms only and introduce two formal differential series

(2.12)
$$v(\sigma, m) = \sqrt{2}\sum_{s=0}^{\infty}(-1)^s m^s u_{2s+1}$$

and

(2.13)
$$\mathrm{w}(\sigma, m) = \sqrt{2} \sum_{s=0}^{\infty} (-1)^s m^s \delta u_{2s+1}.$$

Here δ stands for the variational derivative w.r.t. σ and m is considered as a formal parameter.

Important fact. The series $\mathrm{w}(\sigma, m)$ satisfies the following identity

(2.14)
$$m\mathrm{w}_{xxx}(\sigma, m) + 2\sigma \mathrm{w}_x(\sigma, m) + \sigma_x \mathrm{w}(\sigma, m) = 0.$$

This identity is essentially known and comes from the theory of local infinite-dimensional Poisson structures. It can be understood as the fact that $\mathrm{v}(\sigma, m)$ is a formal Casimir of the Poisson structure defined by the operator $mD^3 + 2\sigma D + \sigma_x$ (for details and proof see e.g. [**23**, Proposition 2.1] and also [**16**, Theorem 2.4] for n-component systems). One can also view (2.14) as a way of applying the Magri-Lenard scheme to produce commuting symmetries for the Harry Dym equation. However, we do not need such an interpretation and will use identity (2.14) as it is.

We will also use the following crucial observation by Gelfand and Dikii [**13**] (see also [**3**]). Multiplying the l.h.s. of (2.14) by $\mathrm{w}(\sigma, m)$ and integrating in x we get another important identity for w (cf. the differential constraint from (2.6)):

(2.15)
$$m\,\mathrm{w}_{xx}\mathrm{w} - \frac{1}{2}\,m\,\mathrm{w}_x^2 + \sigma\,\mathrm{w}^2 = 1,$$

which, in particular, allows us to reconstruct all the terms of (2.13) step by step starting from the first term $\sqrt{2}\,\delta u_1 = \frac{1}{\sqrt{\sigma}}$.

Step 2. For $m(\lambda)$ and $\sigma(\lambda)$ defined by (2.2) and (2.4) respectively, consider the formal differential series (in any local coordinates $u^1, \ldots, u^n$ on M^n) with components depending on parameter λ

(2.16)
$$\mathrm{v}(\lambda) = \mathrm{v}(\sigma(\lambda; u), m(\lambda)), \quad \mathrm{w}(\lambda) = \mathrm{w}(\sigma(\lambda; u), m(\lambda)),$$

obtained by replacing σ with $\sigma(\lambda) = \sigma(\lambda; u^1, \ldots, u^n)$ and m with $m(\lambda)$ in (2.12) and (2.13). Next, introduce the formal n-component vector field

(2.17)
$$\xi(\lambda) = \mathrm{w}(\lambda)_{xxx} (L - \lambda\,\mathrm{Id})^{-1}\zeta + \mathrm{w}(\lambda)(L - \lambda\,\mathrm{Id})^{-1}u_x.$$

This vector field is naturally related to the r.h.s. of the first equation of system (2.6) of Type 1. In fact, (2.6) can be equivalently rewritten as $u_t = \xi(\lambda)$. It follows from the fact that the differential series $\mathrm{w}(\lambda)$ satisfies the same differential relation (2.15) as the function q (see second equation of (2.6)) and can be uniquely reconstructed from it (see 'Important fact' in Step 1).

Step 3. For each root λ_i of the polynomial $m(\lambda)$, expand both $m(\lambda)$ and $\sigma(\lambda, u)$ into Taylor series in powers of $\varepsilon = \lambda - \lambda_i$, i.e.,

$$m(\lambda_i + \varepsilon) = \sum_{s=1}^{\infty} m_{s,\lambda_i}\varepsilon^s, \quad \sigma(\lambda_i + \varepsilon, u) = \sum_{s=0}^{\infty} \sigma_{s,\lambda_i}(u)\varepsilon^s$$

Substitution $m(\lambda) = \sum_{s=1}^{\infty} m_{s,\lambda_i}\varepsilon^s$ and $\sigma(\lambda; u) = \sum_{s=0}^{\infty} \sigma_{s,\lambda_i}(u)\varepsilon^s$ "transforms" $\mathrm{v}(\lambda) = \mathrm{v}(\sigma(\lambda; u), m(\lambda))$ and $\mathrm{w}(\lambda) = \mathrm{v}(\sigma(\lambda, u), m(\lambda))$, as well as $\xi(\lambda)$ defined by (2.17), into series in powers of ε :

(2.18)
$$\mathrm{v}\big(\sigma(\lambda_i + \varepsilon, u), m(\lambda_i + \varepsilon)\big) = \sum_{s=0}^{\infty} v_{s,\lambda_i}\varepsilon^s, \quad \text{with } v_{0,\lambda_i} = 2\sqrt{\sigma(\lambda_i)}.$$

$$(2.19) \qquad \mathrm{w}(\sigma(\lambda_i + \varepsilon), m(\lambda_i + \varepsilon)) = \sum_{s=0}^{\infty} \mathrm{w}_{s,\lambda_i} \varepsilon^s, \quad \text{with} \quad \mathrm{w}_{0,\lambda_i} = \frac{1}{\sqrt{\sigma(\lambda_i)}}.$$

(2.20)

$$\xi = \sum_{s=0}^{\infty} \xi_{s,\lambda_i} \varepsilon^s, \text{with} \ \ \xi_{0,\lambda_i} = \left(\frac{1}{\sqrt{\sigma(\lambda_i)}} \right)_{xxx} (L - \lambda \operatorname{Id})^{-1} \zeta + \frac{1}{\sqrt{\sigma(\lambda_i)}} (L - \lambda \operatorname{Id})^{-1} u_x.$$

Similarly for $\lambda = \infty$, in the above construction we replace $\sigma(\lambda)$ and $m(\lambda)$ with $\bar{\sigma}(\lambda) = (-\lambda)^d \sigma\left(\frac{1}{\lambda}\right)$, $\bar{m}(\lambda) = (-\lambda)^d m\left(\frac{1}{\lambda}\right)$ and also $L - \lambda \operatorname{Id}$ with $\operatorname{Id} - \lambda L$. In particular, we set

$$\bar{\mathrm{v}}(\lambda) = \mathrm{v}(\bar{\sigma}(\lambda), \bar{m}(\lambda)), \quad \bar{\mathrm{w}}(\lambda) = \mathrm{w}(\bar{\sigma}(\lambda), \bar{m}(\lambda)),$$

$$\bar{\xi}(\lambda) = \bar{\mathrm{w}}(\lambda)_{xxx} (\operatorname{Id} - \lambda L)^{-1} \zeta + \bar{\mathrm{w}}(\lambda)(\operatorname{Id} - \lambda L)^{-1} u_x.$$

Then if $\bar{m}(0) = 0$, we substitute $\bar{\sigma}(\varepsilon)$, $\bar{m}(\varepsilon)$ into these relations and expand in powers of ε to get[1]

$$(2.21) \quad \bar{\mathrm{v}}(\varepsilon) = \sum_{s=0}^{\infty} \mathrm{v}_{s-1,\infty} \varepsilon^s, \quad \text{with} \ \ \mathrm{v}_{-1,\infty} = 2\sqrt{\bar{\sigma}(0)} = 2 \text{ and } \mathrm{v}_{0,\infty} = -2f(u);$$

$$(2.22) \qquad \bar{\mathrm{w}}(\varepsilon) = \sum_{s=0}^{\infty} \mathrm{w}_{s-1,\infty} \varepsilon^s, \quad \text{with} \ \ \mathrm{w}_{-1,\infty} = 1 \text{ and } \mathrm{w}_{0,\infty} = \frac{1}{2}f(u);$$

(2.23)

$$\bar{\xi}(\varepsilon) = \sum_{s=0}^{\infty} \xi_{s-1,\infty} \varepsilon^s, \quad \text{with } \xi_{-1,\infty} = u_x \text{ and } \xi_{0,\infty} = \tfrac{1}{2} f_{xxx} \zeta + (L + \tfrac{1}{2} f \operatorname{Id}) u_x$$

where f is defined from $\bar{\sigma}(\varepsilon) = 1 - \varepsilon f + \ldots$, i.e., $f = \frac{d}{d\varepsilon}|_{\varepsilon=0} \bar{\sigma}(\varepsilon) = \operatorname{tr} L_0 - \sum_{j=1}^{N} \ell_j \operatorname{tr} L_j$.

The coefficients of the ε-expansions (2.18), (2.20), (2.21) and (2.23) define hierarchies of common commuting symmetries and conservation laws for the above introduced equations of types I–IV. Namely, we have

THEOREM 1. Let $\lambda_1, \ldots, \lambda_r$ be the roots of the polynomial $m(\lambda)$ (including ∞ when appropriate). Then the evolutionary PDE systems

$$u_t = \xi_{s,\lambda_i}, \quad i = 1, \ldots, r, \ \ s = 0, 1, \ldots.$$

are commuting symmetries and the differential polynomials

$$\mathrm{v}_{s,\lambda_i}, \quad i = 1, \ldots, r, \ \ s = 0, 1, \ldots$$

are conservation law densities for equations (2.6)–(2.9) of Types I–IV. Moreover, the equations (2.7, Type II) and (2.9, Type IV) take the form $u_t = \xi_{0,\lambda_i}$ for $\lambda_i \neq \infty$ and $\lambda_i = \infty$ respectively.

Thus, for a Nijenhuis operator L (decomposed into differentially non-degenerate blocks), Theorem 1 gives a series of multi-component integrable systems and provide, for each of them, commuting symmetries and conservation laws that can be constructed by an explicit iterative procedure.

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[1]In these power series, we shift indices of coefficients by 1. The reason is that the first terms of these expansions are trivial and can be ignored. This shift also allows us to keep notation consistent with the case of $\lambda_i \neq \infty$.

3. General construction

3.1. Parameters of the general construction. Let L be a Nijenhuis operator on M^n and $f : M^n \to \mathbb{R}$ be a function such that the 1-form $L^* \mathrm{d} f$ is closed so that locally $L^* \mathrm{d} f = \mathrm{d} f_1$ for some function f_1. Then (see Section 3 in [**19**]) there exists an infinite sequence of functions f_k, $k = 1, \ldots$, such that $\mathrm{d} f_k = (L^*)^k \mathrm{d} f$. We refer to f as a *conservation law* for the Nijenhuis operator L. The name comes from the fact that f provides a conservation law in the sense of (1.5) for the quasilinear system $u_t = L u_x$. The above observation means that f is a conservation laws for every power of L or, equivalently, generates a hierarchy of conservation laws for L.

In particular, this implies that $(L^* - \lambda \, \mathrm{Id})^{-1} \mathrm{d} f$ is also closed for any λ. Thus, there locally exists a function $g(\lambda; u)$ depending on λ as a parameter and such that $\mathrm{d} g = (L^* - \lambda \, \mathrm{Id})^{-1} \mathrm{d} f$. Here $u = (u^1, \ldots, u^n)$ are coordinates on M^n and d denotes the differential of a function w.r.t. u. One of the main ingredients of our construction is the function $\sigma(\lambda; u) = e^{g(\lambda; u)}$ satisfying the identity

$$(3.1) \qquad \big(L^* - \lambda \, \mathrm{Id}\big) \mathrm{d} \sigma = \sigma \, \mathrm{d} f, \qquad \lambda \in \mathbb{C}.$$

If $L(u) - \lambda \, \mathrm{Id}$ is invertible, i.e. $\lambda \notin \mathrm{Spectrum}\, L(u)$, then $\sigma(\lambda; u)$ is analytic in λ, otherwise the point (λ, u) may be singular (like pole or zero, or branching point).

Next, assume that there exist a vector field ζ on M^n and constant $C \in \mathbb{R}$ such that[2]

$$(3.2) \qquad \mathcal{L}_\zeta \big(\sigma(\lambda; u)\big) + C \sigma(\lambda; u) = m(\lambda),$$

that is, the left hand side does not depend on u and hence is a certain function of λ (in the examples discussed below, $m(\lambda)$ is always a polynomial of degree $\leq n = \dim M$). This triple, i.e., Nijenhuis operator L, conservation law f and vector field ζ are parameters of the construction.

Notice that $\sigma(\lambda; u)$ and $m(\lambda)$ satisfying (3.1), (3.2) are defined up to simultaneous multiplication by an arbitrary function $c(\lambda)$. This kind of scaling is not important for the construction below and we will treat it as a trivial transformation.

3.2. Main theorems. Fix a triple L, f and ζ. Construct $\sigma(\lambda; u)$ and $m(\lambda)$ by formulas (3.1) and (3.2). Using them construct infinite differential series $\mathrm{v}(\lambda)$ and $\mathrm{w}(\lambda)$ by (2.12) and (2.13).

Recall that the series $\mathrm{w}(\lambda)$ satisfies the Gelfand-Dikii identity (2.15):

$$(3.3) \qquad m(\lambda) \left(\mathrm{w}_{xx}(\lambda) \mathrm{w}(\lambda) - \frac{1}{2} (\mathrm{w}_x(\lambda))^2 \right) + \sigma(\lambda)(\mathrm{w}(\lambda))^2 = 1.$$

Based on this information, we introduce an n-component system with a differential constraint[3]

$$(3.4) \quad \begin{aligned} u_{t_\lambda} &= \mathrm{w}_{xxx}(\lambda)\big(L - \lambda \, \mathrm{Id}\big)^{-1} \zeta + \mathrm{w}(\lambda)\big(L - \lambda \, \mathrm{Id}\big)^{-1} u_x, \\ 0 &= m(\lambda) \left(\mathrm{w}_{xx}(\lambda) \mathrm{w}(\lambda) - \frac{1}{2} (\mathrm{w}_x(\lambda))^2 \right) + \sigma(\lambda)(\mathrm{w}(\lambda))^2 - 1. \end{aligned}$$

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[2]This condition is quite non-trivial so that the existence of a non-zero ζ depends on L and f. However, for $\zeta = 0$ the construction still makes sense but reduces to a hydrodynamic type system, see Corollary 3.1.

[3]The equation of Type I from Introduction is exactly of this kind with $q = \mathrm{w}(\lambda)$ for a specific choice of parameters L, f and ζ.

In this construction one can naturally make sense of $\lambda = \infty$. Namely, we define $\bar{\sigma}(\lambda) = c(\lambda)\sigma(\frac{1}{\lambda})$ and $\bar{m}(\lambda) = c(\lambda)m(\frac{1}{\lambda})$, where $c(\lambda)$ is a suitable scaling factor. The function $\bar{\sigma}(\lambda)$ does not satisfy (3.1), but does satisfy a very similar relation

$$(3.5) \qquad (\mathrm{Id} - \lambda L)\,\mathrm{d}\,\bar{\sigma}(\lambda) = -\lambda\bar{\sigma}(\lambda)\mathrm{d}\,f.$$

This implies that $c(\lambda)$ can be chosen in such a way that $\bar{\sigma}(\lambda)$ is analytic in λ in a neighbourhood of zero and moreover, $\bar{\sigma}(\lambda) = 1 - \lambda f + \ldots$, where dots denote higher order terms in λ. We will assume that $c(\lambda)$ is chosen in this way. Then we set $\bar{\mathrm{v}}(\lambda) = \mathrm{v}(\bar{\sigma}(\lambda; u), \bar{m}(\lambda))$ and $\bar{\mathrm{w}}(\lambda) = \mathrm{w}(\bar{\sigma}(\lambda; u), \bar{m}(\lambda))$ and rewrite the family of equations (3.4) in the following equivalent form obtained by replacing λ with $\frac{1}{\lambda}$:

$$(3.6) \qquad \begin{aligned} u_{\bar{t}_\lambda} &= \bar{\mathrm{w}}_{xxx}(\lambda)\big(\mathrm{Id} - \lambda L\big)^{-1}\zeta + \bar{\mathrm{w}}(\lambda)\big(\mathrm{Id} - \lambda L\big)^{-1}u_x, \\ 0 &= \bar{m}(\lambda)\left(\bar{\mathrm{w}}_{xx}(\lambda)\bar{\mathrm{w}}(\lambda) - \frac{1}{2}(\bar{\mathrm{w}}_x(\lambda))^2\right) + \bar{\sigma}(\lambda)(\bar{\mathrm{w}}(\lambda))^2 - 1. \end{aligned}$$

More precisely, we have $\bar{\mathrm{w}}(\lambda) = \frac{1}{\sqrt{c(\lambda)}}\mathrm{w}(\frac{1}{\lambda})$ so that $u_{\bar{t}_\lambda}$ coincides with $u_{t_{\lambda^{-1}}}$ up to appropriate rescaling (with a factor depending on λ).

This transformation allows us to set $\lambda = 0$ in (3.6) which will naturally correspond to $\lambda = \infty$ in (3.4). In particular, we set

$$(3.7) \qquad \mathrm{v}(\infty) = \bar{\mathrm{v}}(0) \quad \text{and} \quad \mathrm{w}(\infty) = \bar{\mathrm{w}}(0).$$

However, $\bar{\sigma}(0) = 1$ leading to $\bar{\mathrm{w}}(0) = 1$ and hence to the trivial evolutionary equation $u_{\bar{t}_\lambda} = u_x$. To get a non-trivial equation 'at infinity', we may consider the derivative of (3.6) at $\lambda = 0$, namely we set

$$\begin{aligned} u_{t_\infty} &= \lim_{\lambda \to 0}\frac{1}{\lambda}(u_{\bar{t}_\lambda} - u_{\bar{t}_0}) = \\ &= \frac{1}{\lambda}\left(\bar{\mathrm{w}}_{xxx}(\lambda)\big(\mathrm{Id} - \lambda L\big)^{-1}\zeta + \bar{\mathrm{w}}(\lambda)\big(\mathrm{Id} - \lambda L\big)^{-1}u_x - u_x\right) = \\ &= q_{xxx}\zeta + (L + q\,\mathrm{Id})u_x, \end{aligned}$$

where $\bar{\mathrm{w}}(\lambda) = 1 + \lambda q + $ (higher order terms in λ). In other words, q is the derivative of $\bar{\mathrm{w}}(\lambda)$ w.r.t. λ at zero. Substituting $\bar{\mathrm{w}}(\lambda) = 1 + \lambda q + \ldots$ into the second equation of (3.6) we obtain the following constraint for q:

$$\bar{m}(0)q_{xx} + 2q - f(u) = 0.$$

To summarise, for $\lambda = \infty$ we consider the following evolutionary PDE system with a constraint:

$$(3.8) \qquad \begin{aligned} u_{t_\infty} &= q_{xxx}\zeta + (L + q\,\mathrm{Id})u_x, \\ 0 &= \bar{m}(0)q_{xx} + 2q - f(u). \end{aligned}$$

Equations (3.4) and (3.8) (related to $\lambda \neq \infty$ and $\lambda = \infty$ respectively) are now understood as a parametric family with $\lambda \in \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The main property of this family of PDEs is as follows.

THEOREM 2. Let L be a Nijenhuis operator and f a conservation law of L. Consider $\sigma(\lambda; u)$ constructed from (3.1), and a vector field ζ satisfying (3.2) for a certain function $m(\lambda)$. Then for any $\lambda, \mu \in \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the differential series $\mathrm{v}(\mu)$ defined by (2.12), (3.7) is a conservation law density for the evolutionary flow u_{t_λ} with a differential constraint defined by (3.4), (3.8). Moreover, if f is generic in the sense that $\mathrm{d}\,f, L^*\mathrm{d}\,f, \ldots, (L^{n-1})^*\mathrm{d}\,f$ are linearly independent, then the flows u_{t_λ}'s pairwise commute.

As a straightforward corollary of this construction, we may consider the 'trivial' case when $\zeta = 0$ and $m(\lambda) = 0$. In this situation, the first term in (3.4) disappears, but our construction still gives a non-trivial series of integrable quasilinear systems.

COROLLARY 3.1. Let $L(u)$ be a Nijenhuis operator, $f(u)$ a conservation law of L and $\sigma(\lambda; u)$ denote the function satisfying (3.1). Then the evolutionary flows

$$u_{t_\lambda} = \sigma(\lambda; u)\big(L(u) - \lambda\,\mathrm{Id}\big)^{-1} u_x$$

pairwise commute for all λ's. Moreover, the functions $\dfrac{1}{\sigma(\mu, x)}$ are common conservation law densities for these flows (for all λ and μ).

REMARK 3.1. In the assumptions of Theorem 2, we obtain the flows of the form $\mathrm{w}(\lambda)(L - \lambda\,\mathrm{Id})^{-1} u_x$ with $\mathrm{w}(\lambda) = \sigma(\lambda)^{-1/2}$ for $m = 0$. This exponent $-\frac{1}{2}$, however, is not very essential. Indeed, if $\sigma(\lambda)$ satisfies (3.1), then $\sigma(\lambda)^c$ satisfies (3.1) also with f replaced with $\tilde{f} = c \cdot f$, so that Corollary 3.1 can be easily obtained by an appropriate rescaling. Of course, this corollary admits a direct proof without using Theorem 2.

We also note that Corollary 3.1 can be understood as a λ-version of the construction by F. Magri suggested in [19] and then developed in [17]. If $f = \mathrm{tr}\, L$ and $\lambda \to \infty$, then we obtain the system studied by E. Ferapontov and M. Pavlov in [12] (see also [6]) and for $f = c \cdot \mathrm{tr}\, L$, $c \in \mathbb{R}$ we obtain the so-called ε-systems studied by M. Pavlov [21].

We will need another corollary from Theorem 2. Consider a formal PDE

$$(3.9) \qquad\qquad u_{t_\lambda} = \xi(\lambda)$$

obtained from (3.4) by resolving the constraint w.r.t. $\mathrm{w}(\lambda)$, i.e., expressing $\mathrm{w}(\lambda)$ as a formal differential series and substituting it into the first equation of (3.4). As a result, the r.h.s. of (3.9) becomes a formal differential series in the derivatives $u_x, u_{xx}, \dots$ whose coefficients are functions in λ and u.

We now fix λ and, in a small neighbourhood of it, expand $\xi(\lambda + \varepsilon)$ in powers of ε:

$$(3.10) \qquad\qquad \xi(\lambda + \varepsilon) = \sum_{s=0}^{\infty} \xi_{s,\lambda}\varepsilon^s.$$

In the same way we defines ε-expansions for $\mathrm{v}(\lambda)$ and $\mathrm{w}(\lambda)$:

$$(3.11) \qquad\qquad \mathrm{v}(\lambda + \varepsilon) = \sum_{s=0}^{\infty} \mathrm{v}_{s,\lambda}\varepsilon^s, \quad \mathrm{w}(\lambda + \varepsilon) = \sum_{s=0}^{\infty} \mathrm{w}_{s,\lambda}\varepsilon^s.$$

Notice that by construction, each coefficient $\xi_{s,\lambda}$, $\mathrm{v}_{s,\lambda}$ or $\mathrm{w}_{s,\lambda}$ is still a formal differential series in $u_x, u_{xx}, \dots$.

COROLLARY 3.2. In the settings of Theorem 2, assume that the conservation law f is generic in the sense that $\mathrm{d}\,f, L^*\mathrm{d}\,f, \dots, (L^{n-1})^*\mathrm{d}\,f$ are linearly independent. Then the (formal) evolutionary flows defined by the (formal) vector fields $\xi_{s,\lambda}$ ($\lambda \in \bar{\mathbb{C}}$, $s = 0, 1, 2, \dots$) pairwise commute. Moreover, $v_{r,\mu}$ are common (formal) conservation law densities for all of them ($\mu \in \bar{\mathbb{C}}$, $r = 0, 1, 2, \dots$).

The next theorem is closely related to Corollary 3.2 and deals with degeneration of the differential constraints that we observed in Section 2 for Type I and Type III, but now in the general case.

THEOREM 3. In the settings of Theorem 2, let λ_i be a zero of $m(\lambda)$, i.e., $m(\lambda_i) = 0$ ($i \in \{1, 2, \ldots, k\}$). Then all the coefficients ξ_{s,λ_i}, v_{s,λ_i} and w_{s,λ_i} of ε-expansions (3.10) and (3.11) at the point λ_i are differential polynomials so that $u_t = \xi_{s,\lambda_i}$ is a usual evolutionary equation as in (1.1). In particular, for $s = 0$ these equations have the following form
(3.12)

$$u_{t_{\lambda_i}} = \xi(\lambda_i) = \left(\frac{1}{\sqrt{\sigma(\lambda_i)}} \right)_{xxx} (L - \lambda_i \,\mathrm{Id})^{-1} \zeta + \frac{1}{\sqrt{\sigma(\lambda_i)}} (L - \lambda_i \,\mathrm{Id})^{-1} u_x, \text{ for } \lambda_i \neq \infty$$

and

$$(3.13) \qquad u_{t_\infty} = \xi(\infty) = \tfrac{1}{2} f_{xxx}\, \zeta + \left(L + \tfrac{1}{2} f \,\mathrm{Id} \right) u_x, \qquad \text{for } \lambda_i = \infty.$$

Summarising the statements of Theorems 2, 3 and Corollary 3.2 we come to the following conclusion. For each $\lambda \in \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ we define an evolutionary multi-component PDE system (3.4), (3.8) with a differential constraint as in (1.2). The corresponding (formal) evolutionary flows $u_{t_\lambda} = \xi(\lambda)$ pairwise commute and admit an infinite family of common (formal) conservation laws also parameterised by $\lambda \in \bar{\mathbb{C}}$. For some special values of the parameter λ, namely for the zeros $\lambda_1, \lambda_2, \ldots$ of the function $m(\lambda)$ (∞ is also allowed when appropriate), the corresponding PDE equations $u_{t_{\lambda_i}} = \xi(\lambda_i)$ are *usual* evolutionary multi-component PDEs whose r.h.s. are differential polynomials as in (1.1). Each λ_i generates hierarchies of commuting *non-formal* symmetry fields ξ_{s,λ_i} and *non-formal* conservation laws v_{s,λ_i} for the whole family $u_{t_\lambda} = \xi(\lambda)$ of *formal* PDE systems. Moreover, the members of these hierarchies are defined by means of an explicit iterative procedure in terms of the function $\sigma(\lambda; u)$ and vector field ζ.

Thus, Theorems 2 and 3 give a recipe for constructing multi-component integrable PDEs starting from a Nijenhuis operator L and its conservation law f satisfying certain conditions. However, in order to construct a specific example of such a system, we need to find a function $\sigma(u, \lambda)$ and a vector field $\zeta(u)$ satisfying (3.1) and (3.2), that is, to solve a (possibly, infinite) system of PDEs. It is straightforward to check that the function σ and vector field ζ given by (2.4) and (2.5) in Section 2 are solutions of (3.1) and (3.2). The construction from Theorems 2, 3 applied to these σ and ζ gives the integrable systems of Types I–IV from Section 2 so that Theorem 1 immediately follows.

The next theorem shows that in the differentially non-degenerate case, σ and ζ given by (2.4) and (2.5) provide the only non-trivial solution of (3.1) and (3.2).

THEOREM 4. Let L be a differentially non-degenerate Nijenhuis operator and f a conservation law of L such that at a point $p \in M^n$ the 1-forms

$$\mathrm{d}\,f, L^* \mathrm{d}\,f, \ldots, (L^{n-1})^* \mathrm{d}\,f$$

are linearly independent. Assume that there exist $\sigma(\lambda; u)$, $m(\lambda)$ and ζ satisfying (3.1) and (3.2) with $m(\lambda) \neq 0$. Then, in a small neighborhood of p, the Nijenhuis operator L, functions[4] $\sigma(\lambda; u)$, $m(\lambda)$ and vector field ζ are as in Section 2, see (2.3), (2.4), (2.2) and (2.5) respectively.

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[4]We recall that $\sigma(\lambda; u)$ and $m(\lambda)$ are defined up to simultaneous multiplication by an arbitrary function $c(\lambda)$ and this freedom is assumed here.

As already mentioned above, Theorem 1 follows directly from Theorems 2, 3 by taking $\sigma(\lambda)$ and ζ given by (2.4) and (2.5). Theorems 2 and 3 are proved in Section 4 and Theorem 4 in Section 5.

3.3. Parameters corresponding to known integrable systems. In this section we show that for particular choice of the parameters, Type I – IV equations from Section 2 contain many famous integrable systems so that our approach allows one to generate a vast amount of different integrable systems in a unifying manner.

EXAMPLE 3.1 (KdV, Camassa-Holm, Dullin-Gottwald-Holm and their generalisations). In dimension $n = 1$, the differentially non-degenerate Nijenhuis operator is $L = u$. Due to Theorem 4, the only possible $\sigma(\lambda)$ is $u - \lambda$ and then $\zeta = m_0 + m_1 u$. Notice that

$$\mathcal{L}_\zeta \sigma(\lambda) = m_0 + m_1 u = m_0 + \lambda m_1 + m_1(u - \lambda) = m(\lambda) + m_1 \sigma(\lambda),$$

as required by (3.2).

The Type I equation in this case is

$$u_t = q_{xxx} \frac{m_0 + m_1 u}{u - \lambda} + q \frac{u_x}{u - \lambda},$$

$$0 = (m_0 + \lambda m_1) q_{xx} q - \frac{1}{2}(m_0 + \lambda m_1)(q_x)^2 + (u - \lambda) q^2 - 1.$$

This is a three-parameter (m_0, m_1 and λ) family of integrable evolutionary PDEs with differential constraint.

If $m_1 \neq 0$, then taking $\lambda_0 = -\frac{m_0}{m_1}$ we get Type II equation

$$u_t = m_1 \left(\frac{1}{\sqrt{u - \lambda_0}} \right)_{xxx} + \frac{u_x}{(u - \lambda_0)^{3/2}}.$$

This is a two-parameter (λ_0 and $m_1 \neq 0$)[5] family of equations. For $\lambda_0 = 0$ it yields (after rescaling) the reduction of the coupled Harry Dym equation [**2**, eqn. 26a]. It also appeared in [**9**] as the first flow of the inverse Camassa-Holm hierarchy (flow $m_t^{(0)}$ in Section "Bihamiltonian structure" from [**9**]).

The Type III equation takes the form

$$u_t = q_{xxx}(m_0 + m_1 u) + (u + q)u_x,$$

$$\frac{u}{2} = -\frac{m_1}{2} q_{xx} + q.$$

This is a two-parameter family of the PDEs with a constraint. Differentiating the constraint we get an expression $m_1 q_{xxx} = 2q_x - u_x$. Substituting it into the first equation and renaming the coefficient we obtain Dullin-Gottwald-Holm equation [**11**]. The case $m_0 = 0$ gives the Camassa-Holm equation.

Finally, the Type IV equation corresponds to $m_1 = 0$, leading to the celebrated KdV equation

$$u_t = \frac{m_0}{2} u_{xxx} + \frac{3}{2} u u_x.$$

EXAMPLE 3.2 (Coupled KdV and Harry Dym, Kaup-Boussinesq and Ito systems). Take an arbitrary n and consider the (differentially non-degenerate) Nijenhuis operator $L = L_{\mathsf{comp}}$ given by the first formula of (2.1). In the notation of

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[5]The parameter λ_0 is not essential unless we consider the limit as $\lambda_0 \to \infty$.

Section 2, we take $N = 0$, $n = \ell_0 = d$ and $m(\lambda) = m_n\lambda^n + m_{n-1}\lambda^{n-1} + \cdots + m_0$. Then

$$\sigma(\lambda) = \det(L - \lambda\,\mathrm{Id}) = (-1)^n(\lambda^n - u^1\lambda^{n-1} - \cdots - u^n),$$

$$\zeta = (-1)^{n+1}\left((m_{n-1} + m_n u^1)\frac{\partial}{\partial u^1} + \cdots + (m_0 + m_n u^n)\frac{\partial}{\partial u^n}\right).$$

It is easy to check that $\mathcal{L}_\zeta\sigma(\lambda) = m(\lambda) - (-1)^n m_n\sigma(\lambda)$. For every root λ_i of $m(\lambda)$, the Type II equation is

$$u_t = \left(\frac{1}{\sqrt{\det(L - \lambda_i\,\mathrm{Id})}}\right)_{xxx}(L - \lambda_i\,\mathrm{Id})^{-1}\zeta + \frac{1}{\sqrt{\det(L - \lambda_i\,\mathrm{Id})}}(L - \lambda_i\,\mathrm{Id})^{-1}u_x.$$

This is a $(n+1)$-parameter family of integrable equations (with m_i's as parameters involved in the formula for ζ above). For the rather special case $m_n = 1$ and $m_0 = \cdots = m_{n-1} = 0$, we get $\lambda_i = 0$ and taking $L = L_{\mathsf{comp}}$ as in (2.1) we come to coupled Harry Dym equations

$$u_t = \left(\frac{1}{\sqrt{u^n}}\right)_{xxx}e_1 + \left(\frac{1}{\sqrt{u^n}}\right)L_{\mathsf{comp}}^{-1}u_x, \quad e_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \quad u = \begin{pmatrix}u^1\\u^2\\\vdots\\u^n\end{pmatrix},$$

introduced in [2] by M. Antonowicz and A. Fordy.

If $m_n = 0$, then $m(\lambda)$ has a root at infinity and Type IV equation is

$$u_t = \tfrac{1}{2}\left(\mathrm{tr}\,L\right)_{xxx}\zeta + \left(L + \tfrac{1}{2}\,\mathrm{tr}\,L\,\mathrm{Id}\right)u_x.$$

This is a family of integrable multi-component PDE systems with n parameters $m_0, \ldots, m_{n-1}$. More specifically, for $L = L_{\mathsf{comp}}$ given by (2.1) we get

$$u_t = \tfrac{1}{2}u^1_{xxx}\zeta + \left(L_{\mathsf{comp}} + \tfrac{1}{2}u^1\,\mathrm{Id}\right)u_x, \quad \zeta = \begin{pmatrix}m_{n-1}\\\vdots\\m_1\\m_0\end{pmatrix} = \sum_{i=1}^n m_{n-i}e_i, \ m_i \in \mathbb{R}.$$

For $\zeta = e_i$, $i = 1, \ldots, n$, we get n different systems known as coupled KdV systems and introduced by Antonowicz and Fordy in [1].

The latter have two important examples for $n = 2$. For $m_2 = m_1 = 0, m_0 \neq 0$ after coordinate change $u^1 \to -u^1, u^2 \to -u^2$ we get the Kaup-Boussinesq system [22, eqn. (4)]:

$$u^1_t = u^2_x - \frac{3}{2}u^1u^1_x,$$

$$u^2_t = \frac{m_0}{2}u^1_{xxx} - u^2u^1_x - \frac{1}{2}u^1u^2_x.$$

For $m_0 = m_2 = 0, m_1 \neq 0$ after coordinate change $u^1 \to -u^1, u^2 \to -u^2$ the same formula yields Ito system [22, eqn. (25)]:

$$u^1_t = \frac{m_1}{2}u^1_{xxx} - \frac{3}{2}u^1u^1_x + u^2_x,$$

$$u^2_t = -u^2u^1_x - \frac{1}{2}u^1u^2_x.$$

EXAMPLE 3.3 (Marvan-Pavlov system). Now consider a pair of differentially non-degenerate Nijenhuis operators L_0, L_1 in dimensions n_0 and n_1. Assume $n_0 - n_1 = d \geq 0$ and consider coordinates $u^1, \ldots, u^{n_0}$ and $v^1, \ldots, v^{n_1}$ in which L_0 and L_1 are given by the first formula of (2.1). In the notations of Section 2, take $N = 1$ and $\ell_1 = 1$. We get $m(\lambda) = m_d \lambda^d + m_{d-1} \lambda^{d-1} + \cdots + m_0$. In these coordinates

$$\sigma(\lambda) = (-1)^d \frac{\lambda^{n_0} - u^1 \lambda^{n_0-1} - \cdots - u^{n_0}}{\lambda^{n_1} - v^1 \lambda^{n_1-1} - \cdots - v^{n_1}}$$

and

$$\zeta = (-1)^d m_d \left(\sum_{r=1}^{n_1} v^r \frac{\partial}{\partial u^r} - \sum_{j=1}^{n_0} u^j \frac{\partial}{\partial u^j} \right) + \sum_{s=1}^{d} (-1)^d m_{d-s} \left(-\frac{\partial}{\partial u^s} + \sum_{j=1}^{n_1} v^j \frac{\partial}{\partial u^{j+s}} \right) \quad .$$

By direct computation we have $\zeta(\sigma(\lambda)) = m(\lambda) - (-1)^d \sigma(\lambda)$. For every root λ_i of $m(\lambda)$, the Type II equation is

$$u_t = \left(\sqrt{(-1)^d \frac{\lambda_i^{n_1} - \sum_{r=1}^{n_1} v^r \lambda_i^{n_1-r}}{\lambda_i^{n_0} - \sum_{j=1}^{n_0} u^j \lambda_i^{n_0-j}}} \right)_{xxx} (L - \lambda_i \operatorname{Id})^{-1} \zeta +$$

$$+ \sqrt{(-1)^d \frac{\lambda_i^{n_1} - \sum_{r=1}^{n_1} v^r \lambda_i^{n_1-r}}{\lambda_i^{n_0} - \sum_{j=1}^{n_0} u^j \lambda_i^{n_0-j}}} (L - \lambda_i \operatorname{Id})^{-1} u_x.$$

This is a d-parameter family of integrable equations (with the coefficients of $m(\lambda)$ as parameters involved into the formula for ζ). If $m_d = 0$ and the infinity is a root of $m(\lambda)$, we get the Type IV equation

$$u_t = \frac{1}{2} \left(\operatorname{tr} L_1 - \operatorname{tr} L_2 \right)_{xxx} \zeta + \left(L + \frac{1}{2} \left(\operatorname{tr} L_1 - \operatorname{tr} L_2 \right) \operatorname{Id} \right) u_x.$$

This is a d-parameter family of integrable equations. Taking $m_0 \neq 0$ and all $m_i = 0, i \geq 1$ yields example by M. Marvan and M. Pavlov [20, 22].

EXAMPLE 3.4 (Two-component Camassa-Holm and Dullin-Gottwald-Holm systems). Fix $n = 2$ and consider the Nijenhuis operator of the from

$$L = \begin{pmatrix} 2u^1 & u^2 \\ u^2 & 0 \end{pmatrix}.$$

This operator is related to left-symmetric algebras and plays an important role in the linearization problem (see [15] for details). We take

$$\sigma(\lambda) = \det(L - \lambda \operatorname{Id}) = \lambda^2 - 2u^1 \lambda - (u^2)^2, \quad m(\lambda) = m_2 \lambda^2 + m_1 \lambda + m_0.$$

The vector field ζ is

$$\zeta = -\left(\frac{m_1}{2} + m_2 u^1 \right) \frac{\partial}{\partial u^1} - \left(\frac{m_0}{2u^2} + \frac{m_2}{2} u^2 \right) \frac{\partial}{\partial u^2}.$$

We get $\mathcal{L}_\zeta(\sigma(\lambda)) = m(\lambda) - m_2 \sigma(\lambda)$. For $m_2 \neq 0$, the Type III system in this setting is

$$u_t^1 = q_{xxx} \left(-\frac{m_1}{2} - m_0 u^1 \right) + 2u^1 u_x^1 + u^2 u_x^2 + q u_x^1,$$

$$u_t^2 = q_{xxx} \left(-\frac{m_2}{2u^2} - \frac{m_0}{2} u^2 \right) + u^2 u_x^1 + q u_x^2,$$

$$u^1 = \frac{m_2}{2} q_{xx} + q.$$

Differentiating the last equation and rearranging terms, we get $-\frac{m_2}{2}q_{xxx} = q_x - u_x^1$. Substituting it into the first, we get the equivalent form of the previous PDE with constraint

$$u_t^1 = \frac{m_1}{2}q_{xxx} + 2u^1 q_x + qu_x^1 + u^2 u_x^2,$$

$$u_t^2 = -\frac{m_0}{4}\frac{q_{xxx}}{u^2} + (qu^2)_x,$$

$$u^1 = q + \frac{m_2}{2}q_{xx}.$$

This is a 3-parameter family of integrable systems. For $m_2 = 0$ we obtain the two-component Dullin-Gottwald-Holm equation [**24**, (3)]. If, in addition, $m_1 = 0$, then we get two-component Camassa-Holm equation [**10**, (3) and (4)].

EXAMPLE 3.5. We actually can generalise the equations of type I and III to the case of k constraints, $k > 1$. Consider the expansion (2.22) up to ε^2 :

$$\bar{w}(\varepsilon) = 1 + \varepsilon w_{0,\epsilon} + \varepsilon^2 w_{1,\infty} + \ldots.$$

If $\bar{m}(0) \neq 0$, then $w_{0,\infty}$ and $w_{1,\infty}$ are formal differential series. Now substitute the decompositions for $\bar{w}(\varepsilon), \bar{\sigma}(\varepsilon)$ and $\bar{m}(\varepsilon)$ into the Gelfand-Dikii identity (3.3). Renaming $q^1 = w_{0,\infty}$, $q^2 = w_{1,\infty}$ we get

$$0 = \bar{m}(\varepsilon)\left(\bar{w}_{xx}(\varepsilon)\bar{w}(\varepsilon) - \frac{1}{2}(\bar{w}_x(\varepsilon))^2\right) + \bar{\sigma}(\varepsilon)(\bar{w}(\varepsilon))^2 - 1 = \varepsilon\left(\bar{\sigma}_1 + 2q^1 + \bar{m}_0 q_{xx}^1\right) +$$

$$+ \varepsilon^2\left(\bar{\sigma}_2 + 2q^2 + \bar{m}_0 q_{xx}^2 + \bar{m}_1 q_{xx}^1 + \bar{m}_0 q^1 q_{xx}^1 + 2\bar{\sigma}_1 q^1 + (q_1)^2 - \frac{1}{2}\bar{m}_0(q_x^1)^2\right) + \ldots.$$

This yields differential relations for q^1, q^2. Differential operator $\mathrm{Id} + D^2$ is formally invertible, so these constraints imply that q^1 is a differential series in $\bar{\sigma}_1$ and its derivatives and q^2 is a differential series in $\bar{\sigma}_1, \bar{\sigma}_2$ and their derivatives.

Now consider the expansion (2.23) up to ε^2:

$$\bar{\xi}(\varepsilon) = \xi_{-1,\infty} + \varepsilon\xi_{0,\infty} + \varepsilon^2\xi_{1,\infty} + \cdots =$$

$$= u_x + \varepsilon\left(q_{xxx}^1 \zeta + (L + q^1 \mathrm{Id})u_x\right)$$

$$+ \varepsilon^2\left(q_{xxx}^2 \zeta + q_{xxx}^1 L\zeta + (L^2 + q^1 L + q^2 \mathrm{Id})u_x\right) + \ldots.$$

Taking $\xi_{1,\infty}$ we get a PDE with two differential constraints

$$u_t = q_{xxx}^2 \zeta + q_{xxx}^1 L\zeta + (L^2 + q^1 L + q^2 \mathrm{Id})u_x,$$

$$0 = \bar{\sigma}_1 + 2q^1 + \bar{m}_0 q_{xx}^1,$$

$$0 = \bar{\sigma}_2 + 2q^2 + \bar{m}_0 q_{xx}^2 + \bar{m}_1 q_{xx}^1 + \bar{m}_0 q^1 q_{xx}^1 + 2\bar{\sigma}_1 q^1 + (q_1)^2 - \frac{1}{2}\bar{m}_0(q_x^1)^2.$$

For L differentially non-degenerate, $N = 0$ and $m(t) = t^n$ this yields the general form of Camassa-Holm equation $\mathrm{CH}(n,2)$ from [**14**]. Taking the expansion up to a higher order, one obtains a greater number of differential constraints.

EXAMPLE 3.6. In the previous examples, we have shown that many notable integrable systems are special cases of the systems from Section 2. We now describe one of the simplest *new* examples. By construction, integrable systems we deal with are written in invariant form that is independent on the choice of a local coordinate chart. In particular, in order to make our system more symmetric, we may choose local coordinates related to the roots of the polynomial $m(\lambda)$.

The next 3-component example is build starting with $\sigma(\lambda) = \det(L - \lambda\,\mathrm{Id})$ and $m(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)$ and choosing local coordinates (u^1, u^2, u^3) to be

$$u^i = \det(\lambda_i\,\mathrm{Id} - L)\prod_{s \neq i}\frac{1}{\lambda_i - \lambda_s}.$$

In particular, u^i is proportional to $\sigma(\lambda_i)$ with some constant factor which is not essential. Moreover, $\zeta = -(u^1, u^2, u^3)^\top$ and $(L - \lambda_i\,\mathrm{Id})^{-1}\zeta = e_i$. Now if we take an arbitrary linear combination of the (commuting) evolutionary PDEs (2.7) of Type II, we get the following integrable system:

$$(3.14) \qquad \begin{pmatrix} u_t^1 \\ u_t^2 \\ u_t^3 \end{pmatrix} = \begin{pmatrix} c_1\left(1/\sqrt{u^1}\right)_{xxx} \\ c_2\left(1/\sqrt{u^2}\right)_{xxx} \\ c_3\left(1/\sqrt{u^3}\right)_{xxx} \end{pmatrix} + A(u)\begin{pmatrix} u_x^1 \\ u_x^2 \\ u_x^3 \end{pmatrix}.$$

where c_1, c_2, c_3 are arbitrary constants and $A(u)$ is the 3×3 matrix with the components

$$A_i^j = \frac{u^j}{\lambda_i - \lambda_j}\left(\frac{c_i}{(u^i)^{3/2}} - \frac{c_j}{(u^j)^{3/2}}\right), \quad j \neq i, \quad \text{and} \quad A_i^i = -\frac{c_i}{(u^i)^{3/2}} - \sum_{j \neq i} A_i^j.$$

One can also write it as follows:

$$u_t^j = c_j\left(\left(\frac{1}{\sqrt{u^j}}\right)_{xxx} - \frac{u_x^j}{(u^j)^{3/2}}\right) + \sum_{i \neq j}\frac{u^j u_x^i - u^i u_x^j}{\lambda_i - \lambda_j}\left(\frac{c_i}{u_i^{3/2}} - \frac{c_j}{u_j^{3/2}}\right).$$

Here the first term represents the system of three *uncoupled* Harry Dym type equations (see Example 3.1), but the second term mixes all the variables.

The recursion formula for the conservation laws gives the following explicit formula for the first six of them. The first two corresponding to the root λ_1 are

$$(3.15) \qquad \sqrt{u^1} \quad \text{and} \quad \frac{\left(u^1 u_{xx}^1 - \frac{5}{4}\left(u_x^1\right)^2\right) - 2\left(u^1\right)^2\left(\frac{u^1 + u^2}{\lambda_1 - \lambda_2} + \frac{u^1 + u^3}{\lambda_1 - \lambda_3} - 1\right)}{(u^1)^{3/2}}.$$

The other four correspond to the roots λ_2, λ_3 and can be obtained from the above formulas by cyclic permutation of indices 1,2,3.

One can also find, using the procedure described in Section 2, the formulas for commuting flows. Actually, the commuting flows of the lowest order are (3.14) with arbitrarily chosen c_1, c_2, c_3. Notice that this example can be naturally generalised to the case of an arbitrary number of components.

4. Proofs of Theorems 2 and 3

We start with the following Lemma.

LEMMA 4.1. Under the assumptions of Theorem 2, we have

$$(L_\lambda^*)^{-1}\mathrm{d}\,\sigma(\mu) = \frac{1}{\mu - \lambda}\left(\mathrm{d}\,\sigma(\mu) - \frac{\sigma(\mu)}{\sigma(\lambda)}\mathrm{d}\,\sigma(\lambda)\right).$$

PROOF. Condition (3.1) reads

(4.1) $L^* \mathrm{d}\, \sigma(\mu) = \sigma(\mu)\mathrm{d}\, f + \mu \mathrm{d}\, \sigma(\mu) = \sigma(\mu)\mathrm{d}\, f + (\mu - \lambda)\mathrm{d}\, \sigma(\mu) + \lambda \mathrm{d}\, \sigma(\mu).$

Recall that by construction $(L_\lambda^*)^{-1}\mathrm{d}\, f = \frac{1}{\sigma(\lambda)}\mathrm{d}\, \sigma(\lambda)$. Rearranging the terms and multiplying both sides of (4.1) by $(L_\lambda^*)^{-1}$, we get the statement of Lemma. $\qquad\square$

Now let us recall some basic formulas and introduce some notations. We denote the derivative coordinates of order j by $u_{x^j}^\alpha$ and set $u_{x^0}^\alpha = u^\alpha$. Consider a formal evolutionary vector field ξ with components ξ^i. The (Lie) derivative of a formal differential series w along ξ is defined by

$$\mathcal{L}_\xi \mathrm{w} = \sum_{j=0}^{\infty} \frac{\partial \mathrm{w}}{\partial u_{x^j}^\alpha} D^j(\xi^\alpha)$$

with summation over α assumed, $\alpha = 1, \ldots, n$.

Let $\xi = \xi(\lambda)$ be the formal vector field defined by (3.9) and associated with the PDEs from Theorem 2. We have:

$$\mathcal{L}_{\xi(\lambda)}\sigma(\mu) = \frac{\partial \sigma(\mu)}{\partial u^\alpha}\xi^\alpha(\lambda) = \frac{\partial \sigma(\mu)}{\partial u^\alpha}\Big(\mathrm{w}_{xxx}(\lambda)\big(L - \lambda \,\mathrm{Id}\,\big)^{-1}\zeta$$

$$+ \mathrm{w}(\lambda)\big(L - \lambda\,\mathrm{Id}\,\big)^{-1}u_x\Big)^\alpha =$$

$$= \mathrm{w}_{xxx}(\lambda)\frac{\partial \sigma^\mu}{\partial u^\alpha}\Big(L_\lambda^{-1}\Big)_q^\alpha \zeta^q + \mathrm{w}(\lambda)\frac{\partial \sigma^\mu}{\partial u^\alpha}\Big(L_\lambda^{-1}\Big)_q^\alpha u_x^q$$

$$= \frac{1}{\mu - \lambda}\mathrm{w}_{xxx}(\lambda)\Big(m(\mu) - \frac{\sigma(\mu)}{\sigma(\lambda)}m(\lambda)\Big)+$$

$$+ \frac{1}{\mu - \lambda}\mathrm{w}(\lambda)\Big(\sigma_x(\mu) - \frac{\sigma(\mu)}{\sigma(\lambda)}\sigma_x(\lambda)\Big).$$

The last step follows from Lemma 4.1. Further rearranging terms and using (2.14), we get
(4.2)
$$\mathcal{L}_{\xi(\lambda)}\sigma(\mu) =$$

$$= \frac{1}{\mu - \lambda}\Big(m(\mu)\mathrm{w}_{xxx}(\lambda) + \mathrm{w}(\lambda)\sigma_x(\mu) - \frac{\sigma(\mu)}{\sigma(\lambda)}\Big(m(\lambda)\mathrm{w}_{xxx}(\lambda) + \mathrm{w}(\lambda)\sigma_x(\lambda)\Big)\Big) =$$

$$= \frac{1}{\mu - \lambda}\Big(m(\mu)\mathrm{w}_{xxx}(\lambda) + \mathrm{w}(\lambda)\sigma_x(\mu) - \frac{\sigma(\mu)}{\sigma(\lambda)}\Big(-2\sigma(\lambda)\mathrm{w}_x(\lambda)\Big)\Big) =$$

$$= \frac{1}{\mu - \lambda}\Big(m(\mu)\mathrm{w}_{xxx}(\lambda) + 2\sigma(\mu)\mathrm{w}_x(\lambda) + \sigma_x(\mu)\mathrm{w}(\lambda)\Big).$$

Now let us proceed with the proof. Consider a pair of formal differential series $\mathrm{w}_1, \mathrm{w}_2$. We use notation $\mathrm{w}_1 \sim \mathrm{w}_2$, if there exists a formal differential series u, such that $\mathrm{w}_1 - \mathrm{w}_2 = D\mathrm{u}$. In particular, the Leibniz rule for D implies, that $\mathrm{w}_1 D(\mathrm{w}_2) \sim -D(\mathrm{w}_1)\mathrm{w}_2$. More generally, it yields the formula

$$\mathrm{w}_1 D^j(\mathrm{w}_2) \sim (-1)^j D^j(\mathrm{w}_1)\mathrm{w}_2.$$

Using the chain rule $\frac{\delta v(\mu)}{\delta u^\alpha} = w(\mu)\frac{\partial \sigma(\mu)}{\partial u^\alpha}$, we get the following sequence of relations

$$\mathcal{L}_{\xi(\lambda)}v(\mu) = \sum_{j=0}^{\infty} \frac{\partial v(\mu)}{\partial u_{x^j}^\alpha} D^j(\xi^\alpha(\lambda)) \sim \sum_{j=0}^{\infty}(-1)^j D^j\left(\frac{\partial v(\mu)}{\partial u_{x^j}^\alpha}\right)\xi^\alpha(\lambda)$$

$$= \frac{\delta v(\mu)}{\delta u^\alpha}\xi^\alpha(\lambda) =$$

$$= w(\mu)\frac{\partial \sigma(\mu)}{\partial u^\alpha}\xi^\alpha(\lambda) = w(\mu)\mathcal{L}_{\xi(\lambda)}\sigma(\mu)$$

$$= \frac{1}{\mu - \lambda}m(\mu)w(\mu)w_{xxx}(\lambda) + \frac{2}{\mu - \lambda}w(\mu)w_x(\lambda)\sigma(\mu)+$$

$$+ \frac{1}{\mu - \lambda}w(\mu)w(\lambda)\sigma_x(\mu) \sim -\frac{1}{\mu - \lambda}m(\mu)w_{xxx}(\mu)w(\lambda)$$

$$- \frac{2}{\mu - \lambda}\big(w(\mu)\sigma(\mu)\big)_x w(\lambda)+$$

$$+ \frac{1}{\mu - \lambda}w(\mu)\sigma_x(\mu)w(\lambda) = -\frac{1}{\mu - \lambda}\bigg(m(\mu)w_{xxx}(\lambda)$$

$$+ 2\sigma(\mu)w_x(\lambda) + \sigma_x(\mu)w(\lambda)\bigg)w(\lambda) = 0$$

Here we used formula (4.2) and, again, identity (2.14). Thus, we get $\mathcal{L}_{\xi(\lambda)}v(\mu) \sim 0$, meaning that $v(\mu)$ is a formal conservation law for the flow $\xi(\lambda)$. For $\lambda = \infty$ the proof is essentially the same.

Now let us proceed to the commuting flows. We will need the following Lemma.

LEMMA 4.2. Assume that $\sigma(\mu)$ and $w(\mu)$ are related by Gelfand-Dikii identity (3.3) and $\mathcal{L}_{\xi(\lambda)}\sigma(\mu)$ is given by (4.2). Then

(4.3) $$\mathcal{L}_{\xi(\lambda)}w(\mu) = \frac{1}{\mu - \lambda}\Big(w_x(\mu)w(\lambda) - w(\mu)w_x(\lambda)\Big).$$

PROOF. We start with applying $\xi(\lambda)$ to Gelfand-Dikii identity (3.3) and multiplying the result by $w(\mu)$ (we also take into account the fact that $\mathcal{L}_\xi$ commute with D):
(4.4)
$$0 = w(\mu)\mathcal{L}_{\xi(\lambda)}\left(m(\mu)\left(w_{xx}(\mu)w(\mu) - \frac{1}{2}(w_x(\mu))^2\right) + \sigma(\mu)w^2(\mu)\right) =$$

$$= m(\mu)\Big(w^2(\mu)\mathcal{L}_{\xi(\lambda)}w_{xx}(\mu) + w_{xx}(\mu)w(\mu)\mathcal{L}_{\xi(\lambda)}w(\mu) - w_x(\mu)w(\mu)\mathcal{L}_{\xi(\lambda)}w_x(\mu)\Big)+$$

$$+ 2\sigma(\mu)w^2(\mu)\mathcal{L}_{\xi(\lambda)}w(\mu) + w^3(\mu)\mathcal{L}_{\xi(\lambda)}\sigma(\mu) =$$

$$= \Big(m(\mu)\big(w^2(\mu)D^2 - w_x(\mu)w(\mu)D + w_{xx}(\mu)w(\mu)\big) + 2\sigma(\mu)w^2(\mu)\Big)\mathcal{L}_{\xi(\lambda)}w(\mu)+$$

$$+ w^3(\mu)\mathcal{L}_{\xi(\lambda)}\sigma(\mu) = \mathcal{R}\big(\mathcal{L}_{\xi(\lambda)}w(\mu)\big) + w^3(\mu)\mathcal{L}_{\xi(\lambda)}\sigma(\mu)$$

with

$$\mathcal{R} = m(\mu)\Big(w^2(\mu)D^2 - w_x(\mu)w(\mu)D + w_{xx}(\mu)w(\mu)\,\mathrm{Id}\Big) + 2\sigma(\mu)w^2(\mu)\,\mathrm{Id} =$$

$$= \big(2 - m(\mu)w(\mu)w_{xx}(\mu) + m(\mu)w_x^2(\mu)\big)\,\mathrm{Id} - m(\mu)w(\mu)w_x(\mu)D + m(\mu)w^2(\mu)D^2,$$

where in the latter relation we substitute

$$\sigma(\mu)\mathrm{w}^2(\mu) = 1 - m(\mu)\left(\mathrm{w}_{xx}(\mu)\mathrm{w}(\mu) - \frac{1}{2}(\mathrm{w}_x(\mu))^2\right)$$

from (3.3). Thus, we have the identity

(4.5) $$\mathcal{R}\big(\mathcal{L}_{\xi(\lambda)}\mathrm{w}(\mu)\big) = -\mathrm{w}^3(\mu)\mathcal{L}_{\xi(\lambda)}\sigma(\mu).$$

Note that $\mathcal{R}$ is a (formally) invertible differential operator. Therefore, it suffices to verify that $\mathcal{L}_{\xi(\lambda)}\mathrm{w}(\mu)$ defined by (4.3) satisfies (4.5) or, equivalently,

$$\mathcal{R}\big(\mathrm{w}_x(\mu)\mathrm{w}(\lambda) - \mathrm{w}(\mu)\mathrm{w}_x(\lambda)\big) + (\mu - \lambda)\mathrm{w}^3(\mu)\mathcal{L}_{\xi(\lambda)}\sigma(\mu) = 0.$$

Computing the l.h.s. of this relation gives:

$$m(\mu)\mathrm{w}^2(\mu)\Big(\mathrm{w}_x(\mu)\mathrm{w}(\lambda) - \mathrm{w}(\mu)\mathrm{w}_x(\lambda)\Big)_{xx}$$
$$+ m(\mu)\mathrm{w}_{xx}(\mu)\mathrm{w}(\mu)\Big(\mathrm{w}_x(\mu)\mathrm{w}(\lambda) - \mathrm{w}(\mu)\mathrm{w}_x(\lambda)\Big) -$$
$$- m(\mu)\mathrm{w}_x(\mu)\mathrm{w}(\mu)\Big(\mathrm{w}_x(\mu)\mathrm{w}(\lambda) - \mathrm{w}(\mu)\mathrm{w}_x(\lambda)\Big)_x$$
$$+ 2\sigma(\mu)\mathrm{w}^2(\mu)\Big(\mathrm{w}_x(\mu)\mathrm{w}(\lambda) - \mathrm{w}(\mu)\mathrm{w}_x(\lambda)\Big) +$$
$$+ (\mu - \lambda)\mathrm{w}^3(\mu)\mathcal{L}_{\xi(\lambda)}\sigma(\mu) =$$
$$= m(\mu)\mathrm{w}^2(\mu)\Big(\mathrm{w}_{xxx}(\mu)\mathrm{w}(\lambda) - \mathrm{w}(\mu)\mathrm{w}_{xxx}(\lambda)\Big)$$
$$+ 2\sigma(\mu)\mathrm{w}^2(\mu)\Big(\mathrm{w}_x(\mu)\mathrm{w}(\lambda) - \mathrm{w}(\mu)\mathrm{w}_x(\lambda)\Big)$$
$$+ (\mu - \lambda)\mathrm{w}^3(\mu)\mathcal{L}_{\xi(\lambda)}\sigma(\mu).$$

Adding and subtracting $\sigma_x(\mu)\mathrm{w}(\lambda)\mathrm{w}^3(\mu)$ we arrive to the identity

$$\Big((\mu - \lambda)\mathcal{L}_{\xi(\lambda)}\sigma(\mu) - m(\mu)\mathrm{w}_{xxx}(\lambda) - 2\sigma(\mu)\mathrm{w}_x(\lambda) - \sigma_x(\mu)\mathrm{w}(\lambda)\Big)\mathrm{w}^3(\mu) +$$
$$+ \Big(m(\mu)\mathrm{w}_{xxx}(\mu) + 2\sigma(\mu)\mathrm{w}_x(\mu) + \sigma_x(\mu)\mathrm{w}(\mu)\Big)\mathrm{w}^2(\mu)\mathrm{w}(\lambda) = 0,$$

where the first term vanishes due to (4.2) and the second due to (2.14), completing the proof. □

As we deal with evolutionary vector fields, it is enough to check that $\xi(\lambda)$ and $\xi(\nu)$ commute, acting on coordinate functions. Fix three pairwise distinct λ, μ, ν.

From Lemma 4.2, we get

$$\mathcal{L}_{\xi(\nu)}\mathcal{L}_{\xi(\lambda)}\sigma(\mu) = \frac{1}{\mu - \lambda}\mathcal{L}_{\xi(\nu)}\Big(m(\mu)\mathrm{w}_{xxx}(\lambda)$$

$$+ 2\sigma(\mu)\mathrm{w}_x(\lambda) + \sigma_x(\mu)\mathrm{w}(\lambda)\Big) =$$

$$= \frac{m(\mu)}{(\mu - \lambda)(\lambda - \nu)}\Big(\mathrm{w}_{xxxx}(\mu)\mathrm{w}(\lambda) + 2\mathrm{w}_{xxx}(\mu)\mathrm{w}_x(\lambda)$$

$$- \mathrm{w}(\mu)\mathrm{w}_{xxxx}(\lambda) - 2\mathrm{w}_x(\mu)\mathrm{w}_{xxx}(\lambda)\Big) +$$

$$+ \frac{2}{(\mu - \lambda)(\mu - \nu)}\Big(m(\mu)\mathrm{w}_{xxx}(\nu) + 2\sigma(\mu)\mathrm{w}_x(\nu) + \sigma_x(\mu)\mathrm{w}(\nu)\Big)\mathrm{w}_x(\lambda) +$$

$$+ \frac{1}{(\mu - \lambda)(\lambda - \nu)}\Big(2\sigma(\mu)\Big(\mathrm{w}_{xx}(\mu)\mathrm{w}(\nu) - \mathrm{w}(\mu)\mathrm{w}_{xx}(\nu)\Big)$$

$$+ \sigma_x(\mu)\Big(\mathrm{w}_x(\mu)\mathrm{w}(\nu) - \mathrm{w}(\mu)\mathrm{w}_x(\nu)\Big)\Big) +$$

$$+ \frac{1}{(\mu - \lambda)(\mu - \nu)}\Big(m(\mu)\mathrm{w}_{xxxx}(\nu) + 2\sigma_x(\mu)\mathrm{w}_x(\nu)$$

$$+ 2\sigma(\mu)\mathrm{w}_{xx}(\nu) + \sigma_{xx}(\mu)\mathrm{w}(\nu) + \sigma_x(\mu)\mathrm{w}_x(\nu)\Big)\mathrm{w}(\lambda)$$

The identity

$$\frac{1}{(\mu - \lambda)(\mu - \nu)} - \frac{1}{(\mu - \lambda)(\lambda - \nu)} + \frac{1}{(\mu - \nu)(\lambda - \nu)} = 0.$$

implies that the r.h.s. of formula for $\mathcal{L}_{\xi(\nu)}\mathcal{L}_{\xi(\lambda)}\sigma(\mu)$ is symmetric in λ, ν. Thus, $\mathcal{L}_{\xi(\nu)}$ and $\mathcal{L}_{\xi(\lambda)}$ commute on $\sigma(\mu)$.

Now recall that $\sigma(\mu)$ is constructed from a generic conservation law f. This implies that in the expansion

$$\sigma(\mu) = \sigma_0 + \mu\sigma_1 + \dots$$

the differentials of $\sigma_0, \dots, \sigma_{n-1}$ are linearly independent almost everywhere. Thus, one can take them as coordinates $u^i = \sigma_{i-1}$ and in these coordinates $(\mathcal{L}_{\xi(\nu)}\mathcal{L}_{\xi(\lambda)} - \mathcal{L}_{\xi(\lambda)}\mathcal{L}_{\xi(\nu)})u^i = 0$, as required. This completes the proof of Theorem 2.

To verify the statement of Theorem 3, we first need to show that the coefficients ξ_{s,λ_i}, v_{s,λ_i} and w_{s,λ_i} of the ε-expansions (3.10) and (3.11) are well defined and are differential polynomials in $u^1, \dots, u^n$. Indeed, by definition,

$$(4.6) \qquad \mathrm{v}(\lambda) = \mathrm{v}(\sigma(\lambda), m(\lambda)) = \sqrt{2}\sum_{s=0}^{\infty}\big(m(\lambda)\big)^s v_s(\lambda),$$

where $v_s(\lambda)$ is a differential polynomial obtained from the homogeneous differential polynomial $u_{2s+1}(\sigma, \sigma_x, \sigma_{xx}, \dots)$ of degree $2s$ by substitution $\sigma = \sigma(\lambda; u)$.

We are interested in the expansion of $\mathrm{v}(\lambda_i + \varepsilon) = \sum \mathrm{v}_{s,\lambda_i}\varepsilon^s$ under the assumption that $m(\lambda_i) = 0$. Since $m(\lambda_i + \varepsilon) = a_1\varepsilon + a_2\varepsilon^2 + \dots$ and, therefore, $\big(m(\lambda_i + \varepsilon)\big)^s = a_1^s\varepsilon^s + \dots$, we see from expansion (4.6) that v_{s,λ_i} is defined from the first $s + 1$ coefficients $v_0, \dots, v_s$. Hence, v_{s,λ_i} is a non-homogeneous differential polynomial of degree at most $2s$.

The proof for w_{s,λ_i} is literally the same. The conclusion for w_{s,λ_i} immediately follows from the explicit formula of $\xi(\lambda)$ in terms of $\mathrm{w}(\lambda)$, see (3.4) and (3.9).

The explicit form (3.12) of the flows $u_{t_{\lambda_i}} = \xi(\lambda_i)$ for a root $\lambda_i \in \mathbb{C}$ of $m(\lambda)$ is straightforward. Indeed, setting $m(\lambda_i) = 0$ in (3.4) gives $\sigma(\lambda_i)(\mathrm{w}(\lambda_i))^2 - 1 = 0$, or equivalently, $\mathrm{w}(\lambda_i) = \frac{1}{\sqrt{\sigma(\lambda_i)}}$. Substituting this expression into the first equation of (3.4) gives (3.12), as required. Similarly, for $\lambda_i = \infty$ we set $\bar{m}(0) = 0$ in (3.8) to get $q = \frac{1}{2}f$, which after substitution into the first equation of (3.8) immediately gives (3.13), completing the proof of Theorem 3.

5. Proof of Theorem 4

Since L is differentially non-degenerate, this operator is diagonalisable almost everywhere. At "non-diagonalisable" points, the conclusion of Theorem 4 can be derived by continuity arguments. So w.l.o.g. we assume that $L = \mathrm{diag}(x^1, \ldots, x^n)$. Recall that f from equation (3.1) is a conservation law for L. Then f is a sum of n functions such that the ith function depends on x^i only. Therefore, for every i the function $f_i := \frac{\partial f}{\partial x^i}$ depends on x^i only.

Next, we consider relation (3.1). In coordinates, it reads:

$$\frac{\partial \ln(\sigma)}{\partial x^1} = \frac{f_1(x^1)}{x^1 - \lambda}$$

$$\vdots$$

$$\frac{\partial \ln(\sigma)}{\partial x^n} = \frac{f_n(x^n)}{x^n - \lambda}.$$

Hence, the system (3.1) of n PDEs is actually a system of n ODEs in different variables. Its solution must be of the form $\sigma = c(\lambda) \cdot \sigma_1 \ldots \sigma_n$ with

$$(5.1) \qquad \sigma_i(\lambda, x^i) = \exp\left(\int_{s_i}^{x^i} \frac{f_i(s)}{s - \lambda} \mathrm{d}\, s\right)$$

where $c(\lambda)$ is an arbitrary function and $(s_1, \ldots, s_n)$ is an arbitrary point; we assume that all $s_i \neq 0$.

Next, consider relation (3.2). For our $\sigma(\lambda) = c(\lambda) \cdot \sigma_1 \ldots \sigma_n$ it reads

$$(5.2) \qquad \left(C + \sum_{i=1}^{n} \zeta^i \frac{f_i}{x^i - \lambda}\right) \sigma_1 \ldots \sigma_n = \frac{m(\lambda)}{c(\lambda)} := \widehat{m}(\lambda).$$

In the left hand side of this relation, f_i and ζ_i are smooth functions in x which are independent on λ, whereas the r.h.s. is a function independent of x. The following statement shows that under these conditions, f_i's have to be constants and, moreover, very special.

LEMMA 5.1. The functions f_i are integer constants different from zero and no greater than 1. Moreover, for every i such that $f_i \neq 1$ we have $\zeta_i = 0$.

PROOF. Integration by parts gives

$$\sigma_i(\lambda, x^i) = \exp\left(\int_{s_i}^{x^i} \frac{f_i(s)}{s - \lambda} \mathrm{d}\, s\right)$$

$$= \exp\left(f_i(x^i) \ln(x^i - \lambda) - f_i(s_i) \ln(s_i - \lambda) - \int_{s_i}^{x^i} f_i'(s) \ln(s - \lambda) \mathrm{d}\, s\right)$$

implying $\sigma(x, \lambda) = (x^1 - \lambda)^{f_1(x^1)}(x^2 - \lambda)^{f_2(x^2)} \ldots (x^n - \lambda)^{f_n(x^n)} \tilde{\sigma}(\lambda, x)$ where the function $\tilde{\sigma}$ has neither zeros nor poles. The equation (5.2) reads then

(5.3)
$$\left(C + \sum_{s=1}^n \frac{\zeta^s f_s(x^s)}{x^s - \lambda} \right) (x^1 - \lambda)^{f_1(x^1)}(x^2 - \lambda)^{f_2(x^2)} \ldots (x^n - \lambda)^{f_n(x^n)} \tilde{\sigma}(\lambda, x) = \widehat{m}(\lambda).$$

Note that the function $\int_{s_i}^{x^i} f_i'(s) \ln(s - \lambda) \mathrm{d}\, s$ is locally bounded, so the function $\tilde{\sigma}(\lambda, x)$ is bounded for small $x^s - \lambda$ and is not zero.

Assume for a certain i that f_i is not integer at a point $(x^1, ..., x^n)$. Substituting $\lambda = x^i$ (for generic x^i) makes $\widehat{m}(\lambda) = 0$ or $\widehat{m}(\lambda) = \infty$ leading to a contradiction. Indeed, the first factor of (5.3) has integer order of zeros and poles, so to compensate it f_i must be integer for each x^i and therefore constant. Note that the case $\widehat{m}(\lambda) \equiv \infty$ is not allowed since the left hand side is finite almost everywhere.

Thus, all f_i are integer. Then the function $\tilde{\sigma}(\lambda, x)$ depends on λ only. Further we assume that it is equal to 1 since we can "hide" it in $\widehat{m}(\lambda)$ (we keep the same notation). Our equation (5.3) then becomes:

(5.4)
$$\left(C + \sum_{s=1}^n \frac{\zeta^s f_s}{x^s - \lambda} \right) (x^1 - \lambda)^{f_1}(x^2 - \lambda)^{f_2} \ldots (x^n - \lambda)^{f_n} = \widehat{m}(\lambda),$$

where f_i are integer constants. If $\zeta^i \neq 0$ then the first factor of (5.4) has a pole of order 1 implying $f_i = 1$. If $\zeta^i = 0$ and $f_i > 0$, then (5.4) has zero for $\lambda = x^i$ which again leads to contradiction. It remains to notice that $f_i \neq 0$ since $\mathrm{d}\, f, L^* \mathrm{d}\, f, \ldots, L^{n-1} \mathrm{d}\, f$ are linearly independent by our assumption. $\square$

W.l.o.g. we assume that $f_i = 1$ for $i = 1, ..., k$ and the other f_i are negative integers; we set $\ell_{k+1} = -f_{k+1}, \ldots, \ell_n = -f_n$. The equation (5.4) reads then

(5.5)
$$\left(C + \sum_{s=1}^k \frac{\zeta^s}{x^s - \lambda} \right) = \frac{(x^{k+1} - \lambda)^{\ell_{k+1}} \ldots (x^n - \lambda)^{\ell_n}}{(x^1 - \lambda)(x^2 - \lambda) \ldots (x^k - \lambda)} \widehat{m}(\lambda).$$

Notice that the expression in the l.h.s. can be written as a rational function in λ of the form $\frac{P(\lambda)}{\prod(x^s - \lambda)}$, where $P(\lambda) = C(-\lambda)^k + \ldots$ is a polynomial of degree at most k. Similarly, the r.h.s. is $\frac{F(\lambda)}{\prod(x^s - \lambda)}$ where $F(\lambda) = (x^{k+1} - \lambda)^{\ell_{k+1}} \ldots (x^n - \lambda)^{\ell_n} \widehat{m}(\lambda)$. Since $P(\lambda) = F(\lambda)$ we conclude that $\widehat{m}(\lambda) = m_d \lambda^d + \ldots$ is a polynomial of degree at most $d = k - \ell_{k+1} - \cdots - \ell_n$. In particular, $d \geq 0$ and $m_d = (-1)^d C$.

Finally, it remains to notice that ζ^i can be found from (5.5) by using the partial fraction decomposition theorem which gives:

(5.6)
$$\zeta^i = \frac{(x^{k+1} - x^i)^{\ell_{k+1}} \ldots (x^n - x^i)^{\ell_n} \widehat{m}(x^i)}{\prod_{s=1, s \neq i}^k (x^s - x^i)}, \qquad i = 1, \ldots, k.$$

Summarising this discussion we conclude that for $L = \mathrm{diag}(x^1, \ldots, x^n)$, we have (up to scaling with a factor $c(\lambda)$)

$$f(x) = \sum_{i=1}^k x^i - \sum_{i=k+1}^n \ell_i x^i$$

$$\sigma(\lambda, x) = \frac{\prod_{i=1}^k (x_i - \lambda)}{\prod_{i=k+1}^n (x_i - \lambda)^{\ell_i}}$$

and ζ given by (5.6).

To get the conclusion of Theorem 4 in its final form, we only need to combine the coordinates x^i's, $i = k + 1, \ldots, n$ into groups depending on the values of the exponents ℓ_i (x^i and x^j go to the same group iff $\ell_i = \ell_j$). After appropriate renumbering, we come to the desired description of L, σ, m and ζ.

6. Conclusion

In our paper, we constructed new explicit families of integrable multi-component evolutionary equations with and without differential constraints, see Section 2. The equations $u_{t_\lambda} = \xi(\lambda; u, u_x, u_{xx}, \ldots)$ within each family are parametrised by $\lambda \in \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The corresponding (formal) evolutionary flows pairwise commute for all values of parameters and admit a family of common (formal) conservation laws $\mathrm{v}(\mu; u, u_x, u_{xx}, \ldots)$ also parametrised by $\mu \in \bar{\mathbb{C}}$. For specific values of λ (namely, roots $\lambda_1, \ldots, \lambda_d$ of a certain polynomial) the above equation generates an hierarchy of usual (i.e. non-formal) commuting flows ξ_{s,λ_i}, $s = 0, 1, \ldots$, defined by means of a differential polynomial of degree $2s + 3$. The equation $u_{t_{\lambda_i}} = \xi(\lambda_i; u, u_x, u_{xx}, u_{xxx})$, initial term of this hierarchy, is a third order PDE system given by an elegant explicit formula. All the subsequent terms can be found by means of explicit recurrent formulas. Similar for conservation laws: $\mathrm{v}(\lambda_i)$ generates an hierarchy of common polynomial conservation laws v_{s,λ_i}, $s = 0, 1, \ldots$, for all the flows, where v_{s,λ_i} is a differential polynomial of degree $2s$ that can be found explicitly by an iterative procedure. These families of integrable equations, for a simple choice of parameters, include and generalise many known examples of integrable systems. Some of multi-component evolutionary equations we constructed are essentially new and they have no low-component analogues.

The construction is based on a new approach, which is rather differential-geometric than algebraic (in contrast to many other constructions of integrable systems which are often based on algebraic or algebraic-geometric concepts). Our results have been naturally obtained within the Nijenhuis Geometry programme initiated in [8]. This suggests that Nijenhuis Geometry might be a convenient framework for studying further properties of the constructed systems and generalizing them. Because of its differential-geometric nature, our constructions are invariant with respect to the choice of coordinates on M^n; that is, the systems behave covariantly if we change unknown functions u by a diffeomorphism $u_{\mathsf{new}} = u_{\mathsf{new}}(u_{\mathsf{old}})$. One can use this fact in the search of applications of our systems in natural sciences. Examples discussed in Section 3.3 actually suggest that 'physically relevant' variables correspond to those coordinates on M^n in which the Nijenhuis operator has a nice form, e.g., the 'companion' and 'diagonal' forms from (2.1), or the form in which the components of L are linear in coordinates.

The famous integrable systems that we generalise to an arbitrary number of components (such as KdV, Camassa-Holm, Dullin-Gottwald-Holm, Harry Dym, Kaup-Boussinesq) have been intensively studied for decades; for these studies, a number of non-trivial geometric, algebraic and analytical methods were invented and successfully applied. The next natural step would be to figure out how to adapt these methods to new systems. In particular, it would be interesting to construct Lax representations for new systems, to find explicit solutions by the inverse scattering method, to construct a recursion operator and, of course, to find

physically relevant models that are described by new systems. We invite our fellow mathematicians and physicists to join this research.

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